# On uniqueness of moving average representations of heavy-tailed stationary processes

Gouriéroux, Christian and Zakoian, Jean-Michel

University of Toronto, CREST

31 March 2014

# On uniqueness of moving average representations of heavy-tailed stationary processes

BY CHRISTIAN GOURIÉROUX

*CREST and University of Toronto. E-Mail: gourieroux@ensae.fr*

AND JEAN-MICHEL ZAKOÏAN

*EQUIPPE (University Lille 3) and CREST, 15 boulevard Gabriel Péri, 92245 Malakoff Cedex, France. E-mail: zakoian@ensae.fr*

SUMMARY

We prove the uniqueness of linear i.i.d. representations of heavy-tailed processes whose distribution belongs to the domain of attraction of an $\alpha$-stable law, with $\alpha < 2$. This shows the possibility to identify nonparametrically both the sequence of two-sided moving average coefficients and the distribution of the heavy-tailed i.i.d. process.

*Some key words*: $\alpha$-stable distribution; Domain of attraction; Infinite moving average; Linear process; Mixed causal/noncausal process; Nonparametric identification; Unobserved component model.

## 1. INTRODUCTION

By definition, a real linear process $(X_t)$ can be written as a two-sided moving average of a strong white noise:

$$X_t = \sum_{j=-\infty}^{\infty} a_j \epsilon_{t-j}, \quad t \in \mathbb{Z}, \tag{1}$$

where $(\epsilon_t)$ is a sequence of independent and identically distributed (i.i.d.) real random variables, and $(a_j)$ is a sequence of Moving-Average (MA) coefficients.

Provided that such a process exists, an important issue is the uniqueness of the MA representation in (1). Uniqueness of the linear representation of non-Gaussian i.i.d. processes plays an important role in time series, for instance in the analysis of time reversibility (see Hallin, Lefèvre and Puri (1988), Breidt and Davis (1992)). From a statistical point of view, this problem is equivalent to the nonparametric identification of the sequence $(a_i)$ and of the common distribution of the variables $\epsilon_t$. See Rosenblatt (2000) for a review of statistical applications of non-Gaussian linear processes.

While Gaussian linear processes generally admit several moving average representations, non-Gaussian linear processes have been shown to admit an essentially unique MA representation, under different regularity conditions on the MA coefficients and the moments of the i.i.d. process. The literature dealing with uniqueness of MA representations of non-Gaussian processes includes Lii and Rosenblatt (1982), Findley (1986, 1990), Cheng (1992), Breidt and Davis (1992). The sufficient uniqueness conditions obtained in these articles are summarized in Table 1. They require at least finite variance of the independent process $(\epsilon_t)$, except in Breidt and Davis (1992) (see their Remark 2, p. 286). In the present article, we provide a direct proof for the identification of the MA representation when the errors belong to the domain of attraction of a stable law, with index $\alpha < 2$. This case is especially important in view of the increased interest, in the recent time series literature, in heavy-tailed distributions. In particular, it has been noted that mixed causal/noncausal processes of the form (1) were adequate for modeling the speculative bubbles observed on fat-tailed financial series, for instance when the noise and the observed process do not have a finite expectation (see Gouriéroux and Zakoïan (2013)). Finally, we extend the identification result to the multivariate framework, especially to unobserved component models.

## 2. Uniqueness results for univariate processes

We start by recalling conditions ensuring the existence of a MA representation of a strictly stationary process.

### 2·1. *Existence*

The existence of infinite MA processes, as in (1), is generally established under the assumption that the variables $\epsilon_t$ belong to $L^1$ or $L^2$. An extension was established by Cline (Theorem 2.1, 1983), who gave conditions for the existence of infinite MA with i.i.d. random variables admitting regularly varying tails. The following is a straightforward consequence of Proposition 13.3.1 in Brockwell and Davis (1991).

PROPOSITION 1. *Consider the two-sided moving average*

$$X_t = \sum_{j=-\infty}^{\infty} \alpha_h Z_{t-h}, \tag{2}$$

*written on a process $(Z_t)$ such that, for some $s \in (0,1)$:*

$$\sup_t E|Z_t|^s < \infty \quad \textit{and} \quad \sum_{h=-\infty}^{\infty} |\alpha_h|^s < \infty.$$

*Then, the series $X_t$ converges absolutely with probability one. If, in addition, $(Z_t)$ is a strictly stationary process, then $(X_t)$ is also strictly stationary.*

We now turn to the uniqueness problem, first in the case where $X_t = \epsilon_t^*$ is an i.i.d. process.

### 2·2. *Uniqueness of the MA representation of an heavy-tailed i.i.d. process*

Let $(\epsilon_t)$ and $(\epsilon_t^*)$ denote two i.i.d. processes. Suppose that

$$\epsilon_t^* = \sum_{j=-\infty}^{\infty} a_j \epsilon_{t-j}, \quad t \in \mathbb{Z}, \tag{3}$$

and, for some $s \in (0,1)$,

$$E|\epsilon_t|^s < \infty \quad \text{and} \quad \sum_{j=-\infty}^{\infty} |a_j|^s < \infty. \tag{4}$$

By Proposition 1, the latter conditions entail the existence of the infinite MA appearing in the right-hand side of (3). Note also that this entails $E|\epsilon_t^*|^s < \infty$, using the inequality $(x+y)^s \leq x^s + y^s$ for $x, y \geq 0$. Let $\Psi$, $\Psi^*$ denote the characteristic functions of $\epsilon$ and $\epsilon^*$, respectively.

The characteristic function of $(\epsilon_t^*, \epsilon_{t-h}^*)$ with $h \neq 0$ is, for $u, v \in \mathbb{R}$,

$$E\left(e^{i(u\epsilon_t^* + v\epsilon_{t-h}^*)}\right) = \Psi^*(u)\Psi^*(v) = E\left(\prod_{j=-\infty}^{\infty} e^{i(ua_j + va_{j+h})\epsilon_{t-j}}\right) = \prod_{j=-\infty}^{\infty} \Psi(ua_j + va_{j+h}).$$

The last equality holds by the dominated convergence theorem, which applies because the products $\prod_{j=-n}^{n} e^{i(ua_j+va_{j+h})\epsilon_{t-j}}$ have unit norm.

Hence, for $u, v \in \mathbb{R}$,

$$\log|\Psi^*(u)| + \log|\Psi^*(v)| = \sum_{j=-\infty}^{\infty} \log|\Psi(ua_j + va_{j+h})|. \tag{5}$$

For expository purpose, we first consider stable variables.

**a) Stable variables**

Let us recall that the distribution of a stable variable $X$ with parameters $(\alpha, \beta, \sigma, \mu)$, denoted $X \sim \mathcal{S}(\alpha, \beta, \sigma, \mu)$, has the characteristic function

$$\Psi(s) = \begin{cases} \exp[-\sigma^\alpha |s|^\alpha \left\{1 - i\beta \,(\text{sign } s) \tan\left(\frac{\pi\alpha}{2}\right)\right\} + i\mu s], & \text{if } \alpha \neq 1, \\ \exp[-\sigma|s| \left\{1 + i\beta \,(\text{sign } s)\, \frac{2}{\pi} \log|s|\right\} + i\mu s], & \text{if } \alpha = 1. \end{cases}$$

For a stable distribution with index of stability $\alpha \in (0, 2)$, the moment condition in (4) is satisfied for $s \in (0, \min\{\alpha, 1\})$. See Samorodnitsky and Taqqu (1994) for further details on stable variables.

PROPOSITION 2. *If $(\epsilon_t)$ and $(\epsilon_t^*)$ have stable laws, $\epsilon_t \sim \mathcal{S}(\alpha, \beta, \sigma, \mu)$ and $\epsilon_t^* \sim \mathcal{S}(\alpha_*, \beta_*, \sigma_*, \mu_*)$ with $\alpha, \alpha_* < 2$, if $\sum_{j=-\infty}^{\infty} |a_j|^s < \infty$ for $s \in (0, \min\{\alpha, 1\})$, then all the MA coefficients $a_j$'s except one are equal to zero in the MA representation (3).*

**b) Variables in the domain of attraction of a stable law**

By definition, a variable $X$ belongs to the domain of attraction of an $\alpha-$stable law, denoted $X \in \text{DA}(\alpha)$, with $\alpha < 2$, if for any i.i.d. sequence $(X_i)$, with $X_1 \overset{d}{=} X$, there exist sequences of constants $A_n \in \mathbb{R}$ and $B_n > 0$ such that

$$\frac{X_1 + \cdots + X_n - A_n}{B_n} \overset{d}{\to} Z, \tag{6}$$

where $\overset{d}{\to}$ denotes the convergence in distribution, and the distribution of $Z$ is $\alpha$-stable. The tails of variables belonging to DA$(\alpha)$ have the following characterization: $X \in \text{DA}(\alpha)$ if and only if its cumulative distribution function is such that:

$$F(-x) \sim \frac{c_1 L(x)}{x^\alpha}, \qquad 1 - F(x) \sim \frac{c_2 L(x)}{x^\alpha}, \quad \text{when } x \to \infty,$$

where $L$ is a slowly varying function[1] and $c_1, c_2 \geq 0$, $c_1 + c_2 \neq 0$ (see for instance Embrechts, Klüppelberg, Mikosch (1997), Theorem 2.2.8). Hence, if $X \in \text{DA}(\alpha)$, $E|X|^s < \infty$ for $s \in (0, \alpha)$. Note that we can have $E|X|^\alpha < \infty$ which is not possible for $\alpha-$stable variables.

Let us start with a simple extension of Proposition 2.

[1] that is, $L$ is measurable positive, defined on $(0, \infty)$, such that $\lim_{x\to\infty} L(tx)/L(x) = 1$ for all $t > 0$.

PROPOSITION 3. *If $\epsilon_t^* \in DA(\alpha_*)$ and $\epsilon_t$ has a stable distribution with index $\alpha$, with $\alpha, \alpha_* < 2$, if $\sum_{j=-\infty}^{\infty} |a_j|^s < \infty$ for $s < \alpha$, then all the MA coefficients $a_j$'s except one are equal to zero in (3).*

Let us now extend the identification result, when both i.i.d. processes $(\epsilon_t)$ and $(\epsilon_t^*)$ are in the attraction domain of a stable law.

PROPOSITION 4. *If $\epsilon_t \in DA(\alpha)$ and $\epsilon_t^* \in DA(\alpha_*)$ are such that*

$$|E\left(e^{iu\epsilon_t}\right)| = e^{-|u|^{\alpha} L_1(1/u)}, \quad |E\left(e^{iu\epsilon_t^*}\right)| = e^{-|u|^{\alpha_*} L_1^*(1/u)}, \tag{7}$$

*with $\alpha, \alpha_* < 2$, where $L_1, L_1^*$ are slowly varying functions, if for any $\tau > 0$ there exist constants $M, K > 0$ such that for all $t > 0$,*

$$\sup_{|x|>M} \left| \frac{L_1(tx)}{L_1(x)} \right| \leq K(1 + t^{\tau}), \tag{8}$$

*and if $\sum_{j=-\infty}^{\infty} |a_j|^s < \infty$ for $s < \alpha$, then all the MA coefficients $a_j$'s except one are equal to zero in (3).*

### 2·3. *Uniqueness of the MA representation of an heavy-tailed linear process*

We now study the uniqueness of representation (1). Assume that

$$X_t = \sum_{j=-\infty}^{\infty} a_j \epsilon_{t-j} = \sum_{j=-\infty}^{\infty} a_j^* \epsilon_{t-j}^*, \quad t \in \mathbb{Z} \tag{9}$$

where $(\epsilon_t)$ and $(\epsilon_t^*)$ are two sequences of i.i.d. real random variables, and $(a_j)$, $(a_j^*)$ are two sequences of MA coefficients with, for some $s \in (0, 1)$:

$$\sum_{h=-\infty}^{\infty} |a_h|^s < \infty \quad \text{and} \quad \sum_{h=-\infty}^{\infty} |a_h^*|^s < \infty. \tag{10}$$

The random series in (9) converging a.s., we may define $a(B)\epsilon_t := \sum_{j=-\infty}^{\infty} a_j \epsilon_{t-j}$ and $a^*(B)\epsilon_t^* := \sum_{j=-\infty}^{\infty} a_j^* \epsilon_{t-j}^*$ where $B$ stands for the backward shift operator. The next proposition shows that under appropriate assumptions on the i.i.d. sequence and the MA coefficients, the representation (1) of $X_t$ is essentially unique.

PROPOSITION 5. *Assume that $\epsilon_t \in DA(\alpha)$ and $\epsilon_t^* \in DA(\alpha_*)$, satisfying conditions (7)-(8), that $a^*(B)$ is invertible, with $\{a^*(B)\}^{-1} = \sum_{j=-\infty}^{\infty} c_j B^j$ such that $\sum_{j=-\infty}^{\infty} |c_j|^s < \infty$, and that $\sum_{j=-\infty}^{\infty} |a_j|^s < \infty$, for $s < \alpha$. Then, if (9)-(10) hold, we have*

$$\epsilon_t^* = c\epsilon_{t-\ell} \quad \textit{and} \quad a_j = ca_{j+\ell}^*, \quad \forall j \in \mathbb{Z}$$

*for some constants $c \in \mathbb{R}$ and $\ell \in \mathbb{Z}$.*

From Proposition 5, we see that the MA representation is identifiable up to a change of scale and a drift of the time index on $\epsilon_t$.

## 3. UNIQUENESS RESULTS FOR MULTIVARIATE HEAVY-TAILED LINEAR PROCESSES

The literature on the identification of multivariate MA processes is rather limited. Sets of conditions for identification have been derived in Chan, Ho and Tong (2006), when the $p$-dimensional observed process and the $q$-dimensional noise processes are square integrable, when

$p \geq q$ (i.e. an order condition), and when the $q$ components of the noise are i.i.d. (see their Condition 3). The order condition is required in the proof to be able to apply the approach by Cheng (1992, 1999).

This section will complete such results by considering processes with fat tails. We start by an extension of Proposition 3.

### 3·1. *Uniqueness of the decomposition of an heavy-tailed i.i.d. process as a sum of independent stable MA*

The next result extends Proposition 3 by considering the decomposition of a non-Gaussian stable one-dimensional noise as the sum of independent stable MA processes:

$$\epsilon_t^* = \sum_{k=1}^{K} \sum_{j=-\infty}^{\infty} a_{k,j} \epsilon_{k,t-j}, \quad t \in \mathbb{Z} \tag{11}$$

where $(\epsilon_{k,t})$ are $K$ independent sequences of stable i.i.d. real random variables, and the $\{a_{k,j}; j \in \mathbb{Z}\}$ are $K$ sequences of real numbers such that, for some $s \in (0,1)$,

$$0 < \sum_{j=-\infty}^{\infty} |a_{k,j}|^s < \infty.$$

PROPOSITION 6. *If $\epsilon_t^* \in DA(\alpha_*)$ with $\alpha_* < 2$, the variables $\epsilon_{k,t}$ have stable laws, $\epsilon_{k,t} \sim \mathcal{S}(\alpha_k, \beta_k, \sigma_k, \mu_k)$ with $\alpha_k \in (s,2)$ for $k = 1, \ldots, K$, then in each sequence $\{a_{k,j}; j \in \mathbb{Z}\}$ all the $a_{k,j}$'s except one are equal to zero in (11).*

### 3·2. *Uniqueness of the MA representation of a vector process with fat tails*

We now show that, under appropriate conditions, a vector process $(\boldsymbol{X}_t)$ has at most one representation of the form

$$\boldsymbol{X}_t = \boldsymbol{A}(B)\boldsymbol{\epsilon}_t, \tag{12}$$

where is $(\boldsymbol{\epsilon}_t)$ is a vector i.i.d. process, with independent components following stable distributions with distinct tail indices. More precisely, consider a $p$-dimensional process $\boldsymbol{X}_t = (X_{1t}, \ldots, X_{pt})'$ admitting two MA($\infty$) representations given by

$$\boldsymbol{X}_t = \boldsymbol{A}(B)\boldsymbol{\epsilon}_t = \boldsymbol{A}^*(B)\boldsymbol{\epsilon}_t^*, \quad t \in \mathbb{Z}, \tag{13}$$

where $(\boldsymbol{\epsilon}_t), (\boldsymbol{\epsilon}_t^*)$ are two $K$-dimensional i.i.d. processes with stable and independent components, and $\boldsymbol{A}(B) = (a_{i,k}(B))_{i=1,\ldots,p;k=1,\ldots,K}$, $\boldsymbol{A}^*(B) = (a_{i,k}^*(B))_{i=1,\ldots,p;k=1,\ldots,K}$ are infinite-order lag polynomials with matrix coefficients. Let $a_{i,k}(B) = \sum_{j=-\infty}^{\infty} a_{i,k,j} B^j$ and $a_{i,k}^*(B) = \sum_{j=-\infty}^{\infty} a_{i,k,j}^* B^j$. Assume that, for some $s \in (0,1)$,

$$0 < \sum_{j=-\infty}^{\infty} |a_{i,k,j}|^s < \infty, \qquad 0 < \sum_{j=-\infty}^{\infty} |a_{i,k,j}^*|^s < \infty.$$

PROPOSITION 7. *Assume that $(\boldsymbol{\epsilon}_t)$ and $(\boldsymbol{\epsilon}_t^*)$ are two $K$-dimensional i.i.d. processes satisfying (13), with stable and independent components $\epsilon_{k,t}$ and $\epsilon_{k,t}^*$ having stable laws, $\epsilon_{k,t} \sim \mathcal{S}(\alpha_k, \beta_k, \sigma_k, \mu_k)$ and $\epsilon_{k,t}^* \sim \mathcal{S}(\alpha_k^*, \beta_k^*, \sigma_k^*, \mu_k^*)$, with $\alpha_k, \alpha_k^* \in (s,2)$ for $k = 1, \ldots, K$ and $\alpha_1 < \ldots < \alpha_K$. Suppose that, for any $i, k$, the lag polynomial $a_{ik}^*(B)$ is invertible, with $\{a_{i,k}^*(B)\}^{-1} = \sum_{j=-\infty}^{\infty} c_{i,k,j} B^j$ such that $\sum_{j=-\infty}^{\infty} |c_{i,k,j}|^s < \infty$. Then, for $i = 1, \ldots, p$, and*

$k = 1, \ldots, K$

$$a_{i,k,j} = c_{i,k} a^*_{i,k,j+\ell_{i,k}}, \quad \forall j \in \mathbb{Z},$$

*for some constants $c_{i,k} \in \mathbb{R}$ and $\ell_{i,k} \in \mathbb{Z}$.*

## 4. CONCLUDING REMARKS

The identification results of this paper can be used in both parametric and nonparametric analyses of linear processes. The proofs also suggest a nonparametric estimation method, for a linear process $X_t$ based on an i.i.d. sequence of heavy-tailed variables $\epsilon_t$, as in (1), by the following steps.

**i) Estimation of $\alpha$.** Since $X_t \in \mathrm{DA}(\alpha)$ when $\epsilon_t \in \mathrm{DA}(\alpha)$ (see for instance Embrechts, Klüppelberg, Mikosch (1997), Theorem A3.26), the tail index $\alpha$ can be consistently estimated by a standard approach. For instance, the Hill estimator can be used (for its main properties under various assumptions see Embrechts, Klüppelberg, Mikosch (1997) Theorem 6.4.6).

**ii) Estimation of the MA coefficients.** The analysis of the joint characteristic function of $(X_t, X_{t-h})$ for $h \neq 0$ and $v = tu$ in a neighborhood of $u = 0$, provides estimation of $\sum_{j=-\infty}^{\infty} |a_j + t a_{j+h}|^\alpha$ when $\epsilon_t \sim \mathcal{S}(\alpha, \beta, \sigma, \mu)$. Estimates of (a finite number of) MA coefficients can be deduced by truncating the infinite sum and solving the moment restrictions.

**iii) Estimation of the distribution of $\epsilon_t$.** Having estimated the lag polynomial $a(B)$, approximations of the variables $\epsilon_t$ can be deduced by computing the residuals $\hat{\epsilon}_t = \hat{a}(B)^{-1} X_t$, where the $X_u$ for $u < 0$ are replaced by 0. Then, the common distribution of the $\epsilon_t$'s will be estimated from the residuals.

## APPENDIX 1

*Uniqueness of non-Gaussian MA representations in the literature*

Table 1 gives a summary of the main conditions obtained in the literature for the uniqueness of the MA representation in (1). Apart from such conditions, it is generally also assumed that the spectral density of $X_t$ is positive almost everywhere.

Note that the condition given by Breidt and Davis (Remark 2, p.386, 1992), as an alternative proof to their Proposition 3.1, is based on the following result:

THEOREM 1 (KAGAN ET AL. (1973, P.94)). *Let $\epsilon_j$ be a sequence of independent random variables, and $\{a_j\}$, $\{b_j\}$ be two sequences of real constants such that*

i) *The sequences $\{a_j/b_j : a_j b_j \neq 0\}$ and $\{b_j/a_j : a_j b_j \neq 0\}$ are both bounded;*
*$\sum a_j \epsilon_j$ and $\sum b_j \epsilon_j$ converge a.s. to random variables $U$ and $V$ respectively;*
ii) *$U$ and $V$ are independent.*

*Then, for every $j$ such that $a_j b_j \neq 0$, $\epsilon_j$ is normally distributed.*

In the context of MA processes, this result can be used as follows. Suppose that (3)-(4) hold, where $(\epsilon_t)$ and $(\epsilon_t^*)$ are two i.i.d. processes. Then the variables

$$\epsilon_t^* = \sum_{j=-\infty}^{\infty} a_j \epsilon_{t-j} \quad \text{and} \quad \epsilon_{t-h}^* = \sum_{j=-\infty}^{\infty} a_{j+h} \epsilon_{t-j}$$

are independent. It follows that if the sequences

$$\{a_j/a_{j+h} : a_j a_{j+h} \neq 0\} \quad \text{and} \quad \{a_{j+h}/a_j : a_j a_{j+h} \neq 0\} \quad \text{are both bounded,} \tag{A1}$$

Table 1. *Uniqueness conditions for the non-Gaussian MA representation (1)*

| Papers | Condition on $\epsilon_t$ | Condition on MA coefficients |
|---|---|---|
| Lii and Rosenblatt (1982) | All moments finite | $\sum j\lvert a_j\rvert < \infty$ |
| Findley (1986) | All moments finite | $\sum a_j^2 < \infty$ |
| Findley (1990) | Finite non-zero $r$th cumulant | $\sum a_j^2 < \infty$ |
| Cheng (1992) | Finite variance | $\sum a_j^2 < \infty$ |
| Breidt and Davis (Prop. 3.1, 1992) | Finite variance | Fractionally Integrated ARMA |
| Breidt and Davis (Rem. (2) p. 386, 1992) | $E \log^+ \lvert\epsilon_t\rvert < \infty$ | Fractionally Integrated ARMA |

then either the variables $\epsilon_t$ and $\epsilon_t^*$ are normally distributed, or all the $a_j$'s except one are equal to zero. The conclusion thus coincides with that of our Proposition 4, but is obtained under (a) less restrictive distributional assumptions on the i.i.d. processes, but (b) more restrictive assumptions on the sequence of coefficients $a_j$. Breidt and Davis (Remark 2, p. 386, 1992) showed that (A1) is satisfied for a MA polynomial of the form $\mathcal{A}(z) = \Psi(z)/\Phi(z) = \sum_{j=0}^{\infty} a_j z^j$, where $\Psi(z)$ and $\Phi(z)$ are finite-order polynomials under standard assumptions. For more general processes, the boundedness assumption can be very restrictive. For instance, it precludes sequences recursively defined by $a_j = 0$ for $j \leq 0$ and $a_j = \lambda_j a_{j-1}$ for $j \geq 1$, where $(\lambda_j)$ is a sequence converging to zero or to infinity.

## Appendix 2

### *Proof of Proposition* 2

In view of (5) with $v = 0$, we have, for all $u \in \mathbb{R}$,

$$-\sigma_*^{\alpha_*}|u|^{\alpha_*} = -\sum_{j=-\infty}^{\infty} \sigma^\alpha |a_j|^\alpha |u|^\alpha.$$

Thus $\alpha = \alpha_*$ and, without loss of generality, we can take $\sigma = \sigma_*$ and $\sum_{j=-\infty}^{\infty} |a_j|^\alpha = 1$. We then have, by taking $v = tu$ in (5),

$$\sum_{j=-\infty}^{\infty} |a_j + t a_{j+h}|^\alpha = 1 + |t|^\alpha, \qquad \forall t \in \mathbb{R}. \tag{A2}$$

The right-hand side of this equality is a differentiable function of $t$ except at $t = 0$. Let us study the differentiability of the left-hand side. Suppose that there exists $j_0$ such that $a_{j_0} \neq 0$ and $a_{j_0+h} \neq 0$. The function $t \mapsto |a_{j_0} + t a_{j_0+h}|^\alpha$ is everywhere differentiable except at $t_0 = -a_{j_0}/a_{j_0+h}$, with $t_0 \neq 0$. The contribution of all terms of the infinite sum displaying non differentiability at $t_0$ is

$$\sum_{j \in J} |a_j + t a_{j+h}|^\alpha = |t - t_0|^\alpha \sum_{j \in J} |a_{j+h}|^\alpha,$$

where $J = \{j \in \mathbb{Z} \mid a_j + t_0 a_{j+h} = 0\}$. This contribution must vanish; otherwise, we would have in the right-hand side of (A2) a function which is differentiable at $t_0$ while the left-hand side is not. Hence $\sum_{j \in J} |a_{j+h}|^\alpha = 0$, in contradiction with the fact that this sum is bounded below by $|a_{j_0+h}|^\alpha \neq 0$. Therefore, there exists no integer $j_0$ such that $a_{j_0} \neq 0$ and $a_{j_0+h} \neq 0$, which establishes the property.

□

### *Proof of Proposition* 3

By Ibragimov and Linnik (Theorem 2.6.5, 1971), for a variable $\epsilon^* \in \mathrm{DA}(\alpha_*)$, where the limiting stable distribution has location parameter $\mu = 0$, we have

$$|E\left(e^{iu\epsilon^*}\right)| = e^{-|u|^{\alpha_*} L_1^*(1/u)}, \tag{A3}$$

in the neighborhood of the origin, for some slowly varying function $L_1^*$. This equality also holds if $\mu \neq 0$, the limit law in (6) being uniquely determined up to positive affine transformations. By (5) with $v = 0$, we thus have

$$-|u|^{\alpha_*} L_1^*\left(\frac{1}{|u|}\right) = -\sum_{j=-\infty}^{\infty} \sigma^\alpha |a_j|^\alpha |u|^\alpha, \tag{A4}$$

in the neighborhood of 0. Thus $\alpha = \alpha_*$ and $L_1^*$ is constant in the neighborhood of 0. It is not restrictive to assume $\sigma^\alpha = L_1^*(1/|u|)$ in the neighborhood of 0 and $\sum_{j=-\infty}^{\infty} |a_j|^\alpha = 1$. We thus are lead to the proof of Proposition 2.

□

*Proof of Proposition* 4

By (5) with $v = 0$ and (A3), we have in the neighborhood of 0,

$$-|u|^{\alpha_*} L_1^*\left(\frac{1}{|u|}\right) = -\sum_{\substack{j=-\infty \\ a_j \neq 0}}^{\infty} |a_j|^\alpha |u|^\alpha L_1\left(\frac{1}{|a_j u|}\right). \tag{A5}$$

i) Let us first show that $\alpha_* = \alpha$. If $\alpha_* < \alpha$, the left-hand side term in

$$|u|^{\alpha_* - \alpha} L_1^*\left(\frac{1}{|u|}\right) = -\sum_{\substack{j=-\infty \\ a_j \neq 0}}^{\infty} |a_j|^\alpha L_1\left(\frac{1}{|a_j u|}\right),$$

tends to 0 when $u \to 0$, while the right-hand side term does not. We also get a contradiction if $\alpha_* < \alpha$, hence $\alpha = \alpha_*$.

ii) Now we will show that

$$L_1^*\left(\frac{1}{|u|}\right) \sim \left(\sum_{j=-\infty}^{\infty} |a_j|^\alpha\right) L_1\left(\frac{1}{|u|}\right), \quad \text{when } u \to 0. \tag{A6}$$

We will use the next property (see for instance Embrechts, Klüppelberg, Mikosch (1997), Theorem A3.2).

LEMMA 1. *If $L$ is a slowly varying function, for any $a, b > 0$, the convergence $L(tx)/L(x) \to 1$ when $x \to \infty$ is uniform on the segment $[a, b]$.*

Using Lemma 1 and (A5) with $\alpha = \alpha^*$ we have, in the neighborhood of 0, for some constants $a, b > 0$ to be chosen later,

$$\begin{aligned}
&\left| \frac{L_1^*\left(\frac{1}{|u|}\right)}{L_1\left(\frac{1}{|u|}\right)} - \sum_{j=-\infty}^{\infty} |a_j|^\alpha \right| \\
&\leq \sum_{\substack{j=-\infty \\ a_j \neq 0}}^{\infty} |a_j|^\alpha \left| \frac{L_1\left(\frac{1}{|a_j u|}\right)}{L_1\left(\frac{1}{|u|}\right)} - 1 \right| \\
&\leq \left(\sum_{j=-n}^{n} |a_j|^\alpha\right) \sup_{t \in [a,b]} \left| \frac{L_1\left(\frac{t}{|u|}\right)}{L_1\left(\frac{1}{|u|}\right)} - 1 \right| + \sum_{\substack{|j|>n \\ a_j \neq 0}} |a_j|^\alpha \left| \frac{L_1\left(\frac{1}{|a_j u|}\right)}{L_1\left(\frac{1}{|u|}\right)} \right| + \sum_{|j|>n} |a_j|^\alpha \\
&:= S_{1,n}(u) + S_{2,n}(u) + S_{3,n}.
\end{aligned}$$

We have, in the neighborhood of 0, for $\tau = \alpha - s$,

$$S_{2,n}(u) = \sum_{\substack{|j|>n \\ a_j \neq 0}} |a_j|^s |a_j|^\tau \left| \frac{L_1\left(\frac{1}{|a_j u|}\right)}{L_1\left(\frac{1}{|u|}\right)} \right| \leq \sum_{\substack{|j|>n \\ a_j \neq 0}} |a_j|^s |a_j|^\tau K \left(1 + |a_j|^{-\tau}\right),$$

which is smaller than an arbitrarily small $\varsigma > 0$ for $n$ sufficiently large, $n > N$ say. The same property holds for $S_{3,n}$. Now let $[a, b] = \left[\min_{|j|\leq N}\{|a_j|^{-1}, a_j \neq 0\}, \max_{|j|\leq N}\{|a_j|^{-1}, a_j \neq 0\}\right]$ in $S_{1,n}(u)$. We have

$$S_{1,n}(u) \leq \left(\sum_{j=-\infty}^{\infty} |a_j|^\alpha\right) \sup_{t\in[a,b]} \left| \frac{L_1\left(\frac{t}{|u|}\right)}{L_1\left(\frac{1}{|u|}\right)} - 1 \right|,$$

which tends to zero using the uniform convergence of $L_1$.

Thus we have established (A6) and, without generality loss, we can make a scale change on the $a_j$'s to get, for $u \neq 0$,

$$L_1^*\left(\frac{1}{|u|}\right) = \left(\sum_{j=-\infty}^{\infty} |a_j|^\alpha\right) L_1\left(\frac{1}{|u|}\right). \tag{A7}$$

iii) Using (A7), Equation (5) can now be written, for $u, v$ in the neighborhood of 0, $uv \neq 0$, as

$$\begin{aligned} |u|^\alpha L_1^*\left(\frac{1}{|u|}\right) + |v|^\alpha L_1^*\left(\frac{1}{|v|}\right) &= \sum_{\substack{j=-\infty \\ a_j u + a_{j+h} v \neq 0}}^{\infty} |a_j u + a_{j+h} v|^\alpha L_1\left(\frac{1}{|a_j u + a_{j+h} v|}\right) \\ &= \sum_{j=-\infty}^{\infty} |a_j u|^\alpha L_1\left(\frac{1}{|u|}\right) + |a_{j+h} v|^\alpha L_1\left(\frac{1}{|v|}\right). \end{aligned}$$

Hence, for $v = tu$, $t > 0$, we get, for $u$ in the neighborhood of 0,

$$\begin{aligned} &\sum_{\substack{j=-\infty \\ a_j + t a_{j+h} \neq 0}}^{\infty} |a_j + t a_{j+h}|^\alpha L_1\left(\frac{1}{|a_j + t a_{j+h}||u|}\right) \\ &= \sum_{j=-\infty}^{\infty} |a_j|^\alpha L_1\left(\frac{1}{|u|}\right) + |t a_{j+h}|^\alpha L_1\left(\frac{1}{t|u|}\right). \end{aligned}$$

By the same argument as before, using the uniform convergence of the slowly varying function $L_1$ (see Lemma 1), the asymptotic behavior of these sums in the neighborhood of $u = 0$ is the same if we replace terms of the form $L_1\left(\frac{1}{x|u|}\right)$, with $x > 0$, by $L_1\left(\frac{1}{|u|}\right)$. We thus have, for any $t > 0$,

$$\sum_{j=-\infty}^{\infty} |a_j + t a_{j+h}|^\alpha = \sum_{j=-\infty}^{\infty} \left(|a_j|^\alpha + |t a_{j+h}|^\alpha\right),$$

and we are led to the discussion of Proposition 2.

□

*Proof of Proposition* 5

We use Theorem 2.1 in Kokoszka (1996), which establishes sufficient conditions for two infinite-order lag polynomials whose coefficients may not be absolutely summable to commute. From the assumptions made on the coefficients of the polynomials $a(B)$ and $\{a^*(B)\}^{-1}$, we have $\epsilon_t^* = \{a^*(B)\}^{-1} a(B)\epsilon_t$, where the random series $\{a^*(B)\}^{-1} a(B)\epsilon_t$ converges absolutely a.s. By Proposition 4, it follows that

$\{a^*(B)\}^{-1}a(B) = cB^\ell$ for some constant $c$ and some integer $\ell \in \mathbb{Z}$. Thus $a(B) = ca^*(B)B^\ell$ and the proof is complete.

□

*Proof of Proposition* 6

Similarly to (5), we have for $u, v \in \mathbb{R}$,

$$\log|\Psi^*(u)| + \log|\Psi^*(v)| = \sum_{k=1}^{K}\sum_{j=-\infty}^{\infty} \log|\Psi(ua_{k,j} + va_{k,j+h})|, \tag{A8}$$

by the independence between $\epsilon_{k,t}$ and $\epsilon_{\ell,t'}$ for any $(k,t) \neq (\ell, t')$. By the arguments leading to (A4) we thus have

$$-|u|^{\alpha_*}L_1^*\left(\frac{1}{|u|}\right) = -\sum_{k=1}^{K}\sum_{j=-\infty}^{\infty} \sigma_k^{\alpha_k}|a_{k,j}|^{\alpha_k}|u|^{\alpha_k}, \tag{A9}$$

in the neighborhood of 0. Thus $\alpha_k = \alpha_* =: \alpha$ for all $k$, and $L_1^*$ is constant in the neighborhood of 0,

$$L_1^*\left(\frac{1}{|u|}\right) = \sum_{k=1}^{K}\sum_{j=-\infty}^{\infty} \sigma_k^{\alpha}|a_{k,j}|^{\alpha}.$$

Now by taking $v = tu$ in the neighborhood of $u = 0$, we find that, similarly to (A2),

$$\sum_{k=1}^{K}\sum_{j=-\infty}^{\infty} \sigma_k^{\alpha}|a_{k,j} + ta_{k,j+h}|^{\alpha} = (1 + |t|^{\alpha})\sum_{k=1}^{K}\sum_{j=-\infty}^{\infty} \sigma_k^{\alpha}|a_{k,j}|^{\alpha}, \qquad \forall t \in \mathbb{R}. \tag{A10}$$

Now we can use the following elementary lemma.

LEMMA 2. *For $\alpha > 0$, let $f_\gamma : t \in \mathbb{R} \mapsto f_\gamma(t) = |1 + \gamma t|^\alpha$ and let $f_* : t \in \mathbb{R} \mapsto f_*(t) = |t|^\alpha$. Let $\Gamma = \{\gamma_i\}_{i\in\mathcal{I}}$ a family of distinct real numbers, with $\mathcal{I} \subset \mathbb{Z}$. The family of functions $\{f_*, f_\gamma; \gamma \in \Gamma\}$ is linearly independent.*

It follows from (A10) and Lemma 2 that for any $(k, j)$ either $a_{k,j}$ or $a_{k,j+h}$ is equal to zero. Thus, for any $k$, the set $\{a_{k,j}, j \in \mathbb{Z}\}$ reduces to a singleton. The conclusion follows.

□

*Proof of Proposition* 7

By the independence between $\epsilon_{k,t}^*$ and $\epsilon_{\ell,t'}^*$ in the one hand, and $\epsilon_{k,t}$ and $\epsilon_{\ell,t'}$ in the other hand, for any $(k,t) \neq (\ell, t')$, we get from the $i$-th equation of (13)

$$\sum_{k=1}^{K}\sum_{j=-\infty}^{\infty} \log|\Psi^*(ua_{i,k,j}^* + va_{i,k,j+h}^*)|, = \sum_{k=1}^{K}\sum_{j=-\infty}^{\infty} \log|\Psi(ua_{i,k,j} + va_{i,k,j+h})|.$$

Thus, by arguments already used,

$$\sum_{k=1}^{K}\sum_{j=-\infty}^{\infty} (\sigma_k^*)^{\alpha_k^*}|a_{i,k,j}^*|^{\alpha_k^*}|u|^{\alpha_k^*} = \sum_{k=1}^{K}\sum_{j=-\infty}^{\infty} \sigma_k^{\alpha_k}|a_{i,k,j}|^{\alpha_k}|u|^{\alpha_k},$$

in the neighborhood of 0. It follows that $\{\alpha_k; k = 1, \ldots, K\} = \{\alpha_k^*; k = 1, \ldots, K\}$ and thus $\alpha_1 = \alpha_1^* < \ldots < \alpha_K = \alpha_K^*$ and $\sum_{j=-\infty}^{\infty}(\sigma_k^*)^{\alpha_k}|a_{i,k,j}^*|^{\alpha_k^*}|u|^{\alpha_k^*} = \sum_{j=-\infty}^{\infty}\sigma_k^{\alpha_k}|a_{i,k,j}|^{\alpha_k}|u|^{\alpha_k}$ for $k = 1, \ldots, K$. It is thus not restrictive to assume $\sigma_k = \sigma_k^*$ and $\sum_{j=-\infty}^{\infty}|a_{i,k,j}^*|^{\alpha_k^*}|u|^{\alpha_k^*} = \sum_{j=-\infty}^{\infty}|a_{i,k,j}|^{\alpha_k}|u|^{\alpha_k}$. Now, similarly to (A10),

$$\sum_{k=1}^{K}\sum_{j=-\infty}^{\infty}(\sigma_k^*)^{\alpha_k}|a_{i,k,j}^*+ta_{i,k,j+h}^*|^{\alpha_k}=\sum_{k=1}^{K}\sum_{j=-\infty}^{\infty}\sigma_k^{\alpha_k}|a_{i,k,j}+ta_{i,k,j+h}|^{\alpha_k},\quad \forall t\in\mathbb{R},$$

which, using the fact that the powers $\alpha_k$ are different, entails, for $k=1,\dots,K$,

$$\sum_{j=-\infty}^{\infty}|a_{i,k,j}^*+ta_{i,k,j+h}^*|^{\alpha_k}=\sum_{j=-\infty}^{\infty}|a_{i,k,j}+ta_{i,k,j+h}|^{\alpha_k},\quad \forall t\in\mathbb{R}. \tag{A11}$$

Let $(\eta_t)$ be an i.i.d. sequence of symmetric $\alpha_k$- stable distributed random variables, $\eta_t\sim\mathcal{S}(\alpha_k,0,1,0)$. The left hand side of (A11) characterizes the distribution of the process $(a_{i,k}^*(B)\eta_t)$. It follows that $a_{i,k}^*(B)\eta_t\stackrel{d}{=}\frac{1}{\sigma_k}a_{i,k}(B)\epsilon_{k,t}$. Therefore, we have $\eta_t\stackrel{d}{=}\frac{1}{\sigma_k}\{a_{i,k}^*(B)\}^{-1}a_{i,k}(B)\epsilon_{k,t}$ and we can conclude as in the proof of Proposition 5.

□

## REFERENCES


Breidt, F.J. and R.A., Davis (1992) Time-reversibility, identifiability and independence of innovations for stationary time-series. *Journal of Time Series Analysis* 13, 377–390.

Brockwell, P., and R., Davis (1991) *Time Series : Theory and Methods*. Springer Verlag, New-York.

Chan, K.-S., Ho, L.-H. and H. Tong A note on time-reversibility of multivariate linear processes. *Biometrika* 93, 221–227.

Cheng, Q. (1992) On the unique representation of non-Gaussian linear processes. *Annals of Statistics* 20, 1143–1145.

Cheng, Q. (1999) On time-reversibility of linear processes. *Biometrika* 86, 483–486.

Cline, D.B.H. (1983) Infinite series of random variables with regularly varying tails. Unpublished document, The University of British Columbia.

Embrechts, P., Klüppelberg, C. and T. Mikosch (1997) *Modelling extremal events.* Springer.

Findley, D. (1986) The uniqueness of moving average representation with independent and identically distributed random variables for non-Gaussian stationary time series. *Biometrika* 73, 520–521.

Findley, D. (1990) Amendments and corrections: The uniqueness of moving average representation with independent and identically distributed random variables for non-Gaussian stationary time series. *Biometrika* 77, 235.

Gouriéroux, G. and J-M. Zakoian (2013) Explosive Bubble Modelling by Noncausal Process. CREST Discussion Paper 2013-04.

Hallin, M., Lefèvre, C. and M. Puri (1988) On time-reversibility and the uniqueness of moving average representations for non-Gaussian stationary time series. *Biometrika* 75, 170–171.

Ibragimov, I.A. and Yu. V. Linnik (1971) *Independent and stationary sequences of random variables.* Wolters-Noordoff, Groningen.

Kagan, A., Linnik, Yu. V. and C. Rao (1973) *Characterization problems in mathematical statistics*, Wiley: New-York.

Kokoszka, P. S. (1996) Prediction of infinite variance fractional ARIMA. *Probability and Mathematical statistics* 16, 65–83.

Lii, K.S. and M. Rosenblatt (1982) Deconvolution and estimation of transfer function phase and coefficients for non-Gaussian linear processes. *Annals of Statistics* 10, 1195–1208.

Rosenblatt, M. (2000) *Gaussian and non-Gaussian linear time series and random fields.* Springer–Verlag, New York.

Samorodnitsky G. and M. S. Taqqu (1994) *Stable Non-Gaussian Random Processes*, Chapman & Hall, London.