A constant elasticity of profit production function

Beard, Rodney

University of Alberta

November 2007

A constant elasticity of profit production function

Rodney Beard

Department of Marketing, Business Economics and Law

School of Business

University of Alberta T6G 2R6

Canada

e-mail: Rodney.Beard@ualberta.ca

Abstract

Impact analysis of changes in production inputs may be simplified if one can apply a constant adjustment factor to profit. In particular, if a production function can be found for which the elasticity of profit is constant and this function has desirable properties, then one can use the input elasticity of profit to study the impact of input changes on profit. In this paper such a production function is derived from first principles.

*J*EL Classification: D2,M2,Q10

1 Introduction

Elasticity of profit is frequently used in managerial economics in discussions of operating leverage. Operating leverage is the percentage change in revenue due to the percentage change in sales. In other words the degree of operating leverage (DOL) is given by:

$$DOL = \frac{(P - AVC)Q}{(P - AVC)Q - TFC}$$

where ϵ is the elasticity of profit, P is the output price, AVC average variable costs, TFC total fixed costs and Q the quantity of output.

If costs and demand are both linear then this becomes [2]:

$$\frac{(MR - MC)Q}{(MR - MC)Q - TFC} = \epsilon$$

MR and MC refer to marginal revenue and marginal cost respectively.

Other applications of profit elasticities include [3] where it is applied to the analysis of the size economy of firms. Shen [3] attributes it's use to Steindl (1946) [4].

Frequently however, one is not interested in the impact of sales on profit but rather the impact of some input on profit. This is particularly the case in agricultural applications where one would like to be able to analyse the impact of policy on for example farm profit, in circumstances where little is known about production technology. For example input and output data may not be available in sufficient quantities to properly estimate a production function. Under these circumstances it is useful to be able to apply some constant adjustment to profits resulting from a change in inputs. For example what would the impact of a 1% reduction in inputs be on farm profit? If the elasticity of profit with respect to inputs can be assumed to be constant, then a 10% elasticity of profit would imply that a 1% reduction in inputs would lead to a 10% reduction in profit. This is legitimate if the elasticity of production with respect to inputs is constant[1]. However, to what extent such an assumption is acceptable in terms of the implied production function does not appear to have been studied.

In this note the production function implied by a constant elasticity of profit with respect to inputs is derived as a solution to a particular ordinary differential equation. It is shown that this production function does in fact have some desirable properties and is capable of capturing technologies with both increasing and decreasing returns to scale and that returns to scale are related to the elasticity of profit.

In the next section the problem is formulated and the constant input elasticity of profit production function is derived. In section 3 scale properties of the production function are studied.

2 Derivation

Profit expressed in terms of inputs is expressed as:

$$\Pi(N) = pq(N) - cN$$

where p is the output price, c is the purchase price of a unit of input. N is a decision variable representing the quantity of inputs employed.

The elasticity of profit is given by:

$$\epsilon = \frac{d\Pi}{dN}\frac{N}{\Pi(N)}$$

$$= \left[pq^{'}(N) - c\right]\frac{N}{pq(N) - cN}$$

[1]Care needs to be taken when using this concept because it may well be that firms are operating at levels of production above profit maximizing levels consequently, a reduction in inputs would lead to an increase in profit, the sign of the impact will depend on where on the profit function one lies and this in turn requires one to know something about the production function

If this elasticity is constant then this may be rearranged to obtain:

$$q^{'}(N) - \epsilon \frac{q(N)}{N} - \frac{c}{p}(1-\epsilon) = 0$$

This is a first-order non-homogeneous ordinary differential equation the solution of which gives a production function that satisfies constant input elasticity of profit.

The solution of this differential equation is given by:

$$q(N) = \frac{Nc}{p} + N^{\epsilon} c_1$$

where c_1 is a constant of integration. Differentiating and substituting back into the differential equation allows one to verify that this is indeed a solution.

Note that the constant of integration may be determined by setting $N = 0$ which on substituting gives, $q(0) = c_1 = 0$. Consequently, a constant input elasticity of profit production function reduces to a linear production function or is not defined in the origin with zero inputs. This suggests that the assumption of constant elasticity of profit should not be used in impact studies where inputs are zero. Picking some positive but small input quantity $N = c_2$ as an initial point,

$$q(c_2) = \frac{c_2 c}{p} + (c_2)^{\epsilon} c_1$$

may be rearranged to obtain:

$$c_1 = \frac{q(c_2) - \frac{c_2 c}{p}}{c_2^{\epsilon}}$$

This gives

$$q(N) = \frac{Nc}{p} + N^{\epsilon} \frac{q(c_2) - \frac{c_2 c}{p}}{c_2^{\epsilon}}$$

for $N \geq c_2$.

This production function satisfies constant input elasticity of profit. It differs in another way from other production functions in that it depends on input cost and output price it is therefore not a purely technical specification. This is not necessarily a disadvantage as it is often argued that as relative prices change, new technologies are selected. The constant input elasticity of profit production function, captures this directly.

A disadvantage of this production function and therefore of constant adjustments to profit is that this is not compatible with the hypothesis of profit maximization in a perfectly competitive market. It is however compatible with a sales maximization approach. for a discussion of sales maximisation see Baumol [1]. The production function is consistent with the behavior of a profit maximising monopolist. Although, this would reuire numerical solution.

3 Returns to scale

Increasing inputs by a factor k, results in:

$$\frac{kNc}{p} + (kN)^{\epsilon} c_1$$

$$k\frac{Nc}{p} + k^{\epsilon} N^{\epsilon} c_1$$

which is clearly a non-homogeneous production function possessed of variable returns to scale that depend on the elasticity of profit ϵ.

Note also that marginal product of the input is given by

$$q^{'}(N) = \frac{c}{p} + \epsilon N^{\epsilon - 1} c_1$$

and the average product of the input is

$$\frac{q(N)}{N} = \frac{c}{p} + N^{\epsilon-1}c_1$$

where $\frac{q(c_2)-\frac{c_2 c}{p}}{c_2^{\epsilon}}$.

4 Multivariate profit function

In the multivariate case the corresponding elasticity will be a partial elasticity. Profit is now

$$\Pi = pq(N_1, \ldots, N_m) - \sum_{i=1}^{m} c_i N_i$$

where m is the number of factors of production.

this gives the elasticity:

$$\epsilon_i = \frac{(p\frac{\partial q}{\partial N_i} - c_i)N_i}{pq(N_1, \ldots, N_m) - \sum_{i=1}^{m} c_i N_i}$$

Proceeding as before one may derive a set of linear first-order partial differential equations for production:

$$\frac{\partial q}{\partial N_i} - \epsilon_i \frac{q(N_1, \ldots, N_m)}{N_i} - \frac{c}{p} - \epsilon_i \frac{\sum_{i=1}^{m} c_i N_i}{pN_i} = 0, i = 1, \ldots, m$$

This is not a coupled system of differential equations in the usual sense, because, the solution of each equation does not depend on the solution of the other equations. Rather the equations are linked by virtues of the exogenous parameter N_i[2]. Consequently, each equation is coupled in terms of multiple exogenous parameters. The problem is similar to that of solving differential equations with multiple time scales.

[2]The term parameter is used here following the usage in differential equations.

For the applied economist, it should be noted that the system could be solved numerically using common numerical solution procedures,for example the problem is highly amenable to solution by method of lines. These should produce a solution surface for the production function.

5 Conclusion

In this note a new production function is derived that has the property that the input elasticity of profit is constant. Consequently in applying a percentage adjustment to profit due to changes in inputs implicitly assumes the production function derived here. Although the derivation presented is complete for the single input case, the multivariate case is also discussed and it is shown that this leads to a particular coupled system of linear first-order partial differential equations, whose solution is best addressed by numerical methods. The results are important for applied economists engaged in policy impact studies.

References

[1] Baumol, W. *Business Behavior and Growth*, New York 1957.

[2] Rives, J.M (1991) An appropriate DOL formula for managerial economics, *Atlantic Economic Journal*, vol. 19, No. 4, p. 60.

[3] Shen, T.Y. (1970) Economies of scale, Penrose effect, Growth of plants and their size distribution, *Journal of Political Economy*, Vol. 78, No.4, pp. 702-716.

[4] Steindl, J. *Small and Big Business*, Oxford University Press, Oxford, 1946.