Investment under Uncertainty and the Recipient of the Entry Cost

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Abstract

A typical model of investment under uncertainty where firms incur an irreversible cost in order to produce is studied with a novel focus on the reciever of this cost ("the source"). The source is modeled as a firm or a government that sells a resource or a right that are necessary for the production of the final good. We study in detail how the source sets its resource's price. We find that this price is a decreasing function of the elasticity of the demand for the final good. We also find that when this demand is sufficiently low, the source does not lower its price accordingly, and the producers of the final good delay their purchases of the resource. The reason is that the source expects demand to be higher in the future and does not want to be committed then to a low price for its resource.

Keywords: Investment, Uncertainty

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1. Introduction

The literature on investment under uncertainty typically deals with the case of a firm that has to incur an irreversible cost in order to be able to produce. This article looks at the part of this setting that has not yet been analyzed – the receiver of the cost. We construct a model in which the resource N is necessary for the production of the durable good X. The producers of X face the typical investment under uncertainty problem studied in the literature, as the cost of each unit of N is an irreversible cost from their point of view. The sellers of N face a problem not yet studied – at what level should they set the cost of N. The dilemma is not just about the trade-off between "selling more but for a lower price" vs. "selling less but for a higher price". In fact, setting a sufficiently large price for the resource would lead to a period of no sales at all, until the demand for X sufficiently rises to make the X producers want to purchase N and start producing X.

For simplicity, we assume that N is sold by a monopolist, named "the source" henceforth. Two cases are studied: In the first case the source is interested in maximizing the value of its sales and in the second case the source is a government interested in welfare. The case of a government is of particular relevancy to this setting because the government is indeed a major seller of resources such as land, broadcasting frequencies and all sorts of franchises. We assume in the case of a government that it is using its income from the sales of N to finance its welfare enhancing activities in other markets suffering from failures. We show in the article that since it also cares about the welfare in the X market, the government sells N at a lower cost compared to the case where N is sold by a firm that maximizes the value of its the profits from its sales.

A key assumption in the model is that the source sets the price for the resource *N* in a once-and-for-all decision. Qualitatively, it is only a simplifying assumption as the forces in action here are the same ones that are active in the more realistic case where the source can change the price it charges, but does not do so continuously due to physical constraints or menu costs. The main implication of the inability to change the price of *N* smoothly is that it is possible that a fall in the demand for *X* would send the producers of *X* into a period in which they prefer to delay their purchases of *N* and production of *X*. Even more interestingly, it might be the case that the source itself, sets the price of *N* at a level that is sufficiently high to make the *X* producers delay their purchases of *N*. The source thus gives up on revenues from immediate sales in order to acquire larger amounts in the future when the demand for *X* rises and the *X* producers start to purchase the resource *N* in order to produce more *X*.

Another important assumption in the model is that the source charges the producers of *X* not just for their purchases of *N* but also for a share of the stream of profits they make by selling the *X* that they produce from this *N*. This assumption fits the case of a government very well, as the government taxes firms' profits. Such contracts exist nonetheless in the private sector as well. This assumption strengthens the role that the elasticity of the demand for *X* plays in setting the price of *N*. Thus for example, if the demand is sufficiently inelastic, the source would opt to avoid lowered tax revenues by preventing any increase in the quantity in the *X* market, setting therefore an infinite price for the resource *N*.[1] At the other end of the scale, if demand is sufficiently elastic, the source would promote increased quantity in the *X* market by setting a negative price for the resource *N*, i.e., subsidizing it.

[1] Since *X* is a durable good, sales of *N* imply additions to the already existing quantity in the *X* market.

The X market in this article is the same market studied by Leahy (1993). This is true although the current paper introduces an endogenous determination of the cost of the resource that the X producers must buy to produce. This endogenous determination concerns only the source firm and the X producers in the model analyzed here take the cost of the resource as given just the same as the producers in Leahy's model do. Since it is the same model, all the relevant results already proven by Leahy are used here without re-proving them.

The article is organized as follows. In section 2 the model is presented and the value of the source is analyzed. In section 3 the source's choice of the cost of N and the resulting immediate market situation - sales or inaction - is analyzed for the case where the source maximizes the value of the revenues it extracts from selling N and taxing the X market. In section 4 the same choice is analyzed, this time for the case of a government that wishes to extract revenues from the X market, but also cares about the welfare in it. Section 5 offers some concluding remarks.

2. The Model

Consider the market for good X. Production of X requires the resource N. The seller of N is a monopoly that fixes the price k per each unit of N. In addition, in each point in time the X producers have to pay the fraction t of their instantaneous revenues to the seller of N. We assume that the X producers can buy N any time they choose and that when such a firm purchases N it must transform it to X immediately. All the X producers are risk-neutral and have the same production process: a unit of N is transformed to a unit of X at the cost w, i.e., the production function of firm i is $Q_i \leq N_i$, where Q_i is firm i's output and N_i is the amount of N that firm i has. Thus, the supply of good X is $Q = N$, where Q is the aggregate amount of good X supplied in the market and $N = \sum_{i=1}^{\infty} N_i$.

The demand for X is given by:

(1) $$P = \frac{A}{Q^{\alpha}},$$

where P is the price of X. A is a geometric Brownian motion that the following rule describes its dynamics:

(2) $$dA = \mu A dt + \sigma A dZ,$$

where Z is the standard Wiener process satisfying:

(3) $$E(dZ) = 0, \qquad E[(dZ)^2] = 1,$$

μ and σ are constants and $\sigma > 0$. By Itô's lemma and (1), when Q is unchanged the evolution of P is governed by:

(4) $$dP = \mu P dt + \sigma P dZ,$$

which means that P is also a geometric Brownian motion. By Itô's lemma, when Q is unchanged the after-tax price, $\tilde{P} = (1-t)P$, is also a geometric Brownian motion, evolving according to:

(4') $d\tilde{P} = \mu\tilde{P}dt + \sigma\tilde{P}dZ$

We denote the discount rate relevant to the *X* producers and to the source firm by *r*. Following Dixit (1989) we assume that $r > \mu$, an assumption that makes the expected rate of growth of $\tilde{P}$, the instantaneous profit, smaller than the discount rate, preventing thus the value of the firms that produce *X* from going to infinity.

Under this modeling, the *X* market is the same market studied by Leahy (1993). As Leahy (1993) shows, under this setup there is a threshold price, $\tilde{P}_H$, that characterizes the optimal policy of each single *X* producer: when $\tilde{P} < \tilde{P}_H$ the *X* producer does nothing, when $\tilde{P}$ hits $\tilde{P}_H$ the *X* producer buys some *N* and produces more *X* from it. This optimal policy is the same for all the *X* producers since they are identical. Firms' purchases of *N* increase the supply of *X* and prevent $\tilde{P}_H$ from rising above $\tilde{P}_H$. As Leahy (1993) shows, the value of $\tilde{P}_H$ is:[2]

(5) $$\tilde{P}_H = \frac{\beta}{\beta - 1}(r - \mu)(k + w),$$

where β is the positive root of the quadratic:

[2]Throughout most of the article, Leahy (1993) studies a more general case then the one presented here. In page 1119, though, the analysis takes several assumptions that make it the same as the model in the current article. The second equation in that page is equation (5) of the current article. Some notational differences should be mentioned: the instantaneous profit is denoted by $\tilde{P}$ here and by *P* there; the investment threshold is denoted here by $\tilde{P}_H$ and by $\overline{P}$ there; the irreversible cost of producing a unit is denoted by *k* in Leahy (1993) while under the notations here make it *k* + *w*; the positive root of equation (5) here is denoted by Leahy as α. All the other relevant notations are identical.

(6) $$\tfrac{1}{2}\sigma^2 Y^2 + (\mu - \tfrac{1}{2}\sigma^2)Y - r = 0.$$

Applying $Y = 0$ and then $Y = 1$ and using the assumption that $r > \mu$ shows that one root of this quadratic (denoted γ) is negative and the other one, denoted β, exceeds unity.

Dividing by $1 - t$, yields the corresponding level of the pre-tax price, P:

(7) $$P_H = \frac{\beta}{\beta - 1} \cdot \frac{r - \mu}{1 - t} \cdot (k + w),$$

Given the initial values of A and Q the source firm sets a value of k that maximizes its expected present value. We now study the two possible cases. We start with the case where this value of k is sufficiently high to make the X producers delay their purchases of N, i.e., the case where $A/Q^\alpha \le P_H(k)$. Applying (7) in $A/Q^\alpha \le P_H$, in this case the source chooses a value of k in the range:

(8) $$k \ge \frac{(\beta - 1)(1 - t)A}{\beta(r - \mu)Q^\alpha} - w \equiv k^*$$

Next we analyze the case where the source firm sets a value of k that is in the range $-w < k < k^*$. This choice leads to $A/Q^\alpha > P_H(k)$ and therefore provokes immediate purchases of N by the X producers, purchases that increase Q until $A/Q^\alpha = P_H(k)$.

2.1 Delaying purchases of *N*

In this case, the *X* producers delay their purchases of *N* because the source sets a value of *k* that makes the market price $P = A/Q^{\alpha}$ smaller than the threshold P_H.

Let $V(A, Q, k)$ denote the value of the source firm in the range defined by (8) given the current levels of *A* and *Q* and given a value of *k*. By Itô's lemma,

$$dV(A, Q, k) = \left[V_A(A,Q,k)\mu A + \tfrac{1}{2}V_{AA}(A,Q,k)\sigma^2 A^2\right] dt + V_A(A,Q,k)\sigma A dZ \tag{9}$$

and due to (3):

$$\frac{E[dV(A,Q,k]}{dt} = V_A(A,Q,k)\mu A + \tfrac{1}{2}V_{AA}(A,Q,k)\sigma^2 A^2 \tag{10}$$

(10) Captures the source's expected capital gain due to the change in *A* over time. The no-arbitrage condition implies that this expected capital gain, together with the instantanous revenue from taxing the *X* market, should equal the normal return to the source firm's value. This implies:

$$\frac{E[dV(A,Q,k]}{dt} + tPQ = rV(A, Q, k) \tag{11}$$

Applying (10) and (1) in (11) and rearanging yields:

$$V_A(A,Q,k)\mu A + \tfrac{1}{2}V_{AA}(A,Q,k)\sigma^2 A^2 - rV(A, Q, k) + t\frac{A}{Q^{\alpha}}Q = 0 \tag{12}$$

(12) is a second-order non-homogenous differential equation. Trying a solution of the form $V(A, Q, k) = C(Q, k)A^{Y}$ to its homogenous part yields the quadratic captured by (6). Recal that the two roots of this quadratic satidfy $\gamma < 0$ and $\beta > 1$.

We now turn to finding a particular solution for (12). Trying a solution of the form $V(A, Q, k) = L(Q, k)A$ yields:

(13) $$L(Q, k) = \frac{tQ^{1-\alpha}}{r-\mu}$$

Combining the solution to the homogenous part of (12) and the particular solution to (12) yields:

(14) $$V(A, Q, k) = H(Q, k)A^{\gamma} + B(Q, k)A^{\beta} + \frac{tAQ^{1-\alpha}}{r-\mu}$$

where $H(Q, k)$ and $B(Q, k)$ will be determined now using two benchmark requirements. To that end, notice that by the standard properties of Brownian motions:

(15) $$E\left[\int_0^{\infty} e^{-rt} AQ^{1-\alpha} dt\right] = \frac{AQ^{1-\alpha}}{r-\mu}$$

Thus, the last addendum in the RHS of (14) is the expected value of tax revenues in the case that Q never changes, given on the initial levels of A and Q. The two addendums preceding it therefore in the RHS of (14) capture therefore the value of future sales of Q that occur each time A is sufficiently high that the price of X hits the investment threshold P_H. However, if A is close to 0 then the probability of A ever rising that high is zero too. In that case therefore the source's value is merely the expected value of the tax revenues, the current quantity, Q, generates. Formally:

(16) $$\lim_{A\to 0} V(A,Q,k) = \frac{tQ^{1-\alpha}}{r-\mu}$$

Since γ is negative, (16) implies that $H(Q, k) = 0$.

We now turn to the determination of $B(Q, k)$. As appendix A shows, the condition for a no-arbitrage evaluation of the source's value in the time instants when there are changes in Q, i.e., when $A/Q^{\alpha} = P_H$, is:

(17) $$V_Q(A, Q, k) = -k.$$

Thus, by (14), (17) and $H(Q, k) = 0$:

(18) $$B_Q(Q, k)A^{\beta} + \frac{(1-\alpha)tAQ^{-\alpha}}{r-\mu} = -k.$$

Applying $A/Q^{\alpha} = P_H$ in (18) and rearranging it yields:

(18') $$B_Q(Q, k) = -\frac{\frac{(1-\alpha)tP_H}{r-\mu}+k}{Q^{\alpha\beta}{P_H}^{\beta}}.$$

Straightforward integration of $B_Q(Q, k)$ leads to:

(19) $$B(Q, k) = \frac{\frac{(1-\alpha)tP_H}{r-\mu} + k}{(\alpha\beta - 1)Q^{\alpha\beta-1}{P_H}^{\beta}} + C$$

Applying (7) in (19) and simplifying yields:

(19') $$B(Q, k) = \frac{(1-\alpha)\beta tw + (\beta - 1 - \alpha t\beta + t)k}{(1-t)(\beta-1)(\alpha\beta-1)Q^{\alpha\beta-1}{P_H}^{\beta}} + C$$

As Q goes to infinity P goes to 0 and the probability of P ever reaching P_H goes to zero too. This implies that the source firm is not going to sell any N in the future and its value should therefore spring merely from the future tax revenues, i.e.:

(20) $$Lim_{Q\to\infty}B(Q, k) = 0.$$

This benchmark dictates a distinction between two cases based on the value of α. We start with a case in which $\alpha < 1/\beta$. In that case, Q in the denominator at the first term at the RHS of (19') is raised by a positive power and as it goes to infinity the entire term goes to infinity as well. This, taken together with (20), implies that C is infinitely large, and therefore that $B(Q, k)$ and $V(A, Q, k)$ also are infinitely large for each finite level of Q, a case that is not at the focus of this article.

The economic logic underlying the infinite value of the source value in this case is based on the relation between α and the elasticity of demand which is $-1/\alpha$. The smaller α, the larger the demand elasticity and therefore the larger the increase in Q each time that P hits P_H. Thus, the smaller α, the faster the process of sales of the resource N and the less heavily discounted are its revenues. In addition, the larger elasticity due to a smaller α also makes the tax revenues increase by more as Q is enlarged. The two positive effects of a smaller α - faster sales of N and greater tax revenues - drive the value of future sales of N to infinity, when α is sufficiently small, namely - below $1/\beta$. Since this case is not in the focus of this article, the rest of the article refers to the case where $\alpha > 1/\beta$.

Returning to (19') and (20), now with $\alpha > 1/\beta$, the first term at the RHS of (19') now goes to zero as Q goes to infinity, implying that $C = 0$. Applying (7), C=0, (19') and $H(Q, k) = 0$ in (14) yields:

(21) $$V(A, Q, k) = D\frac{(\beta-1-\alpha\beta t+t)k+(1-\alpha)\beta tw}{(k+w)^{\beta}} + \frac{tAQ^{1-\alpha}}{(r-\mu)},$$

where:

(22) $$D \equiv \frac{(\beta-1)^{\beta-1}(1-t)^{\beta-1}A^{\beta}}{(\alpha\beta-1)\beta^{\beta}(r-\mu)^{\beta}Q^{\alpha\beta-1}},$$

and $D > 0$. The following *Proposition 1* shows some of the properties of $V(A, Q, k)$.

<u>*Proposition 1*</u>:

(a) If $\alpha > (\beta\text{-}1 + t)/\beta t \equiv \alpha^*$ than $V_k(A, Q, k) > 0$ throughout the range $k>-w$.

(b) If $\alpha<\alpha^*$ than there exists a single value of k, denoted k_1, that maximizes $V(A, Q, k)$.

(c) k_1 is in the range where $V(A, Q, k)$ represents the source firm's value (the range $k>k^*$) iff A/Q^{α} is sufficiently small.

Proof: By (21):

(23) $$V_k(A,Q,k) = D(\beta-1)\frac{(1-t)\beta w + [(\alpha\beta-1)t-(\beta-1)](k+w)}{(k+w)^{\beta+1}}.$$

If $\alpha > \alpha^*$ then the term $(\alpha\beta-1)t-(\beta-1)$ is positive in the range $k > -w$ and therefore so is $V_k(A, Q, k)$. This completes the proof of part (a).

If $\alpha < \alpha^*$ then $(\alpha\beta-1)t-(\beta-1)$ is negative in the range $k > -w$ implying that in that range $V_k(A, Q, k)$ is positive for sufficiently small values of k and vice verse. Thus, in this case there is a single value of k that maximizes $V(A, Q, k)$. This completes the proof of part (b). Solving $V_k(A, Q, k) = 0$ yields that this value satisfies:

(24) $$k = \frac{(1-t)-(1-\alpha)t\beta}{(\beta-1)(1-t)+(1-\alpha)t\beta} w \equiv k_1.$$

Applying (24) in (8) shows that k_1 is in the range $k > k^*$, in which $V(A, Q, k)$ represents the source firm's value, iff the following condition holds:

(25) $$\frac{A}{Q^\alpha} < \frac{\beta^2(r-\mu)w}{(\beta-1)^2(1-t)+(1-\alpha)t\beta(\beta-1)} \equiv P^*.$$

This proves part (c). □

(25) implies that the source firm will set a value of k that is sufficiently large to make the X producers delay their purchases N if the current demand at the X market is sufficiently low.

From continuity it follows, by applying (7) in (12), that at $A/Q^{\alpha} = P_H$:

$$V(A, Q, k) = \frac{k[(1-t)+t\alpha\beta]+t\alpha\beta w}{(1-t)(\alpha\beta-1)}Q. \tag{26}$$

2. Provoking immediate purchases of *N*

The X producers immediately purchase N when the market price, $P = A/Q^{\alpha}$, exceeds the threshold P_H. Applying (7) in $A/Q^{\alpha} \geq P_H$, this range becomes:

$$k < \frac{(\beta-1)(1-t)A}{\beta(r-\mu)Q^{\alpha}} - w \equiv k^*. \tag{27}$$

In this range immediate investment increases Q to Q_1 so that the price after the investment is done is $P=A/Q_1{}^{\alpha}=P_H$. The source firm receives from this increase in Q:

$$k(Q_1 - Q) = k\left[\left(\frac{A}{P_H}\right)^{\frac{1}{\alpha}} - Q\right]. \tag{28}$$

Let $G(A, Q, k)$ denote the value of the source firm in the range defined by (27) given the current levels of A and Q and also for a given value of k. Equation (29) below shows $G(A, Q, k)$ as the sum of two factors: First, the immediate proceeds described by (28); Second, the value of the source firm after the quantity immediately becomes Q_1, as described by (26).

(29) $$G(A, Q, k) = k(Q_1 - Q) + \frac{k[(1-t) + t\alpha\beta] + t\alpha\beta w}{(1-t)(\alpha\beta - 1)} Q_1,$$

which can be simplified to:

(29') $$G(A, Q, k) = -kQ + \alpha\beta \frac{k + wt}{(\alpha\beta - 1)(1-t)} Q_1.$$

Applying $Q_1 = (A/P_H)^{1/\alpha}$ and (7) in (29') yields:

(30) $$G(A, Q, k) = CA^{\frac{1}{\alpha}} f(k) - kQ.$$

where:

(31.a) $$C \equiv \frac{(1-t)^{\frac{1}{\alpha}-1}(\beta - 1)^{\frac{1}{\alpha}}}{(\alpha\beta - 1)\beta^{\frac{1}{\alpha}-1}(r - \mu)^{\frac{1}{\alpha}}} > 0,$$

(31.b) $$f(k) \equiv \alpha \frac{k + wt}{(k + w)^{\frac{1}{\alpha}}}.$$

The following *Proposition 2* shows some important properties of $G(A, Q, k)$

Proposition 2:

(a) There exists a single value of k that brings $G(A, Q, k)$ to a maximum;

(b) This value of k, denoted by k_2, is an increasing concave function of A/Q^{α};

(c) k_2 is in the range $k < k^*$, the range in which $G(A, Q, k)$ represents the source firm's value, iff $\alpha < \alpha^*$ and $A/Q^{\alpha} > P^*$;

Proof: In the appendix. □

3. The optimal *k* when the source firm is maximizing its sales value

Based on the analysis in the previous sections the source firm's value as a function of A, Q and k can be defined and denoted by:

(32) $$VG(A,Q,k) \equiv \begin{cases} G(A,Q,k) & if \quad -w < k < k^*(A,Q) \\ V(A,Q,k) & otherwise \end{cases}$$

Three cases should be analyzed now: The case where $\alpha > \alpha^*$; The case where $\alpha < \alpha^*$ and $A/Q^{\alpha} < P^*$; The case where $\alpha < \alpha^*$ and $A/Q^{\alpha} > P^*$.

3.1. When $\alpha > \alpha^*$

In this case $k_2 > k^*$ for each value of A/Q^{α} as part (c) of *Proposition 2* shows. Thus, in the range $k < k^*$ the source firm's value, represented by $G(A, Q, k)$, is increasing in k. From part (a) of *Proposition 1* it follows that also in the range $k > k^*$ the source firm's value, now represented by $V(A, Q, k)$, is increasing in k. Note that

$V(A, Q, k^*) = G(A, Q, k^*)$ as follows from applying (8) in (21) and then in (30). Thus in that case the source firm's value, $VG(A, Q, k)$, is an increasing function of k and it is optimal for the source firm to push the value of k to infinity. The economic logic in action here is that when α is sufficiently large demand is sufficiently inelastic to make it optimal for the source firm to prevent increases in quantity in order to keep tax revenues from falling. Note from (21) that in that case $VG(A, Q, k)$ approaches $tAQ^{1-\alpha}/(r-\mu)$, which is the expected value of the tax collection, when Q is fixed over time at its current level.

3.2. When $\alpha < \alpha^*$ and $A/Q^\alpha < P^*$

In this case $k_2 > k^*$ as follows from part (c) of *Proposition 2*. Thus, in the range $k < k^*$ the source firm's value, represented by $G(A, Q, k)$, is increasing in k. From parts (b) and (c) of *Proposition 1* it follows that in the range $k > k^*$ the source firm's value, now represented by $V(A, Q, k)$, reaches a maximum in $k = k_1$. Thus, since $V(A, Q, k^*) = G(A, Q, k^*)$, the source firm's value, $VG(A, Q, k)$, reaches its maximum in $k = k_1$.

The line marked with circles in *figure 1* below presents $VG(A, Q, k)$ in that case. The thin line shows $V(A, Q, k)$ and the thick line shows $G(A, Q, k)$.

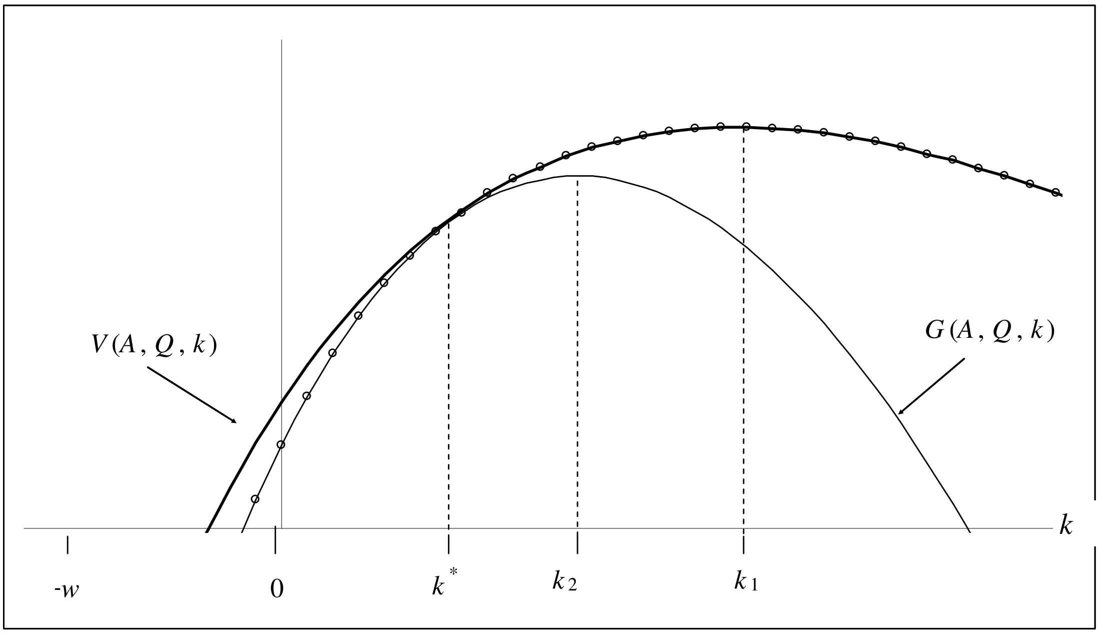

Figure 1: The source firm's value, $VG(A, Q, k)$, when $\alpha < \alpha^*$ and $A/Q^\alpha < P^*$. The Thick line shows $V(A, Q, k)$, the thin line shows $G(A, Q, k)$ and the circles indicates $VG(A, Q, k)$. In this case $VG(A, Q, k)$ is maximized at $k = k_1 > k^*$ implying that the source firm sets a value of k sufficiently high to delay immediate purchases of N by the X producers.

3.3. When $\alpha < \alpha^*$ and $A/Q^\alpha > P^*$

In that case in the range $k < k^*$ the source firm's value, represented by $G(A, Q, k)$ reaches a maximum in $k = k_2$ as follows from parts (a) and (c) of *Proposition 2*. Also in that case, $k_1 < k^*$ as follows from parts (b) and (c) of *Proposition 1*. Thus, in the range $k > k^*$ the source firm's value, represented now by $V(A, Q, k)$, decreases in k. Therefore, since $V(A, Q, k^*) = G(A, Q, k^*)$, the source firm's value reaches its maximum in $k = k_2$.

The line marked with circle in *figure 2* below presents $VG(A, Q, k)$ in that case. The thin line show $V(A, Q, k)$ and the thick line shows $G(A, Q, k)$.

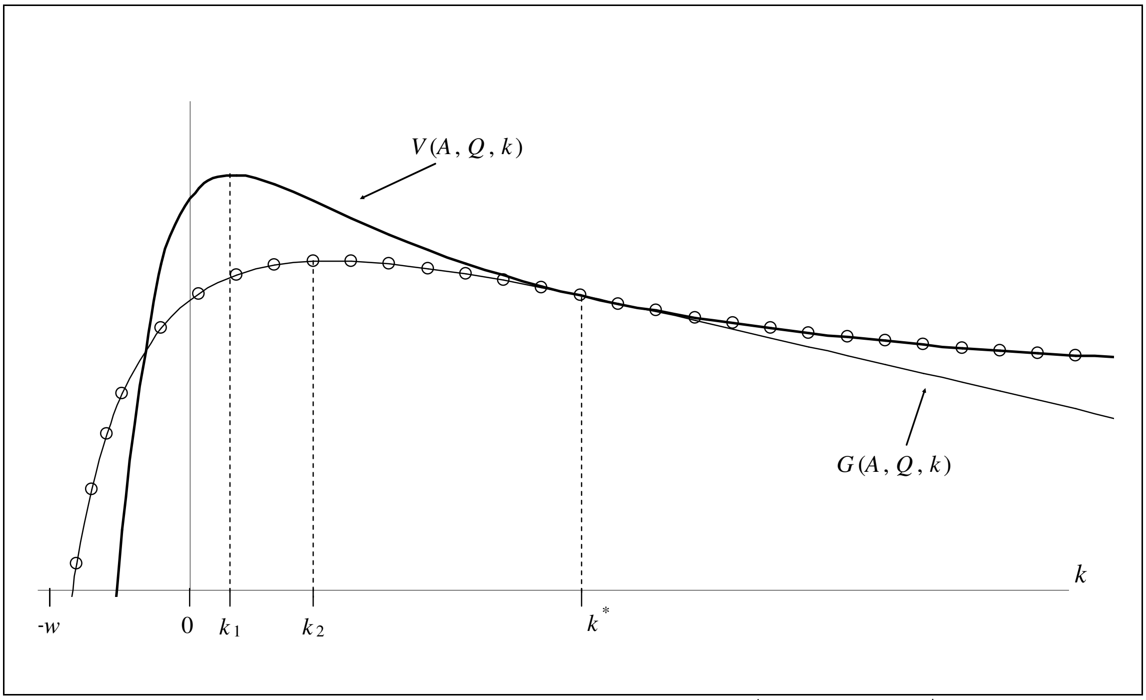

Figure 2: The source firm's value, $VG(A, Q, k)$, when $\alpha < \alpha^*$ and $A/Q^\alpha > P^*$. The Thick line shows $V(A, Q, k)$, the thin line shows $G(A, Q, k)$ and the circles indicate $VG(A, Q, k)$. In this case $VG(A, Q, k)$ is maximized at $k = k_2 < k^*$ implying that the source sets a value of k sufficiently low to promote immediate purchases of N by the X producers.

Based on the analysis of the two previous sub-sections, figure 3 below shows the optimal k as a function of A/Q^α for the case when $\alpha < \alpha^*$.

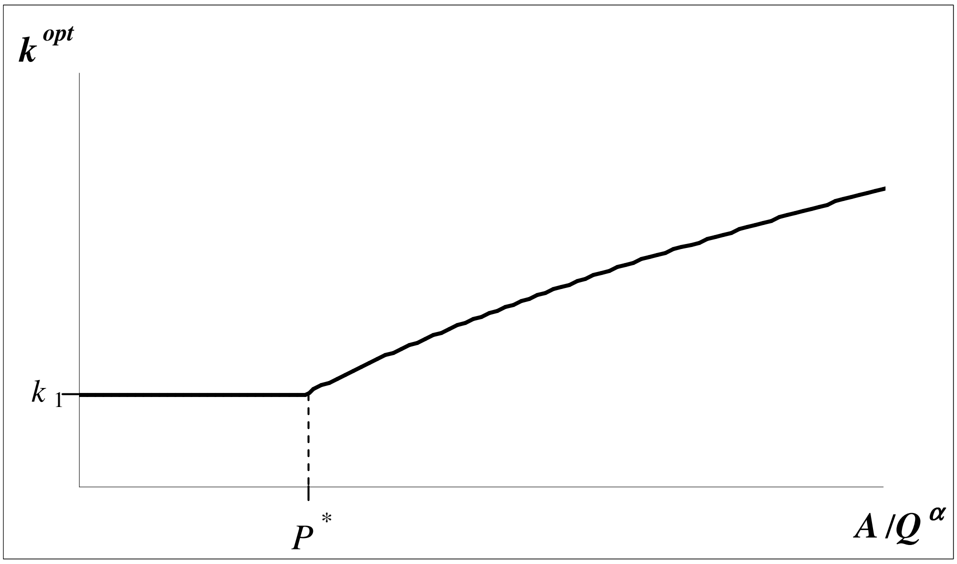

Figure 3: the optimal k as a function of A/Q^α for the case when $\alpha < \alpha^*$.

The increasing part in this function is concave and the entire function may be below zero, as the following proposition establishes.

Proposition 3: If $\alpha < t$ then the optimal value of k is negative.

Proof: In the appendix.

4. The optimal *k* in the case of a welfare objective

Assume now that the source firm is a government that is not interested in maximizing the value of its potential sales of the resource N, namely $VG(A, Q, k)$. Instead we assume now that the government cares about welfare in the X market, but also want to use it for financing its activities in other markets, markets that suffer from market failures and government intervention in them is welfare increasing. Specifically, we assume that the government balances these two contradicting targets by setting an objective of bringing the value $VG(A, Q, k)$ to a certain level M which is below the maximal level of $VG(A, Q, k)$. As established in section 3, in the case where $\alpha < \alpha^*$ the function $VG(A, Q, k)$ has an inverse-U shape and therefore there are two values of k that yields the value M that the government seeks. For that case we assume that the government, wishing to harm welfare in the X market as little as possible, chooses the lower level of the two values of k that solve:

(33) $$VG(A, Q, k) = M$$

As in section 3, three cases will be analyzed next: The case where $\alpha > \alpha^*$; The case where $\alpha < \alpha^*$ and $A/Q^{\alpha} < P^*$; The case where $\alpha < \alpha^*$ and $A/Q^{\alpha} > P^*$.

4.1. When $\alpha > \alpha^*$

As established in section 3.1, in this case $VG(A, Q, k)$ is a monotonically increasing function of k that converges to the value $tAQ^{1-\alpha}/(r-\mu)$ as k approaches infinity. In addition it approaches $-\infty$ as k approaches $-w$, as follows from (30) and (31). Thus, there is single value that solves (33) for every level of M that is smaller than $tAQ^{1-\alpha}/(r-\mu)$.

Note that in contrast to the case where the source is maximizing the value of its potential sales of the resource N, here the value of k that is chosen is not necessarily infinite. The reason for that is that in the current case the source is a government that cares not just about the revenues from selling N but also about the welfare in the market for X. The only possibility for the government to set an infinitely large level of k is if the M it want s to extract from the x market is above $tAQ^{1-\alpha}/(r-\mu)$.

By (21), (32) and an implicitly differentiation of (33):

$$\frac{dk}{dM} = -\frac{-1}{V_K(A,Q,k)} > 0 \tag{34}$$

where the inequality follows from the result that in this case $V_K(A, Q, k) < 0$, as established in the proof of part (a) of *proposition 1.* Thus, the larger the value of the revenues that the government wants to extract from the X market the larger the level of k it sets. In a similar manner it can be shown that in this case k is decreasing in A and increasing in Q.

We denote the value of $V(A, Q, k)$ at the end of its definition range:

(35) $V^*(A, Q) \equiv V[A, Q, k^*(A, Q)] = G[A, Q, k^*(A, Q)]$

where k^* is a function of A and Q by (8). Applying (8) in (21) yields $V^*(A, Q)$ explicitly. Based on the analysis of the properties of $VG(A, Q, k)$ in section 3.1, if M is smaller than $V^*(A, Q)$ then the government chooses is below k^*, implying an immediate purchases of N and production of X. Otherwise, the k that the government chooses is above k^*, a choice that sends the market to a period of inaction until A is sufficiently large so that $P = P_H$.

4.2. When $\alpha < \alpha^*$ and $A/Q^\alpha < P^*$

Based on *figure 1* and the analysis in section 3.2, in this case $VG(A, Q, k)$ has an inverse-U shape maximized at $k = k_1 > k^*$. In the range $-w < k < k^*$ the function $VG(A, Q, k)$ is based on $G(A, Q, k)$ and for higher levels of k it is based on $V(A, Q, k)$. The value of $VG(A, Q, k)$ at its maximum satisfies:

(36) $V^1(A, Q) \equiv V[A, Q, k_1(A, Q)]$.

Note that k_1 is a function of A and Q by (24). Applying (24) in (21) explicitly yields $V^1(A, Q)$, which is the maximal level of M that the government can extract form the market in this case. If the government is interested in a level of M that is satisfying: $V^*(A, Q) < M \leq V^1(A, Q)$ then the level of k that the government chooses, based on (33), is above k^*, implying inaction until A is sufficiently large so that $P = P_H$. If, on the other hand, level of M that the government seeks satisfies $M < V^*(A, Q)$ then the government chooses a value of k that is smaller than k^*, implying immediate purchases of N and construction of X.

An important difference from the case where the source maximizes the value of $VG(A, Q, k)$, is that in that case the value of k was constant at k_1 whereas here it is increasing in Q and decreasing in A.

4.3. When $\alpha < \alpha^*$ and $A/Q^\alpha > P^*$

Based on *figure 2* and the analysis in section 3.3, in this case $VG(A, Q, k)$ has an inverse-U shape maximized at $k = k_2 < k^*$. In the range $-w < k < k^*$ the function $VG(A, Q, k)$ is based on $G(A, Q, k)$ and for higher levels of k it is based on $V(A, Q, k)$. The value of $VG(A, Q, k)$ at its maximum satisfies:

$$(37) \qquad G^2(A, Q) \equiv G[A, Q, k_2(A, Q)]$$

where k_2 is function of A and Q by *proposition 2*.

$G^2(A, Q)$ is the maximal level of M that the government can extract form the market in this case. If the government is interested in a level of M that is $G^2(A, Q)$ then the level of k that the government chooses, based on (33), satisfies $k < k^2 < k^*$, implying an immediate purchases of N and construction of X.

As in the case where the source maximizes the value of $VG(A, Q, k)$, the value of k that the government chooses is increasing in Q and decreasing in A.

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Appendix

A. Establishing condition (17)

In this appendix we derive the benchmark condition (17) for the value of the source firm at the time instants in which P hits P_H. For that end we use the discrete approximation of a Brownian Motion presented in Dixit (1991). Since it is more convenient to perform this approximation for a Brownian Motion, rather than to a Geometric Brownian Motion, the analysis is based on the function:

(A.1) $$F(a, Q, k) \equiv V(A, Q, k)$$

where $a \equiv \ln A$. Due to this definition, to prove that By Itô's lemma, a is a Brownian Motion since A is a Geometric Brownian Motion. The drift and variance parameters of a are denoted here by μ_a and σ_a^2. To approximate the motion of a we divide time to small intervals of length τ and the variable a space into steps of size ξ. The variable a now ranges over a discrete set of values a_i such that:

(A.2) $$a_{i+1} - a_i = \xi \quad \text{for all } i.$$

Starting at state a_i, time τ later the variable a takes with probability p a step down to the value of a_{i-1}, or takes with probability $q = 1 - p$ a step up to the value of a_{i+1}. Two conditions relating τ, ξ, p and q to μ_a and σ_a should be used in order to make this process an approximation of the original Brownian Motion. First:

(A.3) $$\mu\tau = q\xi + p(-\xi),$$

which leads to:

(A.4) $$q = \frac{1}{2}\left(1 + \frac{\mu\tau}{\xi}\right), \qquad p = \frac{1}{2}\left(1 - \frac{\mu\tau}{\xi}\right)$$

The condition regarding the variance of the process is:

(A.5) $$\sigma^2\tau = q(\xi - \mu\tau)^2 + p(-\xi - \mu\tau)^2 = \xi^2 + 2\mu\tau\xi(p - q) + (\tau\mu)^2 = \xi^2 - \mu^2\tau^2$$

eliminating the term with τ^2 leaves:

(A.6) $$\sigma^2\tau = \xi^2$$

When $P = \frac{A}{Q^\alpha}$ is at the investment threshold P_H then, by (1):

(A.7) $$Q = \left(\frac{A}{P_H}\right)^{\frac{1}{\alpha}} = \left(\frac{e^{a_i}}{P_H}\right)^{\frac{1}{\alpha}}$$

if time τ later a takes a step up the endogenous investment by the X producers raises Q such that P remains at P_H, this implies that Q is raised to the level:

(A.8) $$\left(\frac{e^{a_i+\xi}}{P_H}\right)^{\frac{1}{\alpha}} = \left(\frac{e^{a_i}}{P_H}\right)^{\frac{1}{\alpha}} + \left(\frac{e^{a_i}}{P_H}\right)^{\frac{1}{\alpha}} \frac{1}{\alpha}\xi + \frac{\left(\frac{e^{a_i}}{P_H}\right)^{\frac{1}{\alpha}} \frac{1}{\alpha^2}\xi^2}{2} + \frac{\left(\frac{e^{a_i}}{P_H}\right)^{\frac{1}{\alpha}} \frac{1}{\alpha^3}\xi^3}{6} + \ldots$$

The change in Q during that time is therefore:

(A.9) $$\Delta Q = \left(\frac{e^{a_i+\xi}}{P_H}\right)^{\frac{1}{\alpha}} - \left(\frac{e^{a_i}}{P_H}\right)^{\frac{1}{\alpha}} = \left(\frac{e^{a_i}}{P_H}\right)^{\frac{1}{\alpha}} \frac{\xi}{\alpha} + \mathrm{o}(\xi),$$

where $\mathrm{o}(\xi)$ collects all the terms that go to zero faster than ξ, such that $\mathrm{o}(\xi)/\xi \to 0$ as $\xi \to 0$. Note from (A.6) that τ too falls under the category of $\mathrm{o}(\xi)$.

The Bellman equation for the value of the source when a_i and Q are such that $P = P_H$ is:

(A.10) $$F(a_i, Q, k) = tP_H Q\tau + e^{-r\tau}\,[pF(a_{i-1}, Q, k) + qF(a_{i+1}, Q + \Delta Q, k) + qk\Delta Q]$$

(A.10) shows the value of the source in that situation as the sum of the immediate tax revenue and the time τ later value of the source discounted by $e^{-r\tau}$. With probability p the variable a takes a step down and the source's value becomes $F(a_{i-1}, Q, k)$. With probability q the variable a takes a step up. In that case endogenous firm's investment raises Q by ΔQ and the source's value becomes $F(a_{i+1}, Q + \Delta Q, k)$. In addition, in that case the source also gains $k\Delta Q$ from sales to the X producers.

Expanding the term $e^{-r\tau}$ to a Taylor series, it becomes:

(A.11) $$e^{-r\tau} = 1 + (-r\tau) + \frac{(-r\tau)^2}{2} + \frac{(-r\tau)^3}{6} + \ldots = 1 + o(\xi)$$

Applying this in (A.10) and expanding terms of (A.10) to Taylor series yields:

(A.12) $$F(a_i, Q, k) = tP_H Q\tau + p[F(a_i, Q, k) + F_a(a_i, Q, k)(-\xi) + o(\xi)]$$
$$+q[F(a_i, Q, k) + F_a(a_i, Q, k)(\xi) + F_Q(a_i, Q, k)\Delta Q + o(\xi) + k\Delta Q]$$

Using $p + q = 1$ and the result that τ is $o(\xi)$ by itself helps simplify (A.12) to:

(A.13) $$0 = (q - p)F_a(a_i, Q, k)\xi + q\, F_Q(a_i, Q, k)\Delta Q + qk\Delta Q + o(\xi)$$

By (A.4), $(q - p)\xi = \mu\tau = o(\xi)$ which simplifies (A.13) into:

(A.14) $$0 = F_Q(a_i, Q, k)\Delta Q + k\Delta Q + o(\xi)$$

Dividing by ΔQ and applying (A.9) yields:

(A.15) $$F_Q(a_i, Q, k) = -k - \frac{o(\xi)}{\left(\frac{e^{a_i}}{P_H}\right)^{\frac{1}{\alpha}} \frac{\xi}{\alpha} + o(\xi)}$$

By the definition of $o(\xi)$, as $\xi \to 0$ the second addendum on the RHS of (A.15) approaches 0 as well. This, together with $F_Q(a, Q, k) \equiv V_Q(A, Q, k)$, which follows from the definition of $F(a_i, Q, k)$ in (A.1), concludes establishing (17).

B. Proof of *Proposition 2*

By (30) the first order condition for a maximum is

(B.1) $$G_k(A, Q, k) = - Q + CA^{1/\alpha} f'(k) = 0,$$

and by (31.b):

(B.2) $$f'(k) = \frac{(k+w)(\alpha-1)+w(1-t)}{(k+w)^{\frac{1+\alpha}{\alpha}}}.$$

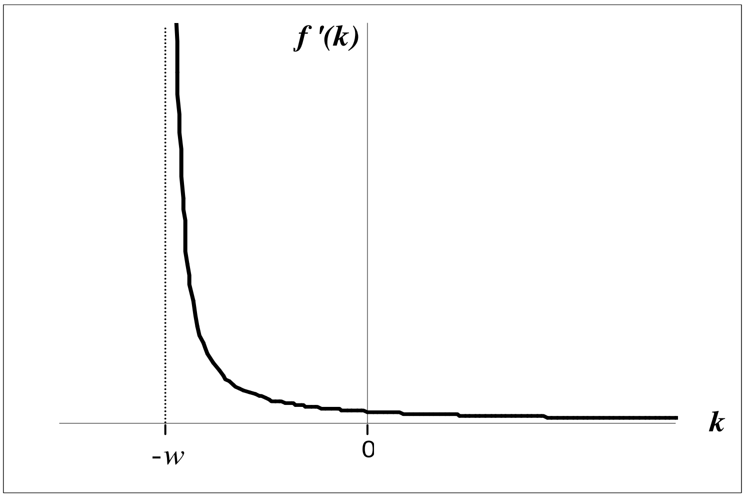

Figure 4: $f'(k)$ when $\alpha > 1$.

When $\alpha > 1$ straightforward differentiation shows that $f'(k) > 0$, $f''(k) < 0$, and $f'''(k) > 0$ in all of the range in which $f(k)$ is defined, the range $-w < k$. In addition $f(k)$ approaches infinity when k approaches $-w$ and approaches 0 when k goes to infinity. Thus there exists a single value of k for which $f'(k)$ equals the positive value $Q/CA^{1/\alpha}$

and for which, therefore, (B.1) holds. Since f''(k) < 0 throughout the definition range this k brings $G(A, Q, k)$ to a maximum. *Figure 4* shows f'(k) in that case.

When $\alpha < 1$ the function f'(k) > 0 only in the range $-w < k < -w(\alpha - t)/(\alpha - 1)$. In addition, in that range f''(k) < 0, and f'''(k) > 0 , as straightforward differentiation shows, and f'(k) approaches infinity when k approaches $-w$. Thus, in that case too, there exists a single value of k for which f'(k) = $Q/CA^{1/\alpha}$ and (B.1) holds. Since this value of k is in the where f''(k) < 0 it brings $G(A, Q, k)$ to a maximum. *Figure 5* shows f'(k) in that case.

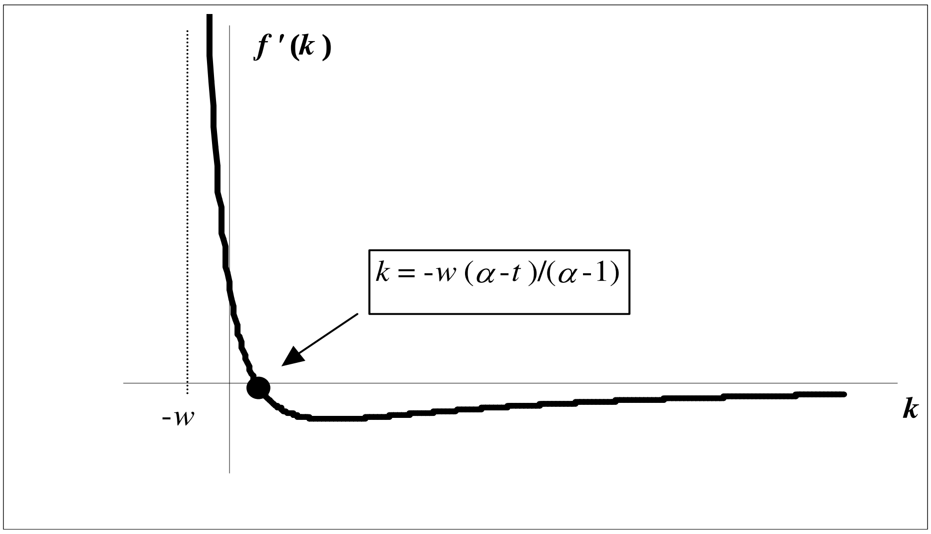

Figure 5: f'(k) when $\alpha < 1$.

(b). Rearranging (B.1) yields that the value of k in the maximum point, k_2, satisfies:

(B.3) $$1 = C\left(\frac{A}{Q^{\alpha}}\right)^{\frac{1}{\alpha}} f'(k_2).$$

(B.3) defines k_2 as an implicit function of A/Q^{α}. By implicit differentiation of (B.3):

(B.4) $$\frac{\partial k_2}{\partial\left(\frac{A}{Q^\alpha}\right)} = -\frac{f'(k_2)}{\alpha\left(\frac{A}{Q^\alpha}\right)f''(k_2)} = -\frac{1}{\alpha C\left(\frac{A}{Q^\alpha}\right)^{\frac{1+\alpha}{\alpha}} f''(k_2)} > 0,$$

where the second equality follows from (B.3) and the inequality sign follows from $f'(k_2) < 0$, as shown in the proof of part (a). Concavity of k_2 in A/Q^α follows from:

(B.4) $$\frac{\partial^2 k_2}{\partial\left(\frac{A}{Q^\alpha}\right)^2} = \frac{(1+\alpha)C\left(\frac{A}{Q^\alpha}\right)^{\frac{1}{\alpha}} f''(k_2)^2 + f'''(k_2)}{\alpha^2 C^2\left(\frac{A}{Q^\alpha}\right)^{2\frac{1+\alpha}{\alpha}} f''(k_2)^3} < 0,$$

where the inequality sign follows from $f''(k_2) < 0$ and $f'''(k_2) > 0$ which were established in the proof of part (a).

(c). Applying (8) and (B.2) in (B.3) shows that $k_2 = k^*$ iff $A/Q^\alpha = P^*$. Yet, iff $\alpha > \alpha^*$ then $k_2 > k^*$ for all values of A/Q^α, as the following analysis shows. (B.4) can be presented as:

(B.5) $$\frac{\partial k_2}{\partial\left(\frac{A}{Q^\alpha}\right)} = \left[\frac{k_2 + w}{(A/Q^\alpha)}\right]^{\frac{1+\alpha}{\alpha}} \frac{(k_2 + w)}{C[(\alpha-1)(k_2+w) + w(1-t)(1+\alpha)]}.$$

From (B.2) and (B.3) it follows that when A/Q^α approaches zero, $f'(k)$ goes to infinity implying that k approaches $-w$. Thus, by Lhopital's rule, as A/Q^α approaches zero the left term in the RHS approaches the term on the RHS leading to:

(B.6) $$\lim_{A/Q^{\alpha}\to 0} \frac{\partial k_2}{\partial\left(\frac{A}{Q^{\alpha}}\right)} = \lim_{k_2\to -w} \left\{\frac{C[(\alpha-1)(k_2+w)+w(1-t)(1+\alpha)]}{(k_2+w)}\right\}^{\alpha} = \infty.$$

Note from (8) that when A/Q^{α} approaches zero k^* approaches $-w$, just the same as k_2. Also note from (8) that k^* is a linear function of A/Q^{α}. Thus, $k_2 > k^*$ at least for sufficiently small values of A/Q^{α}.

From (B.2) and (B.3) it follows that when A/Q^{α} *go*es to infinity, $f\,'(k_2)$ approaches zero. If $\alpha < 1$ this implies that k_2 goes to $-w(\alpha - t)/(\alpha - 1)$ and that $f\,''(k_2)$ is finite. Thus, by (B.4) the slope of k_2 as a function of A/Q^{α} approaches zero as A/Q^{α} goes to infinity. Therefore, for the case of $\alpha < 1$ it holds that $k_2 < k^*$ iff $A/Q^{\alpha} > P^*$, since by part (a) of this proposition k_2 is a concave function of A/Q^{α}.

In the case where $\alpha > 1$ as $f\,'(k)$ approaches zero, k_2 goes to infinity and $f\,''(k_2)$ goes to zero. An analysis similar to the one that leads to (B.6) yields that:

(B.7) $$\lim_{A/Q^{\alpha}\to \infty} \frac{\partial k_2}{\partial\left(\frac{A}{Q^{\alpha}}\right)} = \lim_{k_2\to \infty} \left\{\frac{C[(\alpha-1)(k_2+w)+w(1-t)(1+\alpha)]}{(k_2+w)}\right\}^{\alpha} =$$

$$= [C(\alpha\text{-}1)]^{\alpha} = \left(\frac{\alpha-1}{\alpha\beta-1}\frac{\beta}{1-t}\right)^{\alpha}\left(\frac{1-t}{\beta}\frac{\beta-1}{r-\mu}\right).$$

If $\alpha > \alpha^*$ than $\frac{\alpha-1}{\alpha\beta-1}\frac{\beta}{1-t} > 1$, implying that the slope of k_2 as a function of A/Q^{α} approaches, as A/Q^{α} goes to ∞, the slope of k^* as a function of A/Q^{α}. In that case $k_2 > k^*$ for all values of A/Q^{α}. If, on the other hand, $\alpha < \alpha^*$ than as A/Q^{α} goes to infinity the slope of k_2 as a function of A/Q^{α} is below the slope of k^* as a function of A/Q^{α}. In that case it holds that $k_2 < k^*$ iff $A/Q^{\alpha} > P^*$, exactly as in the case of $\alpha < 1$. □

C. Proof of *Proposition 3*

If $\alpha < t$ then $\alpha < \alpha^*$, implying that the source firm's value is maximized either by k_1 or by k_2, depending on whether $A/Q^{\alpha} > P^*$ or not. As shall be shown now, in that case both k_1 and k_2 are negative.

To prove that $k_1 < 0$ in that case, note that if $\alpha < t$ then α is also smaller then 1. Thus, if $\alpha < t$ the denominator of k_1 is always positive since it satisfies:

(C.1) $$(\beta - 1)(1 - t) + (1 - \alpha)t\beta > (\beta - 1)(1 - t) > 0.$$

Next, note that if $\alpha < t$ the numerator of k_1 is negative since it satisfies:

(C.2) $$1 - t - (1 - \alpha)t\beta < 1 - t - (1 - t)\alpha\beta = (1 - t)(1 - \alpha\beta) < 0,$$

since we also assume $\alpha > 1/\beta$.

To prove that $k_2 < 0$ when $\alpha < t$, note once again that in that case α is below unity. As was shown in part (a) of *Proposition 2*, if $\alpha < 1$ then $k_2 < -w(\alpha - t)/(\alpha - 1)$ leading to $k_2 < 0$. □