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1. Introduction

Two interesting results encountered in the literature concerning the Poisson and the negative binomial distributions are due to MORAN (1952) and PATIL and SESHADRI (1964) respectively.

MORAN's result provided a fundamental property of the Poisson distribution and can be stated as follows:

Let (X, Y) be a random vector of non-negative integer-valued components such that

$$P(Y=r, X=n)=P_n s(r, n), \quad r=0, 1, \ldots, n; \quad n=0, 1, \ldots \tag{1.1}$$

where $\{P_n; n=0, 1, \ldots\}$ and $\{s(r, n); r=0, 1, \ldots n)$ are discrete probability distributions for each $n \geqq 0$. Suppose that Y and $X-Y$ are non-degenerate and independent and that there exists at least one integer i such that

$$P(Y=i)>0, \quad P(X-Y=i)>0 .$$

Then it follows that for every n with $P_n>0$ the distribution $\{s(r, n);\ r=0, 1, \ldots, n\}$ is binomial corresponding to n trials if and only if (iff) $\{P_n\}$ is a Poisson distribution. (i.e. iff Y and $X-Y$ are Poisson random variables.)

Note that, as was pointed out by SHANBHAG and PANARETOS (1979), it is essential to assume that $s(r, n)$ is binomial with index n (something that has not been specified in MORAN's paper); otherwise the result is not valid. This is evident in the following example.

Let Y, Z be two non-negative integer-valued independent random variables such that

$$P(Y=r)=\begin{cases} p & \text{if } r=1 \\ 1-p & \text{if } r=0 \end{cases}$$

where $0<p<1$ and Z has a non-Poisson distribution $\{g_n : n=0, 1, \ldots\}$ with every g_n positive. Then the conditional distribution of Y given that $X=n$ (where $X=$ $=Y+Z$) for every $n \geqq 1$ is

$$P(Y=r \mid X=n)=\begin{cases} \dfrac{(1-p)\, g_n}{pg_{n-1}+(1-p)\, g_n} & \text{if } r=0 \\[2ex] \dfrac{pg_{n-1}}{pg_{n-1}+(1-p)\, g_n} & \text{if } r=1 \end{cases}$$

Since this conditional distribution has the support $\{0, 1\}$ it follows that it is binomial (corresponding to one trial). For $n=0$ the conditional distribution of $Y \mid X=n$ is degenerate at zero and hence trivially binomial (we call this the binomial distribution corresponding to zero trials).

One may observe that in the above example we have Y and $Z=X-Y$ meeting more than all the requirements of MORAN's theorem. (It is evident that MORAN (1952) does not assume his binomial distribution to be non-degenerate when the given values of X are non-zero.) However neither Y nor Z is Poisson.

Motivated by MORAN's theorem PATIL and SESHADRI (1964) obtained a general result, a special case of which provided a characterization for the negative binomial distribution in the following way.

Let (X, Y) be as in MORAN's set-up. The distribution $\{s(r, n) : r=0, 1, \ldots, n$ is negative hypergeometric with parameters m, ϱ iff Y, $X-Y$ follow negative binomial distributions with parameters m, ϱ respectively.

In this paper we examine the results of MORAN and PATIL and SESHADRI, in the case where the conditional distribution $s(r, n)$ is truncated at an arbitrary point $k-1$ $(k=1, 2, \ldots)$. In fact we attempt to answer the question as to whether MORAN's property of the Poisson distribution and subsequently PATIL and SESHADRI's property of the negative binomial distribution can be extended, in one form or another, to the case where $Y \mid X$ is binomial truncated at $k-1$ and negative hypergeometric truncated at $k-1$, respectively. To answer these questions we will make use of a characterization proved by RAO and RUBIN (1964), a generalization of their result by SHANBHAG (1977), and a characterization by PANARETOS (1976). These results will provide the necessary background for the solutions required.

In Section 2 we quote the theorems of RAO and RUBIN (1964), SHANBHAG (1977) and PANARETOS (1976). Then in Sections 3 and 4 we derive two lemmas which help us to show that MORAN's theorem as well as PATIL and SESHADRI's result cannot be extended to the truncated case.

2. Some Results Related to MORAN's Characterization of the Poisson Distribution

MORAN's result has been extended by RAO and RUBIN (1964). They have shown that if $P_0 < 1$ and

$$s(r, n) = \binom{n}{r} p^r q^{n-r}, \quad r = 0, 1, \ldots, n \tag{2.1}$$

for every n with $P_n > 0$ and with p a fixed number in (0, 1) and independent of n, then we have that

$$P(Y=r) = P(Y=r \mid X=Y) = P(Y=r \mid X>Y), \quad r=0, 1, \ldots \tag{2.2}$$

iff $\{P_n\}$ is a Poisson distribution.

Using an argument from the renewal theory SHANBHAG (1977) extended RAO and RUBIN's result to a very general situation. Roughly speaking, he has shown that if $P_0 < 1$ and

$$s(r, n) = \frac{a_r b_{n-r}}{c_n}, \quad r=0, 1, \ldots, n \tag{2.3}$$

for every n with $P_n > 0$ (where $a_n > 0$ for all $n \geqq 0$, $b_0, b_1 > 0$ and $b_n \geqq 0$, $n \geqq 2$, and c_n is the convolution of a_n, b_n), then (2.2) is valid iff

$$\frac{P_n}{c_n} = \frac{P_0}{c_0} \vartheta^n, \quad n = 1, 2, \ldots \quad \text{for some} \quad \vartheta > 0 .$$

(Note that RAO and RUBIN's result follows a special case of this.)

An interesting by-product of SHANBHAG's result is that, if (2.3) is satisfied, condition (2.2) implies that Y and $X-Y$ are independent. One may also observe that (2.1) is of the form (2.3) and hence MORAN's result confirms that of RAO and RUBIN's. In addition, as was pointed out by PANARETOS (1976), the negative hypergeometric distribution is also of the form (2.3) and hence PATIL and SESHADRI's result is also confirmed. In the same paper PANARETOS provided a method for characterizing truncated distributions as follows.

Theorem 2.1. Suppose that $s(r, n)$ is of the form

$$s(r, n)=\frac{a_r b_{n-r}}{c_n}, \quad r=0, 1, \ldots, n; \quad n=k, k+1, \ldots \tag{2.4}$$

with $a_n>0$ for $n \geqq k$, b_n, c_n as previously, $P_k<1$, and X taking values $\geqq k$ only. In this case the condition

$$P(Y=r \mid Y \geqq k)=P(Y=r \mid X=Y), \quad r=k, k+1, \ldots \tag{2.5}$$

is valid iff

$$\frac{P_n}{c_n}=\frac{P_k}{c_k} \vartheta^{n-k}, \quad n=k, k+1, \ldots \quad \text{for some} \quad \vartheta>0. \tag{2.6}$$

This latter result provided the basis for the following characterizations based on truncated forms of $s(r, n)$.

Corollary 2.1. Suppose that the distribution $s(r, n)$ is binomial truncated at $k-1$ i.e.

$$P(Y=r \mid X=n)=\frac{\binom{n}{r} p^r q^{n-r}}{\sum_{r=k}^{\infty}\binom{n}{r} p^r q^{n-r}} \quad \begin{array}{l} r=k, \quad k+1, \ldots, n \\ n=k, \quad k+1, \ldots \\ 0<p<1, \quad q=1-p \end{array} \tag{2.7}$$

Then condition (2.5) holds iff

$$P(X=n)=\frac{e^{-\mu} \sum_{r=k}^{n}\binom{n}{r} \lambda^r \mu^{n-r}}{n! \sum_{n=k}^{\infty} \frac{\lambda^n}{n!}} \quad \begin{array}{l} \text{for some } \lambda, \quad \mu>0 \\ n=k, \quad k+1, \ldots \\ \lambda/(\lambda+\mu)=p \end{array} \tag{2.8}$$

i.e., iff the r.v. X has the distribution of a convolution of a Poisson (μ) with a Poisson (λ) truncated at $k-1$.

Corollary 2.2. Suppose that the distribution $s(r, n)$ is negative hypergeometric truncated at $k-1$, i.e.,

$$P(Y=r \mid X=n)=\frac{\binom{m+r-1}{r}\binom{\varrho+n-r-1}{n-r}}{\sum_{r=k}^{n}\binom{m+r-1}{r}\binom{\varrho+n-r-1}{n-r}} \quad r=k, \quad k+1, \ldots; \quad r \leqq n. \tag{2.9}$$

Then condition (2.5) holds iff the distribution P_n is the convolution of a negative binomial (p, m) and a truncated at $k-1$ negative binomial (p, ϱ) i.e., iff

$$P(X=n)=\frac{\sum_{r=k}^{n}\binom{m+r-1}{r}\binom{\varrho+n-r-1}{n-r} q^n p^{\varrho}}{\sum_{r=k}^{\infty}\binom{m+r-1}{r} q^r} \quad n=k, \quad k+1, \ldots \tag{2.10}$$

3. The Truncated Case of MORAN's Result

The corrected version of MORAN's result has been stated in Section 1. It is now interesting to see whether the result remains valid if one considers the binomial distribution truncated at $k-1$ as the distribution of $Y \mid X$. To do this one may observe first that the truncated binomial distribution is not the only distribution of the form (2.4) for which the result of Corollary 3.1 is valid. This is shown in the following lemma.

Lemma 3.1. Let us assume that P_n corresponds to the distribution given by (2.8) and $s(r, n)$ is of the form $\dfrac{a_r b_{n-r}}{c_n}$, $r=k, k+1, \ldots$

Then, condition (2.5) does not imply uniquely that the distribution of $Y \mid X$ is truncated binomial.

Proof. From Theorem 2.1 we know that condition (2.5) is equivalent to

$$c_n = c_k \frac{P_n}{P_k} \vartheta^{-n+k}, \quad n=k, \quad k+1, \ldots; \quad \text{for some} \quad \vartheta > 0 .$$

which in our particular case implies that

$$c_n = c_k \frac{\left[\sum_{r=k}^{n} \binom{n}{r} \lambda^r \mu^{n-r}\right]/n!}{\lambda^k/k!} \vartheta^{-n+k}, \quad n=k, \quad k+1, \ldots \tag{3.1}$$

It can now be checked that c_n will be of the form (3.1) if the sequences a_n, b_n have the following forms

$$a_r = \begin{cases} \alpha_k \dfrac{\lambda_0^r}{r!} & r=k, \quad k+1, \ldots; \quad \lambda>0; \quad \alpha_k \quad \text{a constant}; \quad \lambda_0=\lambda/\vartheta \\ 0 & r=0, 1, \ldots, k-1 \end{cases} \tag{3.2}$$

$$b_n = \beta_0 \frac{\mu_0^n}{n!} \quad n=0, 1, \ldots; \quad \mu>0; \quad \beta_0 \quad \text{a constant}; \quad \mu_0=\mu/\vartheta . \tag{3.3}$$

However, (3.2) and (3.3) are not the only forms that a_n, b_n can have respectively in order that their convolution be of the same form as (3.1). Take for example the following sequences.

$$a'_r = \begin{cases} \dfrac{e^{-\mu_0} \sum_{m=k}^{r} \binom{r}{m} \lambda_0^m \mu_0^{r-m}}{r! \sum_{n=k}^{\infty} \lambda_0^n/n!} & r=k, k+1, \ldots; \quad \lambda_0=\lambda/\vartheta, \quad \mu_0=\mu/\vartheta \\ 0 & r=0, 1, \ldots, k-1 \end{cases} \tag{3.4}$$

$$b'_n = e^{-\mu_0} \frac{\mu_0^n}{n!} \quad n=0, 1, \ldots; \quad \mu_0=\mu/\vartheta \tag{3.5}$$

i.e., take a_n' to be the convolution of a Poisson with a truncated Poisson and b_n' to be Poisson. It is clear that the convolution of a_n' and b_n', $\{a_n'\}*\{b_n'\}$ is given by (3.1) since

$$\begin{aligned}\{a_n'\}*\{b_n'\} &\sim (\text{Poisson } * \text{ truncated Poisson}) * \text{Poisson}\\ &\sim (\text{Poisson } * \text{ Poisson}) * \text{truncated Poisson}\\ &\sim \text{Poisson } * \text{ truncated Poisson.}\end{aligned}$$

This means that we can find a pair of sequences, other than (3.2), (3.3), whose convolution is of the form (3.1). So, the decomposition of (3.1) is not unique, and hence the truncated binomial is not the only distribution of the form (2.4) satisfying condition (2.5). This completes the proof of the lemma.

Our problem can now be stated in the form of the following theorem.

Theorem 3.1. Let (X, Y) be a random vector of integer-valued components with $X \geqq Y \geqq k$ such that

$$P(Y=r, X=n)=P_n s(r, n), \quad r=k, k+1, \ldots, n; \quad n=k, \quad k+1, \ldots$$

where $\{P_n : n=k, \ k+1, \ldots\}$ and $\{s(r, n) : r=k, k+1, \ldots, n\}$ are discrete probability distributions for each $n \geqq k$. Suppose that Y and $X-Y$ are nondegenerate and independent with $P(Y=i)>0$, $P\,(X-Y=j)>0$ for at least one integer $i>k$ and one integer $j>0$. Then, the condition that Y is Poisson truncated at $k-1$ and $X-Y$ is Poisson, is necessary but not sufficient for the distribution $s(r, n)$ to be binomial truncated at $k-1$.

Proof. Evidently the "necessary" part is a side result of the previous lemma. As for the "sufficient" part we have already seen in Corollary 2.1 that if $s(r, n)$ is truncated binomial, condition (2.5) implies that X is the convolution of a Poisson with a truncated Poisson. But, as was shown in Lemma 3.1, if condition (2.5) holds then the truncated Poisson for Y and the Poisson for $X-Y$ are not the only distributions for which X is Poisson convoluted with a truncated Poisson.

Since (2.4) and (2.5) imply that Y and $X-Y$ are independent the argument is established.

4. The Truncated Case of PATIL and SESHADRI's Result

Lemma 4.1. Suppose that the distribution P_n is the convolution of a negative binomial and a truncated negative binomial at $k-1$ as in (2.10). Let us also assume that the conditional distribution of $Y \mid X$ is of the form (2.4). Then condition (2.5) does not imply uniquely that the distribution of $Y \mid X$ is negative hypergeometric truncated at $k-1$.

Proof. The proof is similar to that of Lemma 3.1. We can arrive at it again by making use of the fact that, under the assumptions made, Theorem 2.1 can be applied. Hence, condition (2.5) is valid iff $c_n = c_k \dfrac{P_n}{P_k} \vartheta^{-n+k}$ which, for P_n as in (2.10), will eventually become

(4.1)
$$c_n=\sum_{r=k}^{n} a_r b_{n-r}=\frac{\sum_{r=k}^{n}\binom{m+r-1}{r}\binom{\varrho+n-r-1}{n-r}(q/\vartheta)^n(1-q/\vartheta)^{\varrho}}{\sum_{r=k}^{\infty}\binom{m+r-1}{r}(q/\vartheta)^r}; \quad n=k, \quad k-1, \ldots$$

for some $0<q/\vartheta<1$ and a_n, b_n probability distributions.

It is evident that if a_n is negative binomial truncated at $k-1$ and b_n is negative binomial, (4.1) is satisfied. However, this is not the only solution. For, consider for example

$$(4.2) \qquad a_r'=\begin{cases}\dfrac{\sum_{i=k}^{r}\binom{m+i-1}{i}\binom{\varrho+r-i-1}{r-i}p_0^{\varrho}q_0^{r}}{\sum_{i=k}^{\infty}\binom{m+i-1}{i}q_0^{i}} & r=k, \quad k=1, \ldots \\ & q_0=q/\vartheta, \quad p_0=1-q_0 \\ 0 & r=0, 1, \ldots, k-1\end{cases}$$

(i.e. a_r' is the convolution of a truncated at $k-1$ negative binomial distribution and a negative binomial distribution), and

$$(4.3) \qquad b_n'=\binom{\varrho+n-1}{n}p_0^{\varrho}q_0^{n} \quad n=0, 1, \ldots; \quad q_0=q/\vartheta, \quad p_0=1-q_0$$

(i.e., b_n' is a negative binomial distribution).

Then

$$\begin{aligned}\{c_n'\}=\{a_n'\}*\{b_n'\}&\sim(\text{negative binomial} * \text{truncated negative binomial}) \\ &\qquad * \text{negative binomial} \\ &\sim(\text{negative binomial} * \text{negative binomial}) \\ &\qquad * \text{truncated negative binomial} \\ &\sim \text{negative binomial} \quad * \text{truncated negative binomial}\end{aligned}$$

Clearly $\{c_n'\}$ is of the same form as $\{c_n\}$ in (4.1). However, the distribution $a_r'b_{n-r}'/c_n'$, with a_r', b_n', c_n' given by (4.2), (4.3) and (4.1), respectively, is not truncated negative hypergeometric. Hence, the lemma is proved.

The question now, concerning the extension of PATIL and SESHADRI's result to the truncated case, is answered in the following theorem.

Theorem 4.1. Suppose that Y, X are as in Theorem 3.1. Then the condition that the distribution of Y is negative binomial truncated at $k-1$ and the distribution of $X-Y$ is negative binomial is necessary but not sufficient for the distribution $s(r, n)$ to be negative hypergeometric truncated at $k-1$.

Proof. This is obtained by using an argument identical to the one employed in Theorem 3.1 and by making use of Lemma 4.1.

Hence it was shown that MORAN's result, as well as PATIL and SESHADRI's result, does not remain valid when the distribution $s(r, n)$ is truncated at $k-1$.

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