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Abstract

This paper analyzes the solution of linear mixed-type functional differential equations with either predetermined or non-predetermined variables. Conditions characterizing the existence and uniqueness of a solution are given and related to the local stability and determinacy properties of the steady state. In particular, it is shown that the relationship between the uniqueness of the solution and the stability of the steady-state is more subtle than the one that holds for ordinary differential equations, and gives rise to new dynamic configurations.

Keywords: Functional differential equations · Local dynamics · Existence · Determinacy

JEL Classification Numbers C61 · C62

1 Introduction

Mixed-type functional differential equations (MFDEs) allow us to describe the dynamics of a variable whose time derivative depends on its past and future values. A great number of dynamic economic problems in continuous time could be written with an MFDE; however, some simplifying assumptions are commonly used in order to reduce the problem to a system of ordinary differential equations (ODE). As an illustration, the unrestricted form of dynamics of an overlapping generations model[1] results in an MFDE except in the case where exponential forms are retained for the survival, discount, and endowment functions (Blanchard [6]). Similarly, models that consider lagged price contracts (Whelan [24]) or vintage capital[2] generally have dynamics characterized by an MFDE. The purpose of this article is to put forward conditions for the uniqueness of the solution and the asymptotic stability of such MFDEs. These conditions are the equivalent for the MFDEs of the Blanchard and Kahn conditions that apply to finite-dimensional systems (Blanchard and Kahn [7], Buiter [11]).

The Blanchard and Kahn conditions are based upon the set of initial conditions of the system and a spectral decomposition of the characteristic equation. More precisely, by comparing the dimensions of the space of predetermined variables with those of the stable eigenspace (or equivalently, by comparing the dimension of the space of non-predetermined variables and those of the unstable eigenspace) they characterize the local uniqueness and stability in the neighborhood of a steady state. On the other hand, for functional differential equations, some of the spaces are infinite-dimensional and the Blanchard and Kahn conditions do not apply. This is the case for the stable eigenspace for delay differential equations (DDE). In the standard case where the variables are predetermined and continuous, there is at most one solution to this type of equation (Diekmann et al. [15]). Otherwise, multiple solutions may arise if the dimension of the space of predetermined variables is greater than that of the

[1] Demichelis [12], Boucekkine et al. [8], d'Albis and Augeraud-Véron [1] and Edmond [13].
[2] Benhabib and Rustichini [5] and Boucekkine et al. [9].

unstable eigenspace (d'Albis et al. [3], [4]). On the contrary, advance differential equations (ADE) are characterized by an unstable eigenspace of infinite dimension and it is necessary to compare the dimension of the stable eigenspace with that of the predetermined variables.

The difficulty with MFDEs, which contain both delays and advances, is that both the stable manifold and the unstable manifold are infinite-dimensional. In order to establish our results, we use and extend the results of Mallet-Paret and Verduyn Lunel [20]. This approach consists of analyzing a factorization of the characteristic equation of the MFDE in question, written as the product of two characteristic equations associated with a DDE and an ADE, respectively. Existence and uniqueness of solutions to either differential equation depend on the number of misplaced roots of the respective characteristic function. Taking the difference between these numbers, we are able to provide conditions for existence and uniqueness of solutions to the MFDE, and to characterize the stability properties and degree of indeterminacy in the neighborhood of a steady state. We then extend these results to certain algebraic equations of mixed type.

The advantage of the technique we propose is that it is simple enough to implement, as we illustrate in three examples. It is an alternative to the existing procedures based on a formulation in discrete time (Gautier [14]) or to numerical methods (Collard et al. [10]), which are well suited when the characteristic functions are complicated.

Most importantly, the theoretical analysis of MFDEs gives rise to new and interesting dynamic configurations and reconsiders the link between the local uniqueness of a solution and the stability of a steady state. In particular, we show that the dynamics of a predetermined variable may be both stable and indeterminate, while with an ODE stability implies uniqueness. In addition, the dynamics of a non-predetermined variable may be both stable and determinate, while with an ODE stability implies indeterminacy. Finally, we show that a non-predetermined variable does not generally jump to its steady state value.

The article is organized as follows. In Section 2, we begin with a simple economic model illustrating the equations we are going to study. Then, we explain why the presence of advances and the definition of initial conditions imply that the mathematical problem is ill-posed, and why this may lead to the non-existence or the multiplicity of solutions. This section also allows us to relate our contribution to existing literature. In Section 3, we present our results on the existence and uniqueness of solutions of MFDEs as well as two examples that we solve in order to illustrate our theorems. In Section 4, we extend our results to algebraic equations of mixed type and solve an example. We also put forward a linearization theorem in order to apply our results to non-linear MFDEs. We conclude in Section 5.

2 Presentation of the problem

2.1 A simple economic model

To introduce the equations we are going to study, let us consider a model where the investment goods follow a “one-hoss shay" depreciation rule. Such goods contribute to the capital stock throughout their lifetime before falling to a zero scrap value. Let $I(s)$ be the investment implemented at date $s \leq t$ and let $\tau \in \mathbb{R}_+$ be the lifespan of the investment goods. The capital stock is therefore at date t equal to the sum of all investments made between dates $t-\tau$ and t:

$$K(t) = \int_{t-\tau}^{t} I(s)\, ds. \tag{1}$$

If one considers both a Solowian framework, where the investment chosen at time t is proportional to the demand received by the firms in the same period (i.e. $I(t) = \alpha Y(t)$), and an equilibrium on the goods market that equalizes demand and production such that $Y(t) = F(K(t))$, equation (1) can be rewritten as:

$$K(t) = \alpha \int_{t-\tau}^{t} F(K(s))\, ds. \tag{2}$$

Differentiating with respect to time, one obtains a DDE:

$$K'(t) = \alpha F(K(t)) - \alpha F(K(t-\tau)). \tag{3}$$

If one considers a more sophisticated framework where the investment chosen at time t is proportional to the demand that firms expect to receive throughout the lifetime of the investment good, equation (1) can be rewritten as:

$$K(t) = \alpha \int_{t-\tau}^{t} \int_{s}^{s+\tau} F(K(v))\, dv ds, \tag{4}$$

which is an algebraic equation of mixed type. Differentiating (4) with respect to time yields an MFDE:

$$K'(t) = \alpha \int_{t}^{t+\tau} F(K(v))\, dv - \alpha \int_{t-\tau}^{t} F(K(v))\, dv. \tag{5}$$

In Section 4.1.3, we solve equation (4) in the particular case of a linear production function given by $F(K(t)) = AK(t)$. This is, of course, an illustrative example, as MFDEs usually arise in models with more relevant microeconomic foundations. But in all cases, delays are due to the vintage structure of a stock variable (e.g. capital, population, price or wage contracts, etc.) whereas advances are due to the forecasts made by the agents. Comprehensive presentations of the use of MFDEs in economics are given in Collard et al. [10] for vintage capital models and in d'Albis and Augeraud-Véron [2] for overlapping generation models.

2.2 The mathematical problem

Let $t \in \mathbb{R}_+$ be the time index. We consider the following scalar linear MFDE:

$$x'(t) = \int_{-a}^{b} x(u+t)\, d\mu(u), \tag{6}$$

where $(a, b) \in \mathbb{R}_+^2$ and where μ is a measure on $[-a, b]$. Equation (6) characterizes dynamics for which the time-derivative at time t of a variable x, denoted $x'(t)$, depends on both the delayed values of x over the interval $[t-a, t)$ and on the advanced values of x over the interval $(t, t+b]$.

Due to the delay, initial conditions are defined over an interval. But, as usual, initial conditions may be of two different types. First, variable x can be predetermined (sometimes referred to as backward-looking), with initial condition of equation (6) written as:

$$x(\xi) = x_0(\xi) \text{ for } \xi \in [-a, 0], \tag{7}$$

where $x_0 \in \mathcal{C}([-a, 0])$, the space of continuous functions on $[-a, 0]$. Second, for a non-predetermined (or forward-looking) variable, the initial condition can be written as:

$$x(\xi) = x_0(\xi) \text{ for } \xi \in [-a, 0), \tag{8}$$

where $x_0 \in \mathcal{C}^b([-a, 0])$, the space of continuous functions on $[-a, 0)$ such that $x_0(0^-)$ exists. Here, $x(0^+)$ is not given and may be different from $x_0(0^-)$.

Note that equation (6) admits a unique steady state, namely $x = 0$, but may also be solved by functions that grow at a constant growth rate, thereby exhibiting a balanced growth path (BGP). This allows us to characterize the local stability of either a steady state or a BGP.

Let us first define a solution to the problem being considered.

Definition 1. *A solution with maximal growth rate $\eta \in \mathbb{R}$ is a function $x : [-a, +\infty) \rightarrow \mathbb{R}$ with $\|x\|_\eta := \sup_{t \in \mathbb{R}_+} e^{-\eta t} \mid x(t) \mid < \infty$ that is continuous on $\mathbb{R}_+$ and satisfies (6) together with either (7) or (8).*

It is worth noting that solutions to MFDEs are defined according to their asymptotic growth properties and that parameter η can be used to investigate various situations. For each problem, an appropriate η is chosen and fixed for the subsequent analysis. For dynamics that are analyzed in the neighborhood of the steady state $x = 0$, the appropriate η will be fixed at ε, with $0 < \varepsilon \ll 1$. For dynamics that admit a BGP, η will be chosen to be just above the asymptotic growth rate. To simplify things, a solution with maximal growth rate η is hereafter referred to as a solution. In addition, according to Definition 1 and condition (8), a discontinuity at time $t = 0$ is permitted for problems with a non-predetermined variable. Finally, a convergence condition is introduced in Definition 1; this condition is not necessary for models that include a transversality condition.

The main relevant information about equation (6) is obtainable from its characteristic function

$$\Delta_L(\lambda) = \lambda - \int_{-a}^{b} e^{\lambda u} d\mu(u), \tag{9}$$

where L denotes the linear operator, defined as $Lx(t) = \int_{-a}^{b} x(u+t)\, d\mu(u)$. The characteristic roots (i.e. the roots of $\Delta_L(\lambda) = 0$) have been studied in Rustichini [23], Hupkes and Verduyn Lunel [18], and Hupkes [16]. They prove that the roots are isolated, and that for any line $\mathrm{Re}(z) = \eta$ in the complex plane, each of the half-planes $\{z \in \mathbb{C} : \mathrm{Re}(z) < \eta\}$ and $\{z \in \mathbb{C} : \mathrm{Re}(z) > \eta\}$ generically contains infinitely many roots. Because of this double infinity it is not possible to use a spectral projection formula to prove the existence and uniqueness of solutions, as done for DDE by d'Albis et al. [4].

In the following, we limit our analysis to dynamics with no center manifold.

Assumption 1. $\Delta_L(\lambda) = 0$ *has no roots that satisfy* $\mathrm{Re}(\lambda) = \eta$.

With Assumption 1, we therefore exclude MFDEs that may give rise to bifurcations.

Let us now explain why equations (7) or (8) are not sufficient initial conditions for a well-posed problem. Regarding the dynamics of a predetermined variable, it is natural to consider initial conditions in $C([-a,b])$. However, as shown by Rustichini [23], there always exist initial conditions in $C([-a,b])$ that do not lead to a continuous function on $[-a,\infty)$ that satisfies (6). To solve this problem, it is necessary to obtain a decomposition of $C([-a,b])$ as a direct sum decomposition $C([-a,b]) = Q_L(\eta) \oplus P_L(\eta)$, such that initial conditions in space $Q_L(\eta) \subset C([-a,b])$ lead to trajectories with maximal growth rates η, while initial conditions in space $P_L(\eta) \subset C([-a,b])$ lead to functions defined on $(-\infty, b]$ that satisfy $\sup_{t\in\mathbb{R}_-} e^{-\eta t}|x(t)| < \infty$. This decomposition, obtained by Rustichini [23] allows for the definition of a stable manifold and an unstable manifold –both in $C([-a,b])$– that corresponds to a saddle-path configuration. Similarly, in the case of a non-predetermined variable, d'Albis et al. [3] show that the natural state space would be $C([-a,0]) \times C([0,b])$, a space of functions that are continuous over $[-a,0]$ and $[0,b]$. These functions are multivalued at 0 as both $x_0(0^-)$ and the jump $x_0(0^+)$ are considered. A decomposition of $C([-a,0]) \times C([0,b])$, similar to the one of Rustichini [23], can then be obtained.

However, those decompositions are not enough to solve equation (6) with initial condition (7) or (8), as those initial conditions are just functions in $C([-a,0])$ or $\mathcal{C}^b([-a,0])$. Therefore, one has to go beyond and analyze the properties of the restriction operator that maps $Q_L(\eta)$ to $C([-a,0])$, as it is done by Mallet-Paret and Verduyn Lunel [20] and [21]. More precisely, these authors check whether any initial condition over $[-a,0]$ can be extended continuously over $[-a,b]$ as a function that belongs to $Q_L(\eta)$. When such an extension over $[0,b]$ is not available, the non-existence of a solution to the MFDE can be established. When the extension is available, it constitutes, together with the initial condition over $[-a,0]$, a function in the natural state space that leads to a unique solution. Because the extension over $[0,b]$ may not necessarily be unique, multiple solutions may emerge. D'Albis et al. [3] analyze the case of non-predetermined variables and check whether any initial condition in $\mathcal{C}^b([-a,0])$ can be extended over $[0,b]$. They use a modified restriction operator that maps any elements of $\mathcal{C}([-a,0]) \times \mathcal{C}([0,b])$ that lead to a solution into $\mathcal{C}([-a,0]) \times \mathcal{C}([0,+\infty))$. As for predetermined variables, the solution may not exist and, if it does, may not be unique.

The multiplicity of solutions that may appear for such dynamic problems happens to be related to certain definitions of indeterminacy that are used in economics. More precisely, we use the following definition, proposed by Polemarchakis [22].

Definition 2. *A solution has $K \in \mathbb{N}$ degrees of indeterminacy if the set of distinct solutions contains a K-dimensional open set*[3]*.*

According to this definition, the solution is unique if $K = 0$ and indeter-

[3] A K-dimensional open set is the image of a continuously differentiable, one-to-one function with domain an open neighborhood in K-dimensional euclidean space (Polemarchakis [22]).

minate for $K \geq 1$. When there are infinitely many solutions, the degree of indeterminacy allows for defining families of solutions that are parameterized by $\alpha \in \mathbb{R}^K$. Note that we cannot use a definition of indeterminacy that is based on the comparison between the numbers of roots with positive real parts and missing initial conditions (as in Buiter [11] for finite-dimensional systems and d'Albis et al. [4] for DDE). Indeed, as shown above, the missing initial condition lies in an infinite-dimensional set as it is a continuous function on $(0, b]$ or $[0, b]$.

To compute the degree of indeterminacy according to Definition 2, we are going to use a theoretical concept put forth by Mallet-Paret and Verduyn Lunel [20] that is based on "misplaced" characteristic roots. More precisely, they show that characteristic function (9) can be factorized as:

$$(\lambda - \lambda_0)\Delta_L(\lambda) = \Delta_{L_-}(\lambda)\Delta_{L_+}(\lambda), \tag{10}$$

where $\lambda_0 \in \mathbb{R}$, $\Delta_{L_-}(\lambda)$ is the characteristic equation of a DDE, and $\Delta_{L_+}(\lambda)$ is the characteristic equation of an ADE. Let us note that factorization (10) is not uniquely defined as one can swap roots from one factor to another. Furthermore, an explicit factorization is generally difficult to obtain. Mallet-Paret and Verduyn Lunel then define an integer, denoted $n_L^\sharp(\eta)$, that can be written as:

$$n_L^\sharp(\eta) = n_{L_+}^-(\eta) - n_{L_-}^+(\eta) + n_0(\eta), \tag{11}$$

where $n_{L_+}^-(\eta)$ is the number of characteristic roots of $\Delta_{L_+}(\lambda) = 0$ such that $\mathrm{Re}(\lambda) < \eta$, $n_{L_-}^+(\eta)$ is the number of characteristic roots of $\Delta_{L_-}(\lambda) = 0$ such that $\mathrm{Re}(\lambda) > \eta$, and $n_0(\eta)$ is equal to 1 if $\lambda_0 > \eta$ and to 0 if $\lambda_0 < \eta$. The localization of roots of DDE or ADE implies that both $n_{L_-}^+(\eta)$ and $n_{L_+}^-(\eta)$ are finite, which consequently implies that $n_L^\sharp(\eta)$ is also finite. Depending on whether the variable is predetermined or not, the positivity of $n_{L_-}^+(\eta)$ and $n_{L_+}^-(\eta)$ would be an indication of the non-existence of the solution of the differential equations related to $\Delta_{L_-}(\lambda)$ and $\Delta_{L_+}(\lambda)$ (see d'Albis et al. [3]). The difference between $n_{L_-}^+(\eta)$ and $n_{L_+}^-(\eta)$ is thus an indication of the "misplaced" characteristic roots of the factorization.

Under the assumption that the measure μ is atomic at both at $u = -a$ and $u = b$, Mallet-Paret and Verduyn Lunel [20] proved that $n_L^\sharp(\eta)$ is an invariant of equation (6). For the models in economics that we are aware of, however, this assumption is too restrictive. Hupkes and Augeraud-Véron [17] have extended Mallet-Paret and Verduyn Lunel's results by showing that it is enough to assume the atomicity asymptotically. It is thus sufficient to make the following assumption:

Assumption 2. *There exist $s_\pm \in \mathbb{R}_+$ and $J_\pm \in \mathbb{R}_*$ such that the following asymptotic expansions hold true:*

$$\Delta_L(\lambda) = \begin{cases} \lambda^{-s_+} e^{\lambda b}(J_+ + o(1)) \text{ as } \lambda \to +\infty, \\ \lambda^{-s_-} e^{-\lambda a}(J_- + o(1)) \text{ as } \lambda \to -\infty. \end{cases} \tag{12}$$

In the next section, we demonstrate how to relate the value of $n_L^\sharp(\eta)$ to the existence and uniqueness of the solution. However, first let us show how to compute the invariant integer in the most likely case, where explicit factorization is not available. The method, developed by Hupkes and Augeraud-Véron [17], consists of building a continuous path $\Gamma(\mu)$ for $\mu \in [0, 1]$, which allows a family of operators associated with MFDEs to be defined. Such a path is built so that $\Gamma(1)$ is operator L while $\Gamma(0)$ is an operator for which the characteristic equation can be explicitly factorized. Let us then suppose the following:

Assumption 3. *$\Delta_{\Gamma(\mu)}(\lambda) = 0$ has roots with $\mathrm{Re}(\lambda) = \eta$ for only a finite number of values of $\mu \in (0, 1)$, while $\Delta_{\Gamma(0)}(\lambda) = 0$ and $\Delta_{\Gamma(1)}(\lambda) = 0$ have no roots with $\mathrm{Re}(\lambda) = \eta$.*

Assumption 3 is not restrictive as a continuous path $\Gamma(\mu)$ always exists. The example in section 3.3.1 illustrates how to build a path that satisfies the properties exhibited in Assumption 3. In the following, we use this assumption to prove that the explicit computation of the invariant integer of equation (6) is always possible.

Lemma 1. *Let Assumptions 1, 2, and 3 prevail. The invariant integer* $n_L^\sharp(\eta)$ *of equation (6) can be computed explicitly.*

Proof. See Appendix.

We conclude this presentation of the problem with a discussion of the stability of the steady state (or, equivalently, of the BGP). As mentioned above, even if the variable is a scalar, the configuration is a saddle path with stable and unstable manifolds that are infinite-dimensional. Within this configuration, and by analogy with finite-dimensional problems (such as those involving ODEs), the stability of the steady state may be defined as follows:

Definition 3. *The steady state is said to be stable if at least one solution with a non-positive growth rate exists and unstable if this is not the case. The steady state is said to be saddle-point stable if this solution is unique.*

Definition 3 links the existence and uniqueness of a solution to the stability of the steady state. If any initial condition can be extended in the natural state space, one obtains functions that initiate solutions which converge to the steady state, which is thus stable. The saddle path configuration implies that any initial condition can also be extended to initiate functions that diverge. In order to keep the analogy with finite-dimensional problems, we nevertheless restrict the notion of saddle-point stability to cases where a unique solution is initiated by any function defined in the natural state space. Alternatively, if for any initial condition it is not possible to find an extension, all initiated functions are divergent and the steady state becomes unstable. As we are in a saddle path configuration, initial conditions that lead to a solution always exist, but those conditions are not generic. Definition 3 extends to BGP by considering solutions with maximal growth rate η.

3 Main theorems

3.1 Equations with predetermined variables

Consider the following problem with a predetermined variable:

$$\begin{cases} x'(t) = \int_{-a}^{b} x(u+t)\, d\mu(u), \\ \\ x(\xi) = x_0(\xi) \ \text{ for } \xi \in [-a, 0]. \end{cases} \tag{13}$$

Theorem 1. *Let Assumptions 1, 2, and 3 prevail. If $n_L^\sharp(\eta) \geq 1$, the degree of indeterminacy of problem (13) is equal to $n_L^\sharp(\eta) - 1$ and the steady state, or the BGP, is stable. If $n_L^\sharp(\eta) < 1$, problem (13) has no solution and the steady state, or the BGP, is unstable.*

Proof. See Appendix.

Theorem 1, which is a corollary of Theorem 6.2 in Mallet-Paret and Verduyn Lunel [20], relates $n_L^\sharp(\eta)$ to the existence and uniqueness of solutions to equation (13) and can thus be used to asses the determinacy and stability properties of any model characterized by such an equation.

In particular, for $n_L^\sharp(\eta) > 1$, any initial condition in $\mathcal{C}([-a,0])$ can be extended in $\mathcal{C}([-a,b])$ to initiate a solution. These extensions are not unique. Any initial condition can lead to many solutions that converge to a steady state, or a BGP, which are thus both stable and indeterminate. Contrary to finite-dimensional systems (e.g. with ODEs) and to DDEs, the dynamics of a predetermined variable in the neighborhood of the steady state, or the BGP, can be simultaneously stable and indeterminate. There are an infinite number of solutions and the degree of indeterminacy indicates the number of missing real parameters in the equation of the solution. More precisely, the solutions are parameterized by a vector of dimension $n_L^\#(\eta) - 1$, namely $(\alpha_i)_{i=1..n_L^\#(\eta)-1}$. For one solution, all other solutions may be written as:

$$x_\alpha(t) = x_{(0,..,0)}(t) + \sum_{i=1}^{n_L^\sharp(\eta)-1} \alpha_i g_i(t), \tag{14}$$

where $g_i(t)$ can be explicitly computed using only the initial conditions.

If $n_L^\sharp(\eta) = 1$, any initial condition in $\mathcal{C}([-a,0])$ can be extended in $\mathcal{C}([-a,b])$ to initiate a solution, but this extension is unique. The steady state, or the BGP, is both stable and determinate or, according to Definition 3, saddle-point stable. In order to solve equation (6) with initial condition (7), one needs the explicit factorization (10) for which we know there exists a DDE associated to function $\Delta_{L_-}(\lambda)$, if λ_0 is not a root of $\Delta_{L_-}(\lambda) = 0$ or otherwise associated to $\Delta_{L_-}(\lambda) / (\lambda - \lambda_0)$. The solution is the one obtained by solving the DDE in initial conditions (7). According to Diekmann et al. [15], the solution can be written asymptotically as:

$$x(t) = \sum_{j=1}^{l} p_j(t) e^{\lambda_j t} + o\left(e^{\beta t}\right) \text{ for } t \to +\infty, \tag{15}$$

where $\beta \in \mathbb{R}$ and $(\lambda_j)_{j=1..l}$ are the roots of either equation $\Delta_{L_-}(\lambda) = 0$ or $\Delta_{L_-}(\lambda) / (\lambda - \lambda_0) = 0$ such that $\mathrm{Re}(\lambda_j) > \beta$, and $p_j(t)$ are polynomials in t.

If $n_L^\sharp(\eta) < 1$, there exist initial conditions in $\mathcal{C}([-a,0])$ that cannot be extended within $\mathcal{C}([-a,b])$ to initiate a solution. However, there exists a non-empty subspace of $\mathcal{C}([-a,0])$ that contains initial conditions leading to a solution. The codimension of this subspace is finite and equal to $1 - n_L^\sharp(\eta)$.

The figures below represent the beginning of trajectories that converge to (Figures (1a) and (1b)) or diverge from (Figure (1c)) steady state $x = 0$, given the different values of $n_L^\sharp(0)$. Figure (1b) represents the dynamic configuration that is specific to MFDEs and that cannot be found when an ODE is considered.

Figures 1a, 1b, 1c about here

3.2 Equations with non-predetermined variables

Now consider the problem with a non-predetermined variable

$$\begin{cases} x'(t) = \int_{-a}^{b} x(u+t)\, d\mu(u), \\ x(\xi) = x_0(\xi) \text{ for } \xi \in [-a, 0). \end{cases} \tag{16}$$

Theorem 2. *Let Assumptions 1, 2, and 3 prevail. If $n_L^\sharp(\eta) \geq 0$, the degree of indeterminacy of problem (16) is equal to $n_L^\sharp(\eta)$ and the steady state, or the*

BGP, is stable. If $n_L^\sharp(\eta) < 0$, problem (16) has no solution and the steady state, or the BGP, is unstable.

<u>Proof.</u> See Appendix.

Theorem 2 is the counterpart of Theorem 1 for non-predetermined variables. The permitted initial jump provides extra degrees of freedom, which explains why the critical value of the invariant integer is now 0 rather than 1. The determinacy and stability properties are outlined below.

If $n_L^\sharp(\eta) > 0$, the solution is indeterminate. In the case where $n_L^\sharp(\eta) = 1$, for every $x(0^+) \in \mathbb{R}$, a unique extension that leads to a solution exists. This is an indeterminacy of degree 1, corresponding to the case of a stable steady state in a problem described by an ODE. When $n_L^\sharp(\eta) > 1$, for every $x(0^+) \in \mathbb{R}$, families of solutions defined on $[0, \infty)$ exist. The height of the initial jump, given by $x(0^+) - x_0(0^-)$, is not determined and starting from $x(0^+)$ there are an infinity of solutions converging to the steady state or the BGP. This kind of dynamic is specific to MFDEs and does not exist with scalar ODEs.

If $n_L^\sharp(\eta) = 0$, any initial condition within $\mathcal{C}^b([-a, 0])$ can be extended in $\mathcal{C}([0, b])$ to initiate a solution and this extension is unique. A unique solution that converges to the steady state or the BGP exists. According to Definition 3, this corresponds to saddle-point stability. In this case, the non-predetermined variable does not jump onto the steady state but onto a path that converges to it.

Finally, if $n_L^\sharp(\eta) < 0$, there are initial conditions in $\mathcal{C}^b([-a, 0])$ that cannot be extended to initiate a solution and there is no generic solution to (16).

The figures below represent the initial trajectories of the dynamics that converge to steady state $x = 0$. Figure (2a) represents a problem for which the invariant integer is 0. Here, there is an initial jump and a unique trajectory converging to the steady state. Note that the solution does not consist of a jump to the steady state. Figure (2b) represents an indeterminacy of degree 1, where the initial jump is not determined. Figure (2c) represents an indeterminacy of a degree larger than 1, where the initial jump is not determined and a family of trajectories is possible for each jump. This latter case is specific to MFDEs.

Figures 2a, 2b, 2c about here

A summary of Theorems 1 and 2 is presented in Table 1.

Table 1: Existence and uniqueness of solutions

	Predetermined variable	Non-predetermined variable
$n^\sharp < 0$	Non-existence	Non-existence
$n^\sharp = 0$	Non-existence	Existence and uniqueness
$n^\sharp = 1$	Existence and uniqueness	Indeterminacy of degree 1
$n^\sharp > 1$	Indeterminacy of degree $n^\sharp - 1$	Indeterminacy of degree $n^\sharp$

The cases where $n^\sharp > 1$ are specific to MFDEs and lead to dynamic configurations that do not exist with ODEs.

3.3 Examples

Let us now analyze two MFDEs that clearly illustrate Theorems 1 and 2. In the first example, we consider an equation in which the factorization of the characteristic equation is not explicit. We show how to compute the invariant integer by using Lemma 1. In the second example, we show that the solution of a scalar MFDE can display an indeterminacy of degree 2, corresponding to a situation that cannot occur with an ODE.

3.3.1 An application of Lemma 1

Let us consider the following equation:

$$x'(t) = x(t+1) + x(t-1), \tag{17}$$

which was first analyzed in Rustichini [23]. He shows that a solution may not exist for a given initial condition in $\mathcal{C}([-1,1])$. We focus on the existence and uniqueness of solutions that converge to the unique steady state $x = 0$. Variable x can be predetermined (i.e. $x(\xi)$ is given in $\mathcal{C}([-1,0])$), or non-predetermined (i.e. $x(\xi)$ is given in $\mathcal{C}^b([-1,0])$). Our results can be summarized as follows:

Proposition 1. *If x is predetermined, equation* (17) *has no solution. If x is non-predetermined, equation* (17) *has a unique solution.*

These results are proven by characterizing the roots of $\Delta_L(\lambda) = 0$, where:

$$\Delta_L(\lambda) = \lambda - e^{\lambda} - e^{-\lambda}. \tag{18}$$

First of all, we show that Assumption 1 is satisfied for $\eta = 0$ (as we consider trajectories that converge to the steady state).

Lemma 2. $\Delta_L(\lambda) = 0$ *has no pure imaginary root.*

Proof. See Appendix.

Next, we need to factorize the characteristic function. Let us denote:

$$\Delta_{L_+}(\lambda) = \lambda - 1 - e^{\lambda}, \tag{19}$$

$$\Delta_{L_-}(\lambda) = \lambda - 1 - e^{-\lambda} + e^{-1}. \tag{20}$$

We obtain

$$\Delta_{L_+}(\lambda)\,\Delta_{L_-}(\lambda) = (\lambda - 1)\left[\Delta_L(\lambda) + e^{-1} - 1 - \int_0^1 e^{(\lambda-1)u}du\right], \tag{21}$$

which does not have the same form as (10). In order to apply Lemma 1 we define a path operator, denoted $\Gamma(\mu)$, whose characteristic equation is given by:

$$\Delta_{\Gamma(\mu)}(\lambda) = \Delta_L(\lambda) + (1-\mu)\left(e^{-1} - 1 - \int_0^1 e^{(\lambda-1)u}du\right), \tag{22}$$

implying that $\Delta_{\Gamma(1)}(\lambda) = \Delta_L(\lambda)$ and $(\lambda - 1)\,\Delta_{\Gamma(0)}(\lambda) = \Delta_{L_+}(\lambda)\,\Delta_{L_-}(\lambda)$. The characteristic function $\Delta_{\Gamma(0)}(\lambda)$ can thus be explicitly factorized as in (10) with $\lambda_0 = 1$. Furthermore, it has no pure imaginary roots.

Lemma 3. $\Delta_{\Gamma(\mu)}(\lambda) = 0$ *has no pure imaginary root.*

Proof. See Appendix.

As a consequence, Lemma 1 applies to equation (17) and the associated invariant integer $n_L^{\sharp}(0)$ can be computed.

Lemma 4. $n_L^{\sharp}(0) = 0$.

Proof. See Appendix.

Using Theorems 1 and 2, Proposition 1 is deduced from Lemma 4.

3.3.2 An equation that gives rise to an indeterminacy of degree 2

Consider the following MFDE:

$$x'(t) = \alpha x(t) - \beta \int_{t-1}^{t} e^{\alpha(u-t)} x(u)\,du + \gamma\left[e^{-\alpha} x(t+1) - x(t)\right] \\ + \beta\gamma \int_{t}^{t+1} \int_{s}^{s+1} e^{-\alpha(u-t)} x(u)\,duds, \tag{23}$$

where $(\alpha, \beta, \gamma) \in \mathbb{R}^3_{++}$ and $\beta \int_{-1}^{0} u e^{-\alpha u} du > -1$. The characteristic function can be written as:

$$\Delta_L(\lambda) = \lambda - \alpha - \beta\left(\frac{e^{-(\lambda-\alpha)} - 1}{\lambda - \alpha}\right) - \gamma\left(e^{\lambda-\alpha} - 1\right) \\ + \beta a \left(\frac{e^{\lambda-\alpha} - 1}{\lambda - \alpha}\right)\left(\frac{e^{-(\lambda-\alpha)} - 1}{\lambda - \alpha}\right). \tag{24}$$

This function can be explicitly factorized as:

$$\Delta_L(\lambda) = (\lambda - \alpha)\,\Delta_{L_+}(\lambda)\,\Delta_{L_-}(\lambda), \tag{25}$$

where

$$\Delta_{L_+}(\lambda) = \lambda - \alpha - \gamma\left(e^{\lambda-\alpha} - 1\right) \text{ and } \Delta_{L_-}(\lambda) = \lambda - \alpha - \beta\left(\frac{e^{-(\lambda-\alpha)} - 1}{\lambda - \alpha}\right). \tag{26}$$

Using Theorems 1 and 2, the localization of the roots allows us to establish the existence and uniqueness of a solution that converges to zero as t tends to infinity. The next Lemma proposes a characterization of the roots.

Lemma 5. *If* $\gamma \int_0^1 e^{-\alpha u} du < 1$*, each root of* $\Delta_{L_+}(\lambda) = 0$ *has a non-negative real part. If* $\gamma \int_0^1 e^{-\alpha u} du > 1$, $\Delta_{L_+}(\lambda) = 0$ *has at least one negative real root.*
Proof. See Appendix.

Lemma 6. *If* $\beta \int_{-1}^{0} e^{-\alpha u} du > \alpha$*, each root of* $\Delta_{L_-}(\lambda) = 0$ *has a non-positive real part. If* $\beta \int_{-1}^{0} e^{-\alpha u} du < \alpha$, $\Delta_{L_-}(\lambda) = 0$ *has one positive real root.*
Proof. See Appendix.

With these two lemmas the following result can be established:

Proposition 2. *If* $\gamma \int_0^1 e^{-\alpha u} du > 1$ *and* $\beta \int_{-1}^{0} e^{-\alpha u} du > \alpha$*, then* $n^-_{L_+}(0) = 1$, $n^+_{L_-}(0) = 0$, $n_0(0) = 1$ *and thus* $n^\sharp_L(0) = 2$.

If $\gamma \int_0^1 e^{-\alpha u} du < 1$ *and* $\beta \int_{-1}^0 e^{-\alpha u} du > \alpha$, *then* $n_{L_+}^- (0) = 0$, $n_{L_-}^+ (0) = 0$, $n_0 (0) = 1$ *and thus* $n_L^\sharp (0) = 1$.

If $\gamma \int_0^1 e^{-\alpha u} du < 1$ *and* $\beta \int_{-1}^0 e^{-\alpha u} du < \alpha$, *then* $n_{L_+}^- (0) = 0$, $n_{L_-}^+ (0) = 1$, $n_0 (0) = 1$ *and thus* $n_L^\sharp (0) = 0$.

Depending on whether $x(t)$ is a predetermined or not, and on the values of the parameters, the solution may or may not exist and, upon existence, may be unique or indeterminate. In particular, if $x(t)$ is a non-predetermined variable with initial condition $x(\xi) = x_0(\xi)$ for $\xi \in \mathcal{C}^b([-1, 0])$, an indeterminacy of degree 2 (i.e. for each jump, there is a one-parameter family of solutions that solve the equation) may occur.

4 Extensions

4.1 Algebraic equations of mixed type

As we have seen in the example presented in Section 2.1, algebraic equations of mixed type, like (4), may arise in economic models. Below we consider algebraic equations that reduce to differential equations when differentiated a finite number of times. Theorems 1 and 2 may not, however, be applied immediately to these equations as the differentiation creates some extra roots that have to be eliminated in order to compute the invariant integer.

4.1.1 Equations with predetermined variables

We now extend Theorem 1 to scalar algebraic equations of mixed type. These can be written as:

$$\int_{-a}^{b} x(u+t)\, d\nu(u) = \int_{-a}^{b} x(u+t)\, d\mu(u) \tag{27}$$

with initial condition (7). The characteristic equation of (27) is

$$\int_{-a}^{b} e^{\lambda u} d\nu(u) = \int_{-a}^{b} e^{\lambda u} d\mu(u). \tag{28}$$

We are only concerned with algebraic equations that satisfy the following condition:

Assumption 4. *There exist $n \in \mathbb{N}^*$, for all $\gamma \in \mathbb{R}$, such that the characteristic function of (27), denoted $\delta(\lambda)$, satisfies $\delta(\lambda)(\lambda-\gamma)^n = \Delta_{L_\gamma}(\lambda)$ where where $\Delta_{L_\gamma}(\lambda)$ is the characteristic function of a scalar MFDE whose operator is denoted L_γ.*

For all $j = 1, ..., n$ let L_γ^j denote the operator associated with the algebraic equation whose characteristic function is $\delta(\lambda)(\lambda-\gamma)^j = \Delta_{L_\gamma^j}(\lambda)$; in particular $L_\gamma^n = L_\gamma$. Note that for $\gamma = 0$, Assumption 4 states that differentiating equation (27) n times leads to an MFDE with associated operator L_0.[4] Contrary to differential equation (6), an algebraic equation of type (27) would display n additional constraints at time $t = 0$. For $n > 1$, the constraints are written as:

$$\begin{cases} \int_{-a}^{b} x(u)\, d\nu(u) = \int_{-a}^{b} x(u)\, d\mu(u), \\ L_\gamma^j x(0) = 0, \text{ for all } j = 1..n-1. \end{cases} \tag{29}$$

For $n = 1$, which corresponds to the case where equation (27) reduces to

$$x(t) = \int_{-a}^{b} x(u+t)\, d\mu(u), \tag{30}$$

the initial constraint is one-dimensional and is given by

$$x_0(0) = \int_{-a}^{b} x(u)\, d\mu(u). \tag{31}$$

We characterize a solution with maximal growth rate η as in Definition 1 except that it now refers to equation (27) instead of (6). Note that it is equivalent to say that x is a solution to (27) or to say that x is a solution of an MFDE defined by an operator L_γ that satisfies the initial condition $x(\xi) = x_0(\xi)$ for $\xi \in [-a, 0]$, the growth condition $\|x\|_\eta < \infty$, and the constraints given by equation (29). The associated problem is stated as:

$$\begin{cases} \int_{-a}^{b} x(u+t)\, d\nu(u) = \int_{-a}^{b} x(u+t)\, d\mu(u), \\ x(\xi) = x_0(\xi) \text{ for } \xi \in [-a, 0]. \end{cases} \tag{32}$$

Theorem 3. *Let Assumption 4 prevails. Let us consider γ such that $\gamma > \eta$, no root of $\delta(\lambda) = 0$ satisfies $\mathrm{Re}(\lambda) \in [\eta, \gamma)$, and Assumptions 1, 2 and 3 prevail*

[4] Note that L_γ^j corresponds to the equation that arise after having applied $(d/dt - \lambda)^j$ to (27).

for L_γ. If $n^\sharp_{L_\gamma}(\eta) \geq 1$, the degree of indeterminacy of problem (32) is equal to $n^\sharp_{L_\gamma}(\eta) - 1$ and the steady state, or the BGP, is stable. If $n^\sharp_{L_\gamma}(\eta) < 1$, problem (32) has no solution and the steady state, or the BGP, is unstable.

Proof. See Appendix.

The relevant integer is now the one associated with the MFDE whose operator is L_γ. As detailed in Assumption 4, γ appears as an extra root of the characteristic equation with multiplicity n. Theorem 3 shows that the projection of the initial conditions on these extra roots is the null vector. As a consequence, the determinacy and stability properties of the algebraic equation can be assessed with the invariant integer $n^\sharp_{L_\gamma}(\eta)$. As the latter is associated with an MFDE characterizing a predetermined variable, the critical value for the integer is the same as in Theorem 1.

4.1.2 Equations with non-predetermined variables

Let us now extend Theorem 2 to scalar algebraic equations of mixed type that can be written as (27), with initial condition (8). The problem can now be written as:

$$\begin{cases} \int_{-a}^{b} x(u+t)\, d\nu(u) = \int_{-a}^{b} x(u+t)\, d\mu(u), \\ x(\xi) = x_0(\xi) \text{ for } \xi \in [-a, 0). \end{cases} \tag{33}$$

As for predetermined variables, we are only concerned with algebraic equations that satisfy Assumption 4 and establish the following:

Theorem 4. *Let Assumption 4 prevails. Let us consider γ such that $\gamma > \eta$, no root of $\delta(\lambda) = 0$ satisfies $\mathrm{Re}(\lambda) \in [\eta, \gamma)$, and Assumptions 1, 2 and 3 prevail for L_γ. If $n^\sharp_{L_\gamma}(\eta) \geq 0$, the degree of indeterminacy of problem (33) is equal to $n^\sharp_{L_\gamma}(\eta)$ and the steady state, or the BGP, is stable. If $n^\sharp_{L_\gamma}(\eta) < 0$, problem (33) has no solution and the steady state, or the BGP, is unstable.*

Proof. See Appendix.

The interpretation of Theorem 4 is similar to that of Theorem 3. As we consider a non-predetermined variable, the only difference lies in the critical number for the invariant integer that is now zero.

4.1.3 An example

The simple example presented in Section 2.1 can be used to illustrate Theorem 3. With a linear production function, equation (4) can be rewritten as:

$$K(t) = \alpha A \int_{t-\tau}^{t} \int_{s}^{s+\tau} K(v)\, dv ds, \tag{34}$$

for all $t \in \mathbb{R}_+$, while the initial condition can be written as:

$$K(\xi) = K_0(\xi) \text{ for } \xi \in [-\tau, 0]. \tag{35}$$

We will now assume that $\alpha A \tau^2 < 1$.

Proposition 3. *Problem (34)-(35) has a unique solution.*

This result is proven by characterizing the zeros of the characteristic function of (34) that satisfies

$$\delta(\lambda) = 1 - \alpha A \int_{-\tau}^{0} \int_{s}^{s+\tau} e^{\lambda v} dv ds. \tag{36}$$

The function $\delta(\lambda)$ is concave and symmetric with respect to the imaginary axis. Given the assumption we made about parameters, $\delta(\lambda) = 0$ has two real roots, $\bar{g} > 0$ and $-\bar{g}$. We now study the existence and uniqueness of a solution with asymptotic growth rate $\bar{g}$. Writing

$$\delta_-(\lambda) = 1 + i\sqrt{\alpha A} \int_0^{\tau} e^{-\lambda s} ds \text{ and } \delta_+(\lambda) = 1 + i\sqrt{\alpha A} \int_0^{\tau} e^{\lambda s} ds, \tag{37}$$

we have

$$\delta_-(\lambda)\,\delta_+(\lambda) = \delta(\lambda) + i\sqrt{\alpha A}\left[\int_0^{\tau} e^{-\lambda s} ds + \int_0^{\tau} e^{\lambda s} ds\right]. \tag{38}$$

We consider the function $\Delta_L(\lambda) = (\lambda - \gamma)\,\delta(\lambda)$, where it is assumed that $\gamma > \bar{g}$. Thus, there are two functions, $\Delta_+(\lambda) = (\lambda - \gamma)\,\delta_+(\lambda)$ and $\Delta_-(\lambda) = (\lambda - \gamma)\,\delta_-(\lambda)$, the characteristic functions of an ADE and a DDE, which satisfy

$$\Delta_+(\lambda)\,\Delta_-(\lambda) = (\lambda - \gamma)\left[\Delta_L + i\sqrt{\alpha A}\,(\lambda - \gamma)\left[\int_0^{\tau} e^{-\lambda s} ds + \int_0^{\tau} e^{\lambda s} ds\right]\right]. \tag{39}$$

We define a path operator, denoted $\Gamma(\mu)$, whose characteristic function is

$$\Delta_{\Gamma(\mu)}(\lambda) = \Delta_L(\lambda) + (1-\mu)\, i\sqrt{\alpha A}\,(\lambda-\gamma)\left[\int_0^\tau e^{-\lambda s} ds + \int_0^\tau e^{\lambda s} ds\right], \quad (40)$$

implying that $\Delta_{\Gamma(1)}(\lambda) = \Delta_{L_\gamma}(\lambda)$ and $\Delta_{\Gamma(0)}(\lambda) = (\lambda-\gamma)^{-1}\Delta_+(\lambda)\Delta_-(\lambda)$. Note that, unlike $\Delta_{\Gamma(1)}(\lambda)$, $\Delta_{\Gamma(0)}(\lambda)$ has an explicit factorization. As in the previous examples, the objective is to compute the invariant integer $n^\sharp_{\Gamma(1)}(\eta)$ for $\eta = \bar{g} + \varepsilon$, where $0 < \varepsilon \ll 1$. This is done in three steps. First, we use the symmetry of functions $\delta_+(\lambda)$ and $\delta_-(\lambda)$ to compute the invariant integer associated with $\Delta_{\Gamma(0)}(\lambda) = 0$ for $\eta = 0$. This yields:

Lemma 7. $n^\sharp_{\Gamma(0)}(0) = 0$.

Proof. See Appendix.

Next, we compute the invariant integer associated with $\Delta_{\Gamma(1)}(\lambda) = 0$ for $\eta = 0$. This is done by counting the number of roots that cross the line $\operatorname{Re}\lambda$ as μ goes from 0 to 1.

Lemma 8. $n^\sharp_{\Gamma(1)}(0) = 0$.

Proof. See Appendix.

Finally, we compute the invariant integer associated with $\Delta_{\Gamma(1)}(\lambda) = 0$ for $\eta = \bar{g} + \varepsilon$. This is done by counting the roots so that $\operatorname{Re}\lambda \in [0, \bar{g}]$.

Lemma 9. $n^\sharp_{\Gamma(1)}(\bar{g}+\varepsilon) = 1$.

Proof. See Appendix.

Proposition 3 is then deduced from Theorem 3 and Lemma 9.

4.2 A linearization theorem

Let us now consider a non-linear MFDE

$$x'(t) = Lx(t) + g(Mx), \quad (41)$$

where L is as before, M is defined by $Mx(t) = \int_{-a}^{b} x(u+t)\, d\nu(u)$, ν being a measure on $[-a, b]$.

We impose the following conditions on the nonlinearity g, which basically states that L contains all the linear terms when linearizing (41) around the steady state.

Assumption 5. *The function $g : \mathbb{R} \to \mathbb{R}$ is C^k-smooth with $g(0) = g'(0) = 0$.*

We denote by $\widehat{Q}_L(\eta) \subset C([-a,0],\mathbb{R}) \times C([0,b],\mathbb{R})$ the set of initial conditions leading to a solution as defined by Definition 1, that is for the linear equation $x'(t) = Lx(t)$. As described in d'Albis et al. [3], it is possible to define a projection

$$\Pi_{\widehat{Q}_L(\eta)} : C([-a,0]) \times C([0,b]) \to \widehat{Q}_L(\eta) \tag{42}$$

so that $\mathrm{Range}\,\Pi_{\widehat{Q}_L(\eta)} = \widehat{Q}_L(\eta)$. Finally, we introduce the subset:

$$\widehat{\mathcal{V}}_\delta(\eta) = \left\{ \psi \in \widehat{Q}_L(\eta) : ||\psi|| < \delta \right\}. \tag{43}$$

Theorem 5. *Let Assumptions 1 and 5 prevail for some $\eta \leq 0$. There exist $0 < \delta_* < \delta$ and a C^k-smooth map*

$$u^* : \widehat{\mathcal{V}}_\delta(\eta) \to C([-a,0],\mathbb{R}) \times \left\{ x \in C([0,\infty),\mathbb{R}) : \|x\|_\eta < \infty \right\} \tag{44}$$

with $u^*(0) = 0$ and $Du^*(0) = 0$, such that for any $\phi \in \widehat{\mathcal{V}}_\delta(\eta)$, the function $x = u^*(\phi)$ satisfies (41), with

$$\Pi_{\widehat{Q}_L(\eta)} \left(x_{|[-a,0]} \times x_{|[0,b]} \right) = \phi. \tag{45}$$

Moreover, every solution x to (41) with $|x(t)| < \delta_* e^{-\eta t}$ for all $t \geq -a$ can be written in this form.

Proof. See Appendix.

5 Conclusion

In this article, we presented the conditions for the existence, uniqueness, and stability of a solution to an MFDE or to mixed-type algebraic equations. Furthermore, we proposed an explicit and simple method that verifies whether or not these conditions are satisfied in a given model. We hope this method will encourage the use of functional differential equations in economics and permit a better understanding of vintage capital and overlapping generation models.

In future research, we aim to establish the conditions for the existence and uniqueness of solutions to MFDE systems. We also want to develop numerical techniques for computing the invariant integer.

APPENDIX

Proof of Lemma 1. Consider any continuous path $\Gamma : [0,1] \to \mathcal{L}(\mathcal{C}([-a,b]),\mathbb{R})$, such that operators $\Gamma(\mu)$ satisfy Assumption 2 for all $0 \le \mu \le 1$, that $\Delta_{\Gamma(0)}(\lambda) = (\lambda - \gamma)\,\Delta_{L^-}(\lambda)\,\Delta_{L^+}(\lambda)$ and $\Delta_{\Gamma(1)}(\lambda) = \Delta_L(\lambda)$. We then follow the proof of Theorem 2.5 in Hupkes and Augeraud-Véron [17]. First, we know, from Mallet-Paret and Verduyn Lunel [20], that the invariant $n^{\sharp}_{\Gamma(0)}(\eta)$ can be computed (it is generated using equation (10)). Second, we may denote roots of $\Delta_{\Gamma(\mu)}(\lambda) = 0$ as $\lambda(\mu)$ since they continuously depend on parameter μ. Let us analyze how the roots of $\Delta_{\Gamma(\mu)}(\lambda) = 0$ cross the real line $\mathrm{Re}(\lambda) = \eta$ when μ varies from 0 to 1. Let μ^* be the smallest value of μ within $(0,1)$ such that a root of $\Delta_{\Gamma(\mu^*)}(\lambda) = 0$, denoted $\bar\lambda(\mu^*)$, exists and satisfies $\mathrm{Re}\left(\bar\lambda(\mu^*)\right) = \eta$. Assume that for $\mu < \mu^*$, $\mathrm{Re}\left(\bar\lambda(\mu)\right) < \eta$ while for $\mu > \mu^*$, $\mathrm{Re}\left(\bar\lambda(\mu)\right) > \eta$. Let us now compute the integer $n^{\sharp}_{\Gamma(\mu)}(\eta)$ for μ in the neighborhood of μ^*. The easiest way to proceed is to compare $n^{\sharp}_{\Gamma(\mu^*)}(\eta+\varepsilon)$ with $n^{\sharp}_{\Gamma(\mu^*)}(\eta-\varepsilon)$ for $\varepsilon > 0$ small enough. Two situations may occur: either $\bar\lambda(\mu^*)$ belongs to the roots of the delay characteristic equation $\Delta_{\Gamma(\mu^*)^-}(\lambda) = 0$ and $n^{+}_{\Gamma(\mu^*)}(\eta-\varepsilon) = n^{+}_{\Gamma(\mu^*)}(\eta+\varepsilon)+1$, or $\bar\lambda(\mu^*)$ belongs to the roots of the advance characteristic equation $\Delta_{\Gamma(\mu^*)^+}(\lambda) = 0$ and $n^{-}_{\Gamma(\mu^*)}(\eta-\varepsilon) = n^{-}_{\Gamma(\mu^*)}(\eta+\varepsilon)-1$. According to definition (11), we have: $n^{\sharp}_{\Gamma(\mu^*)}(\eta+\varepsilon) = n^{\sharp}_{\Gamma(\mu^*)}(\eta-\varepsilon)+1$. As the roots are isolated, for $\mu < \mu^*$, $n^{\sharp}_{\Gamma(\mu)}(\eta) = n^{\sharp}_{\Gamma(\mu^*)}(\eta+\varepsilon)$ while for $\mu > \mu^*$, $n^{\sharp}_{\Gamma(\mu)}(\eta) = n^{\sharp}_{\Gamma(\mu^*)}(\eta-\varepsilon)$. We conclude that $n^{\sharp}_{\Gamma(\mu^*+\varepsilon)}(\eta) = n^{\sharp}_{\Gamma(\mu^*-\varepsilon)}(\eta)+1$. Replicating this argument when μ varies from μ^* to 1, we obtain $n^{\sharp}_{\Gamma(1)}(\eta) = n^{\sharp}_{\Gamma(0)}(\eta) - \mathrm{cross}(\Gamma,\eta)$, where $\mathrm{cross}(\Gamma,\eta)$ is the number of roots of $\Delta_{\Gamma(\mu)}(\lambda) = 0$–counted with multiplicity– that cross the line $\mathrm{Re}(z) = \eta$ from left to right as μ increases from 0 to 1. $\square$

Proof of Theorem 1. The existence and uniqueness properties of the solution are deduced from the computation of the invariant integer $n^{\sharp}_{L}(\eta)$. We use the result of Mallet-Paret and Verduyn Lunel [20] that the restriction operator presented in Section 2.2, which we denote as $\pi^{-}_{Q_L(\eta)}$ and define as $\pi^{-}_{Q_L(\eta)}$:

$Q_L(\eta) \to \mathcal{C}([-a,0],\mathbb{C}^n)$, $\psi \mapsto \psi_{|[-a,0]}$, plays a major role in the description of the properties of operator L. Most notably, they show that $\dim \operatorname{Ker} \pi^-_{Q_L(\eta)} = \max\{n^\sharp_L(\eta)-1, 0\}$ and that $\operatorname{codim} \operatorname{Range} \pi^-_{Q_L(\eta)} = \max\{1-n^\sharp_L(\eta), 0\}$. $\operatorname{Ker} \pi^-_{Q_L(\eta)}$ is composed of the trivial solution zero and of nonzero solutions (as in Definition 1) where initial conditions on $\mathcal{C}([-a,0])$ are equal to zero. Multiple solutions can exist if $\dim \operatorname{Ker} \pi^-_{Q_L(\eta)} > 0$. Conversely, if $\operatorname{Range} \pi^-_{Q_L(\eta)} \neq \mathcal{C}([-a,0])$, which happens for $\operatorname{codim} \operatorname{Range} \pi^-_{Q_L(\eta)} > 0$, initial conditions $x_0 \in \mathcal{C}([-a,0])$ exist for which the equation has no solution. Thus, if $n^\sharp_{\Gamma(1)}(\eta) < 1$, any initial condition within $\mathcal{C}([-a,0])$ cannot be extended to a solution. Mallet-Paret and Verduyn Lunel [20] show that if $n^\sharp_{\Gamma(1)}(\eta) \geq 2$, at least two solutions with the same initial condition on $[-a,0]$ can be considered. The difference between these two solutions is a solution with an initial condition equal to 0 on $[-a,0]$. They define a prolongation as the restriction of a solution to $[0,+\infty)$ and they show that the prolongation of 0, called $y(t)$, exists if and only if the Laplace transform of the difference, defined as $\hat{y}(\lambda) = \int_0^{+\infty} y(t)\, e^{-\lambda t} dt$, is a function of a polynomial of degree $n^\sharp_{\Gamma(1)}(\eta) - 2$. This gives a space of $n^\sharp_{\Gamma(1)}(\eta) - 1$ coefficients that can be freely chosen to define a prolongation. If $n^\sharp_{\Gamma(1)}(\eta) \geq 2$ we get a multiplicity of extensions and, using Definition 1, an indeterminacy of degree $n^\sharp_{\Gamma(1)}(\eta) - 1$. If $n^\sharp_{\Gamma(1)}(\eta) = 1$, the only prolongation of 0 is the function that takes the value of 0 on $[0,\infty)$. $\square$

Proof of Theorem 2. We define $\widehat{Q}_L(\eta) \subset \mathcal{C}([-a,0]) \times \mathcal{C}([0,b])$ as the space of initial conditions leading to a solution. Next, we formally define the modified restriction operator we presented in Section 2.2 as:

$$\widehat{\pi}^-_{\widehat{Q}_L(\eta)} : \widehat{Q}_L(\eta) \to \mathcal{C}([-a,0]) \times \mathbb{R} \qquad (\psi^l, \psi^r) \mapsto \left(\psi^l, \psi^r(0)\right).$$

Following Hupkes and Augeraud-Véron [17], we have to prove that codim Range $\widehat{\pi}^-_{\widehat{Q}_L(\eta)} = \max\{-n^\sharp_L(\eta), 0\}$ and $\dim \operatorname{Ker} \widehat{\pi}^-_{\widehat{Q}_L(\eta)} = \max\{n^\sharp_L(\eta), 0\}$. We proceed by using the results presented in the proof of Theorem 1. By definition, we have:

$$\operatorname{Ker} \widehat{\pi}^-_{\widehat{Q}_L(\eta)} = \left\{ x = (x_1, x_2) \in \widehat{Q}_L(\eta), \text{ such that } \widehat{\pi}^-_{\widehat{Q}_L(\eta)}(x_1, x_2) = 0 \right\}.$$

Suppose that two solutions, denoted x and $\bar{x}$, in $\operatorname{Ker}\widehat{\pi}^{-}_{\widehat{Q}_L(\eta)}$ exist. This implies that $x_1(\xi) - \bar{x}_1(\xi) = 0$ for all $\xi \in [-a, 0]$ and $x_2(0^+) - \bar{x}_2(0^+) = 0$. Compared to the case with a predetermined variable, there is one additional constraint that implies that $\dim\left(\operatorname{Ker}\widehat{\pi}^{-}_{\widehat{Q}_L(\eta)}\right) = \dim\left(\operatorname{Ker}\pi^{-}_{Q(\eta)}\right) + 1$, where $\pi^{-}_{Q(\eta)}$ is the restriction operator defined in the proof of Theorem 1. Similarly, by definition:

$$\operatorname{Range}\widehat{\pi}^{-}_{\widehat{Q}_L(\eta)} = \left\{\left(x_1(\xi), x_2\left(0^+\right)\right) = \widehat{\pi}^{-}_{\widehat{Q}_L(\eta)}(x_1, x_2),\ \text{with}\ (x_1, x_2) \in \widehat{Q}_L(\eta)\right\}.$$

Thus, there is one additional degree of freedom compared to the predetermined case, and we obtain $\dim\left(\operatorname{Range}\widehat{\pi}^{-}_{\widehat{Q}_L(\eta)}\right) = \dim\left(\operatorname{Range}\pi^{-}_{Q(\eta)}\right) - 1$. The computations made in the proof of Theorem 1 are then sufficient to conclude. □

Proof of Lemma 2. Splitting the real and imaginary parts of $\Delta_L(iq) = 0$, we notice that, if they exist, pure imaginary roots solve: $2\cos(q) = 0$ and $q = 0$. As this is impossible, there are no pure imaginary roots. □

Proof of Lemma 3. Proceed by contradiction and assume that such roots exist. In this case they should solve $\operatorname{Im}\left(\Delta_{\Gamma(\mu)}(iq)\right) = 0$, which is written as: $(1-\mu)\left(\int_0^1 e^{-u}\sin(qu)\,du\right) - q = 0$. First, consider the roots such that $q \neq 0$. They satisfy $(1-\mu)\left(\int_0^1 e^{-u}\frac{\sin(qu)}{q}du\right) - 1 = 0$. As $\sin(qu)/q < u$ we have $(1-\mu)\left(\int_0^1 e^{-u}\frac{\sin(qu)}{q}du\right) - 1 < 0$. To conclude, we have to show that the real root $\lambda = 0$ is not a solution of $\Delta_{\Gamma(\mu)}(\lambda) = 0$, which is obtained by showing that $\Delta_{\Gamma(\mu)}(0) = 4 - 2e^{-1}$. □

Proof of Lemma 4. We are going to show that $n^{\sharp}_{\Gamma(0)}(0) = 0$. Using the result proposed in Lemma 3, we can use Lemma 1 to state that $n^{\sharp}_{\Gamma(0)}(0) = n^{\sharp}_{\Gamma(1)}(0)$ and conclude by computing $n^{\sharp}_{\Gamma(0)}(0)$. To compute $n^{\sharp}_{\Gamma(0)}(0)$, we successively compute the number of "misplaced roots", $n^{-}_{L_+}(0)$ and $n^{+}_{L_-}(0)$ of $\Delta_{L_+}(\lambda) = 0$ and $\Delta_{L_-}(\lambda) = 0$. The equation $\Delta_{L_+}(\lambda) = 0$ admits no real roots, as $\Delta^{(2)}_{L_+}(\lambda) < 0$, with $\lim_{\lambda\to\pm\infty}\Delta_{L_+}(\lambda) = -\infty$ and $\Delta_{L_+}(0) = -2 < 0$. Furthermore, $\Delta_{L_+}(\lambda) = 0$ has no complex root with a negative real part. If such roots, denoted $p + iq$ with $p < 0$ and $q > 0$, were to exist, they would satisfy $\operatorname{Im}(\Delta_+(p + iq)) = 0$, which can be rewritten as $e^{-p} - \sin(q)/q = 0$. This is impossible as $\sin(q)/q < 1$ for $q > 0$ and $e^{-p} > 1$ for $p < 0$. Thus $n^{-}_{L^{+}}(0) = 0$. Regarding $\Delta_{L_-}(\lambda) = 0$, it can be easily shown that $\lambda = 1$ is the only positive real root. Furthermore, analyzing $\mathrm{Im}(\Delta_-(p + iq)) = 0$ allows us to conclude that there are no complex roots denoted $p + iq$, with $p > 0$ and $q > 0$. Thus $n^{+}_{L^{-}}(0) = 1$. Finally, as we consider solutions where $\eta = 0$ and $\lambda_0 = 1 > 0$, we have $n_0(0) = 1$. We conclude by using formula (11). □

Proof of Lemma 5. The roots of $\Delta_{L_+}(\lambda) = 0$ are equivalent to the roots of $\delta_+(\lambda) = 0$, where $\delta_+(\lambda) = 1 - \gamma\int_0^1 e^{(\lambda-\alpha)u}du$, plus an additional root α. As $\delta_+(\lambda)$ is a decreasing function from 1 to $-\infty$, the sign of the only real root of $\delta_+(\lambda) = 0$, denoted λ^+, is given by the sign of $\delta_+(0)$. It remains to be proved that each complex root has a real part greater than λ^+. Let us proceed by contradiction and suppose that $\lambda = p + iq$, with $p < \lambda^+$, exists such that $\delta_{L_+}(\lambda) = 0$. Then, the following inequality holds:

$$1 = \left|\gamma\int_0^1 e^{(\lambda-\alpha)u}du\right| < \gamma\int_0^1 e^{(p-\alpha)u}du, \tag{46}$$

which contradicts the fact that $\delta_{L_+}(\lambda) > 0$ if $p < \lambda^+$. □

Proof of Lemma 6. As $\Delta^{(2)}_{L_-}(\lambda) > 0$ and $\lim_{\lambda\to\pm\infty}\Delta_{L_-}(\lambda) = 0$, there are no positive real roots if $\beta\int_{-1}^0 e^{-\alpha u}du > \alpha$ and $\beta\int_{-1}^0 ue^{-\alpha u}du > -1$. A unique positive real root exists if $\beta\int_{-1}^0 e^{-\alpha u}du < \alpha$ and $\beta\int_{-1}^0 ue^{-\alpha u}du > -1$. We will now prove that $\Delta_{L_-}(\lambda) = 0$ has no complex roots $\lambda = p + iq$ such that $p > 0$ and $q > 0$. Let us proceed by contradiction and suppose that such a root exists. Splitting real and complex parts gives:

$$\begin{cases} p - \alpha + \beta\int_{-1}^0 e^{(p-\alpha)u}\cos(qu)\,du = 0, \\ q + \beta\int_{-1}^0 e^{(p-\alpha)u}\sin(qu)\,du = 0. \end{cases} \tag{47}$$

The second equation can be rewritten as

$$1 + \beta\int_{-1}^0 ue^{(p-\alpha)u}\frac{\sin(qu)}{qu}du = 0. \tag{48}$$

However, $\int_{-1}^0 e^{(p-\alpha)u}\frac{\sin(qu)}{q}du = \int_{-1}^0 ue^{(p-\alpha)u}\frac{\sin(qu)}{qu}du > \int_{-1}^0 ue^{(p-\alpha)u}du$ and according to the conditions on parameters $\Delta'_{L_-}(0) = 1 + \beta\int_{-1}^0 ue^{-\alpha u}du > 0$.

According to the convexity property of $\Delta_{L_-}(\lambda)$, $\Delta'_{L_-}(p) > 0$ for all $p > 0$. Thus:

$$1+\beta\int_{-1}^{0} e^{(p-\alpha)u}\frac{\sin(qu)}{q}du > 0, \tag{49}$$

contradicting the supposition that a complex root, with $p > 0$ and $q > 0$, exists.

□

Proof of Theorem 3. We want to determine the existence and uniqueness of solutions with maximal growth rate η of equation (27), whose characteristic equation is $\delta(\lambda) = 0$. To begin with, note that the roots of $\Delta_{L_\gamma}(\lambda) = 0$ are identical to those of $\delta(\lambda) = 0$ with an extra root $\lambda = \gamma$ with multiplicity n. By Theorem 1, the existence and uniqueness of solutions with maximal growth rate η' of the equation defined by operator L_γ can be determined by computing $n^\sharp_{L_\gamma}(\eta')$. The proof considers the role played by the n extra roots and the n constraints (29).

In the first step, we show that the spectral projection of the solution to characteristic equation $\Delta_{L_\gamma}(\lambda) = 0$ onto the eigenspace $\mathcal{M}_\gamma$ spanned by $\{e^{\gamma t}, te^{\gamma t}, ..., t^{n-1}e^{\gamma t}\}$ gives n expressions that are equal to zero since they correspond exactly to equation (29). In a second step, we link the initial conditions of equation (27), which produce a solution with maximal growth rate η, to the initial conditions of the MFDE with operator L_γ. Applying constraints (29), this yields a solution with maximal growth rate γ.

Step 1. To facilitate legibility, we assume $n = 1$. We rewrite equation (27) as:

$$x(t) = \int_{-a}^{b} x(u+t)\, dm(u). \tag{50}$$

Taking the derivative gives:

$$x'(t) = x(t+b)\, m(b) - x(t-a)\, m(-a) - \int_{-a}^{b} x(u+t)\, dm'(u). \tag{51}$$

We can construct operator L_γ by modifying equation (51) to get:

$$\begin{aligned} x'(t) - \gamma x(t) = {} & x(t+b)\, m(b) - x(t-a)\, m(-a) \\ & - \int_{-a}^{b} x(u+t)\left[dm'(u) + \gamma dm(u)\right]. \end{aligned} \tag{52}$$

This equation can be written as:

$$x'(t) = \int_{-a}^{b} x(u+t)\, dm_*^{\gamma}(u). \tag{53}$$

The characteristic equation of this MFDE is $\Delta_{L_\gamma}(\lambda) = 0$, where:

$$\Delta_{L_\gamma}(\lambda) = \lambda - \int_{-a}^{b} e^{\lambda u} dm_*^{\gamma}(u). \tag{54}$$

However, it will be easier to work with another operator L_0, defined directly using (51). The relationship between the characteristic function Δ_{L_0} and Δ_{L_γ} is given by $(\lambda - \gamma)\Delta_{L_0}(\lambda) = \lambda \Delta_{L_\gamma}(\lambda)$, where Δ_{L_0} is obtained through Δ_{L_γ} by swapping roots γ and 0. According to Lemma 5.4 in Mallet-Paret and Verduyn Lunel [20], a measure m_* exists that can be explicitly computed using m_*^{γ}, such that:

$$\Delta_{L_0}(\lambda) = \lambda - \int_{-a}^{b} e^{\lambda u} dm_*(u). \tag{55}$$

As $(\lambda - \gamma)\delta(\lambda) = \Delta_{L_\gamma}(\lambda)$, we obtain $\lambda\delta(\lambda) = \Delta_{L_0}(\lambda)$. Thus:

$$\Delta_{L_0} = \lambda - e^{\lambda b} m(b) + e^{-\lambda a} m(-a) + \int_{-a}^{b} e^{\lambda u} dm'(u), \tag{56}$$

which implies that $dm_* = \delta(b)\, m(b) - \delta(-a)\, m(-a) - \int_{-a}^{b} dm'(u)$.

We consider the spectral projection associated with the root $\lambda = 0$ of the characteristic equation $\Delta_{L_0}(\lambda) = 0$ for any $x \in \mathcal{C}([-a,b])$. We define $\Pi_{sp} x(\theta)$ as the spectral projection when evaluated at $\theta \in [-a,b]$. As shown by Hupkes and Verduyn Lunel [18], it satisfies:

$$(\Pi_{sp} x)(\theta) = \operatorname{Re}\mathrm{s}_{\lambda=0} e^{\lambda\theta} \Delta_{L_0}(\lambda)^{-1} \left[x(0) + \int_{-a}^{b} e^{\lambda\sigma} \int_{\sigma}^{0} e^{-\lambda\tau} x(\tau)\, d\tau\, dm_*(\sigma) \right]. \tag{57}$$

Using the definition of dm_*, we have:

$$\int_{-a}^{b} e^{\lambda\sigma} \int_{\sigma}^{0} e^{-\lambda\tau} x(\tau)\, d\tau\, dm_*(\sigma) = -\int_{-a}^{b} x(u)\, dm(u). \tag{58}$$

Thus, $(\Pi_{sp} x)(\theta) = 0$. For $n > 1$, the reasoning is similar.

Step 2. Consider γ such that $\gamma > \eta$ and assume that there are no roots of $\delta(\lambda) = 0$ such that $\operatorname{Re}(\lambda) \in [\eta, \gamma)$. For any initial condition of the algebraic equation (and its associated constraint) leading to a solution $x(t)$ with maximal growth rate γ, it is sufficient to prove that the solutions associated with the differential equation related to L_γ with maximal growth rate η are the same as those associated with L_γ with growth rate γ. The solutions imply that the projection of initial conditions on the eigenspace $\mathcal{M}_\gamma$ is null. As the previous computation showed that the equivalence between requiring the initial condition to solve the constraint and proving that the spectral projection is null, the proof is immediate. This allows us to conclude that the existence and uniqueness of the solution of the algebraic equation depends only on $n^\sharp_{L_\gamma}(\gamma)$. $\square$

<u>Proof of Theorem 4.</u> As in Theorem 3, when considering the characteristic equation of a differential equation rather than that of an algebraic equation, we introduce the extra root γ with its multiplicity n. The same proof may be used to show that the additional constraints induced by the algebraic equation at $t = 0$ make projections on the space $\mathcal{M}_\gamma$ null. This proof also implies that the space of initial conditions of the algebraic equation with growth rate η is $\widehat{Q}_{L_\gamma}(\gamma)$. We conclude using Theorem 2 which gives properties of $n^\sharp_{L_\gamma}(\gamma)$. $\square$

<u>Proof of Lemma 7.</u> By definition,

$$n^\sharp_{\Gamma(0)}(0) = n^-_{L_+}(0) - n^+_{L_-}(0) + n_0(0), \tag{59}$$

where

$$\begin{cases} n^+_{L_-}(\eta) = \#\{z \in \mathbb{C} : \det \Delta_-(z) = 0 \text{ and } \operatorname{Re} z > \eta\}, \\ n^-_{L_+}(\eta) = \#\{z \in \mathbb{C} : \det \Delta_+(z) = 0 \text{ and } \operatorname{Re} z < \eta\}, \\ n_0(\eta) = \#\{z \in \mathbb{C} : z - \gamma = 0 \text{ and } \operatorname{Re} z > \eta\}. \end{cases} \tag{60}$$

We use the fact that:

$$\begin{aligned} n^-_{L^+}(0) - n^+_{L^-}(0) = {} & \#\{z \in \mathbb{C} : \det \delta_+(z) = 0 \text{ and } \operatorname{Re} z < 0\} \\ & - \#\{z \in \mathbb{C} : \det \delta_-(z) = 0 \text{ and } \operatorname{Re} z > 0\} \\ & - 1, \end{aligned} \tag{61}$$

and conclude, using the symmetry of δ_+ and δ_-, that $n^-_{L^+}(0) - n^+_{L^-}(0) = -1$. Furthermore, $n_0(0) = 1$. $\square$

Proof of Lemma 8. We study the roots of $\Delta_{\Gamma_\gamma(\mu)}(\lambda) = 0$ when μ moves from 0 to 1 and count how many roots cross the axis $\operatorname{Re}\lambda = 0$. Let

$$\Delta_{\Gamma(\mu)}(\lambda) = (\lambda - \gamma)\,\varphi(\lambda;\mu)\,, \tag{62}$$

where

$$\varphi(\lambda;\mu) = 1 - \alpha A \int_{-\tau}^{0}\int_{s}^{s+\tau} e^{\lambda v}\,dvds + (1-\mu)\,i\sqrt{\alpha A}\left[\int_0^\tau e^{-\lambda s}ds + \int_0^\tau e^{\lambda s}ds\right]. \tag{63}$$

We first notice that as

$$\varphi(iq;\mu) = 1 - \alpha A \int_{-\tau}^{0}\int_{s}^{s+\tau} e^{iqva}\,dvds + 2(1-\mu)\,i\sqrt{\alpha A}\left[\int_0^\tau \cos(qs)\,ds\right], \tag{64}$$

$\operatorname{Re}\varphi(iq;\mu)$ does not depend on μ. We now show that $\operatorname{Re}\varphi(iq;\mu) \neq 0$ for all $q \in \mathbb{R}$. Let us proceed by contradiction and suppose that $\operatorname{Re}\varphi(iq;\mu) = 0$. This implies:

$$1 = \alpha A\left(\int_{-\tau}^{0}\int_{s}^{s+\tau} \cos(qv)\,dvds\right) < \alpha A\tau^2, \tag{65}$$

which contradicts the fact that $1 > \alpha A\tau^2$. $\square$

Proof of Lemma 9. We have to add the real root $\lambda = \bar{g}$ and simply have to prove that $\delta(\lambda) = 0$ has no other root that satisfies $\{\lambda \in \mathbb{C}, \operatorname{Re}\lambda \in [0, \overline{g}]\}$. Let us suppose a complex root exists, denoted $\lambda = p + iq$, which satisfies $p < \bar{g}$. It solves:

$$1 = \alpha A\left(\int_{-\tau}^{0}\int_{s}^{s+\tau} e^{(p+iq)v}\,dvds\right) < \int_{-\tau}^{0}\int_{s}^{s+\tau} e^{pv}\,dvds, \tag{66}$$

which contradicts the fact that $\delta(p) > 0$. $\square$

Proof of Theorem 5. For convenience, take $\eta = 0$. The key here is to notice that for any $x \in C([-a,0],\mathbb{R}) \times \{x \in C([0,\infty),\mathbb{R}) : \|x\|_\infty < \infty\}$, the function $t \mapsto g(Mx)$ lies in $L^\infty([0,\infty),\mathbb{R})$. As in Hupkes and Verduyn Lunel [19], one can easily construct a linear operator:

$$\mathcal{K} : L^\infty([0,\infty),\mathbb{R}) \to \{x \in C([-a,\infty),\mathbb{R}) : \|x\|_\infty < \infty\} \tag{67}$$

in such a way that $v = \mathcal{K}h$ solves:

$$v'(t) = Lv(t) + h(t) \tag{68}$$

for almost all $t \geq 0$. For any function $q \in \widehat{Q}_L(0)$, we can write $\psi = q_{|[-a,0]} \times q_{|[0,b]}$. We then define $u^*(\psi)$ to be the solution of the fixed point problem:

$$x = q + \mathcal{K}f(x). \tag{69}$$

This construction is based on the classical Lyapunov-Perron method and the stable manifold can now be written as the graph of the function u^*. The smoothness of u^* follows in a standard fashion from the implicit function theorem. $\square$

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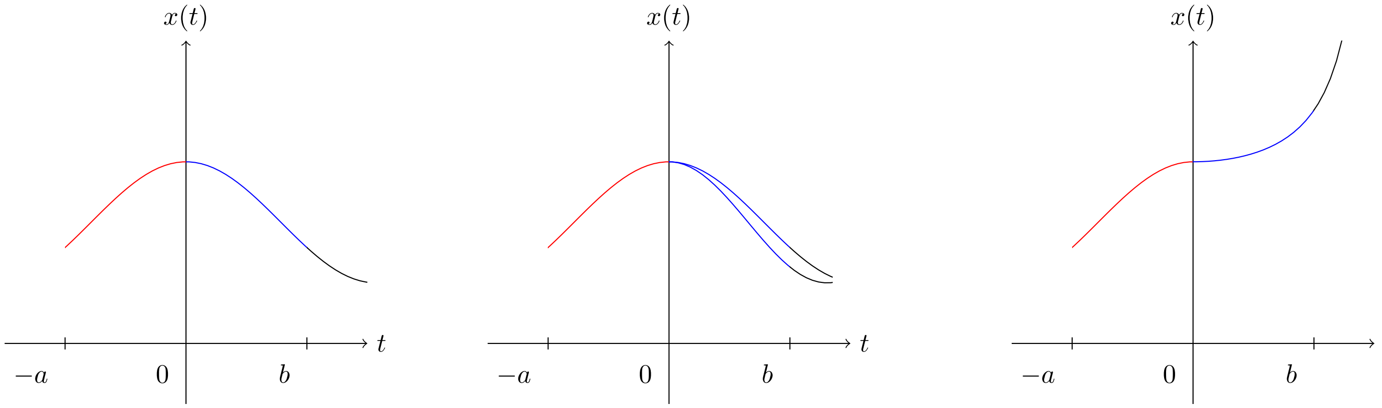
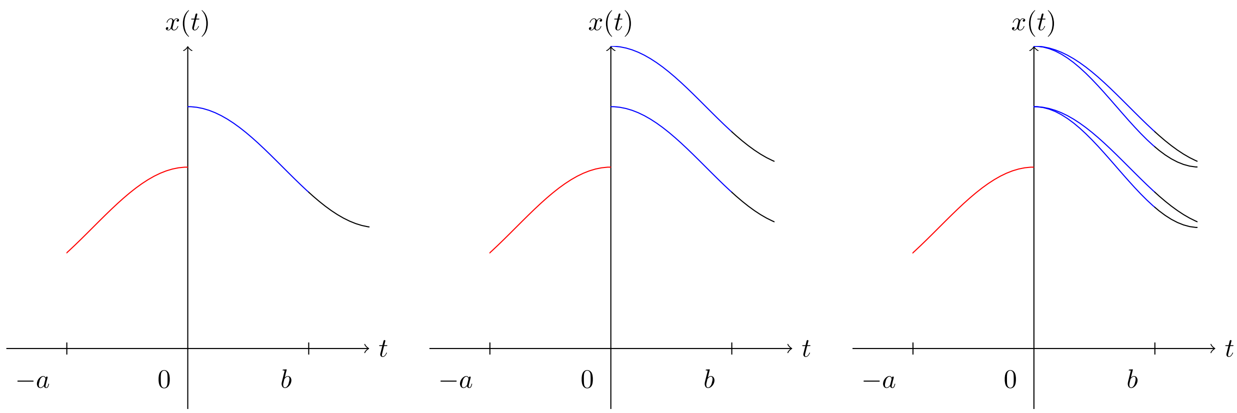


(a) Existence and uniqueness ($n^{\#} = 1$) (b) Existence and indeterminacy ($n^{\#} > 1$) (c) non existence ($n^{\#} < 1$)

Figure 1: Predetermined variables



(a) Existence and uniqueness ($n^{\#} = 0$) (b) Existence and indeterminacy ($n^{\#} = 1$) (c) Existence and indeterminacy ($n^{\#} > 1$)

Figure 2: Non-predetermined variables