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Abstract

A pillage game is a coalitional game that is meant to be a model of *Hobbesian anarchy*. The *spatial pillage game* introduces a *spatial feature* into the pillage game by assuming that players are located in regions. Players can travel from one region to another in one move and can form a coalition and combine their power only with players in the same region. A coalition has power only within its region. Under this spatial restriction, some members of a coalition can pillage less powerful coalitions without any cost. The feasibility of pillages between coalitions determines the *dominance relation*. *Core, stable set, and farsighted core* are adopted as alternative solution concepts.

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1 Introduction

Hobbesian anarchy is a societal state prior to the formation of a government that ensures property rights. Without such an organization, no individuals are safe to secure their wealth. Individuals could be tempted to pillage others whenever possible and beneficial. Although a coalition could be formed to secure their wealth, some members of the coalition may still be tempted to betray others and to take their wealth. Consequently, in Hobbesian anarchy, the possibility of the stable distribution of wealth is questionable.

A substantial amount of literature on *allocation by force* has been devoted to this possibility. Skaperdas (1992) showed that a cooperative outcome is possible in equilibrium if the probability of winning in conflict is sufficiently robust against each

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individual's action. Hirshleifer (1995) found the conditions under which Hobbesian anarchy is stable. Also, Hirshleifer (1991), Konrad and Skaperdas (1998), and Muthoo (1991) studied the situations in which property right is partially secured. These studies analyzed noncooperative models in which the formation of coalitions is limited or not allowed.

In contrast to the previous models, Piccione and Rubinstein (2007) and Jordan (2006) developed models of Hobbesian anarchy that allow the formation of coalitions. Piccione and Rubinstein introduced *the jungle* in which coercion governs economic transactions and they compared the equilibrium allocation of the jungle with the equilibrium allocation of an exchange economy. Jordan introduced *pillage games* and examined stable sets of allocations in which the power of pillaging balances endogenously.

The *spatial pillage game* is an extended version of a pillage game. In most literature on "allocation by force" including the papers reviewed above, there is no restriction on using power. Thus any individual or coalitions can pillage another individual or other coalitions if one is more powerful than others. However, the acts of pillaging and defending are inevitably under spatial restriction. Members of a coalition, if they move together, cannot simultaneously pillage two less powerful coalitions that are far apart from each other. Likewise, two coalitions cannot combine their power to defend themselves together against another powerful coalition unless they are close enough to each other. The spatial pillage game introduces a *space feature*, which conditions power usage based on location, into a Hobbesian anarchy model that allows the formation of coalitions, in the hope of understanding how spatial restriction affects stable distributions of wealth.

The spatial pillage game internalizes the space feature through the following assumptions. There are regions and each player can stay in only one of the regions. Players can change their regions to pillage others. The regions are connected with one another, and thus players can travel from a region to another in one move. Players can form a coalition and combine their power only after getting together in a common region. If coalitions are in different regions, they cannot combine their power. The influence of the power of each coalition is limited within its region. Therefore, a coalition cannot pillage two other coalitions in different regions simultaneously.

The other assumptions in this spatial pillage game are the same as in the original pillage games. A fixed amount of wealth is allocated among a finite number of players. Some players can form a coalition under the spatial restriction. A coalition can pillage less powerful coalitions within its region without any cost. An increase in the wealth of a coalition causes an increase in its power. Since the power of each coalition is endogenously determined, the spatial pillage game cannot have a characteristic function, which exogenously determines the power of each coalition.

The pillage games are characterized by *power functions* that determine the feasibility of pillages between coalitions. Jordan (2006) presented three power functions classified by the degree of their dependence on the sizes of coalitions. *Wealth is* *power* is one of the power functions and specifies the power of each coalition as its total wealth. Therefore, "wealth is power" is characterized as independent of the sizes of coalitions. Only the pillage game with this power function has a stable set in every possible case. Therefore, the spatial pillage game adopts "wealth is power" as a power function so that if there exists solutions in this spatial pillage game, then we can compare it with the solutions in the original pillage game and can find how the spatial restriction affects a stable distribution of wealth.

As criteria for stable distributions of wealth and players, three solution concepts are explored; core, stable set, and farsighted core. First, the core is the collection of states at which pillage is not possible, thus it is one of the most persuasive solution concepts. However, due to its strong requirement, the core is too small to represent stable states as shown in Theorem 1. Second, a stable set is much bigger than the core if it exists, as shown in Theorem 3. A stable set is a collection of states that is both internally stable and externally stable. Internal stability requires that pillage not be possible between states in the collection and external stability requires that pillage at a state outside the collection result in another state inside the collection. In most cases, however, no stable set exists as shown in Theorem 4. Even when they exist, they contain implausible states as shown in Theorem 3. Third, farsighted core, which was introduced by Jordan (2006), solves these problems with stable sets, as shown in Theorem 5. A farsighted core accepts the assumption that a player has a forecasting ability and is defined as a collection of states at which *pillage in expectation* is not possible in the sense that some members of the pillage would end up being worse off, and consequently they would not join the pillage.

In section 2, we search for the core and stable sets. First, the core is characterized. In $N-$region model where $N \geq 2$, as N increases, the size of the core grows. So, the core in $N-$region model for $N \geq 3$ is greater than the core in Jordan's model, which can be considered as a one-region model. This is because as the number of the regions increases, the feasibility of pillages is more limited under the spatial restriction. As a result, the set of states at which pillage is not possible grows. Next, a stable set is studied. A stable set, if it exists, is much bigger than the stable set in Jordan's model and does not show an endogenous balance with respect to the pillaging power as the stable set in Jordan's model does. This is because the limited feasibility of the pillages under the spatial restriction makes the conditions of the stable set, both internal stability and external stability, improper to be requirements for a reasonable solution to the spatial pillage game.

In section 3, we find that there exists the unique farsighted core. This farsighted core is similar to the farsighted core in Jordan's model. So, the farsighted core represents stable distributions of wealth, which shows an endogenous balance with respect to the pillaging power, as the farsighted core in Jordan's model does. Since Jordan's model does not include the spatial concept and induces the result similar to the one in this spatial pillage game, we concluded that if players have the forecasting ability, then the stable distributions of wealth does not change under the spatial restriction.

In section 4, a suggestion for further research is presented.

2 Core and stable set

The environment of a spatial pillage game is defined in Definitions 1 and 2. We normalize the total wealth to unity.

Definition 1 [1] *The finite set I is the set of **players**. A **coalition** is a subset of I. The set $A = \{w \in \mathbb{R}^I : w_i \geq 0 \text{ for all } i \in I \text{ and } \sum_{z \in I} w_z = 1\}$ is the set of **allocations**.*

The definitions below concern the spatial environment.

Definition 2 *The finite set R is the set of **regions** and the Cartesian product R^I is the set of **distributions**. Given a distribution $p \in R^I$, the coalition $p^r = \{i \in I : p_i = r\}$ is the **population** at region r.*

A distribution is short for a population distribution and denotes how players are distributed over the regions. For example, the distribution $p = (1, 1, 2)$ expresses that players 1 and 2 are at region 1 and player 3 is at region 2. Also, it means $p^1 = \{1, 2\}$ and $p^2 = \{3\}$.

A **state** denotes both the allocation and distribution of the status quo.

Definition 3 *The Cartesian product $X = A \times R^I$ is the set of **states**.*

For instance, the ordered pair $(w, p) = ((\frac{1}{2}, \frac{1}{4}, \frac{1}{4}), (1, 1, 2))$ is a state in the three-player and two-region model. The state (w, p) expresses that player 1 has $\frac{1}{2}$ and player 2 has $\frac{1}{4}$ while staying at region 1 and player 3 has $\frac{1}{4}$ while staying at region 2.

The **dominance** relation between states is defined as follows.

Definition 4 *Given states (w, p) and (w', p'), define $W = \{i : w'_i > w_i\}$ and $L = \{i : w'_i < w_i\}$. Suppose for some $r, q \in R$, i) $\{i : w'_i \neq w_i\} \subset p'^r$; ii) $\{i : p_i \neq p'_i\} = \varnothing$ or $\{i : p_i \neq p'_i\} = W \subset p^q$; and iii) $\sum_{i \in W} w_i > \sum_{i \in L} w_i$. Then (w', p') **dominates** (w, p).*

The dominance relation shows the states to which the status quo can move. It must satisfy both *physical* and *spatial conditions.* The physical condition requires that the winning coalition W have enough power to pillage the losing coalition L. Definition 4 presents this condition at *iii*). Jordan (2006) introduced a variety of physical conditions. The condition *iii*) above accords with the physical condition of the *wealth is power* in Jordan (2006). The spatial condition requires that the

[1] I follow notations in Jordan (2006).

act of pillaging satisfy spatial restriction. This condition is expressed at i) and ii) in Definition 4. The condition i) means that transfers of wealth happen only in destination region r where the pillage happens. The condition ii) denotes that only the winners can travel and that they are all from the common region q. That is, the spatial restriction in dominance relation is that W can combine their power only when they are in the common region, and once they combine their power, then they can all together move to another region in order to pillage L.

This concept of spatial restriction is designed to reflect the simplest way of forming a coalition. This study intends to make an initial contribution to understand how spatial restriction affects stable distributions of wealth. So, as a starting point, this study adopts the simplest feature of spatial restriction. An advanced concept of the spatial restriction, which allows a complicated way of forming a coalition, will be presented at the definition of Dominance in Expectation in section 3. Note that if there is only one region, then this definition of dominance relation coincides with the definition in Jordan (2006). So, this definition can be considered as a spatial version of the Jordan's definition.

In this section, we adopt the solution concepts of **core** and **stable set**. The definition stated below follows Lucas (1992) and Jordan (2006).

Definition 5 *The set of undominated states is the* ***core*** C. *For any set* E *of states, let the set* $U(E)$ *be the set of states that are not dominated by any state in* E. *A set* S *of states is a* ***stable set*** *if it satisfies both* $S \subset U(S)$, *which means internal stability, and* $S \supset U(S)$, *which means external stability.*

Therefore, a stable set S is defined by the set of states that satisfies $S = U(S)$.

Theorem 1 embodies the core.

Theorem 1 *The set* $\{(w,p) \in X :$ *for each* i, $w_i = \frac{1}{\#\{r \in R : \Sigma_{j \in p^r} w_j > 0\}}, \frac{1}{2},$ *or* $0\}$ *is the core* C.

Proof. Suppose $(w,p) \in \{(w,p) \in X :$ for each i, $w_i = \frac{1}{\#\{r \in R : \Sigma_{j \in p^r} w_j > 0\}}, \frac{1}{2}$, or $0\}$. If $w_i > 0$, then $w_i \geq \min\{\frac{1}{2}, \frac{1}{\#\{r \in R : \Sigma_{j \in p^r} w_j > 0\}}\}$. If $\frac{1}{\#\{r \in R : \Sigma_{j \in p^r} w_j > 0\}} \geq \frac{1}{2}$, then $\#\{r \in R : \Sigma_{j \in p^r} w_j > 0\} = 1$ or 2 and thus for each i, $w_i = 1, \frac{1}{2}$, or 0. In this case, any coalition W cannot pillage another coalition L such that $W \cap L = \varnothing$ because if $\sum_{i \in L} w_i > 0$, then $\sum_{i \in L} w_i \geq \frac{1}{2}$ and so $\frac{1}{2} \geq \sum_{i \notin L} w_i \geq \sum_{i \in W} w_i$. If $\frac{1}{2} > \frac{1}{\#\{r \in R : \Sigma_{j \in p^r} w_j > 0\}}$, then $\#\{i : w_i > 0\} = \#\{r \in R : \Sigma_{j \in p^r} w_j > 0\}$ since $\#\{i : w_i > 0\} \geq \#\{r \in R : \Sigma_{j \in p^r} w_j > 0\}$ and $\#\{i : w_i > 0\} \times \frac{1}{\#\{r \in R : \Sigma_{j \in p^r} w_j > 0\}} = \#\{i : w_i > 0\} \times \min\{\frac{1}{2}, \frac{1}{\#\{r \in R : \Sigma_{j \in p^r} w_j > 0\}}\} \leq \Sigma_{i \in I} w_i = 1$, and thus for each i, $w_i = \frac{1}{\#\{r \in R : \Sigma_{j \in p^r} w_j > 0\}}$ or 0 since $w_i \geq \frac{1}{\#\{r \in R : \Sigma_{j \in p^r} w_j > 0\}}$ or 0. In this case, we have for each $r \in R$, $\Sigma_{j \in p^r} w_j = \frac{1}{\#\{r \in R : \Sigma_{j \in p^r} w_j > 0\}}$ or 0, and so any coalition W such that $W \subset p^q$ for some $q \in R$ cannot pillage another coalition L such that $W \cap L = \varnothing$ because if $\sum_{i \in L} w_i > 0$, then $\sum_{i \in L} w_i \geq \frac{1}{\#\{r \in R : \Sigma_{j \in p^r} w_j > 0\}}$ and $\frac{1}{\#\{r \in R : \Sigma_{j \in p^r} w_j > 0\}} \geq \sum_{i \in W} w_i$. Therefore, (w,p) is not dominated. Since (w,p) is arbitrary, every state in the set $\{(w,p) \in X : \text{for each } i,\ w_i = \frac{1}{\#\{r \in R : \Sigma_{j \in p^r} w_j > 0\}}, \frac{1}{2}, \text{ or } 0\}$ is not dominated.

Suppose $(w,p) \notin \{(w,p) \in X : \text{for each } i,\ w_i = \frac{1}{\#\{r \in R : \Sigma_{j \in p^r} w_j > 0\}}, \frac{1}{2}, \text{ or } 0\}$. Then there exists i such that $w_i \notin \{0, \frac{1}{\#\{r \in R : \Sigma_{j \in p^r} w_j > 0\}}, \frac{1}{2}, 1\}$. If $w_i > \frac{1}{\#\{r \in R : \Sigma_{j \in p^r} w_j > 0\}}$, then there exists $q \in R$ such that $\frac{1}{\#\{r \in R : \Sigma_{j \in p^r} w_j > 0\}} > \Sigma_{j \in p^q} w_j > 0$ since $\Sigma_{j \in I} w_j = 1$, and thus player i can pillage another player j such that $w_j > 0$ and $p_j = q$ since $w_i > \frac{1}{\#\{r \in R : \Sigma_{j \in p^r} w_j > 0\}} > \Sigma_{k \in p^q} w_k \geq w_j > 0$. If $\#\{r \in R : \Sigma_{j \in p^r} w_j > 0\} = 1$ and $w_i < 1$, then either $1 > w_i > \frac{1}{2}$ or $\frac{1}{2} > w_i > 0$ since $w_i \notin \{\frac{1}{2}, 0\}$, and thus player i can pillage player j such that $w_j > 0$ or the coalition $W = \{k : k \neq i$ and $p_k = p_i\}$ can pillage player i. If $\#\{r \in R : \Sigma_{j \in p^r} w_j > 0\} \geq 2$ and $\frac{1}{\#\{r \in R : \Sigma_{j \in p^r} w_j > 0\}} > w_i$, then $\frac{\Sigma_{j \notin p^{p_i}} w_j}{\#\{r \in R : \Sigma_{j \in p^r} w_j > 0 \text{ and } r \neq p_i\}}$, which denotes the average wealth of regions except the region p_i, is well defined, and thus either $w_i \geq \frac{\Sigma_{j \notin p^{p_i}} w_j}{\#\{r \in R : \Sigma_{j \in p^r} w_j > 0 \text{ and } r \neq p_i\}}$ or $\frac{\Sigma_{j \notin p^{p_i}} w_j}{\#\{r \in R : \Sigma_{j \in p^r} w_j > 0 \text{ and } r \neq p_i\}} > w_i$. If $w_i \geq \frac{\Sigma_{j \notin p^{p_i}} w_j}{\#\{r \in R : \Sigma_{j \in p^r} w_j > 0 \text{ and } r \neq p_i\}}$, then $\Sigma_{j \in p^{p_i}} w_j > \frac{\Sigma_{j \notin p^{p_i}} w_j}{\#\{r \in R : \Sigma_{j \in p^r} w_j > 0 \text{ and } r \neq p_i\}}$, which means that the wealth of the region p_i is greater than the average wealth of regions except the region p_i, since $\frac{1}{\#\{r \in R : \Sigma_{j \in p^r} w_j > 0\}} > w_i \geq \frac{\Sigma_{j \notin p^{p_i}} w_j}{\#\{r \in R : \Sigma_{j \in p^r} w_j > 0 \text{ and } r \neq p_i\}}$, and thus all players in the region p_i can pillage another region q such that $w_i \geq \Sigma_{j \in p^q} w_j > 0$. If $\frac{\Sigma_{j \notin p^{p_i}} w_j}{\#\{r \in R : \Sigma_{j \in p^r} w_j > 0 \text{ and } r \neq p_i\}} > w_i$, then all players in q such that $\Sigma_{j \in p^q} w_j \geq \frac{\Sigma_{j \notin p^{p_i}} w_j}{\#\{r \in R : \Sigma_{j \in p^r} w_j > 0 \text{ and } r \neq p_i\}}$ can pillage the player i. This means that (w,p) is dominated by some state in X. Since (w,p) is arbitrary, every state in $X \setminus \{(w,p) \in X : \text{for each } i,\ w_i = \frac{1}{\#\{r \in R : \Sigma_{j \in p^r} w_j > 0\}}, \frac{1}{2}, \text{ or } 0\}$ is dominated. ■

Theorem 1 shows how the core changes under the spatial restriction. If there are less than three regions, then we have $C = \{(w,p) \in X : \text{for each } i,\ w_i = 1, \frac{1}{2}, \text{ or } 0\}$ which is the core in Jordan (2006). However, if there are three regions or more, then we have the greater core $C = \{(w,p) \in X : \text{for each } i,\ w_i = \frac{1}{\#\{r \in R : \Sigma_{j \in p^r} w_j > 0\}}, \frac{1}{2}, \text{ or } 0\}$. This greater core results from the limited feasibility of the dominance relation under the spatial restriction so that in some specific population distributions, no pillaging movement is feasible.

2.1 Stable set in three-player models

To characterize stable sets, we divide states into four groups according to their distributions and allocations; group 1) all players are in one region; group 2) players have less than halves and occupy two regions; group 3) only one player has a half or more and the player stays alone in his region; and group 4) only one player has a half or more and the player is together with only another player in his region. It is easy to analyze the states in groups 1), 2), and 3) to find a stable set; it is relatively hard in group 4), however. Thus we would devote most of this subsection to analyzing the states in group 4). For simplicity of expression, we call a state in group 4) a *basic*

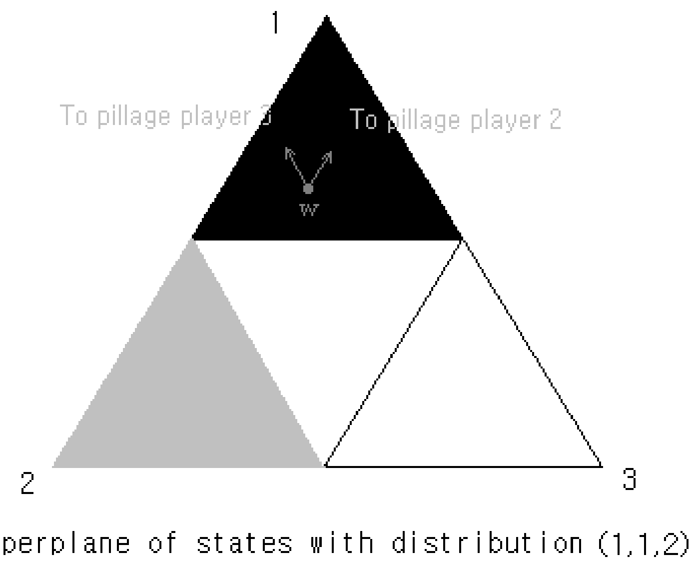

Figure 1: Basic Sets

state and a set of basic states a *basic set.*

Definition 6 formalizes a **basic set** and a **basic state**.

Definition 6 *For any three distinct players i, j, and k, define the set $B(i; j, k)$ of distributions by $B(i; j, k) = \{p \in R^I$: for some region $r \in R$, $p^r = \{i, j\}$ or $\{i, k\}$ $\}$ and define the correspondence $B^i_{j,k} : [\frac{1}{2}, 1] \times R^I \longrightarrow\!\!\!\!\longrightarrow X$ by $B^i_{j,k}(a, \dot{p}) = \{(w, p) \in X : p = \dot{p},\ w_i \geq a,$ and $w_i + w_j + w_k = 1\}$. For each $p \in B(i; j, k)$, the set $B^i_{j,k}(\frac{1}{2}, p)$ of states is called a **basic set.** A state in a basic set is called a **basic state.***

The set $B(i; j, k)$ denotes the set of distributions such that either player i and player j, or player i and player k constitute all population in some region. For example, let $p = (1, 1, 2)$ and $p' = (1, 2, 1)$, then $p, p' \in B(1; 2, 3)$ because player 1 shares region 1 only with player 2 at the distribution p and only with player 3 at the distribution p'. The basic sets are visualized on the hyperplane of states in Figure 1. The black area and the gray area denote the basic set $B^1_{2,3}(\frac{1}{2}, (1, 1, 2))$ and the basic set $B^2_{1,3}(\frac{1}{2}, (1, 1, 2))$, respectively. They are all possible basic sets under the distribution $(1, 1, 2)$.

In Figure 1, consider the basic state $(w, p) = ((\frac{7}{12}, \frac{3}{12}, \frac{1}{6}), (1, 1, 2))$ where players 1 has $\frac{7}{12}$ and player 2 has $\frac{3}{12}$ while staying at region 1 and player 3 has $\frac{1}{6}$ while staying at region 2. Note that player 1 cannot pillage players 2 and 3 simultaneously because players 2 and 3 are in different regions. If player 1 pillages player 3 at (w, p), then the allocation of the state is located on the left arrow in the figure, and the distribution changes from $(1, 1, 2)$ to $(2, 1, 2)$. If player 1 pillages player 2 at (w, p), then the state is located on the right arrow, and the distribution does not change.

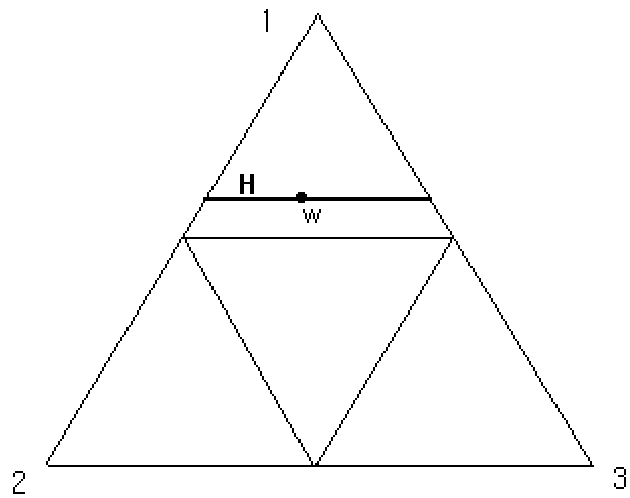

Figure 2: The Set $H^1_{2,3}(\frac{7}{12},(1,1,2))$

For notational simplicity, we define the following set of states.

Definition 7 *For any three distinct players i, j, and k, define the correspondence $H^i_{j,k} : [\frac{1}{2}, 1] \times R^I \longrightarrow\!\!\!\!\!\rightarrow X$ by $H^i_{j,k}(a, \dot{p}) = \{(w,p) \in X : p = \dot{p},\ w_i = a,$ and $w_i + w_j + w_k = 1\}$.*

For each $(a,p) \in [\frac{1}{2},1] \times R^I$, the set $H^i_{j,k}(a,p)$ consists of the states such that $w_i = a$ in $B^i_{j,k}(\frac{1}{2},p)$. In Figure 2, the bold horizontal line and the dot denote $H^i_{j,k}(\frac{7}{12},(1,1,2))$ and $w = (\frac{7}{12}, \frac{3}{12}, \frac{1}{6})$, respectively.

Definition 8 introduces the condition that a stable set has to satisfy. The condition is related to basic sets, and thus we call this condition the **basic condition**. If a set S' of states lacks the basic condition, then S' cannot simultaneously satisfy internal stability and external stability.

Definition 8 *Given a set E of states, for any two distinct states $(w,p), (\dot{w},p) \in E \cap B^i_{j,k}(\frac{1}{2},p)$ such that $p \in B(i;j,k)$ and $1 > \dot{w}_i \geq w_i > \frac{1}{2}$, suppose that i) $0 < \dot{w}_j \leq w_j$ and $0 < \dot{w}_k \leq w_k$; and ii) $\dot{w}_k < w_k$ when $p_i = p_j$ and $\dot{w}_j < w_j$ when $p_i = p_k$. Then the set E of states is said to satisfy the* ***basic condition****.*

We can prove that a stable set satisfies the basic condition by way of contradiction. That is, if we assume that there is a stable set that lacks the basic condition, then we can show that the stable set cannot satisfy internal stability and external stability simultaneously.

Lemma 1 *A stable set satisfies the basic condition.*

Proof. By way of contradiction, suppose that there exists a stable set S that does not satisfy the basic condition. Then for some three distinct players i, j, and k, there exist two distinct states $(w, p), (\dot{w}, p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)$ such that $p \in B(i; j, k)$; $1 > \dot{w}_i \geq w_i > \frac{1}{2}$; if $p_i = p_k$, then $\dot{w}_k > w_k$ or $\dot{w}_j \geq w_j$; and if $p_i = p_j$, then $\dot{w}_k \geq w_k$ or $\dot{w}_j > w_j$. Without loss of generality, we can assume that player i is together with player j in a common region, i.e. $p_i = p_j$. Then we must have that either $\dot{w}_k \geq w_k$ or $\dot{w}_j > w_j$. We can show that in each case, S cannot satisfy internal stability and external stability simultaneously.

Suppose that we have $\dot{w}_k \geq w_k$. We first show that $\dot{w}_k > w_k$. Every state $(\ddot{w}, p)$ in $B^i_{j,k}(\frac{1}{2}, p)$ such that $\ddot{w}_k = w_k$ and $\ddot{w}_i = w_i$ has $\ddot{w}_j = w_j$ since $\ddot{w}_i + \ddot{w}_j + \ddot{w}_k = w_i + w_j + w_k = 1$. Thus we have that $(\ddot{w}, p) = (w, p)$. Therefore, $(\dot{w}, p)$ cannot have $\dot{w}_k = w_k$ and $\dot{w}_i = w_i$ since $(w, p) \neq (\dot{w}, p)$. Every state $(\dddot{w}, p)$ in $B^i_{j,k}(\frac{1}{2}, p)$ such that $\dddot{w}_k = w_k$ and $\dddot{w}_i > w_i$ is the state that results from player i pillaging player j at the state (w, p); that is, such state $(\dddot{w}, p)$ dominates (w, p). By internal stability, S cannot contain such state $(\dddot{w}, p)$ and thus $(\dot{w}, p)$ cannot be $\dot{w}_k = w_k$ and $\dot{w}_i > w_i$. Therefore, we must have that $\dot{w}_k \neq w_k$ and thus that $\dot{w}_k > w_k$.

Let the allocation w' be $w'_j = \dot{w}_j$, $w'_k = w_k$, and $w'_i = 1 - \dot{w}_j - w_k$. Since $\dot{w}_i \geq w_i$ and $\dot{w}_k > w_k$, we have $w_j = 1 - w_i - w_k > 1 - \dot{w}_i - \dot{w}_k = \dot{w}_j$. Thus we have $w'_i = 1 - \dot{w}_j - w_k > 1 - w_j - w_k = w_i$. Since $w'_k = w_k$, $w'_i > w_i$, and $w'_j = \dot{w}_j = 1 - w_k - w'_i = w_j - (w'_i - w_i)$, (w', p) dominates (w, p) by player i pillaging player j. Thus S cannot contain (w', p) according to internal stability. To satisfy external stability, S has to dominate (w', p).

However, we can show that S cannot dominate (w', p). The stable set S can dominate (w', p), only if S contains those states as follows; the states that result from player i pillaging player j at (w', p), the states that result from player i pillaging player k at (w', p), the states that result from players i and j pillaging player k at (w', p), the states that result from player j pillaging player k at (w', p) when $w'_j > w'_k$, or the states that result from player k pillaging player j at (w', p) when $w'_k > w'_j$. Note that player j and player k are in different regions and so player i cannot pillage both of them simultaneously although player i has enough power to do it, i.e. $p_j \neq p_k$ and $w'_i > w'_j + w'_k$. We will show that S cannot contain any state above.

Every state that results from player i pillaging player j at (w', p) dominates (w, p), which is in S according to our assumption. By internal stability, S cannot contain those states. Every state that results from player i pillaging player k at (w', p) dominates $(\dot{w}, p)$, which is in S according to our assumption. Similarly, S cannot contain those states. The states that result from players i and j pillaging player k at (w', p) are all dominated by $((0, ..., w_i = 1, ..., 0), (p_k, ..., p_k))$, which is in the core. Thus S cannot contain those states. The states that result from player j pillaging player k at (w', p) or that result from player k pillaging player j at (w', p) are all dominated by either $((0, ..., w_i = 1, ..., 0), (p_k, ..., p_k))$ or $((0, ..., w_i = 1, ..., 0), (p_j, ..., p_j))$. Thus S cannot contain those states. Therefore, S cannot dominate (w', p) and thus cannot satisfy external stability. This contradiction shows that $\dot{w}_k \geq w_k$ is not possible.

Suppose that we have that $\dot{w}_j > w_j$. Then we can similarly show that S cannot dominate the state (w'', p) such that $w''_j = w_j$, $w''_k = \dot{w}_k$, and $w''_i = 1 - w_j - \dot{w}_k$. Consequently, the stable set S cannot satisfy internal stability and external stability simultaneously. This contradiction completes the proof. ■

Lemma 2 presents another condition that a stable set must follow. Lemma 1 examines the relation between two basic states in a stable set. Lemma 2 examines the relation between a basic state and another state whose distribution results from the move of the player who has a half or more at the basic state.

Lemma 2 *Suppose that $p \in B(i; j, k)$ and $(w, p) \in B^i_{j,k}(\frac{1}{2}, p)$. Let a distribution $\dot{p}$ satisfy that $\dot{p}_z = p_z$ for each $z \neq i$ and $\dot{p}_i \in \{p_j, p_k\}$. Given a stable set S, if $(w, \dot{p}) \in S$, then $(w, p) \in S$.*

Proof. If $\dot{p}_i = p_i \in \{p_j, p_k\}$, then $\dot{p} = p$, and thus this result obviously follows. Now, we have to show that if $\dot{p}_i \neq p_i$ and $(w, \dot{p}) \in S$, then $(w, p) \in S$. Suppose by way of contradiction that $\dot{p}_i \neq p_i$ and $(w, \dot{p}) \in S$, but $(w, p) \notin S$. It suffices to show that S cannot contain any state that dominates (w, p).

Without loss of generality, we assume that $w_j \geq w_k$. Since $(w, \dot{p}) \in B^i_{j,k}(\frac{1}{2}, \dot{p})$, we have that $w_i \geq \frac{1}{2}$ and $w_i + w_j + w_k = 1$. We first show that if $w_j > w_k$ then $w_j + w_k < \frac{1}{2}$. By way of contradiction, suppose not, that is, $w_j > w_k$ and $w_j + w_k = \frac{1}{2}$. Then $(w, \dot{p})$ is dominated by the state $(\ddot{w}, \ddot{p})$ such that $\ddot{w}_i = \ddot{w}_j = \frac{1}{2}$, which is in the core, C, by player j pillaging all wealth of player k at $(w, \dot{p})$. This contradicts internal stability of S since $(w, \dot{p}) \in S$ and $C \subset S$.

Let (w', p') result from player j or players i and j pillaging player k at (w, p). Then we have $p'_j = p'_k$. If players i and j pillage player k at $(w, \dot{p})$, then $w'_i > w_i \geq \frac{1}{2}$ and $w'_j > 0$, and thus player i can deprive the other players of their all wealth by pillage since $p'_j = p'_k$ and $w'_j + w'_k < w'_i$. If player j alone pillages player k at $(w, \dot{p})$, then $w_j > w_k$ and thus $w'_i = w_i > \frac{1}{2}$ since $w_j + w_k < \frac{1}{2}$. Thus player i can also deprive the other players of their all wealth in one move since $p'_j = p'_k$ and $w'_j + w'_k < w'_i$. Therefore, (w', p') is dominated by some state $(\dot{w}', \dot{p}')$ in the core such that $\dot{w}'_i = 1$, and thus S cannot contain (w', p'). Similarly, we can show that S cannot contain any state that results from players i and k pillaging player j at (w, p).

Let (w'', p'') result from player i pillaging player j at (w, p). Then we have that $w''_i > w_i$, $w''_j < w_j$, and $w''_z = w_z$ for each $z \in I \setminus \{i, j\}$. Note that $\{z : p''_z \neq p_z\} \subset \{i\}$ and thus $\{z : p''_z \neq \dot{p}_z\} \subset \{i\}$ since $\dot{p}_z = p_z$ for each $z \neq i$. Therefore, (w'', p'') dominates $(w, \dot{p})$ by player i pillaging player j. Thus S cannot contain (w'', p''). Similarly, we can show that S cannot contain any state that results from player i pillaging player k at (w, p).

Consequently, S cannot contain any state that dominates $(\dot{w}, p)$ and thus cannot satisfy external stability. This contradiction completes the proof. ■

Lemma 3 shows another implication of Lemmas 1 and 2. We will express basic states in a stable set with a function. Lemma 3 provides a basis to define the function that characterizes a stable set.

Lemma 3 *Given a stable set S, $S \cap H^i_{j,k}(a,p)$ has a single element for each $1 \geq a > \frac{1}{2}$ and $p \in B(i;j,k)$.*

Proof. It suffices to show that for each $1 > a > \frac{1}{2}$ and $p \in B(i;j,k)$, $S \cap H^i_{j,k}(a,p)$ has a single element because $H^i_{j,k}(1,p)$ has only one state regardless of p, which is in the core and so in a stable set. Suppose that $(w',p),(w,p) \in S \cap H^i_{j,k}(a,p)$ such that $1 > a > \frac{1}{2}$ and $p \in B(i;j,k)$. Then we have that $(w',p),(w,p) \in S \cap B^i_{j,k}(\frac{1}{2},p)$ and $1 > w'_i = w_i > \frac{1}{2}$. Suppose by way of contradiction that $w' \neq w$. By the basic condition of S, we have that either $w'_j \leq w_j$ and $w'_k < w_k$, or $w'_j < w_j$ and $w'_k \leq w_k$ since $1 > w'_i \geq w_i > \frac{1}{2}$. However, neither case is possible since $w'_i + w'_j + w'_k = w_i + w_j + w_k$. Therefore, we must have that $w' = w$, and thus $S \cap H^i_{j,k}(a,p)$ has at most one state for each $1 > a > \frac{1}{2}$ and $p \in B(i;j,k)$.

We need to show that $S \cap H^i_{j,k}(a,p) \neq \varnothing$ for each $1 > a > \frac{1}{2}$ and $p \in B(i;j,k)$ to complete the proof. By way of contradiction, suppose that there exists a stable set S such that $S \cap H^i_{j,k}(a,p) = \varnothing$ for some $1 > a > \frac{1}{2}$ and $p \in B(i;j,k)$. Without loss of generality, we can assume that player i is together with player j in a common region, i.e. $p_i = p_j$. Let $\bar{w}_j = \sup\{w_j : (w,p) \in S \cap B^i_{j,k}(a,p)\}$ and $\bar{w}_k = \sup\{w_k : (w,p) \in S \cap B^i_{j,k}(a,p)\}$.

We first show that $a + \bar{w}_j + \bar{w}_k \leq 1$. Suppose by way of contradiction that $a + \bar{w}_j + \bar{w}_k > 1$. Then by the definitions of $\bar{w}_j$ and $\bar{w}_k$, there exist states $(\dot{w},p)$ and $(\ddot{w},p)$ such that $(\dot{w},p),(\ddot{w},p) \in S \cap B^i_{j,k}(a,p)$ and $a + \dot{w}_j + \ddot{w}_k > 1$. Since $(\dot{w},p),(\ddot{w},p) \in B^i_{j,k}(a,p)$, we have that $\dot{w}_i, \ddot{w}_i \geq a$ and thus that $\dot{w}_i + \dot{w}_j + \ddot{w}_k > 1$ and $\ddot{w}_i + \dot{w}_j + \ddot{w}_k > 1$. Thus we have that $\dot{w}_j > \ddot{w}_j$ and $\dot{w}_k < \ddot{w}_k$ since $\dot{w}_i + \dot{w}_j + \dot{w}_k = \ddot{w}_i + \ddot{w}_j + \ddot{w}_k = 1$. However, the basic condition of S implies that both $\dot{w}_j \geq \ddot{w}_j$ and $\dot{w}_k \geq \ddot{w}_k$ if $\dot{w}_i \leq \ddot{w}_i$ and both $\dot{w}_j \leq \ddot{w}_j$ and $\dot{w}_k \leq \ddot{w}_k$ if $\dot{w}_i \geq \ddot{w}_i$. This contradiction guarantees that $a + \bar{w}_j + \bar{w}_k \leq 1$.

Define the allocation w to be $w_i = a$, $w_j = \bar{w}_j + \frac{1-(a+\bar{w}_j+\bar{w}_k)}{2}$, and $w_k = \bar{w}_k + \frac{1-(a+\bar{w}_j+\bar{w}_k)}{2}$. Then S cannot contain (w,p) since $(w,p) \in H^i_{j,k}(a,p)$ and $S \cap H^i_{j,k}(a,p) = \varnothing$. To prove that the assertion, $S \cap H^i_{j,k}(a,p) = \varnothing$, is impossible, it suffices to show that S cannot dominate (w,p).

First, we show that every state that results from player i pillaging player j at (w,p) cannot be in S. Let the state (w',p) result from player i pillaging player j at (w,p). Then we have that $w'_i > a$, $w'_j < w_j$, $w'_k = w_k$, and $(w',p) \in B^i_{j,k}(a,p)$; that is, player i increases its wealth through pillaging player j at the state (w,p) and player k maintains its wealth because the pillage does not affect player $k's$ wealth. If $1 > (\bar{w}_j + \bar{w}_k + a)$ then $w'_k > \bar{w}_k$, and thus $(w',p) \notin S$ because $\bar{w}_k$ is the supremum of the wealth that player k can have at states in $S \cap B^i_{j,k}(a,p)$ and $(w',p) \in B^i_{j,k}(a,p)$. If $1 = \bar{w}_j + \bar{w}_k + a$, then $w'_j < w_j = \bar{w}_j$ and $w'_k = w_k = \bar{w}_k$. Thus there exists a state $(w'',p) \in S \cap B^i_{j,k}(a,p)$ such that $w'_j < w''_j \leq \bar{w}_j$ and $w''_k \leq \bar{w}_k$ by the definitions of $\bar{w}_j$ and $\bar{w}_k$. Thus if $(w',p) \in S$, then the basic condition of S means that $w''_i < w'_i$ since $w'_j < w''_j$ and so that $w'_k < w''_k$. Since $w''_k \leq \bar{w}_k = w'_k$, we have that $(w',p) \notin S$. Note that (w',p) is arbitrary such that (w',p) results from player i pillaging player j

at (w,p). Therefore, S cannot contain the states that result from player i pillaging player j at (w,p).

Second, we prove that every state that results from player i pillaging player k at (w,p) cannot be in S. Suppose by way of contradiction that S contains a state (w''',p') that results from player i pillaging player k at (w,p). Then we have that $w'''_i > w_i$, $w'''_k < w_k$, and $w'''_j = w_j$. Consider the state (w''',p). Then we have that $p \in B(i;j,k)$ and $p'_z = p_z$ for each $z \neq i$ and $p'_i = p_k$. Lemma 2 means that $(w''',p) \in S \cap B^i_{j,k}(a,p)$ since $(w''',p') \in S$. Then we have that $\bar{w}_j \geq w'''_j$ according to the definition of $\bar{w}_j$. Since $w'''_j = w_j = \bar{w}_j + \frac{1-(a+\bar{w}_j+\bar{w}_k)}{2} \geq \bar{w}_j$, we have that $w'''_j = \bar{w}_j$ and thus that $1 = a + \bar{w}_j + \bar{w}_k$. By the definition of $\bar{w}_k$, there exists $(w^{(4)},p)$ such that $(w^{(4)},p) \in S \cap B^i_{j,k}(a,p)$ and $w'''_k < w^{(4)}_k < w_k = \bar{w}_k$. The basic condition of S implies that $w'''_j \leq w^{(4)}_j$ since $w'''_k < w^{(4)}_k$. Since $w^{(4)}_j \leq \bar{w}_j = w'''_j$ according to the definition of $\bar{w}_j$, we have that $w'''_j = w^{(4)}_j$. Therefore, we have that $w'''_i > w^{(4)}_i$, $w'''_k < w^{(4)}_k$, and $w'''_j = w^{(4)}_j$. This means that (w''',p') dominates $(w^{(4)},p)$ by player i pillaging player k at $(w^{(4)},p)$. This contradiction assures that $(w''',p') \notin S$. Since (w''',p') is arbitrary, S cannot contain the states that result from player i pillaging player k at (w,p).

Finally, we demonstrate that every state that dominates (w,p) and that is not covered by the two cases above is not in S. Note that these states result from either player j or player k moving to the other regardless of the move of player i. Consequently, player j and player k are in a common region at these states. Therefore, all such states are dominated by some state in the core such that player i has all of the wealth because player i, who has a majority of the power, $w_i \geq a > \frac{1}{2}$, can pillage both players in one move. Therefore, S cannot contain these states.

Consequently, S cannot dominate (w,p). This means that S cannot satisfy internal stability and external stability simultaneously. This contradiction guarantees that we must have that $S \cap H^i_{j,k}(a,p) \neq \varnothing$ for each $1 > a > \frac{1}{2}$ and $p \in B(i;j,k)$. ■

Definition 9 presents the conditions for the function that characterizes a stable set and names the function a **basic function**.

Definition 9 *For any three distinct players i,j, and k, let a function $\beta^i_{j,k} : [0,\frac{1}{2}] \times B(i;j,k) \longrightarrow [0,\frac{1}{2}]$ satisfy that $\beta^i_{j,k}(\lambda,p) \leq \lambda$ for each $(\lambda,p) \in [0,\frac{1}{2}] \times B(i;j,k)$. Define the set $B(\beta^i_{j,k})$ of states by $B(\beta^i_{j,k}) = \{(w,p) \in \bigcup_{p \in B(i;j,k)} B^i_{j,k}(\frac{1}{2},p)$: for some $(\lambda,p) \in [0,\frac{1}{2}] \times B(i;j,k)$, $w_j = \beta^i_{j,k}(\lambda,p)$ and $w_k = \lambda - \beta^i_{j,k}(\lambda,p)\}$. Suppose that $\beta^i_{j,k}$ satisfies the following three conditions; i) $B(\beta^i_{j,k})$ satisfies the basic condition; ii) if $p,\dot{p} \in B(i;j,k)$ satisfy that $\dot{p}_z = p_z$ for each $z \neq i$ and $p_i \neq \dot{p}_i$, then for each $\lambda \in [0,\frac{1}{2}]$, $\beta^i_{j,k}(\lambda,p) = \beta^i_{j,k}(\lambda,\dot{p})$; and for each $p \in B(i;j,k)$, iii) if $\lim_{\lambda \to 1/2} \beta^i_{j,k}(\lambda,p) = \frac{1}{4}$, then $\beta^i_{j,k}(\frac{1}{2},p) = \frac{1}{4}$, otherwise $\beta^i_{j,k}(\frac{1}{2},p) = \frac{1}{2}$. Then $\beta^i_{j,k}$ is called a **basic function**.*

Lemma 4 characterizes the functions that generate the set satisfying the basic condition.

Lemma 4 *Let a function $\beta^i_{j,k}$ be a function from $[0, \frac{1}{2}] \times B(i; j, k)$ to $[0, \frac{1}{2}]$ such that $\beta^i_{j,k}(\lambda, p) \leq \lambda$ for each $(\lambda, p) \in [0, \frac{1}{2}] \times B(i; j, k)$. If $B(\beta^i_{j,k})$ satisfies the basic condition, then $\beta^i_{j,k}(\cdot, p)$ is uniformly continuous and non-decreasing on $[0, \frac{1}{2})$.*

Proof. If $B(\beta^i_{j,k})$ satisfies the basic condition, then for each $\frac{1}{2} > \lambda > \lambda' \geq 0$ and $p \in B(i; j, k)$, we have that $\lambda - \beta^i_{j,k}(\lambda, p) \geq \lambda' - \beta^i_{j,k}(\lambda', p)$ and $\beta^i_{j,k}(\lambda, p) \geq \beta^i_{j,k}(\lambda', p)$. Therefore, Given any $\varepsilon > 0$, we must have that $\varepsilon > \beta^i_{j,k}(\lambda, p) - \beta^i_{j,k}(\lambda', p) \geq 0$ for all $\lambda, \lambda' \in [0, \frac{1}{2})$ and $p \in B(i; j, k)$ such that $\varepsilon > \lambda - \lambda' \geq 0$. This shows that the function $\beta^i_{j,k}(\cdot, p)$ is uniformly continuous and non-decreasing on $[0, \frac{1}{2})$. ■

Corollary 1 shows properties of a basic function.

Corollary 1 *For each $p \in B(i; j, k)$, a basic function $\beta^i_{j,k}(\cdot, p) : [0, \frac{1}{2}] \to [0, \frac{1}{2}]$ is uniformly continuous and non-decreasing on $[0, \frac{1}{2})$.*

Proof. According to Lemma 4, this result follows. ■

Lemma 5 strengthens Lemma 1. More concretely, Lemma 5 shows that a stable set must satisfy three conditions that are reflected on a basic function.

Lemma 5 *Given a stable set S, for any three distinct players i, j, and k, there exists a unique basic function $\beta^i_{j,k}$ such that $(B(\beta^i_{j,k}) \cup C) \cap B^i_{j,k}(\frac{1}{2}, p) = S \cap B^i_{j,k}(\frac{1}{2}, p)$ for each $p \in B(i; j, k)$.*

Proof. According to Lemma 3, $S \cap H^i_{j,k}(a, p)$ has a single state for each $1 \geq a > \frac{1}{2}$ and $p \in B(i; j, k)$. In addition, we have that $S \cap H^i_{j,k}(\frac{1}{2}, p) \neq \varnothing$ for each $p \in B(i; j, k)$ since such a set contains some states in C, at which two players have halves. Therefore, we can define the function $\alpha : [\frac{1}{2}, 1] \times B(i; j, k) \to [0, \frac{1}{2}]$ as follows; *i)* $\alpha(w_i, p) = w_j$ such that $(w, p) \in S \cap \bigcup_{p \in B(i;j,k)} B^i_{j,k}(\frac{1}{2}, p)$; and for each $p \in B(i; j, k)$, *ii)* if there exists $(w, p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)$ such that $w_i = \frac{1}{2}$ and $w_j = \frac{1}{4}$, then $\alpha(\frac{1}{2}, p) = \frac{1}{4}$, otherwise $\alpha(\frac{1}{2}, p) = \frac{1}{2}$. That is, the function α assigns each (w_i, p) the player $j's$ allocation according to $(w, p) \in S \cap \bigcup_{p \in B(i;j,k)} B^i_{j,k}(\frac{1}{2}, p)$.

Define the function $\beta^i_{j,k} : [0, \frac{1}{2}] \times B(i; j, k) \to [0, \frac{1}{2}]$ by $\beta^i_{j,k}(\lambda, p) = \alpha(1 - \lambda, p)$. Then it is easily seen that for each $(\lambda, p) \in [0, \frac{1}{2}] \times B(i; j, k)$, $\beta^i_{j,k}(\lambda, p) \leq \lambda$ and $((B(\beta^i_{j,k}) \cup C) \cap B^i_{j,k}(\frac{1}{2}, p)) \setminus H^i_{j,k}(\frac{1}{2}, p) = (S \cap B^i_{j,k}(\frac{1}{2}, p)) \setminus H^i_{j,k}(\frac{1}{2}, p)$. Next, we will show that for each $p \in B(i; j, k)$, $(B(\beta^i_{j,k}) \cup C) \cap H^i_{j,k}(\frac{1}{2}, p) = S \cap H^i_{j,k}(\frac{1}{2}, p)$.

By the definition of $\beta^i_{j,k}$, we have that $B(\beta^i_{j,k}) \subset S$ and thus that $(B(\beta^i_{j,k}) \cup C) \cap H^i_{j,k}(\frac{1}{2}, p) \subset S \cap H^i_{j,k}(\frac{1}{2}, p)$ for each $p \in B(i; j, k)$. Note that if $(w', p) \in H^i_{j,k}(\frac{1}{2}, p)$ with $w'_j \notin \{0, \frac{1}{4}, \frac{1}{2}\}$, then (w', p) is dominated by some state in C such that two players have halves. Therefore, if $(w, p) \in S \cap H^i_{j,k}(\frac{1}{2}, p)$ for some $p \in B(i; j, k)$, then $w_j = 0$, $\frac{1}{4}$, or $\frac{1}{2}$. And thus $(w, p) \in (B(\beta^i_{j,k}) \cup C) \cap H^i_{j,k}(\frac{1}{2}, p)$ because if $w_j = 0$ or $\frac{1}{2}$ then $(w, p) \in C$ and if $w_j = \frac{1}{4}$ then $(w, p) \in B(\beta^i_{j,k})$.

To complete the proof, we must show that the function $\beta^i_{j,k}$ is a basic function. Since $(B(\beta^i_{j,k}) \cup C) \cap B^i_{j,k}(\frac{1}{2}, p) = S \cap B^i_{j,k}(\frac{1}{2}, p)$ for each $p \in B(i; j, k)$, the set $B(\beta^i_{j,k})$

satisfies the basic condition as S does by Lemma 1. If $p, \dot{p} \in B(i;j,k)$ such that $\dot{p}_z = p_z$ for each $z \neq i$ and $p_i \neq \dot{p}_i$, then $S \cap H^i_{j,k}(w_i, p) = S \cap H^i_{j,k}(w_i, \dot{p})$ for each $1 \geq w_i \geq \frac{1}{2}$ by Lemma 2, and thus $\beta^i_{j,k}(\lambda, p) = \beta^i_{j,k}(\lambda, \dot{p})$ for each $\lambda \in [0, \frac{1}{2}]$. Now, we only need to prove that $\lim_{\lambda \to 1/2} \beta^i_{j,k}(\lambda, p) = \frac{1}{4}$ if and only if $\beta^i_{j,k}(\frac{1}{2}, p) = \frac{1}{4}$ because if $\beta^i_{j,k}(\frac{1}{2}, p) \neq \frac{1}{4}$, then $S \cap H^i_{j,k}(\frac{1}{2}, p)$ has two elements at which player j has either 0 or $\frac{1}{2}$.

First, we prove that for some $p \in B(i;j,k)$, if $\beta^i_{j,k}(\frac{1}{2}, p) = \frac{1}{4}$ then $\lim_{w_i \longrightarrow 1/2} \beta^i_{j,k}(w_i, p) = \frac{1}{4}$. Suppose that for some $p \in B(i;j,k)$, there exists $(w, p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)$ with $w_i = \frac{1}{2}$ and $w_j = \frac{1}{4}$. Without loss of generality, we assume that player i is together with player j in a common region, i.e. $p_i = p_j$. Since $\beta^i_{j,k}(\cdot, p)$ is uniformly continuous on $[0, \frac{1}{2})$ by Lemma 4, $\lim_{w_i \longrightarrow 1/2} \beta^i_{j,k}(w_i, p)$ always exists. Let $b = \lim_{w_i \longrightarrow 1/2} \beta^i_{j,k}(w_i, p)$. Suppose by way of contradiction that $b \neq \frac{1}{4}$.

Let $b > \frac{1}{4}$ first. Then there exists $\dot{w}_i$ such that $\beta^i_{j,k}(1 - \dot{w}_i, p) = \frac{1}{4}$ by the continuity of $\beta^i_{j,k}(\cdot, p)$ on $[0, \frac{1}{2})$, and thus there exists the allocation $\dot{w}$ such that $\dot{w}_j = \frac{1}{4}$ and $(\dot{w}, p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)$. Let $\dot{p} \in R^I$ such that $\dot{p}_z = p_z$ for each $z \neq i$ and $\dot{p}_i = p_k$. Then the state $(\dot{w}, \dot{p})$ dominates (w, p) by player i pillaging player k at (w, p) since $\dot{w}_i > w_i$, $\dot{w}_k < w_k$, and $\dot{w}_j = w_j$. Thus we have that $(\dot{w}, \dot{p}) \notin S$. However, every state that results from player i pillaging either player j or player k at $(\dot{w}, \dot{p})$ dominates $(\dot{w}, p) \in S$ as well. Every state that results from either player j or player k moving his region to pillage, regardless of the movement of player i, is dominated by some state in the core such that player i has the entire wealth. Therefore, S cannot dominate $(\dot{w}, \dot{p})$, and thus S lacks external stability.

Let $b < \frac{1}{4}$ next. Then there exists $\lambda \in [0, \frac{1}{2})$ with $\lambda - \beta^i_{j,k}(\lambda, p) > \frac{1}{4}$, and thus there exists λ'' such that $\lambda'' - \beta^i_{j,k}(\lambda'', p) = \frac{1}{4}$ since $\beta^i_{j,k}(\cdot, p)$ is continuous on $[0, \frac{1}{2})$. Then there exists $(\ddot{w}, p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)$ with $\ddot{w}_k = \frac{1}{4}$ and $\ddot{w}_j = \beta^i_{j,k}(\lambda'', p)$, and $(\ddot{w}, p)$ dominates $(w, p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)$ by player i pillaging player j at (w, p). This shows that S lacks internal stability. Consequently, these contradictions ensure that $b = \frac{1}{4}$.

Lastly, we prove that for some $p \in B(i;j,k)$, if $\lim_{\lambda \longrightarrow 1/2} \beta^i_{j,k}(\lambda, p) = \frac{1}{4}$, then $\beta^i_{j,k}(\frac{1}{2}, p) = \frac{1}{4}$. Suppose that for some $p \in B(i;j,k)$, $\lim_{\lambda \longrightarrow 1/2} \beta^i_{j,k}(\lambda, p) = \frac{1}{4}$. Note that $(w, p) \in B^i_{j,k}(\frac{1}{2}, p)$ with $w_i = \frac{1}{2}$, $w_j = \frac{1}{4}$, and $p_i = p_j$ is dominated only either by player i pillaging another player at (w, p), or by players i and j pillaging player k at (w, p). Note that $\beta^i_{j,k}(\lambda, p)$ and $\lambda - \beta^i_{j,k}(\lambda, p)$ denote player $j's$ allocation and player $k's$ allocation, respectively, when player i has $1 - \lambda$ at the distribution p in the stable set S. Because $\lim_{\lambda \longrightarrow 1/2} \beta^i_{j,k}(\lambda, p) = \frac{1}{4}$, we have that $\lim_{\lambda \longrightarrow 1/2} (\lambda - \beta^i_{j,k}(\lambda, p)) = \frac{1}{4}$. Therefore, the basic condition implies that a state (w'', p) such that $w''_j < \frac{1}{4}$ and $w''_k = \frac{1}{4}$ is not in S. Such a state (w'', p) results from player i pillaging player j at (w, p). Furthermore, the basic condition implies that a state (w''', p) such that $w'''_j = \frac{1}{4}$ and $w'''_k < \frac{1}{4}$ is not in S. By Lemma 2, S cannot contain such a state $(w''', \dot{p})$ that results from player

i pillaging player k at (w, p). Finally, every state that results from players i and j pillaging player k at (w, p) is dominated by some state in the core such that player i has all of the wealth. Therefore, S must contain (w, p) to satisfy external stability, and thus we have that $\beta^i_{j,k}(\frac{1}{2}, p) = \frac{1}{4}$. ■

Jordan (2006) studied the pillage game of "wealth is power" power function without spatial restriction and found the unique stable set, the set of dyadic allocations. Definition 10 introduces a **dyadic state** and the set of dyadic states $\boldsymbol{D}$. Theorem 2 establishes that the set D is the unique stable set in a one-region model. Note that Definition 10 and Theorem 2 are adapted from Jordan (2006) for the spatial pillage game.

Definition 10 *An allocation $w \in A$ is dyadic if for each i, $w_i = 0$ or $(\frac{1}{2})^{k_i}$ for some nonnegative integer k_i. A state (w, p) is **dyadic** if w is dyadic. The set $\boldsymbol{D}$ denotes the set of dyadic states.*

Theorem 2 (Theorem 3.3 in Jordan, 2006) *In a one-region model, the unique stable set is D.*

Lemma 6 reveals another implication of Theorem 2. It applies Theorem 2 to a general model, which possibly can have more than one region.

Lemma 6 *Define the set $\bar{X}$ of states by $\bar{X} = \{(w, p) \in X :$ for each region $r \in R$, $\sum_{i \in p^r} w_i = 0, \frac{1}{2}$, or 1 and if for some region $r' \in R$, $\sum_{i \in p^{r'}} w_i = \frac{1}{2}$, then for some player z, $w_z = \frac{1}{2}\}$. Then D is the unique set that satisfies both internal stability and external stability with respect to $\bar{X}$. In addition, a stable set includes $D \cap \bar{X}$.*

Proof. For any region $r \in R$ and any distribution $\dot{p} \in R^I$, define the set $X(r; \dot{p})$ of states by $X(r; \dot{p}) = \{(w, p) \in X : p = \dot{p}$ and $\sum_{i \in \dot{p}^r} w_i = 1\}$. We first show that $D \cap X(r; p)$ is the unique set that satisfies both internal stability and external stability with respect to $X(r; p)$. By Theorem 2, the unique stable set in a one-region model is the set of dyadic states. Given a region $r \in R$ and a distribution $p \in R^I$, define the function $w^{r,p} : X \longrightarrow [0, 1]^{\# p^r}$ by $w^{r,p}(w, p)_1 = w_{\min p^r}, ..., w^{r,p}(w, p)_{\# p^r} = w_{\max p^r}$; that is, $w^{r,p}$ projects from X onto allocations of players in the region r of the distribution p. Then $\{w^{r,p}(w, p) : (w, p) \in D \cap X(r; p)\}$ is the set of allocations of dyadic states in the $\# p^r$-player one-region model, and thus it is the unique stable set by Theorem 2 in this one-region model. Note that in a one-region model, dominance relation between states is well defined without distributions. Thus it is easily seen that $(w', p) \in X(r; p)$ dominates $(w, p) \in X(r; p)$ if and only if $w^{r,p}(w', p)$ dominates $w^{r,p}(w, p)$ in the $\# p^r$-player one-region model; both mean that $\sum_{z \in \{i : w'_i > w_i\}} w_z > \sum_{z \in \{i : w'_i < w_i\}} w_z$. Therefore, $D \cap X(r; p)$ is the unique set that satisfies both internal stability and external stability with respect to $X(r; p)$ because $\{w^{r,p}(w, p) : (w, p) \in D \cap X(r; p)\}$ is the unique stable set of allocations in the $\# p^r$-player one-region model.

For any region $r \in R$, any distribution $\dot{p} \in R^I$, and any player z with $\dot{p}_z \notin \dot{p}^r$, define the set $X(z, r; \dot{p})$ of states by $X(z, r; \dot{p}) = \{(w, p) \in X : p = \dot{p}, \sum_{i \in \dot{p}^r} w_i = \frac{1}{2}$, and $w_z = \frac{1}{2}\}$. We secondly prove that $D \cap X(z, r; p)$ is the unique set that satisfies both internal stability and external stability with respect to $X(z, r; p)$. Note that $(w', p) \in X(z, r; p)$ dominates $(w, p) \in X(z, r; p)$ if and only if $2w^{r,p}(w', p)$ dominates $2w^{r,p}(w, p)$ in the $\#p^r$-player one-region model. It is easily seen that $\{2w^{r,p}(w, p) : (w, p) \in D \cap X(z, r; p)\}$ is the set of allocations of dyadic states in the $\#p^r$-player one-region model, and thus by Theorem 2, $\{2w^{r,p}(w, p) : (w, p) \in D \cap X(z, r; p)\}$ is the unique stable set. Therefore, $D \cap X(z, r; p)$ is the unique set that satisfies both internal stability and external stability with respect to $X(z, r; p)$.

Third, we check that a state in $X(r; p)$ can be dominated only by another state in $X(r; p)$. If $(w, p) \in X(r; p)$ is dominated by another state (w', p'), then because the coalition $\{i : w'_i > w_i\} \subset p^r$ pillages the coalition $\{i : w'_i < w_i\} \subset p^r$ within region r, we have that $p_i = p'_i = r$ for any $i \in \{i : w'_i \neq w_i\}$. Since the pillage does not affect the coalition $\{i : w'_i = w_i\}$, we have that $p_i = p'_i$ for any $i \in \{i : w'_i = w_i\}$. Since $p' = p$ and $p'_i = r$ for each $i \in \{i : w'_i > 0\}$, we have that $(w', p') \in X(r; p)$.

Suppose that $\bar{S} \subset \bar{X}$ is a set that satisfies both internal stability and external stability with respect to $\bar{X}$. We next demonstrate that $\bar{S} = D \cap \bar{X}$. The set $\bar{S}$ must dominate every state in $X(r, p) \setminus \bar{S}$. However, $X(r, p) \setminus \bar{S}$ can be dominated only by some state in $X(r, p)$, and thus $\bar{S} \cap X(r, p)$ dominates every state in $X(r, p) \setminus \bar{S}$. Since $\bar{S} \cap X(r, p)$ is internally stable, $\bar{S} \cap X(r, p)$ is a set that satisfies both internal stability and external stability with respect to $X(r, p)$. Therefore, we have that $\bar{S} \cap X(r, p) = D \cap X(r; p)$. Since r and p are arbitrary, we have that $\bar{S} \cap \bigcup_{(r,p) \in R \times R^I} X(r, p) = D \cap \bigcup_{(r,p) \in R \times R^I} X(r; p)$. Note that a state in $X(z, r; p)$ can be dominated only by some state in $X(z, r; p) \cup \bigcup_{(r,p) \in R \times R^I} X(r; p)$. No state in $D \cap \bigcup_{(r,p) \in R \times R^I} X(r; p)$ can dominate another state in $X(z, r; p)$ because a state (w', p') that results from player z with $w_z = \frac{1}{2}$ and $p_z \neq r$ pillaging other players at region r at (w, p) in $X(z, r; p)$ has $1 > w'_z > \frac{1}{2}$, and thus $(w', p') \notin D \cap \bar{X}$. Therefore, $\bar{S} \cap X(z, r; p)$ must dominate every state in $X(z, r; p) \setminus \bar{S}$ because $\bar{S}$ dominates every state in $X(z, r; p) \setminus \bar{S}$. Since $\bar{S} \cap X(z, r; p)$ is internally stable, $\bar{S} \cap X(z, r; p)$ satisfies both internal stability and external stability with respect to $X(z, r; p)$. Thus we have that $\bar{S} \cap X(z, r; p) = D \cap X(z, r; p)$ because $D \cap X(z, r; p)$ is the unique set that satisfies both internal stability and external stability with respect to $X(z, r; p)$. Since r, p, and z with $\dot{p}_z \notin \dot{p}^r$ are arbitrary, we have that $\bar{S} \cap \bigcup_{(r,p) \in R \times R^I} (\bigcup_{z \notin p^r} X(z, r; p)) = D \cap \bigcup_{(r,p) \in R \times R^I} (\bigcup_{z \notin p^r} X(z, r; p))$. Since $\bigcup_{(r,p) \in R \times R^I} (X(r; p) \cup \bigcup_{z \notin p^r} X(z, r; p)) = \bar{X}$ and $\bar{S}, D \cap \bar{X} \subset \bar{X}$, we have that $\bar{S} = D \cap \bar{X}$.

Finally, we complete the proof that $D \cap \bar{X}$ is the unique set that satisfies both internal stability and external stability with respect to $\bar{X}$. We have proven that if a set satisfies both internal stability and external stability with respect to $\bar{X}$, then it must be $D \cap \bar{X}$. Therefore, we need to show that $D \cap \bar{X}$ satisfies both internal stability and external stability with respect to $\bar{X}$. Because for any states $(w, p), (w', p') \in D$, we have that $\sum_{z\in\{i:w'_i>w_i\}} w_z \leq \sum_{z\in\{i:w'_i<w_i\}} w_z$ or $\sum_{z\in\{i:w_i>w'_i\}} w'_z \leq \sum_{z\in\{i:w_i<w'_i\}} w'_z$, the set D is internally stable. Note that for each r, p, and z with $p_z \notin p^r$, $D \cap X(r;p)$ and $D \cap X(z,r;p)$ satisfy external stability with respect to $X(r;p)$ and $X(z,r;p)$, respectively. Therefore, $D \cap \bar{X}$ is externally stable with respect to $\bar{X}$. Consequently, $D \cap \bar{X}$ satisfies both internal stability and external stability with respect to $\bar{X}$.

In addition, It is easily seen that a stable set S includes $D \cap \bar{X}$. Note that a state in $\bar{X}$ can be dominated only by another state in $\bar{X}$. Every state (w',p') that results from player z with $w_z = \frac{1}{2}$ being involved in pillaging other players at (w,p) in $\bar{X}$ satisfies that $\sum_{i\in p'^r} w'_i = 1$ for some $r \in R$ and thus that $(w',p') \in \bar{X}$. Every state (w'',p'') that results from players in some region r pillaging other players in the same region r at (w,p) in $\bar{X}$ satisfies that for each $r \in R$, $\sum_{i\in p''^r} w''_i = 0, \frac{1}{2}$, or 1 and that if $\sum_{i\in p''^r} w''_i = \frac{1}{2}$ for some region $r \in R$, then $w''_z = \frac{1}{2}$ for some player z. Thus we have that $(w'',p'') \in \bar{X}$. Since a stable set S dominates every state in $\bar{X} \setminus S$, $S \cap \bar{X}$ dominates every state in $\bar{X} \setminus S$. Since $S \cap \bar{X}$ is internally stable, $S \cap \bar{X}$ satisfies both internal stability and external stability with respect to $\bar{X}$. Therefore, we have that $S \cap \bar{X} = D \cap \bar{X}$ and thus that $D \cap \bar{X} \subset S$. Since S is an arbitrary stable set, a stable set includes $D \cap \bar{X}$. ■

Proposition 1 completely characterizes stable sets in the three-player and two-region model.

Proposition 1 *In the three-player and two-region model, a set S is a stable set if and only if $S = B(\beta^1_{2,3}) \cup B(\beta^2_{3,1}) \cup B(\beta^3_{1,2}) \cup (D \cap \bar{X})$ for some basic functions $\beta^1_{2,3}$, $\beta^2_{3,1}$, and $\beta^3_{1,2}$ where $\bar{X} = \{(w,p) \in X :$ for each region $r \in R$, $\sum_{i\in p^r} w_i = 0, \frac{1}{2}$, or 1 and if for some region $r' \in R$, $\sum_{i\in p^{r'}} w_i = \frac{1}{2}$, then for some player z, $w_z = \frac{1}{2}\}$.*

Proof. We prove the necessary condition first. Suppose that S is a stable set. By Lemma 5, there exist basic functions $\beta^1_{2,3}$, $\beta^2_{3,1}$, and $\beta^3_{1,2}$ such that $B(\beta^1_{2,3}) \cup B(\beta^2_{3,1}) \cup B(\beta^3_{1,2}) \subset S$. By Lemma 6, we have that $D \cap \bar{X} \subset S$. Therefore, we must have that $B(\beta^1_{2,3}) \cup B(\beta^2_{3,1}) \cup B(\beta^3_{1,2}) \cup (D \cap \bar{X}) \subset S$. To show that $B(\beta^1_{2,3}) \cup B(\beta^2_{3,1}) \cup B(\beta^3_{1,2}) \cup (D \cap \bar{X}) \supset S$, it suffices to show that $B(\beta^1_{2,3}) \cup B(\beta^2_{3,1}) \cup B(\beta^3_{1,2}) \cup (D \cap \bar{X})$ is externally stable.

Let $\dot{S} = B(\beta^1_{2,3}) \cup B(\beta^2_{3,1}) \cup B(\beta^3_{1,2}) \cup (D \cap \bar{X})$. If $(\mathring{w},p) \in X \setminus \dot{S}$ with $\mathring{w}_z < \frac{1}{2}$ for each $z \in I$, then $(\mathring{w},p)$ is dominated by some state in $D \cap \bar{X}$ such that two players have halves. Let $\bar{X} = \{(w,p) \in X :$ for each region $r \in R$, $\sum_{i\in p^r} w_i = 0, \frac{1}{2}$, or 1 and if for some region $r' \in R$, $\sum_{i\in p^{r'}} w_i = \frac{1}{2}$, then for some player z, $w_z = \frac{1}{2}\}$. If $(\mathring{w},p) \in X \setminus \dot{S}$ and $(\mathring{w},p) \in \bar{X}$, then by Lemma 6, $(\mathring{w},p)$ is dominated by some state in $D \cap \bar{X}$ such that either two players have halves, or one player has all of the wealth. If $(\mathring{w},p) \in X \setminus \dot{S}$ with $\mathring{w}_i > \frac{1}{2}$ and $p_j = p_k$, then $(\mathring{w},p)$ is dominated by some state in $D \cap \bar{X}$ such that player i has all of the wealth.

Let $(\mathring{w},p) \in X \setminus \dot{S}$ satisfy that $p \in B(i;j,k)$ and $(\mathring{w},p) \in B^i_{j,k}(\frac{1}{2},p)$. Without loss of generality, we assume that $p_i = p_j$. Since $1 > \mathring{w}_i \geq \frac{1}{2}$ and a basic function $\beta^i_{j,k}(\cdot,p)$:

$[0, \frac{1}{2}] \to [0, \frac{1}{2}]$ is uniformly continuous on $[0, \frac{1}{2})$, $\lim_{\lambda \longrightarrow 1-\mathring{w}_i} \beta^i_{j,k}(\lambda, p)$ is well defined. If $\mathring{w}_i > \frac{1}{2}$ and $\mathring{w}_j > \beta^i_{j,k}(1 - \mathring{w}_i, p)$ or $\mathring{w}_i = \frac{1}{2}$ and $\mathring{w}_j > \lim_{\lambda \longrightarrow 1-\mathring{w}_i} \beta^i_{j,k}(\lambda, p)$, both of which mean that $\mathring{w}_j > \lim_{\lambda \longrightarrow 1-\mathring{w}_i} \beta^i_{j,k}(\lambda, p)$, then $\mathring{w}_k < 1 - \mathring{w}_i - \lim_{\lambda \longrightarrow 1-\mathring{w}_i} \beta^i_{j,k}(\lambda, p)$. Thus there exists a state $(w, p) \in B(\beta^i_{j,k})$ with $w_i > \mathring{w}_i$ and $w_k = \mathring{w}_k$. In this case, (w, p) dominates $(\mathring{w}, p)$ by player i pillaging player j. If $\mathring{w}_i > \frac{1}{2}$ and $\mathring{w}_j < \beta^i_{j,k}(1 - \mathring{w}_i, p)$ or $\mathring{w}_i = \frac{1}{2}$ and $\mathring{w}_j < \lim_{\lambda \longrightarrow 1-\mathring{w}_i} \beta^i_{j,k}(\lambda, p)$, both of which mean that $\mathring{w}_j < \lim_{\lambda \longrightarrow 1-\mathring{w}_i} \beta^i_{j,k}(\lambda, p)$, then $(w', \dot{p}) \in B(\beta^i_{j,k})$ such that $w'_j = \lim_{\lambda \longrightarrow 1-\mathring{w}_i} \beta^i_{j,k}(\lambda, p)$, $\dot{p}_z = p_z$ for all $z \neq i$, and $\dot{p}_i \neq p_i$ dominates $(\mathring{w}, p)$ by player i pillaging player k. If $\mathring{w}_i = \frac{1}{2}$ and $\mathring{w}_j = \lim_{\lambda \longrightarrow 1-\mathring{w}_i} \beta^i_{j,k}(\lambda, p) \neq \frac{1}{4}$, then some state in the core such that two players have halves dominates $(\mathring{w}, p)$. Therefore, $\dot{S}$ is externally stable, and thus $\dot{S} = S$.

Next, we prove the sufficient condition, that is, if functions $\beta^1_{2,3}$, $\beta^2_{3,1}$, and $\beta^3_{1,2}$ are basic functions, then the set $B(\beta^1_{2,3}) \cup B(\beta^2_{3,1}) \cup B(\beta^3_{1,2}) \cup (D \cap \bar{X})$ is a stable set. Suppose that $S' = B(\beta^1_{2,3}) \cup B(\beta^2_{3,1}) \cup B(\beta^3_{1,2}) \cup (D \cap \bar{X})$ for some basic functions $\beta^1_{2,3}$, $\beta^2_{3,1}$, and $\beta^3_{1,2}$. Then S' is externally stable as shown above. Now, we need to show that S' is internally stable.

Notice that each $B(i; j, k)$ has four elements and each element p in $B(i; j, k)$ has its counterpart distribution $\dot{p}$ such that $\dot{p} \in B(i; j, k)$, $p_i \neq \dot{p}_i$, $\dot{p}_j = p_j$, and $\dot{p}_k = p_k$. For example, $B(1; 2, 3) = \{(1, 1, 2), (1, 2, 1), (2, 1, 2), (2, 2, 1)\}$ and $(1, 1, 2)$ and $(1, 2, 1)$ are counterpart distributions to $(2, 1, 2)$ and $(2, 2, 1)$, respectively. Therefore, by the second condition of a basic function, if $(w, p), (w', p) \in B(\beta^i_{j,k})$ with $w_i > w'_i > \frac{1}{2}$ and $p_i = p_j$, then $(w, \dot{p}), (w', \dot{p}) \in B(\beta^i_{j,k})$ with $\dot{p}_i = \dot{p}_k$, and vice versa. Then we have that $w_j \leq w'_j$ and $w_k < w'_k$ by the first condition of a basic function since $p_i = p_j$ and $\dot{p}_i = \dot{p}_k$. Similarly, we have that $w_j < w'_j$ and $w_k \leq w'_k$. Consequently, we have that $w_j < w'_j$ and $w_k < w'_k$.

First, we prove that each $B(\beta^1_{2,3})$, $B(\beta^2_{3,1})$, and $B(\beta^3_{1,2})$ is internally stable. Let $(w, p), (w', p') \in B(\beta^i_{j,k})$ such that $(w, p) \neq (w', p')$ and $(w, p) \notin C$, which is the core. Since $\{p_j, p_k\} = \{p'_j, p'_k\} = R$, i.e. players j and k are distributed all over regions at p and p', we have that either $p'_j \neq p_j$ and $p'_k \neq p_k$, or $p'_j = p_j$ and $p'_k = p_k$. Thus if $p'_j \neq p_j$, then $\{z : p_z \neq p'_z\} \nsubseteq p^r$ for each $r \in R$, and so (w', p') does not dominate (w, p).

Suppose that p and p' satisfies that $p'_j = p_j$ and $p'_k = p_k$. If $w_i, w'_i > \frac{1}{2}$, then *i)* $w_j < w'_j$ and $w_k < w'_k$; *ii)* $w_j > w'_j$ and $w_k > w'_k$; or *iii)* $w = w'$. If either *i)* $w_j < w'_j$ and $w_k < w'_k$, or *ii)* $w_j > w'_j$ and $w_k > w'_k$, then $\{z : w'_z \neq w_z\} \nsubseteq p^r$ for each $r \in R$. If $w = w'$ then $\sum_{z \in \{i : w'_i > w_i\}} w_z = \sum_{z \in \{i : w'_i < w_i\}} w_z = 0$. If $w_i > \frac{1}{2}$ and $w'_i = \frac{1}{2}$, then $\sum_{z \in \{i : w'_i > w_i\}} w_z \leq \frac{1}{2} < \sum_{z \in \{i : w'_i < w_i\}} w_z$. If $w_i = \frac{1}{2}$, $w_j = \frac{1}{4}$, and $w'_i > \frac{1}{2}$, then since $\lim_{\lambda \longrightarrow \frac{1}{2}} \beta^i_{j,k}(\lambda) = \frac{1}{4}$, $w_j = \frac{1}{4} > w'_j$ and $w_k = \frac{1}{4} > w'_k$. Thus we have that $\{i : w'_i \neq w_i\} \nsubseteq p^r$ for each $r \in R$. If $w_i = \frac{1}{2}$, $w_j = \frac{1}{4}$, and $w'_i = \frac{1}{2}$, then since $w'_j = \frac{1}{4}$ or $\frac{1}{2}$, we have that $\sum_{z \in \{i : w'_i > w_i\}} w_z = \sum_{z \in \{i : w'_i < w_i\}} w_z = 0$ or $\frac{1}{4}$. Therefore, in these cases, (w', p') does not dominate (w, p). Since $(w, p), (w', p') \in B(\beta^i_{j,k})$ with

$(w,p) \neq (w',p')$ and $(w,p) \notin C$ are arbitrary, each set $B(\beta^1_{2,3})$, $B(\beta^2_{3,1})$, and $B(\beta^3_{1,2})$ is internally stable.

Second, we check internal stability of the set $B(\beta^1_{2,3}) \cup B(\beta^2_{3,1}) \cup B(\beta^3_{1,2})$. Let $(w,p) \in B(\beta^i_{j,k})$ and $(w'',p'') \in B(\beta^j_{k,i})$. Then $w_i \geq \frac{1}{2}$ and $w''_j \geq \frac{1}{2}$, and thus $w_j \leq \frac{1}{2}$ and $w''_i \leq \frac{1}{2}$. If $w_i > w''_i$ then $\sum_{z\in\{y:w''_y>w_y\}} w_z \leq \frac{1}{2} \leq w_i \leq \sum_{z\in\{y:w''_y<w_y\}} w_z$. If $w_i = w''_i$ then $w_i = w''_i = \frac{1}{2}$. Since $w_j \in \{\frac{1}{4}, \frac{1}{2}\}$ by the third condition of a basic function, $\sum_{z\in\{i:w''_i>w_i\}} w_z = \sum_{z\in\{i:w''_i<w_i\}} w_z = 0$ or $\frac{1}{4}$. Therefore, (w,p) does not dominate (w'',p''). Similarly, we can prove that (w'',p'') does not dominate (w,p). Consequently, $B(\beta^1_{2,3}) \cup B(\beta^2_{3,1}) \cup B(\beta^3_{1,2})$ is internally stable.

Finally, we examine internal stability of $S' = B(\beta^1_{2,3}) \cup B(\beta^2_{3,1}) \cup B(\beta^3_{1,2}) \cup (D \cap \bar{X})$. Note that $(B(\beta^1_{2,3}) \cup B(\beta^2_{3,1}) \cup B(\beta^3_{1,2})) \cap \bar{X} \subset C$ and that $C \subset D \cap \bar{X}$. Since a state in $D \cap \bar{X}$ can be dominated only by another state in $\bar{X}$ and $D \cap \bar{X}$ is internally stable, any state in $D \cap \bar{X}$ is not dominated by another state in $B(\beta^1_{2,3}) \cup B(\beta^2_{3,1}) \cup B(\beta^3_{1,2})$. Therefore, it suffices to show that any state in $B(\beta^1_{2,3}) \cup B(\beta^2_{3,1}) \cup B(\beta^3_{1,2})$ is not dominated by another state in $D \cap \bar{X}$. Let $(w,p) \in B(\beta^i_{j,k})$. Then we have that $w_i \geq \frac{1}{2}$. If $w_i = 1$ or $w_i = \frac{1}{2}$ and $w_j = \frac{1}{2}$, then $(w,p) \in C$, and thus (w,p) is not dominated by any state in $D \cap \bar{X}$. If $1 > w_i > \frac{1}{2}$, then by the basic condition of $B(\beta^i_{j,k})$, we have that $w_j > 0$ and $w_k > 0$. Also, if $w_i = \frac{1}{2}$ and $w_j = \frac{1}{4}$, then $w_k = \frac{1}{4}$. In these cases, player i cannot pillage both players j and k simultaneously since $p_j \neq p_k$ and cannot be pillaged by another player since $w_i > \frac{1}{2} > \max\{w_j, w_k\}$. Thus if a state (w''', p''') dominates (w,p), then (w''',p''') satisfies that for some player z, $w_z \notin \{0, \frac{1}{4}, \frac{1}{2}, 1\}$, that is, $(w''',p''') \notin D \cap \bar{X}$. Since (w,p) is arbitrary, any state in $B(\beta^1_{2,3}) \cup B(\beta^2_{3,1}) \cup B(\beta^3_{1,2})$ is not dominated by another state in $D \cap \bar{X}$.

Therefore, S' is a stable set. Since functions $\beta^1_{2,3}$, $\beta^2_{3,1}$, and $\beta^3_{1,2}$ are arbitrary basic functions, $B(\beta^1_{2,3}) \cup B(\beta^2_{3,1}) \cup B(\beta^3_{1,2}) \cup (D \cap \bar{X})$ for any basic functions $\beta^1_{2,3}$, $\beta^2_{3,1}$, and $\beta^3_{1,2}$ is a stable set. ■

Proposition 1 shows that a stable set in the three-player two-region model differs from the stable set D in a one-region model. In this model, a stable set is greater than the stable set in a one-region model, but it is possible that a stable set does not include D. Figures 3 and 4 show one possible stable set S on the hyperplanes. Dots and bold curves in the figures denote states in S at each distribution. In these figures, the stable set does not contain the states $((\frac{1}{2}, \frac{1}{4}, \frac{1}{4}), (1,2,1))$ and $((\frac{1}{4}, \frac{1}{4}, \frac{1}{2}), (2,1,2))$, which are in D. Just like in the case of the core, this change of the stable set results from the limited feasibility of the dominance relation under the spatial restriction.

Theorem 3 generalizes Proposition 1 to the three-player $N-$region models where $N \geq 2$.

Theorem 3 *In a three-player $N-$region model where $N \geq 2$, a set S is a stable set if and only if $S = B(\beta^1_{2,3}) \cup B(\beta^2_{3,1}) \cup B(\beta^3_{1,2}) \cup (D \cap \bar{X}) \cup U(B(\beta^1_{2,3}) \cup B(\beta^2_{3,1}) \cup B(\beta^3_{1,2}) \cup C)$ for some basic functions $\beta^1_{2,3}$, $\beta^2_{3,1}$, and $\beta^3_{1,2}$ where $\bar{X} = \{(w,p) \in X :$ for each region $r \in R$, $\sum_{i\in p^r} w_i = 0, \frac{1}{2},$ or 1 and if for some region $r' \in R$, $\sum_{i \in p^{r'}} w_i = \frac{1}{2}$, then for*

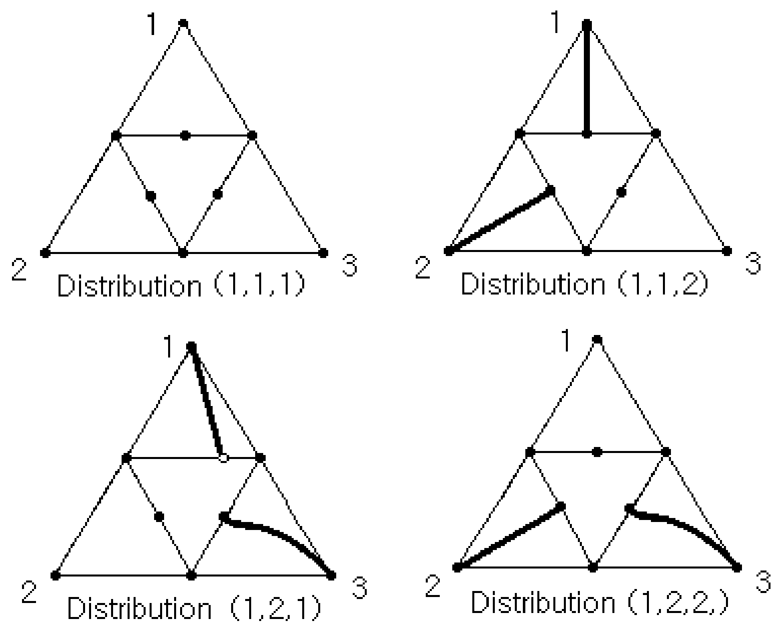

Figure 3: Stable set in the hyperplanes of states with $p_1 = 1$

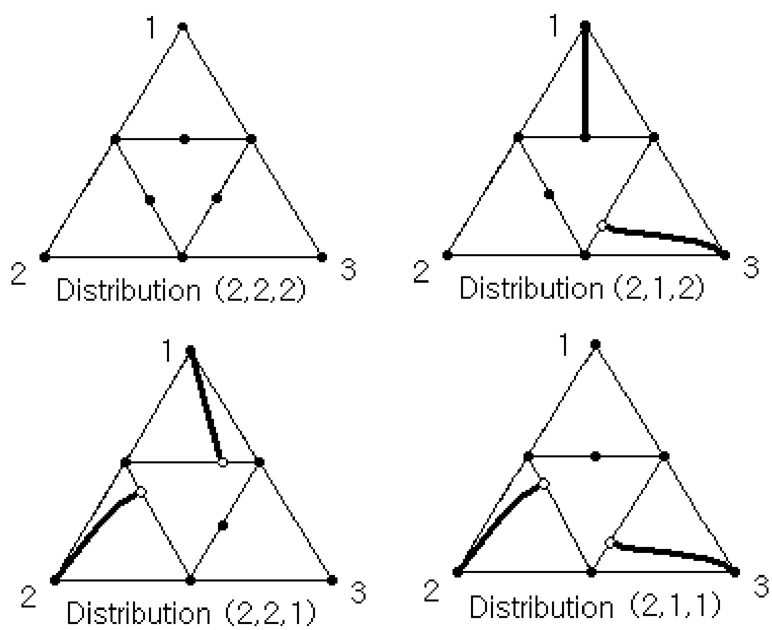

Figure 4: Stable set in the hyperplanes of states with $p_1 = 2$

some player z, $w_z = \frac{1}{2}\}$ and the set $U(B(\beta^1_{2,3}) \cup B(\beta^2_{3,1}) \cup B(\beta^3_{1,2}) \cup C)$ is the set of states that are not dominated by any state in $B(\beta^1_{2,3}) \cup B(\beta^2_{3,1}) \cup B(\beta^3_{1,2}) \cup C$.

Proof. For any two distinct regions $q, r \in R$, define the set $X^{q,r}$ of states by $X^{q,r} = \{(w,p) \in X : \text{for each } i,\ p_i = q \text{ or } r\}$. Then, it is easily seen that a state in $X^{r,q}$ can be dominated only by some state in $X^{r,q}$ because the act of the pillage does not disperse players. If there are more than two regions, then define the set $X^{indiv.}$ of states by $X^{indiv.} = \{(w,p) \in X : \text{for any three distinct regions } o, q, \text{ and } r,\ \{p_1, p_2, p_3\} = \{o, q, r\}\}$. That is, $X^{indiv.}$ is the set of states at which each player occupies its own region alone, i.e. individual region distribution. Note that any state in $X^{indiv.}$ does not dominate any other state in X, however, it can be dominated by some state at which only one region contains two players, whose distribution results from one player pillaging another player. Therefore, a set S is a stable set if and only if i) for any two distinct regions q and r, $S \cap X^{r,q}$ is both internally stable and externally stable with respect to $X^{r,q}$; and ii) S dominates all states in $X^{indiv.}$ except itself $X^{indiv.} \cap S$.

In the three-player and two-region model, by Proposition 1, a set S is a stable set if and only if $S = B(\beta^1_{2,3}) \cup B(\beta^2_{3,1}) \cup B(\beta^3_{1,2}) \cup (D \cap \bar{X})$ for some basic functions $\beta^1_{2,3}$, $\beta^2_{3,1}$, and $\beta^3_{1,2}$. Without loss of generality, given any two distinct regions q and r, we can regard a state (w,p) in $X^{r,q}$ as the state (w,p) in the two-region model and vice versa. Then, it is easily seen that $(w',p') \in X^{r,q}$ dominates $(w,p) \in X^{r,q}$ if and only if (w',p') dominates (w,p) in the two-region model. Therefore, for any two distinct regions q and r, $S \cap X^{r,q}$ is both internally stable and externally stable with respect to $X^{r,q}$ if and only if $S \cap X^{r,q} = \{B(\beta^1_{2,3}) \cup B(\beta^2_{3,1}) \cup B(\beta^3_{1,2}) \cup (D \cap \bar{X})\} \cap X^{r,q}$ for some basic functions $\beta^1_{2,3}$, $\beta^2_{3,1}$, and $\beta^3_{1,2}$. The observation that for any basic functions $\beta^1_{2,3}$, $\beta^2_{3,1}$, and $\beta^3_{1,2}$, the set $B(\beta^1_{2,3}) \cup B(\beta^2_{3,1}) \cup B(\beta^3_{1,2}) \cup C$ dominates all states in $X^{indiv.}$ except $U(B(\beta^1_{2,3}) \cup B(\beta^2_{3,1}) \cup B(\beta^3_{1,2}) \cup C)$ completes the proof. ■

Figures 5 and 6 show one possible stable set S on the hyperplanes. Figure 5 covers distributions where at least two players are in a common region and Figure 6 covers the other distributions, where each player occupies its own region alone. In these figures, dots, bold lines, and the gray area denote states in the stable set S. Note that except for the three corner points and three middle points, the gray area does not contain boundary lines.

2.2 Stable set in $I-$player and $N-$region models where $I = 4$ and $N = 2$, or $I \geq 4$ and $N \geq 3$

A stable set does not exist in an $I-$player and $N-$region model where $I = 4$ and $N = 2$, or $I \geq 4$ and $N \geq 3$. First, we prove that in the four-player and two-region

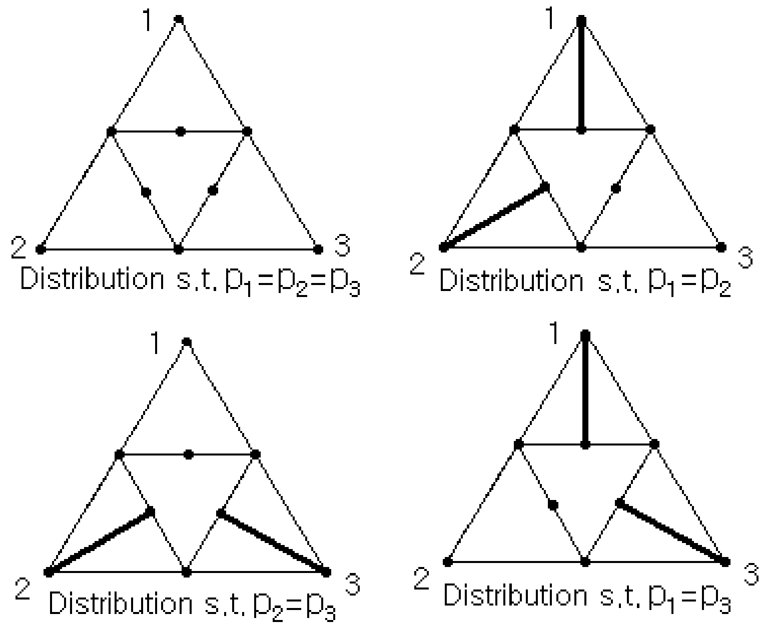

Figure 5: Stable set in the hyperplanes of states such that $p_j = p_k$ for two distinct players j and k.

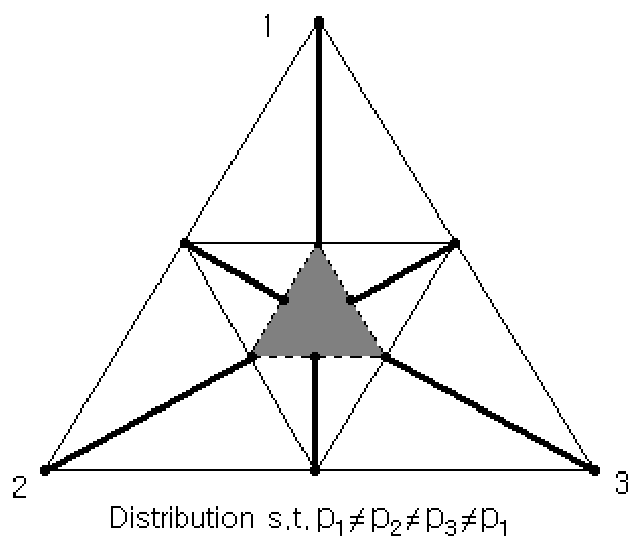

Figure 6: Stable set in the hyperplane of states such that $p_1 \neq p_2 \neq p_3 \neq p_1$

model, a stable set must contain a group of states out of basic sets. Second, we discover some properties of a group of states that are not in basic sets, but are in a stable set. Next, we show that in the four-player and two-region model, if there exists a stable set S, then we can find a state (w,p) such that (w,p) cannot be in S and S cannot dominate (w,p). It is because S contains four states and the properties of these states assure that S dominates every state that dominates (w,p). Finally, we generalize the result in the four-player and two-region model and verify the nonexistence of a stable set in an $I-$player and $N-$region model where $I \geq 4$ and $N \geq 3$.

Lemma 7 shows another property of a stable set. To satisfy both internal and external stabilities, a stable set must contain some states outside the basic sets as well as some basic states. Lemma 7 reveals relation among states that are outside the basic states and belong to some stable set.

Lemma 7 *In the four-player and two-region model, for some player j, let a distribution p satisfy $p^1 = \{j\}$ or $p^2 = \{j\}$. Then given a stable set S, there exists a positive real number a_p such that $[0, a_p] \subset \{w_j : (w,p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)\}$. In particular, if $(w', p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)$, then $[0, w'_j] \subset \{w_j : (w,p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)\}$.*

Proof. Let a distribution $\dot{p} \in B(i;j,k)$ with $\dot{p}_z = p_z$ for each $z \neq i$. Note that $B(i;j,k)$ is the set of distributions at which player i is together with only either player j or player k; that is, at the distribution $\dot{p}$, there exists a region $r \in R$ such that $\dot{p}^r = \{i,j\}$ or $\{i,k\}$. At the distribution p, player j is alone in a region and player k is together with the other players including player i. Therefore, we must have that $\dot{p}_i = \dot{p}_j$ so that player i is together with only one player in a common region. By Lemma 5, there exists a basic function $\beta^i_{j,k}$ with $(B(\beta^i_{j,k}) \cup C) \cap B^i_{j,k}(\frac{1}{2}, \dot{p}) = S \cap B^i_{j,k}(\frac{1}{2}, \dot{p})$. By the first condition of a basic function, there exists $\dot{\lambda} \in (0, \frac{1}{2})$ with $\beta^i_{j,k}(\dot{\lambda}, \dot{p}) > 0$. According to Corollary 1, $\beta^i_{j,k}(\cdot, \dot{p})$ is uniformly continuous on $[0, \frac{1}{2})$. Since $\beta^i_{j,k}(0, \dot{p}) = 0$, the intermediate value theorem implies that $[0, \beta^i_{j,k}(\dot{\lambda}, \dot{p})] \subset \beta^i_{j,k}([0, \dot{\lambda}], \dot{p})$.

To prove the first assertion, it suffices to show that $[0, \beta^i_{j,k}(\dot{\lambda}, \dot{p})] \subset \{w_j : (w,p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)\}$. For any $\delta \in [0, \beta^i_{j,k}(\dot{\lambda}, \dot{p})]$, let $\lambda_\delta = \min\{\lambda \in [0, \dot{\lambda}] : \beta^i_{j,k}(\lambda, \dot{p}) = \delta\}$, which is well defined by the uniform continuity of $\beta^i_{j,k}$ on $[0, \dot{\lambda}]$ because if a function is continuous, then the inverse image of a closed set under the function is a closed set. Given any $\lambda_\delta \in [0, \dot{\lambda}]$, let the allocation w^δ satisfy that $w^\delta_i = 1 - \lambda_\delta$, $w^\delta_j = \beta^i_{j,k}(\lambda_\delta, \dot{p})$, and $w^\delta_k = \lambda_\delta - \beta^i_{j,k}(\lambda_\delta, \dot{p})$. Then we have that $(w^\delta, \dot{p}) \in B(\beta^i_{j,k}) \cap B^i_{j,k}(\frac{1}{2}, \dot{p})$ and thus that $(w^\delta, \dot{p}) \in S$. Suppose by way of contradiction that $(w^\delta, p) \notin S$. Every state that results from player i pillaging either player j or player k at (w^δ, p) dominates $(w^\delta, \dot{p}) \in S$ as well as (w^δ, p). Every state that results from either player j or player k engaging in pillage at (w^δ, p), regardless of player $i's$ participation, is dominated by some state in the core such that player i has the total wealth because players j and k get together in a common region and player i has greater than a half. Therefore, S cannot dominate (w^δ, p), and thus S lacks external stability. This contradiction guarantees that $(w^\delta, p) \in S$ and thus that $\delta = w_j^\delta \in \{w_j : (w,p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)\}$ since $(w^\delta, p) \in B^i_{j,k}(\frac{1}{2}, p)$. Since $\delta \in [0, \beta^i_{j,k}(\dot{\lambda}, \dot{p})]$ is arbitrary, we have that $[0, \beta^i_{j,k}(\dot{\lambda}, \dot{p})] \subset \{w_j : (w,p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)\}$.

In particular, if $(w', p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)$, then by Lemma 2, $(w', \dot{p}) \in S \cap B^i_{j,k}(\frac{1}{2}, \dot{p})$. By the same way as shown above, we can show that for any $\delta \in [0, w'_j]$, $\delta \in \{w_j : (w,p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)\}$ and thus that $[0, w'_j] \subset \{w_j : (w,p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)\}$. ■

Lemmas 8, 9, and 10 strengthen Lemma 7 by revealing relations between states that are outside the basic states and belong to some stable set.

Lemma 8 *In the four-player and two-region model, for some player j, let a distribution p satisfy either $p^1 = \{j\}$ or $p^2 = \{j\}$. Then given a stable set S, for any $a \in (0, \frac{1}{2})$, there exists a state $(w,p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)$ such that $a > w_j + w_k > w_j > 0$ and $[0, w_j] \subset \{w_j : (w,p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)\}$. In addition, if $(w', p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)$ with $w_j > w'_j > 0$, then we have that $w_k \geq w'_k > 0$.*

Proof. By Lemma 5, there exists a basic function $\beta^i_{j,k}$ with $B(\beta^i_{j,k}) \subset S$. Let the distribution $\dot{p} \in B(i; j, k)$ satisfy $\dot{p}_z = p_z$ for each $z \neq i$. Then since $\beta^i_{j,k}(\cdot, \dot{p})$ is defined on $[0, \frac{1}{2}]$, for any $a \in (0, \frac{1}{2})$, we can find the allocation w^a with $w^a_j = \beta^i_{j,k}(a, \dot{p})$, $w^a_k = a - \beta^i_{j,k}(a, \dot{p})$, and $w^a_i = 1 - a$. Then we have that $(w^a, \dot{p}) \in B(\beta^i_{j,k})$. By Lemma 7, there exists a state $(\dot{w}, p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)$ with $\dot{w}_j > 0$ and $[0, \dot{w}_j] \subset \{w_j : (w,p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)\}$. Let a state $(w,p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)$ satisfy that $0 < w_j < \min\{\dot{w}_j, w^a_j\}$. We will show that (w,p) satisfies all required conditions, that is, $a > w_j + w_k > w_j > 0$ and $[0, w_j] \subset \{w_j : (w,p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)\}$.

Since $(w,p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)$, by Lemma 2, we have that $(w, \dot{p}) \in S \cap B^i_{j,k}(\frac{1}{2}, \dot{p})$. Since $w^a_j > w_j > 0$ and $w^a_i > \frac{1}{2}$, the basic condition means that $w_i > w^a_i$ and $w_k > 0$, and thus we have that $a > 1 - w_i = w_j + w_k > w_j > 0$. Since $(w,p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)$, Lemma 7 assures the second condition, $[0, w_j] \subset \{w_j : (w,p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)\}$.

In addition, let $(w', p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)$ with $w'_j \in (0, w_j)$. Then by Lemma 2, we have that $(w', \dot{p}) \in S \cap B^i_{j,k}(\frac{1}{2}, \dot{p})$. Since $w_j > w'_j > 0$ and $w_i > 1 - a > \frac{1}{2}$, the basic condition of S implies that $w_i < w'_i < 1$ and thus that $w_k \geq w'_k > 0$. ■

Lemma 9 *In the four-player and two-region model, for the distinct four players i, j, k, and y, let distributions p and p' satisfy either $p^1 = \{j\}$ and $p'^1 = \{i, j, y\}$, or $p^2 = \{j\}$ and $p'^2 = \{i, j, y\}$. Given a stable set S, suppose that $(w,p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)$ and $(w', p') \in S \cap B^i_{k,y}(\frac{1}{2}, p')$. If $w_i, w'_i > \frac{1}{2}$ and $w_j = w'_k$, then $w_k \geq w_j$ or $w'_y \geq w'_k$.*

Proof. By way of contradiction, suppose that $w_i, w'_i > \frac{1}{2}$, $w_j = w'_k$, $w_j > w_k$, and $w'_k > w'_y$. Lemma 7 implies that $[0, w'_k] \subset \{w_k : (w, p') \in S \cap B^i_{k,y}(\frac{1}{2}, p')\}$. Since $w'_k = w_j > w_k$, there exists a state $(\dot{w}', p') \in S \cap B^i_{k,y}(\frac{1}{2}, p')$ with $\dot{w}'_k = w_k$. Let $\dot{p}' \in B(i; k, y)$ satisfy $\dot{p}'_i \neq p'_i$ and $\dot{p}'_z = p'_z$ for each $z \neq i$. Since $(w', p'), (\dot{w}', p') \in S \cap B^i_{k,y}(\frac{1}{2}, p')$, we have that $(w', \dot{p}'), (\dot{w}', \dot{p}') \in S \cap B^i_{k,y}(\frac{1}{2}, \dot{p}')$ by Lemma 2. Since $w_i, w'_i > \frac{1}{2}$ and $w'_k > w_k = \dot{w}'_k$, the basic condition of S means that $w'_i < \dot{w}'_i$ and thus that $w'_y \geq \dot{w}'_y$. Since $w'_k > w'_y$, we have that $w_j = w'_k > w'_y \geq \dot{w}'_y$ and thus that $\dot{w}'_i = 1 - \dot{w}'_k - \dot{w}'_y > 1 - \dot{w}'_k - w_j = 1 - w_k - w_j = w_i$. Since $w_i + w_j + w_k = 1$, we have that $w_y = 0$. Therefore, we have that $\dot{w}'_i > w_i > \frac{1}{2}$, $\dot{w}'_k = w_k$, and $\dot{w}'_y > w_y = 0$. Note that the distribution p' results from players i and y moving to the region of player j at the distribution p. Therefore, $(\dot{w}', p') \in S$ dominates $(w, p) \in S$ by players i and y pillaging player j at (w, p). This contradiction guarantees that if $w_i, w'_i > \frac{1}{2}$ and $w_j = w'_k$, then $w_k \geq w_j$ or $w'_y \geq w'_k$. ■

Lemma 10 *In the four-player and two-region model, for any player j, let a distribution p satisfy either $p^1 = \{j\}$ or $p^2 = \{j\}$. Given a stable set S, if $(w, p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)$ and $(w', p) \in S \cap B^i_{j,y}(\frac{1}{2}, p)$ with $w_j = w'_j > 0$, then $w_i = w'_i$.*

Proof. Suppose by way of contradiction that $(w, p) \in S \cap B^i_{j,k}(\frac{1}{2}, p)$, $(w', p) \in S \cap B^i_{j,y}(\frac{1}{2}, p)$, $w_j = w'_j > 0$ and $w_i \neq w'_i$. Since $(w, p) \in B^i_{j,k}(\frac{1}{2}, p)$ and $(w', p) \in B^i_{j,y}(\frac{1}{2}, p)$, we have that $w_i \geq \frac{1}{2} \geq w_j + w_k$ and $w'_i \geq \frac{1}{2} \geq w'_j + w'_y$. Since $w_j, w'_j > 0$, we have that $w_i > w_k$ and $w'_i > w'_y$. Therefore, if $w_i > w'_i$ then $(w, p) \in S$ dominates $(w', p) \in S$ by either player i or players i and k pillaging player y at (w', p). Similarly, if $w_i < w'_i$ then $(w', p) \in S$ dominates $(w, p) \in S$. This contradiction completes the proof. ■

Lemma 11 synthesizes the previous results in this subsection and shows that in the four-player and two-region model, a stable set S contains four distinct states that satisfy six conditions introduced in this lemma. We can use these four states to show the nonexistence of stable set. The six conditions guarantee that there exists a state (w, p) that S cannot contain or dominate.

Lemma 11 *In the four-player and two-region model, a stable set S contains four states $(\dot{w}, p')$, $(\ddot{w}, p'')$, $(\dddot{w}, p''')$, and $(\mathring{w}, p''')$ such that for some four distinct players i, j, k, and y, i) distributions p', p'', and p''' satisfy either $p'^1 = \{j\}$, $p''^1 = \{i, j, y\}$, and $p'''^1 = \{k\}$, or $p'^2 = \{j\}$, $p''^2 = \{i, j, y\}$, and $p'''^2 = \{k\}$; ii) $(\dot{w}, p') \in S \cap B^i_{j,k}(\frac{1}{2}, p')$, $(\ddot{w}, p'') \in S \cap B^i_{k,j}(\frac{1}{2}, p'')$, and $(\dddot{w}, p'''), (\mathring{w}, p''') \in S \cap B^i_{k,j}(\frac{1}{2}, p''')$; iii) $\ddot{w}_k > \dot{w}_k \geq \dot{w}_j > 0$; iv) $\ddot{w}_j, \dddot{w}_j < \frac{1}{4}$; v) $\frac{1}{4} > \dddot{w}_k > \dot{w}_k + \dot{w}_j$; and vi) $\mathring{w}_j \geq \mathring{w}_k = \dot{w}_j$.*

Proof. Let distributions p, $p(1)$, $p(2)$, $p(3)$, $p(4)$, and $p(5)$ satisfy that either $p^1 = \{i, j, y\}$, $p(1)^1 = \{j\}$, $p(2)^1 = \{i, j, k\}$, $p(3)^1 = \{k\}$, $p(4)^1 = \{i, k, y\}$, and $p(5)^1 = \{y\}$; or $p^2 = \{i, j, y\}$, $p(1)^2 = \{j\}$, $p(2)^2 = \{i, j, k\}$, $p(3)^2 = \{k\}$, $p(4)^2 = \{i, k, y\}$, and $p(5)^2 = \{y\}$. By Lemma 8, there exist states $(\ddot{w}, p) \in S \cap B^i_{k,j}(\frac{1}{2}, p)$, $(\ddot{w}^{(1)}, p(1)) \in S \cap B^i_{j,y}(\frac{1}{2}, p(1))$, $(\dddot{w}^{(2)}, p(2)) \in S \cap B^i_{y,k}(\frac{1}{2}, p(2))$, $(\dddot{w}^{(3)}, p(3)) \in S \cap B^i_{k,j}(\frac{1}{2}, p(3))$, $(\dddot{w}^{(4)}, p(4)) \in S \cap B^i_{j,y}(\frac{1}{2}, p(4))$, and $(\ddot{w}^{(5)}, p(5)) \in S \cap B^i_{y,k}(\frac{1}{2}, p(5))$ with $0 < \ddot{w}_j, \ddot{w}_k, \ddot{w}^{(1)}_j, \ddot{w}^{(1)}_y, \dddot{w}^{(2)}_k, \dddot{w}^{(2)}_y, \dddot{w}^{(3)}_j, \dddot{w}^{(3)}_k, \dddot{w}^{(4)}_j, \dddot{w}^{(4)}_y, \ddot{w}^{(5)}_k, \ddot{w}^{(5)}_y < \frac{1}{4}$. Lemma 8 also implies that there exist states $(\dot{w}, p) \in S \cap B^i_{k,y}(\frac{1}{2}, p)$, $(\mathring{w}, p) \in S \cap B^i_{k,j}(\frac{1}{2}, p)$, $(\dot{w}^{(1)}, p(1)) \in S \cap B^i_{j,k}(\frac{1}{2}, p(1))$, $(\mathring{w}^{(1)}, p(1)) \in S \cap B^i_{j,y}(\frac{1}{2}, p(1))$, $(\dot{w}^{(2)}, p(2)) \in S \cap B^i_{y,j}(\frac{1}{2}, p(2))$, $(\mathring{w}^{(2)}, p(2)) \in S \cap B^i_{y,k}(\frac{1}{2}, p(2))$, $(\mathring{w}^{(3)}, p(3)) \in S \cap B^i_{k,j}(\frac{1}{2}, p(3))$, and $(\mathring{w}^{(4)}, p(4)) \in S \cap B^i_{j,y}(\frac{1}{2}, p(4))$ such that $0 < \dot{w}_k + \dot{w}_y, \dot{w}^{(1)}_j + \dot{w}^{(1)}_k, \dot{w}^{(2)}_j + \dot{w}^{(2)}_y <$

$\min\{\ddot{w}_k, \ddot{w}_j^{(1)}, \ddot{w}_y^{(5)}, \dddot{w}_y^{(2)}, \dddot{w}_k^{(3)}, \dddot{w}_j^{(4)}\}$ and $\dot{w}_k = \mathring{w}_k = \dot{w}_j^{(1)} = \mathring{w}_j^{(1)} = \dot{w}_y^{(2)} = \mathring{w}_y^{(2)} = \mathring{w}_k^{(3)} = \mathring{w}_j^{(4)} > 0$.

Since $\dot{w}_j^{(1)} = \mathring{w}_j^{(1)} > 0$, Lemma 10 means that $\dot{w}_i^{(1)} = \mathring{w}_i^{(1)}$ and thus that $\dot{w}_k^{(1)} = \mathring{w}_y^{(1)}$. If $\mathring{w}_y^{(1)} \geq \mathring{w}_j^{(1)}$ then $\dot{w}_k^{(1)} = \mathring{w}_y^{(1)} \geq \mathring{w}_j^{(1)} = \dot{w}_j^{(1)}$. Therefore, if $\mathring{w}_y^{(1)} \geq \mathring{w}_j^{(1)}$ and $\mathring{w}_j^{(3)} \geq \mathring{w}_k^{(3)}$, then the states $(\dot{w}^{(1)}, p(1))$, $(\ddot{w}, p)$, $(\dddot{w}^{(3)}, p(3))$, and $(\mathring{w}^{(3)}, p(3))$ satisfy all required conditions; that is, for some four distinct players i, j, k, and y, $i)$ distributions $p(1)$, p, and $p(3)$ satisfy either $p(1)^1 = \{j\}$, $p^1 = \{i, j, y\}$, and $p(3)^1 = \{k\}$, or $p(1)^2 = \{j\}$, $p^2 = \{i, j, y\}$, and $p(3)^2 = \{k\}$; $ii)$ $(\dot{w}^{(1)}, p(1)) \in S \cap B^i_{j,k}(\frac{1}{2}, p(1))$, $(\ddot{w}, p) \in S \cap B^i_{k,j}(\frac{1}{2}, p)$, and $(\dddot{w}^{(3)}, p(3)), (\mathring{w}^{(3)}, p(3)) \in S \cap B^i_{k,j}(\frac{1}{2}, p(3))$; $iii)$ $\ddot{w}_k > \dot{w}_k^{(1)} \geq \dot{w}_j^{(1)} > 0$; $iv)$ $\ddot{w}_j, \dddot{w}_j^{(3)} < \frac{1}{4}$; $v)$ $\frac{1}{4} > \dddot{w}_k^{(3)} > \dot{w}_j^{(1)} + \dot{w}_k^{(1)}$; and $vi)$ $\mathring{w}_j^{(3)} \geq \mathring{w}_k^{(3)} = \dot{w}_j^{(1)}$.

Note that if $\mathring{w}_i = \frac{1}{2}$, then since $\mathring{w}_j + \mathring{w}_k = \frac{1}{2}$ and $\mathring{w}_k < \frac{1}{4}$, we have that $\mathring{w}_j > \frac{1}{4} > \mathring{w}_k$, and thus some state in the core such that players i and j have halves dominates $(\mathring{w}, p) \in S$. This contradiction shows that we must have that $\mathring{w}_i > \frac{1}{2}$. Similarly, we can show that we have that $\mathring{w}_i^{(1)}$, $\mathring{w}_i^{(2)}$, $\mathring{w}_i^{(3)}$, and $\mathring{w}_i^{(4)} > \frac{1}{2}$. Therefore, if $\mathring{w}_y^{(1)} < \mathring{w}_j^{(1)}$, then since $\mathring{w}_i$, $\mathring{w}_i^{(1)}$, and $\mathring{w}_i^{(2)} > \frac{1}{2}$, Lemma 9 implies that $\mathring{w}_j \geq \mathring{w}_k$ and $\mathring{w}_k^{(2)} \geq \mathring{w}_y^{(2)}$. Since $\dot{w}_k = \mathring{w}_k$, by Lemma 10, we have that $\dot{w}_i = \mathring{w}_i$ and thus that $\dot{w}_y = \mathring{w}_j \geq \mathring{w}_k = \dot{w}_k$. In this case, the states $(\dot{w}, p)$, $(\ddot{w}^{(5)}, p(5))$, $(\dddot{w}^{(2)}, p(2))$, and $(\mathring{w}^{(2)}, p(2))$ satisfy all six conditions; that is, for some four distinct players i, k, y, and j, $i)$ distributions p, $p(5)$, and $p(2)$ satisfy $p^1 = \{k\}$, $p(5)^1 = \{i, j, k\}$, and $p(2)^1 = \{y\}$, or $p^2 = \{k\}$, $p(5)^2 = \{i, j, k\}$, and $p(2)^2 = \{y\}$; $ii)$ $(\dot{w}, p) \in S \cap B^i_{k,y}(\frac{1}{2}, p)$, $(\ddot{w}^{(5)}, p(5)) \in S \cap B^i_{y,k}(\frac{1}{2}, p(5))$, and $(\dddot{w}^{(2)}, p(2))$, $(\mathring{w}^{(2)}, p(2)) \in S \cap B^i_{y,k}(\frac{1}{2}, p(2))$; $iii)$ $\ddot{w}_y^{(5)} > \dot{w}_y \geq \dot{w}_k > 0$, $iv)$ $\ddot{w}_k^{(5)}, \dddot{w}_k^{(2)} < \frac{1}{4}$, $v)$ $\frac{1}{4} > \dddot{w}_y^{(2)} > \dot{w}_k + \dot{w}_y$, and $vi)$ $\mathring{w}_k^{(2)} \geq \mathring{w}_y^{(2)} = \dot{w}_k$. Similarly, if $\mathring{w}_j^{(3)} < \mathring{w}_k^{(3)}$, then $(\dot{w}^{(2)}, p(2))$, $(\ddot{w}^{(1)}, p(1))$, $(\dddot{w}^{(4)}, p(4))$, and $(\mathring{w}^{(4)}, p(4))$ satisfy all six conditions. ■

Proposition 2 proves the nonexistence of stable set in the four-player and two-region model.

Proposition 2 *No stable set exists in the four-player and two-region model.*

Proof. By way of contradiction, suppose that there exists a stable set S in the four-player and two-region model. Then by Lemma 11, there exist four states $(\dot{w}, p')$, $(\ddot{w}, p'')$, $(\dddot{w}, p''')$, and $(\mathring{w}, p''')$ such that for some four distinct players i, j, k, and y, $i)$ distributions p', p'', and p''' satisfy either $p'^1 = \{j\}$, $p''^1 = \{i, j, y\}$, and $p'''^1 = \{k\}$, or $p'^2 = \{j\}$, $p''^2 = \{i, j, y\}$, and $p'''^2 = \{k\}$; $ii)$ $(\dot{w}, p') \in S \cap B^i_{j,k}(\frac{1}{2}, p')$, $(\ddot{w}, p'') \in S \cap B^i_{k,j}(\frac{1}{2}, p'')$, and $(\dddot{w}, p''')$, $(\mathring{w}, p''') \in S \cap B^i_{k,j}(\frac{1}{2}, p''')$; $iii)$ $\ddot{w}_k > \dot{w}_k \geq \dot{w}_j > 0$; $iv)$ $\ddot{w}_j, \dddot{w}_j < \frac{1}{4}$; $v)$ $\frac{1}{4} > \dddot{w}_k > \dot{w}_k + \dot{w}_j$; and $vi)$ $\mathring{w}_j \geq \mathring{w}_k = \dot{w}_j$.

Define the set of states $T(\dot{w}, \ddot{w}, \dddot{w}; p') = \{(w, p) : w_i = \frac{1 - \dot{w}_j}{2}, w_j = \dot{w}_j, \min\{\ddot{w}_k, \dddot{w}_k - \dot{w}_j\} \geq w_k > \dot{w}_k, \text{ and } p = p'\}$. Then, we can show that $T(\dot{w}, \ddot{w}, \dddot{w}; p')$ has uncountably many elements. Note that by conditions $iii)$ and $v)$, $\min\{\ddot{w}_k, \dddot{w}_k - \dot{w}_j\} > \dot{w}_k$.

Let $a \in \mathbb{R}^4$ satisfy that $a_i = \frac{1-\dot{w}_j}{2}$, $a_j = \dot{w}_j$, $\min\{\ddot{w}_k, \dddot{w}_k - \dot{w}_j\} \geq a_k > \dot{w}_k$, and $a_y = 1 - a_i - a_j - a_k$. Since $1 > \dot{w}_j > 0$ and $\dot{w}_k > 0$, we have that $a_i, a_j, a_k \in (0, 1)$. Note that $a_y = 1 - \frac{1-\dot{w}_j}{2} - \dot{w}_j - a_k > \frac{1}{2} - \frac{\dot{w}_j}{2} - \dot{w}_k > \frac{1}{4}$, that is, $a_y \in (\frac{1}{4}, 1)$. Since $a_i + a_j + a_k + a_y = 1$, a is an allocation in the four-player model. Since a satisfies all requirements to be an allocation in $T(\dot{w}, \ddot{w}, \dddot{w}; p')$, we have that $(a, p') \in T(\dot{w}, \ddot{w}, \dddot{w}; p')$. It is easily seen that for any $\varepsilon \in (0, a_k - \dot{w}_k)$, $((a_i, a_j, a_k - \varepsilon, a_y + \varepsilon), p') \in T(\dot{w}, \ddot{w}, \dddot{w}; p')$.

We will prove that $T(\dot{w}, \ddot{w}, \dddot{w}; p')$ contains some state that S cannot contain or dominate. First, we show that there exists a state in $T(\dot{w}, \ddot{w}, \dddot{w}; p')$ that S cannot contain. Note that for any distinct states (w, p'), $(w', p') \in T(\dot{w}, \ddot{w}, \dddot{w}; p')$, we have that $w_i = w'_i$, $w_j = w'_j$, and either $w'_y > w_y > w_k > w'_k$ or $w_y > w'_y > w'_k > w_k$. If $w'_y > w_y > w_k > w'_k$, then (w', p') dominates (w', p') by player y pillaging player k. Similarly, if $w_y > w'_y > w'_k > w_k$, then (w, p') dominates (w', p'). Therefore, internal stability of S means that $T(\dot{w}, \ddot{w}, \dddot{w}; p') \cap S$ has at most one element. In addition, for any $(w, p') \in T(\dot{w}, \ddot{w}, \dddot{w}; p')$, there exists a state $(w'', p') \in T(\dot{w}, \ddot{w}, \dddot{w}; p')$ such that $w''_i = w_i$, $w''_j = w_j$, and $w''_y > w_y > w_k > w''_k$; that is, every state in $T(\dot{w}, \ddot{w}, \dddot{w}; p')$ is dominated by another state in $T(\dot{w}, \ddot{w}, \dddot{w}; p')$. Therefore, there exists a state $(w^T, p') \in T(\dot{w}, \ddot{w}, \dddot{w}; p')$ such that (w^T, p') dominates a state in $T(\dot{w}, \ddot{w}, \dddot{w}; p') \cap S$, which can be empty. Then we have that $(w^T, p') \notin S$. Next, we show that S cannot dominate (w^T, p').

Let the set of states $T_1(w^T, p') = \{(w, p) : (w, p)$ results from player i pillaging player j at $(w^T, p')\}$. Note that $w^T_k + w^T_y = 1 - w^T_i - w^T_j = 1 - \frac{1-\dot{w}_j}{2} - \dot{w}_j = \frac{1-\dot{w}_j}{2} = w^T_i$. Let $(w^{T_1}, p) \in T_1(w^T, p')$. Since $w^T_i = w^T_k + w^T_y$, $w^T_j = \dot{w}_j$, and $w^T_k > \dot{w}_k$, we have that $w^{T_1}_i > w^{T_1}_k + w^{T_1}_y$, $w^{T_1}_j < w^T_j = \dot{w}_j$, and $w^{T_1}_k = w^T_k > \dot{w}_k$. Since $(\dot{w}, p') \in S \cap B^i_{j,k}(\frac{1}{2}, p')$, Lemma 7 implies that $[0, \dot{w}_j] \subset \{w_j : (w, p') \in S \cap B^i_{j,k}(\frac{1}{2}, p')\}$. Therefore, $S \cap B^i_{j,k}(\frac{1}{2}, p')$ contains $(\dot{w}^{T_1}, p')$ with $\dot{w}^{T_1}_j = w^{T_1}_j$. Since $\dot{w}_j > \dot{w}^{T_1}_j$, by Lemma 8, we have that $\dot{w}_k \geq \dot{w}^{T_1}_k$ and thus that $w^{T_1}_k > \dot{w}^{T_1}_k$. Therefore, (w^{T_1}, p) in $T_1(w^T, p')$ is dominated by $(\dot{w}^{T_1}, p')$ in $S \cap B^i_{j,k}(\frac{1}{2}, p')$ by player i pillaging players k and y. Since $(w^{T_1}, p) \in T_1(w^T, p')$ is arbitrary, every state in $T_1(w^T, p')$ is dominated by some state in $S \cap B^i_{j,k}(\frac{1}{2}, p')$ through player i pillaging players k and y. Therefore, we must have that $T_1(w^T, p') \cap S = \varnothing$.

Let the set of states $T_2(w^T, p') = \{(w, p) : (w, p)$ results from player i pillaging player k at $(w^T, p')\}$. Then for each $(w^{T_2}, p') \in T_2(w^T, p')$, we have that $w^{T_2}_i > w^{T_2}_k + w^{T_2}_y$, $w^{T_2}_j = \dot{w}_j$, and $w^{T_2}_y = w^T_y > w^T_k > \dot{w}_k$ since $w^T_i = w^T_k + w^T_y$, $w^T_j = \dot{w}_j$, and $w^T_y > \frac{1}{4} > w^T_k$. Therefore, every state in $T_2(w^T, p')$ is dominated by $(\dot{w}, p') \in S \cap B^i_{j,k}(\frac{1}{2}, p')$ through players i and k pillaging player y when $\dot{w}_k > w^{T_2}_k$, through player i pillaging player y when $\dot{w}_k = w^{T_2}_k$, or through player i pillaging players k and y when $\dot{w}_k < w^{T_2}_k$. Therefore, we have that $T_2(w^T, p') \cap S = \varnothing$.

Let the set of states $T_3(w^T, p') = \{(w, p) : (w, p)$ results from player i pillaging player y at $(w^T, p')\}$. Then for each $(w^{T_3}, p') \in T_3(w^T, p')$, we have that $w^{T_3}_i > w^{T_3}_k + w^{T_3}_y$, $w^{T_3}_j = \dot{w}_j$, and $w^{T_3}_k = w^T_k > \dot{w}_k$ since $w^T_i = w^T_k + w^T_y$, $w^T_j = \dot{w}_j$, and $w^T_k > \dot{w}_k$. Therefore, every state in $T_3(w^T, p')$ is dominated by $(\dot{w}, p') \in S \cap B^i_{j,k}(\frac{1}{2}, p')$

through player i either pillaging players k and y when $w_y^{T_3} > 0$, or pillaging player k when $w_y^{T_3} = 0$. Therefore, we have that $T_3(w^T, p') \cap S = \varnothing$.

Let the set of states $T_4(w^T, p') = \{(w, p) : (w, p)$ results from player y pillaging player k at $(w^T, p')\}$. Let $(w^{T_4}, p') \in T_4(w^T, p')$. Since $w_i^T = w_k^T + w_y^T$ and $w_y^T > \frac{1}{4} > \dot{w}_k$, we have that $w_i^{T_4} \geq w_y^{T_4} > \dot{w}_k$ and $w_j^{T_4} = \dot{w}_j$. Note that dominance relation in $T(\dot{w}, \ddot{w}, \dddot{w}; p')$ is transitive; that is, if $(w, p') \in T(\dot{w}, \ddot{w}, \dddot{w}; p')$ is dominated by $(w', p') \in T(\dot{w}, \ddot{w}, \dddot{w}; p')$ and (w', p') is dominated by $(w'', p') \in T(\dot{w}, \ddot{w}, \dddot{w}; p')$, then because $w_i = w_i' = w_i''$, $w_j = w_j' = w_j''$, and $w_y'' > w_y' > w_y > w_k > w_k' > w_k''$, (w, p') is dominated by (w'', p') through player y pillaging player k at (w, p'). Therefore, if $w_k^{T_4} > \dot{w}_k$, then (w^{T_4}, p) is still in $T(\dot{w}, \ddot{w}, \dddot{w}; p')$ and thus since (w^T, p') dominates a state in $T(\dot{w}, \ddot{w}, \dddot{w}; p') \cap S$, which can be empty, (w^{T_4}, p') dominates a state in $T(\dot{w}, \ddot{w}, \dddot{w}; p') \cap S$. This means that $(w^{T_4}, p') \notin S$. If $\dot{w}_k \geq w_k^{T_4} > 0$, then (w^{T_4}, p') is dominated by $(\dot{w}, p')$ in $S \cap B_{j,k}^i(\frac{1}{2}, p')$ either through player i pillaging player y when $\dot{w}_k = w_k^{T_4}$, or through players i and k pillaging player y when $\dot{w}_k > w_k^{T_4}$. If $w_k^{T_4} = 0$, then $w_i^{T_4} = w_y^{T_4} > w_j^{T_4}$, and thus (w^{T_4}, p') is dominated by some state in the core such that players i and y have halves. Since $(w^{T_4}, p') \in T_4(w^T, p')$ is arbitrary, we have that $T_4(w^T, p') \cap S = \varnothing$.

Let the set of states $T_5(w^T, p') = \{(w, p) : (w, p)$ results from player y pillaging player j at $(w^T, p_1)\}$. Then for each $(w^{T_5}, p) \in T_5(w^T, p')$, we have that $w_i^{T_5} + w_k^{T_5} > w_j^{T_5} + w_y^{T_5}$ and $\frac{1}{2} > w_i^{T_5} > w_k^{T_5}$ since $w_i^T > w_y^T$, $w_k^T > w_j^T$ and $\frac{1}{2} > w_i^T > w_k^T$. Therefore, every state in $T_5(w^T, p')$ is dominated by some state in the core such that players i and k have halves. Therefore, we have that $T_5(w^T, p') \cap S = \varnothing$.

Let the set of states $T_6(w^T, p') = \{(w, p) : (w, p)$ results from player k pillaging player j at $(w^T, p')\}$. Then for each $(w^{T_6}, p) \in T_6(w^T, p')$, we have that $w_i^{T_6} + w_y^{T_6} > w_j^{T_6} + w_k^{T_6}$ and $\frac{1}{2} > w_i^{T_6} > w_y^{T_6}$ since $\frac{1}{2} > w_i^T > w_y^T > w_k^T > w_j^T$. Therefore, every state in $T_6(w^T, p')$ is dominated by some state in the core such that players i and y have halves. Therefore, we have that $T_6(w^T, p') \cap S = \varnothing$.

Let the set of states $T_7(w^T, p') = \{(w, p) : (w, p)$ results from players i and y pillaging player k at $(w^T, p')\}$. Then for each $(w^{T_7}, p) \in T_7(w^T, p')$, we have that $w_i^{T_7} > w_k^{T_7} + w_y^{T_7}$, $w_j^T = \dot{w}_j$, and $w_y^{T_7} > \dot{w}_k$ since $w_i^T = w_k^T + w_y^T$ and $w_y^T > \dot{w}_k$. Therefore, every state in $T_7(w^T, p')$ is dominated by $(\dot{w}, p') \in S \cap B_{j,k}^i(\frac{1}{2}, p')$. Therefore, we have that $T_7(w^T, p') \cap S = \varnothing$.

Let the set of states $T_8(w^T, p') = \{(w, p) : (w, p)$ results from players i and y pillaging player j at $(w^T, p')\}$. Let $(w^{T_8}, p'') \in T_8(w^T, p')$. Since $w_i^T > w_j^T + w_y^T$, $\ddot{w}_k \geq w_k^T$, and $w_y^T > \frac{1}{4} > \ddot{w}_j$, we have that $w_i^{T_8} > w_j^{T_8} + w_y^{T_8}$, $\ddot{w}_k \geq w_k^T = w_k^{T_8}$, and $w_y^{T_8} > w_y^T > \ddot{w}_j$. When $w_k^{T_8} = \ddot{w}_k$, (w^{T_8}, p'') is dominated by $(\ddot{w}, p'') \in S \cap B_{k,j}^i(\frac{1}{2}, p'')$ through player i pillaging players j and y when $\ddot{w}_j < w_j^{T_8}$, through player i pillaging player y when $\ddot{w}_j = w_j^{T_8}$, or through players i and j pillaging player y when $\ddot{w}_j > w_j^{T_8}$. Now, we check the case that $\ddot{w}_k > w_k^{T_8}$. Since $(\ddot{w}, p'') \in S \cap B_{k,j}^i(\frac{1}{2}, p'')$, Lemma 7 implies that $[0, \ddot{w}_k] \subset \{w_k : (w, p'') \in S \cap B_{k,j}^i(\frac{1}{2}, p'')\}$. Therefore, $S \cap B_{k,j}^i(\frac{1}{2}, p'')$ contains the state $(\ddot{w}^{T_8}, p'')$ such that $\ddot{w}_k^{T_8} = w_k^{T_8}$. Since $\ddot{w}_k > \ddot{w}_k^{T_8}$, Lemma 8 implies that $\ddot{w}_j \geq \ddot{w}_j^{T_8}$ and thus that $w_y^{T_8} > w_y^T > \frac{1}{4} > \ddot{w}_j \geq \ddot{w}_j^{T_8}$. Therefore, $(w^{T_8}, p'') \in T_8(w^T, p')$ is dominated by the state $(\ddot{w}^{T_8}, p'') \in S \cap B^i_{k,j}(\frac{1}{2}, p'')$ through player i pillaging players j and y when $\ddot{w}_j^{T_8} < w_j^{T_8}$, through player i pillaging player y when $\ddot{w}_j^{T_8} = w_j^{T_8}$, or through players i and j pillaging player y when $\ddot{w}_j^{T_8} > w_j^{T_8}$. Since $(w^{T_8}, p'') \in T_8(w^T, p')$ is arbitrary, we have that $T_8(w^T, p') \cap S = \varnothing$.

Let the set of states $T_9(w^T, p') = \{(w, p) : (w, p)$ results from players i and k pillaging player y at $(w^T, p')\}$. Then for each $(w^{T_9}, p') \in T_9(w^T, p')$, we have that $w_i^{T_9} > w_k^{T_9} + w_y^{T_9}$, $w_j^{T_9} = \dot{w}_j$, and $w_k^{T_9} > \dot{w}_k$ since $w_i^T = w_k^T + w_y^T$ and $w_k^T > \dot{w}_k$. Therefore, every state in $T_9(w^T, p')$ is dominated by $(\dot{w}, p') \in S \cap B^i_{j,k}(\frac{1}{2}, p')$. Therefore, we have that $T_9(w^T, p') \cap S = \varnothing$.

Let the set of states $T_{10}(w^T, p') = \{(w, p) : (w, p)$ results from players i and k pillaging player j at $(w^T, p')\}$. Let $(w^{T_{10}}, p) \in T_{10}(w^T, p')$. Since $w_i^T > w_y^T > \frac{1}{4}$, $\mathring{w}_j \geq \mathring{w}_k = \dot{w}_j = w_j^T$, and $\dddot{w}_k - \dot{w}_j \geq w_k^T > \dot{w}_j$, we have that $w_i^{T_{10}} > w_y^{T_{10}} > \frac{1}{4}$, $\mathring{w}_j \geq w_j^T > w_j^{T_{10}}$, and $\dddot{w}_k > w_k^{T_{10}} > \mathring{w}_k$. Since $(\dddot{w}, p''') \in S \cap B^i_{k,j}(\frac{1}{2}, p''')$, Lemma 7 implies that $[0, \dddot{w}_k] \subset \{w_k : (w, p''') \in S \cap B^i_{k,j}(\frac{1}{2}, p''')\}$. Therefore, $S \cap B^i_{k,j}(\frac{1}{2}, p'')$ contains the state $(\dddot{w}^{T_{10}}, p''')$ with $\dddot{w}_k^{T_{10}} = w_k^{T_{10}}$. Since $\dddot{w}_k > w_k^{T_{10}} > \mathring{w}_k$, Lemma 8 means that $\dddot{w}_j \geq \dddot{w}_j^{T_{10}} \geq \mathring{w}_j$. Since $w_y^{T_{10}} > \frac{1}{4} > \dddot{w}_j \geq \dddot{w}_j^{T_{10}} > w_j^{T_{10}}$, $(\dddot{w}^{T_{10}}, p''') \in S \cap B^i_{k,j}(\frac{1}{2}, p'')$ dominates the state $(w^{T_{10}}, p)$ by players i and j pillaging player y. Since $(w^{T_{10}}, p) \in T_{10}(w^T, p')$ is arbitrary, every state in $T_{10}(w^T, p')$ is dominated by some state in $S \cap B^i_{k,j}(\frac{1}{2}, p''')$. Therefore, we have that $T_{10}(w^T, p') \cap S = \varnothing$.

Let the set of states $T_{11}(w^T, p') = \{(w, p) : (w, p)$ results from players k and y pillaging player j at $(w^T, p')\}$. Then for each $(w^{T_{11}}, p) \in T_{11}(w^T, p')$, we have that $w_i^{T_{11}} < w_j^{T_{11}} + w_k^{T_{11}} + w_y^{T_{11}}$, $\frac{1}{2} > w_y^{T_{11}}$, and $\frac{1}{4} > w_k^{T_{11}} > w_j^{T_{11}}$ since $w_i^T < w_j^T + w_k^T + w_y^T$, $\frac{1}{2} > w_y^T + w_j^T$, and $\frac{1}{4} > \dddot{w}_k \geq w_k^T + w_j^T$. Note that by Lemma 6, $D \cap \bar{X} \subset S$ where $\bar{X} = \{(w, p) \in X :$ for each region $r \in R$, $\sum_{i \in p^r} w_i = 0, \frac{1}{2}$, or 1 and if for some region $r' \in R$, $\sum_{i \in p^{r'}} w_i = \frac{1}{2}$, then for some player z, $w_z = \frac{1}{2}\}$. Therefore, every state in $T_{11}(w^T, p')$ is dominated by some state (w, p) in $D \cap \bar{X}$ such that $w_y = \frac{1}{2}$, $w_j = w_k = \frac{1}{4}$, and $p^{p'_i} = I$ through players j, k, and y pillaging player i. Therefore, we have that $T_{11}(w^T, p') \cap S = \varnothing$.

Finally, let the set of states $T_{12}(w^T, p') = \{(w, p) : (w, p)$ results from players i, k, and y pillaging player j at $(w^T, p')\}$. Let $(w^{T_{12}}, p) \in T_{12}(w^T, p')$. Since $w_i^T > w_y^T + w_j^T$ and $w_i^T > w_y^T > \frac{1}{4} > w_k^T + w_j^T$, we have that $w_i^{T_{12}} > w_y^{T_{12}} > \frac{1}{4} > w_j^{T_{12}} + w_k^{T_{12}}$. If $w_i^{T_{12}} > \frac{1}{2}$, then $(w^{T_{12}}, p)$ is dominated by some state in the core such that player i has the total wealth. If $w_i^{T_{12}} \leq \frac{1}{2}$, then $(w^{T_{12}}, p)$ is dominated by some state in the core at which players i and j have halves. Since $(w^{T_{12}}, p') \in T_{12}(w^T, p')$ is arbitrary, we have that $T_{12}(w^T, p') \cap S = \varnothing$.

Therefore, $(w^T, p) \notin S$ cannot be dominated by any state in S. This contradiction shows that there is no stable set in the four-player and two-region model. ■

Theorem 4 generalizes Proposition 2 to an $I-$player and $N-$region model where $I = 4$ and $N = 2$, or $I \geq 4$ and $N \geq 3$.

Theorem 4 *No stable set exists in an $I-$player and $N-$region model where $I = 4$ and $N = 2$, or $I \geq 4$ and $N \geq 3$.*

Proof. Suppose by way of contradiction that there exists a stable set S. For any four distinct players i, j, k, and y, define the set $F(i,j,k,y)$ of states by $F(i,j,k,y) = \{(w,p) : w_i + w_j + w_k + w_y = 1,\ p^1 \cup p^2 = \{i,j,k,y\}$, and $p^3 = I \setminus \{i,j,k,y\}$ if $N \geq 3\}$. Then any state in $F(i,j,k,y)$ cannot be dominated by another state in $X \setminus F(i,j,k,y)$. Thus $S \cap F(i,j,k,y)$ is externally stable with respect to $F(i,j,k,y)$. Obviously $S \cap F(i,j,k,y)$ is internally stable. Therefore, $S \cap F(i,j,k,y)$ is both internal stable and external stable with respect to $F(i,j,k,y)$. It is easily seen that $S \cap F(i,j,k,y)$ of states can be adapted for a stable set in the four-player and two-region model. This contradicts Proposition 2, which shows nonexistence of stable set in the four-player and two-region model. This contradiction completes the proof. ■

3 Core in expectation

As shown in section 2, the stable set with respect to the dominance relation is not regarded as a plausible solution to the spatial pillage game. In $I-$player and $N-$region models where $I = 4$ and $N = 2$, or $I \geq 4$ and $N \geq 3$, no stable set exists. In three-player models, there exist stable sets. However, they contain implausible states, such as some states in the set of states $X_{\#I} = \{(w,p) :$ for some player i, $1 > w_i > \frac{1}{2}\}$. According to the interpretation about a stable set in Harsanyi (1974), no state in $X_{\#I}$ can be a plausible outcome because one of the players has enough power to pillage the others, so eventually the player will pillage the rest of the wealth. That is, any state in $X_{\#I}$ is directly or indirectly dominated by the core and thus cannot be a stable state.

These problems with the stable set with respect to the dominance relation are caused by the limited feasibility of dominance relation under the spatial restriction. This limited feasibility of dominance relation, in turn, makes the conditions of the stable set, both internal stability and external stability, improper to be requirements for a solution to the spatial pillage game. The external stability requires that any state outside a stable set be directly dominated by some state in the stable set. With respect to this limited dominance relation, some states in $X_{\#I}$ are directly dominated only by other states in $X_{\#I}$, thus a stable set must contain some states in $X_{\#I}$ to satisfy external stability. Also, with respect to this limited dominance relation, the core cannot directly dominate every state in $X_{\#I}$, and thus an internally stable set can include both the core and some states in $X_{\#I}$. This explains why stable sets in three-player models contain some states in $X_{\#I}$.

In $I-$player and $N-$region models where $I = 4$ and $N = 2$, or $I \geq 4$ and $N \geq 3$, if an internally stable set S' includes a set, of states, that dominates every state in $X_{\#I} \setminus S'$, then due to the limited feasibility of the dominance relation, S' contains improperly many states so that there exists some state $(w,p) \notin S'$ such that S' inevitably dominates every state that dominates (w, p). Thus, by the internal stability of S', S' cannot dominate (w, p), which is not in S'. That is, there is no set of states that satisfies both internal stability and external stability. This explains why no stable set exists in these models.

Jordan (2006) introduced a new solution concept, **farsighted core**. This farsighted core is defined based on an advanced concept of dominance relation, **Dominance in Expectation**. In this dominance in expectation, players make an **expectation** about how each state proceeds, and they pillage or defend according to their expectation. Naturally, this advanced concept of dominance relation allows broader feasibility of dominance relation while satisfying the spatial restriction. As a result, this solution concept based on the dominance in expectation solves the problems with the stable set based on the previous dominance relation and provides the unique solution which represents "an endogenous balance of power," as Jordan (2006) mentioned.

Formal definitions of this solution concept are as follows. An *expectation* is a belief that all players have in common and indicates how each state proceeds.

Definition 11 *An **expectation** is a function $f : X \longrightarrow X$ satisfying, for some integer $k \geq 2$, $f^k = f^{k-1}$ where $f^k = f \circ f^{k-1}$. Let $f_w(w, p)$ and $f_p(w, p)$ denote the allocation and the distribution at $f(w, p)$, respectively.*

Jordan (2006) considered only one step expectation where every state reaches its stationary state within one step, i.e. $f = f^2$. Here, the expectation is extended as a finite step expectation where some states take finite steps, possibly more than one step, to reach their stationary states. Based on this extended expectation, this study shows the same result, Corollary 2, as the result in Jordan (2006).

Dominance in Expectation between states indicates the possible states that the present state can change to provided that players follow the expectation after the changes. Just like in the previous dominance relation, both *physical* and *spatial conditions* should be satisfied in order for a *winning coalition in expectation*, who end up being better off, to change its present state through defeating a *losing coalition in expectation*, who end up being worse off. Physical condition is reflected on the conditions *iii*) and *iv*) in Definition 12 and spatial condition is reflected on the conditions *i*) and *ii*).

Definition 12 *Let an expectation f satisfy $f^k = f^{k+1}$. Given states $(\bar{w}, \bar{p})$ and $(w(n), p(n))$, for each $n \in \mathbb{N}$, define $W_f^{(n)} = \{i : f_w^k(\bar{w}, \bar{p})_i > w(n)_i\}$ and $L_f^{(n)} = \{i : f_w^k(\bar{w}, \bar{p})_i < w(n)_i\}$. Then a state $(\bar{w}, \bar{p})$ **dominates** (w, p) **in expectation** if there exists a sequence of states $\{(w(n), p(n))\}_{n=1}^N$ that has $(w(1), p(1)) = (w, p)$ and $(w(N), p(N)) = (\bar{w}, \bar{p})$ such that for each $1 \leq n \leq N-1$ and for some $r \in R$, i) $\{i : w(n+1)_i \neq w(n)_i\} \subset p(n+1)^r$; ii) for all $q \neq r$, $p(n+1)^q = p(n)^q \setminus (W_f^{(n)} \cap p(n+1)^r)$; iii) $\sum_{i \in W_f^{(n)} \cap p(n+1)^r} w(n)_i > \sum_{i \in L_f^{(n)} \cap p(n+1)^r} w(n)_i$; and iv) $\sum_{i \in W_f^{(n)}} w(n)_i > \sum_{i \in L_f^{(n)}} w(n)_i$.*

This dominance relation concept reflects players' ability to forecast how each state proceeds. With this forecasting ability, the players try to maximize their allocations in a final state. Thus, if some players expect that they belong to a losing coalition in expectation, $L_f^{(n)}$, who will be pillaged and so will be worse off in a final state, then they might have an incentive to get together in a common region and combine their powers in order to defend themselves against a winning coalition in expectation, $W_f^{(n)}$, who will be better off in the final state. However, under the condition iv), $L_f^{(n)}$ basically has no power to deter $W_f^{(n)}$ from pillaging $L_f^{(n)}$ even when all members of $L_f^{(n)}$ get together and combine their powers. This is because in this case, $W_f^{(n)}$ can also get together and combine their powers to pillage $L_f^{(n)}$. As a result, under the condition iv), $L_f^{(n)}$ has no incentive to take any defensive action and therefore, this condition is necessary that $W_f^{(n)}$ successfully pillages $L_f^{(n)}$ when the players have the forecasting ability.

However, the condition iv) is not sufficient that $W_f^{(n)}$ practically executes its plan to pillage $L_f^{(n)}$. This is because $W_f^{(n)}$ can exert its power only under the spatial restriction. Together with the condition iv), the conditions i), ii), and iii) represent sufficient conditions that $W_f^{(n)}$ executes its plan to pillage $L_f^{(n)}$ under the spatial restriction. These conditions are similar to the conditions in Definition 4 except the condition ii). So, the condition i) means that in each step of the pillaging process, transfers of wealth happen only in one region r where pillage actually happens. Also, the condition iii) denotes that members of $W_f^{(n)}$ in the region r have enough power to pillage members of $L_f^{(n)}$ in that region r. So, this pillage by the members of $W_f^{(n)}$ is feasible. But, the condition ii) differs from the condition ii) in Definition 4 in that the condition ii) allows more ways of forming a coalition. Under the condition ii), just like under the condition ii) in Definition 4, only members of $W_f^{(n)}$ travel, but in contrast to the condition ii) in Definition 4 they can come from more than one regions.

In addition, this definition differs from the definition of dominance in expectation in Jordan (2006) in that this definition generalizes the number of steps that the dominance relation can take. Jordan (2006) introduced one step dominance in expectation in which every plan to change a state can be completed within one step, i.e. $(w(1), p(1)) = (w, p)$ and $(w(2), p(2)) = (\bar{w}, \bar{p})$. However, in this setting, the players have a forecasting ability, and thus it is also a natural assumption that the players can make a finite step plan to change a state such that the plan can take more than one steps before it ends, i.e. $(w(N), p(N)) = (\bar{w}, \bar{p})$ for some $N \geq 2$. So, Definition 12 can be considered as a general version of the definition in Jordan (2006).

An expectation is *consistent* if it is organized in accord with the relation of dominance in expectation.

Definition 13 *An expectation f is* ***consistent*** *if $f(w,p)$ dominates (w,p) in expectation whenever $f(w,p) \neq (w,p)$ and (w,p) is undominated in expectation whenever*

$f(w,p) = (w,p)$.

Jordan (2006) interpreted consistency as "a rational expectation property." He said that "an expectation is consistent if only rational acts of pillage are expected, and an allocation is expected to persist only if no rational pillage is possible."

Farsighted core and *farsighted supercore*[1] are defined as follows.

Definition 14 *Given a consistent expectation f, the* ***farsighted core*** *under the expectation f is the set of states $K_f = \{(w,p) \in X :$ under the expectation f, no state in X dominates (w,p) in expectation$\}$. The* ***farsighted supercore*** *C_S is the intersection of all farsighted cores.*

A farsighted core is a set of stationary states under some consistent expectation. The farsighted supercore is the set of stationary states for all consistent expectations. Theorem 5 states that for any consistent expectation, the set of dyadic states D is the unique farsighted core and therefore is the farsighted supercore.

Theorem 5 *A consistent expectation f exists and the farsighted core K_f under f is the set of dyadic states, D. Therefore, the farsighted supercore C_S is also D.*

Lemma 12 *For any state (w,p), if an allocation w' satisfies $\sum_{i\in\{i:w'_i>w_i\}} w_i > \sum_{i\in\{i:w'_i<w_i\}} w_i$ and $f(w',p^*) = (w',p^*)$ for every distribution p^*, then there exists a distribution p' such that (w',p') dominates (w,p) in expectation.*

Proof. Suppose that a state (w,p) and an allocation w' satisfy the premise of this lemma. To prove this, it suffices to construct a sequence of states $\{(w(n),p(n))\}_{n=1}^N$ that can make (w',p') dominate (w,p) in expectation for some p'.

Let $W'_f = \{i : w'_i > w_i\}$ and $L'_f = \{i : w'_i < w_i\}$. Select $(w(2),p(2))$ such that $(w(2),p(2))$ results from W'_f's pillaging all members of L'_f in the region $\min\{p_i : i \in L'_f\}$ and also from W'_f's proportioning their wealth to w'. Similarly, select states $(w(n),p(n))$ for $n \in \mathbb{N}$ until $w(N) = w'$ for some N. Then the sequence of the states $\{(w(n),p(n))\}_{n=1}^N$ makes (w',p') dominate (w,p) in expectation for some p'. ■

Lemma 13 (Lemma 3.10 in Jordan, 2006) *For some positive integer k, let w be a dyadic allocation such that for each i, if $w_i > 0$ then $w_i \geq 2^{-(k+1)}$. If an allocation w' satisfies that $\sum_{z\in\{i:w'_i>w_i\}} w_z > \sum_{z\in\{i:w'_i<w_i\}} w_z$, then there exists a dyadic allocation w'' such that $\sum_{z\in\{i:w''_i>w'_i\}} w'_z > \sum_{z\in\{i:w''_i<w'_i\}} w'_z$ and for each i, if $w''_i > 0$ then $w''_i \geq 2^k$.*

Proof of Theorem 5. [2]First, we are going to construct a consistent expectation that has a farsighted core D and then will prove the uniqueness of the farsighted core.

[1] *Farsighted supercore* is named after Roth's (1976) *supercore*.

[2] The proof of Theorem 5 is similar to the proof of Theorem 3.3 in Jordan (2006).

Construct an expectation f as follows. If $(w,p) \in D$, then $f(w,p) = (w,p)$. If $(w,p) \notin D$, then select a dyadic allocation w' such that $\sum_{i\in\{i:w'_i>w_i\}} w_i > \sum_{i\in\{i:w'_i<w_i\}} w_i$. Theorem 2 assures the existence of w'. Let $W'_f = \{i : w'_i > w_i\}$ and $L'_f = \{i : w'_i < w_i\}$. Construct $f(w,p)$ such that $f(w,p)$ results from W'_f's pillaging all members of L'_f in the region $\min\{p_i : i \in L'_f\}$ and also from W'_f's proportioning their wealth to w'. Similarly, construct $f^n(w,p)$ for $n \in \mathbb{N}$ until $f^N_w(w,p) = w'$ for some N.

Now, we need to show the expectation f constructed above is consistent. If $(w,p) \notin D$, then for each $n \leq N-1$, we have that $\sum_{i\in W_f^{(n)}} f^n_w(w,p)_i > \sum_{i\in L_f^{(n)}} f^n_w(w,p)_i$ where $W_f^{(n)} = \{i : w'_i > f^n_w(w,p)_i\}$ and $L_f^{(n)} = \{i : w'_i < f^n_w(w,p)_i\}$. That is, the fourth condition in Definition 12 is satisfied. Also, it is easily seen that the expectation f is designed to satisfy the other three conditions in Definition 12. Consequently, for each $n \leq N$, a state $f^n(w,p)$ dominates $f^{n-1}(w,p)$ in expectation where $f^0(w,p) = (w,p)$. In addition, no state (w,p) in D is dominated in expectation by another state (w',p') in D because $\sum_{i\in\{i:w'_i>w_i\}} w_i \leq \sum_{i\in\{i:w'_i<w_i\}} w_i$ by Theorem 2. That is, if $(w,p) \in D$ and thus $f(w,p) = (w,p)$, then (w,p) is not dominated in expectation. Therefore, the expectation f is consistent.

To prove the uniqueness of a farsighted core, let f be a consistent expectation with the farsighted core K_f. Also, for each non-negative integer n, define $D_n = \{(w,p) \in D : w_i = 0 \text{ or } \geq (\frac{1}{2})^n\}$. Then, we have $D_0 \subset K_f$. Suppose, by way of induction, that for some n, we have $D_n \subset K_f$. Note that if $(w,p) \in D_n$, then $(w,p') \in D_n$ for any distribution p'. Thus, by Lemmas 12 and 13, any state (w',p') that dominates some state $(w,p) \in D_{n+1}$ in expectation is dominated in expectation by some state (w'',p'') in D_n because the allocation w' satisfies that $\sum_{i\in\{i:w'_i>w_i\}} w_i > \sum_{i\in\{i:w'_i<w_i\}} w_i$. Since f is consistent, we have that $D_{n+1} \subset K_f$. By induction, we have $D \subset K_f$. In addition, D dominated all states outside D by Theorem 2 and Lemma 12. Again, since f is consistent, if $(w,p) \notin D$, then $(w,p) \notin K_f$, that is, $K_f \subset D$. Therefore, we have $K_f = D$, and since f is an arbitrary consistent expectation, we have $C_S = D$. ■

This result is similar to the result in Jordan (2006), which stated that D is the unique farsighted core in one region models, which, in turn, do not have the spatial restriction. Clearly, dominance relation with respect to dominance in expectation changes if we introduce the spatial restriction. For example, let's consider the dominance relation between the following two states; $(\bar{w},\bar{p}) = ((1,0),(1,1))$ and $(w,p) = ((\frac{3}{4},\frac{1}{4}),(2,2))$. Then $(\bar{w},\bar{p})$ and (w,p) satisfy the physical condition for the dominance relation because $\sum_{i\in\{i:\bar{w}_i>w_i\}} w_i > \sum_{i\in\{i:\bar{w}_i<w_i\}} w_i$. So, if there is no spatial restriction, then $(\bar{w},\bar{p})$ dominates (w,p) in expectation. But, if we introduce the spatial restriction, then $(\bar{w},\bar{p})$ does not dominate (w,p) in expectation because any possible pillaging movement from (w,p) will results in the distribution $(2,2)$. That is, in this example, dominance relation with respect to dominance in expectation has changed under the spatial restriction. Nevertheless, Theorem 5 shows that if the players have a forecasting ability, then only states in D are expected to persist even when there is the spatial restriction. Therefore, **we conclude that under the far-**

sighted player assumption, the set of stationary states that represents an endogenous balance of power does not change under the spatial restriction.

Theorem 5 also shows that the dominance in expectation selectively reflects the concept of "indirect dominance" which was introduced by Harsanyi (1974) and formalized by Chwe (1994). The indirect dominance concept means that if (w, p) dominates (w', p'), with respect to the dominance relation in Definition 4, and (w', p') dominates (w'', p''), then (w, p) indirectly dominates (w'', p''), and so (w'', p'') cannot be a stable state if (w, p) is a stable state. To see how the dominance in expectation selectively reflects this indirect dominance concept, let (w, p) only indirectly dominate (w'', p''), that is, (w, p) cannot dominate (w'', p'') at once, and there exists a state that is dominated by (w, p) and dominates (w'', p''). If there exists a route that connects from (w'', p'') to (w, p) and through which a winning coalition who prefers (w, p) to (w'', p'') can achieve (w, p) by pillaging a losing coalition who prefers (w'', p'') to (w, p), then (w, p) dominate (w'', p'') in expectation. Otherwise, (w, p) does not dominate (w'', p'') in expectation. Here, the route is a sequence of states in Definition 12 that satisfies four conditions above, and the four conditions are sufficient conditions to change a state when the players have a forecasting ability. Therefore, the dominance in expectation reflects the indirect dominance concept if a dominance relation can be actualized by the players who have a forecasting ability. As a result, this dominance in expectation designates the set of dyadic states D as the unique set of stationary states.

In addition, the set D can be considered as a self-enforcing "standard of behavior," which is an interpretation about a stable set by von Neumann and Morgenstern (1947), in the sense that no state inside D is dominated in expectation by another state in D and every state outside D is dominated in expectation by some state in D. Then, we can conclude that in this concept of dominance in expectation, Harsanyi's indirect dominance concept and von Neumann and Morgenstern's self-enforcing standard of behavior concept are combined and constitute a congruous dominance relation in this spatial pillage game.

Xue (1998) and Konishi and Ray (2003) also introduced solution concepts for a coalitional game. Their solution concepts, similar to the farsighted core, are defined based on a reasonable progress of states that shows how the status quo progresses toward a stationary state under the farsighted player assumption. However, in contrast to the farsighted core, their solution concepts focus mainly on the forecasting ability of a winning coalition, and thus their solution concepts might not capture the fact that a losing coalition also has the forecasting ability and so they can defend themselves according to their expectation. Therefore, the solution concepts by Xue and Konishi and Ray might not designate stable states that are considered as stationary states under the farsighted player assumption. For example, in their solution concepts, the progress of states $((\frac{1}{2}, \frac{1}{4}, \frac{1}{4}), (1, 1, 1)) \longrightarrow ((\frac{3}{4}, 0, \frac{1}{4}), (1, 1, 1)) \longrightarrow ((1, 0, 0), (1, 1, 1))$ is possible. As a result, the state $((\frac{1}{2}, \frac{1}{4}, \frac{1}{4}), (1, 1, 1))$ might not be a stationary state according to their solution concepts. But, since a losing coalition has the forecasting ability, at the state $((\frac{1}{2},\frac{1}{4},\frac{1}{4}),(1,1,1))$, player 3 will help player 2 in order to deter player 1 from pillaging player 2 in expectation that the second state $((\frac{3}{4},0,\frac{1}{4}),(1,1,1))$ will proceed to the third state $((1,0,0),(1,1,1))$. Accordingly, the state $((\frac{1}{2},\frac{1}{4},\frac{1}{4}),(1,1,1))$ shows balanced power among the players and therefore must be considered as a stationary state under the farsighted player assumption as it is under the farsighted core solution concept.

Finally, Corollary 2 states that definitions about the farsighted core in Jordan (2006) can be extended to the definitions in this study.

Corollary 2 *In one region models, a consistent expectation exists, and it has $K_f = D$. Therefore, C_S is also D.*

Jordan (2006) used *one-step expectation,* where every state reaches its stationary state within one step, and *one step dominance in expectation*, where every plan to change a state can be completed within one step, and showed the same result as Corollary 2. Therefore, the definitions in Jordan (2006) can be extended from two aspects; *finite-step expectation,* where some states take finite steps to reach their stationary states, and *finite step dominance in expectation*, where a plan to change a state can take more than one step before it ends.

4 Suggestion for further research

Throughout this paper, we have assumed that regions are connected with one another and thus players can travel from one region to another in one move. The results based on this assumption are meaningful in that they give general understanding of how spatial restriction affects stable distribution of wealth. Also, for applications, when we consider that many countries, which could be regarded as individual regions, are surrounded by the sea and we can travel from one country to another through the sea, the assumption seems to be an approximation to reality.

However, in order to describe real situations more exactly, we can design a general model where some regions are not connected and thus players cannot travel between these regions in one move. A **geography correspondence** G embodies the general models as follows.

Definition 15 *A **geography correspondence** is a correspondence $G : R \longrightarrow\!\!\!\!\longrightarrow R$ satisfying for any $r \in R$, i) $r \in G(r)$; ii) if $r' \in G(r)$ then $r \in G(r')$; and iii) there exists a positive integer k such that $G^k(r) = R$ where $G^k(r) = G^{k-1}(G(r))$.*

For any $r \in R$, $G(r)$ denotes the regions that players at region r can go to in one move. Condition $i)$ means that players can stay in their regions. Condition $ii)$ means that connections between two regions are bilateral. And condition $iii)$ means that there is no separated region where players cannot travel. For example, we can define G as $G(1) = \{1,2\}$, $G(2) = \{1,2,3\}$, and $G(3) = \{2,3\}$, then G describes that three regions are located along a line.

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