Control Charts for the Lognormal Distribution

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1. Introduction

Control charts are the tools of statistical process control for detecting a change in a process. The most known are the Shewhart control charts that where developed for controlling the mean and variance of the distribution our observations follow. Shewhart charts perform satisfactorily for detecting large shifts in the process. However, for small shifts they had poor results. This fact led Page in 1954 to construct CUSUM control chart for detecting small and moderate shifts in the process.

Control charts have found great applications in industry and especially the ones for the normal distribution. Nevertheless, a process may not follow the normal distribution-it may be positive or right skewed. Morrison (1958), Joffe and Sichel (1968) and Kotz and Lovelace (1998) provided examples of processes that follow the lognormal distribution.

Morrison (1958), Kotz and Lovelace (1998) have developed Shewhart control charts for the lognormal distribution. Joffe and Sichel (1968) constructed a control chart for testing sequentially arithmetic means from a lognormal population. In this paper, CUSUM procedures are developed for the lognormal for testing the mean and variance. In addition, a chart based on the sequential probability ratio test (SPRT) is illustrated.

2. Lognormal distribution

A positive random variable X is said to be lognormally distributed with two parameters μ and σ^2 if Y=lnX is normally distributed with mean μ and variance σ^2. The probability density function is

$$f(x)=\begin{cases}\dfrac{1}{\sigma x\sqrt{2\pi}}\exp\left\{-\dfrac{(\ln x-\mu)^2}{2\sigma^2}\right\}, & x>0\\ 0, & x\le 0\end{cases}$$

There are four forms of the lognormal distribution as Crow and Shimizu (1988) present in their book:

(a) The two-parameter distribution $\Lambda(\mu,\ \sigma^2)$ that describe positive skew data with a lower threshold of zero.

(b) The three-parameter distribution $\Lambda(\tau,\ \mu,\ \sigma^2)$ that describe positive skew data with a lower threshold of τ. This form can be reduced to the two-parameter case by the transposition Y=X-τ.

(c) The three-parameter distribution $\Lambda(\theta,\ \mu,\ \sigma^2)$ that describe negative skew data with an upper threshold of θ. This form can be reduced to the two-parameter case by the transposition Y=θ-X.

(d) The four-parameter distribution $\Lambda(\tau,\ \theta,\ \mu,\ \sigma^2)$ that describe skew data with upper and lower thresholds of θ and τ.

In the following, we will deal with the two-parameter distribution. The three-parameter may occur in practice but with the suitable transformations as it is shown in (b), (c) it reduces to the two-parameter one. The four-parameter is not likely to have applications in process control.

3. CUSUM Chart

3.1. The general exponential family.

The probability density for any member of the exponential family with a single parameter θ can be written as:

$$f(y\mid\theta)=\exp\{\alpha(y)b(\theta)+c(y)+d(\theta)\}$$

where θ is the parameter of the distribution and Y is the random variable. The joint density for a random sample of Y, where Y is a member of the exponential family, is given by:

$$f(\mathbf{y}|\theta)=\exp\left(\sum_{i=1}^{n}\alpha(y_i)b(\theta)+\sum_{i=1}^{n}c(y_i)+nd(\theta)\right).$$

Suppose we want to test whether the process has gone from an in-control parameter value θ_0 to an out-of-control value θ_1. We will use Wald's sequential probability ratio test (SPRT) because CUSUM is a sequence of Wald sequential tests [Page (1954)]. Then the variables Z_i are:

$$Z_i=\ln\left(\frac{f_1(Y_i)}{f_0(Y_i)}\right)=\alpha(Y_i)\{b(\theta_1)-b(\theta_0)\}+\{d(\theta_1)-d(\theta_0)\}$$

The CUSUM scheme is:

$$D_n=\max(0,D_{n-1}+Z_n)$$

and it gives an out-of-control signal if

$$D_n>A.$$

Therefore the CUSUM scheme is:

$$D_n=\max(0,D_{n-1}+\alpha(Y_i)\{b(\theta_1)-b(\theta_0)\}+d(\theta_1)-d(\theta_0))$$

and it signals when

$$D_n>A.$$

Let $X_n=\alpha(Y_n)$ and $k=-\frac{d(\theta_1)-d(\theta_0)}{b(\theta_1)-b(\theta_0)}$. If $b(\theta_1)-b(\theta_0)>0$ then we rescale CUSUM by dividing it with this quantity and

$$C_n^+ = \max\left(0, C_{n-1}^+ + X_n - k\right)$$

where $C_n^+ = D_n / \left(b(\theta_1) - b(\theta_0)\right)$ and the CUSUM signals if $C_n^+ > h^+$ where $h^+ = A / \left(b(\theta_1) - b(\theta_0)\right)$. If $b(\theta_1) - b(\theta_0)$<0 then we rescale CUSUM by dividing it with this quantity and

$$C_n^- = \min\left(0, C_{n-1}^- + X_n - k\right)$$

where $C_n^- = D_n / \left(b(\theta_1) - b(\theta_0)\right)$ and the CUSUM signals if $C_n^- < -h^-$ where $h^- = A / \left(b(\theta_1) - b(\theta_0)\right)$.

3.2. CUSUM schemes for the lognormal distribution

3.2.1. σ known

Let σ be fixed and known. The above density function is written in the form of the exponential family of distributions as follows:

$$f(x \mid \mu) = \exp\left\{-\ln\left(\sigma\sqrt{2\pi}\right) - \ln x - \frac{(\ln x)^2}{2\sigma^2} + \frac{\mu}{\sigma^2}\ln x - \frac{\mu^2}{2\sigma^2}\right\}$$

Then $\alpha(x)$=lnx, $b(\mu)$=$\frac{\mu}{\sigma^2}$, $d(\mu)$= $-\frac{\mu}{2\sigma^2}$. Therefore

$$k = -\frac{d(\theta_1) - d(\theta_0)}{b(\theta_1) - b(\theta_0)} = \frac{\mu_1^2 - \mu_0^2}{2(\mu_1 - \mu_0)} = \frac{\mu_1 + \mu_0}{2}$$

The CUSUM scheme for the mean when $\mu_1 > \mu_0$ will be:

$$C_0^+ = 0$$

$$C_{n+1}^+ = \max(0, C_n^+ + \ln X_n - k)$$

$$k^+ = \frac{\mu_1 + \mu_0}{2}$$

The scheme signals when $C_n^+ > h^+$ where h^+ is chosen to give a specified ARL.

The CUSUM scheme for the mean when $\mu_1 < \mu_0$ will be:

$$C_0^- = 0$$

$$C_{n+1}^- = \min(0, C_n^- + \ln X_n - k)$$

$$k^- = \frac{\mu_1 + \mu_0}{2}$$

The scheme signals when $C_n^- < -h^-$.

3.2.2. μ known

Let μ be fixed and known. The density function of the lognormal is written in the form of the exponential family of distributions as follows:

$$f(x \mid \sigma) = \exp\left\{ -\ln\left(x\sqrt{2\pi}\right) - \ln\sigma - \frac{(\ln x - \mu)^2}{2\sigma^2} \right\}$$

Then $\alpha(x)=(\ln x - \mu)^2$, $b(\sigma)=-\frac{1}{2\sigma^2}$, $d(\sigma)=-\ln\sigma$. Therefore

$$k = -\frac{d(\theta_1) - d(\theta_0)}{b(\theta_1) - b(\theta_0)} = -\frac{\ln\sigma_1 - \ln\sigma_0}{\frac{1}{2\sigma_1^2} - \frac{1}{2\sigma_0^2}}$$

The CUSUM scheme for the variance when $\sigma_1 > \sigma_0$ will be:

$$C_0^+ = 0$$

$$C_{n+1}^+ = \max(0, C_n^+ + (\ln X_n - \mu)^2 - k)$$

$$k = -\frac{\ln\sigma_1 - \ln\sigma_0}{\frac{1}{2\sigma_1^2} - \frac{1}{2\sigma_0^2}}$$

The CUSUM scheme for the variance when $\sigma_1 < \sigma_0$ will be:

$$C_0^- = 0$$

$$C_{n+1}^- = \min(0, C_n^- + (\ln X_n - \mu)^2 - k)$$

$$k = -\frac{\ln\sigma_1 - \ln\sigma_0}{\frac{1}{2\sigma_1^2} - \frac{1}{2\sigma_0^2}}$$

If one looks at the CUSUM schemes for the variance, he will notice that it is affected by changes in μ as well as by changes in σ. However, we monitor μ separately, therefore we are able to distinguish changes in the CUSUM chart for the variance caused by μ.

3.3. A general result

In the lognormal CUSUM for monitoring σ, I have shown that the lognormal is a member of the exponential family with parameters α(x)= $(\ln x-\mu)^2$, b(σ)= $-\frac{1}{2\sigma^2}$, d(σ)=-lnσ. Let Z= $(\ln x-\mu)^2$. Then the distribution of Z is:

$$F_Z(z)=P[Z\le z]=P\left[(\ln X-\mu)^2\le z\right]=P\left[|\ln X-\mu|\le\sqrt{z}\right]=P\left[-\sqrt{z}\le\ln X-\mu\le\sqrt{z}\right]=$$

$$=P\left[\mu-\sqrt{z}\le\ln X\le\mu+\sqrt{z}\right]=F_{\ln X}\left(\mu+\sqrt{z}\right)-F_{\ln X}\left(\mu-\sqrt{z}\right).$$

Therefore

$$\begin{aligned}f_Z(z)&=\frac{d\left(\mu+\sqrt{z}\right)}{dz}f_{\ln X}(\mu+\sqrt{z})-\frac{d\left(\mu-\sqrt{z}\right)}{dz}f_{\ln X}(\mu-\sqrt{z})=\\&=\frac{1}{2\sqrt{z}}\left\{\frac{1}{\sigma\sqrt{2\pi}}\exp\left[-\frac{\left(\mu+\sqrt{z}-\mu\right)^2}{2\sigma^2}\right]-\frac{1}{\sigma\sqrt{2\pi}}\exp\left[-\frac{\left(\mu-\sqrt{z}-\mu\right)^2}{2\sigma^2}\right]\right\}=\\&=\frac{\exp\left(-\frac{z}{2\sigma^2}\right)}{\sigma\sqrt{2\pi z}}\end{aligned}$$

But this is a $\Gamma\left(\frac{1}{2},2\sigma^2\right)$ distribution with density function

$$f(x;\alpha,\beta)=\frac{\beta^{-\alpha}x^{\alpha-1}\exp(-x/\beta)}{\Gamma(\alpha)}$$

Consequently, $\alpha(x)\sim\Gamma\left(\frac{1}{2},2\sigma^2\right)$.

3.3.1. Monitoring σ^2 in the lognormal CUSUM

It is known that if Y~Γ(α,β) and Z=aY then Z~Γ(α,aβ). Therefore $\frac{1}{\sigma^2}\alpha(x) \sim \Gamma\left(\frac{1}{2},2\right)$. If we want to monitor σ^2 we just have to monitor for a scale change of the gamma distribution from β to β^+. Hawkins and Olwell (1998) have constructed CUSUM schemes for the gamma distribution.

3.3.2. Monitoring σ in the lognormal CUSUM

From the definition of the lognormal distribution, it is known that lnx~N(μ, σ^2) and as a result $\frac{\ln x - \mu}{\sigma} \sim N(0,1)$. Hence monitoring σ is just the case of monitoring for σ in the standard normal distribution. Hawkins and Olwell (1998) have also constructed CUSUM schemes for the normal distribution.

3.4. Wald's sequential probability ratio test (SPRT)

CUSUM control charts, as Johnson (1961) pointed out, are roughly equivalent to the sequential probability ratio test (SPRT). SPRT, as it will be shown in the sequel, leads to an acceptance plan. This acceptance plan has been used for determining the in and out-of-control limits in CUSUM procedures.

Suppose that we take a sample of m values $x_1, x_2, ..., x_m$, successively, from a population $f(x,\theta)$. Consider two hypotheses about θ, $H_0 : \theta = \theta_0$ and $H_1 : \theta = \theta_1$. The ratio of the probabilities of the sample is:

$$L_m = \frac{\prod_{i=1}^{m} f(x_i, \theta_1)}{\prod_{i=1}^{m} f(x_i, \theta_0)}$$

We select two numbers A and B, which are related to the desired α and β errors in a way we will explain later. The sequential test is set up as follows:

as long as $B<L_m<A$ we continue sampling

at the first i that $L_m \geq A$ we accept H_1

at the first i that $L_m \leq B$ we accept H_0

An equivalent way for computation is to use the logarithm of L_m. Then, the inequality becomes:

$$\log B < \sum_{i=1}^{m} \log f(x_i, \theta_1) - \sum_{i=1}^{m} \log f(x_i, \theta_0) < \log A$$

This family of tests is referred to as sequential probability-ratio tests.

If $z_i = \log\{f(x_i, \theta_1)/f(x_i, \theta_0)\}$ then sampling terminates when

$$\sum z_i \geq \log A$$

or

$$\sum z_i \leq \log B$$

The z_i's are independent random variables with variance, say, $\sigma^2>0$. Obviously $\sum_{i=1}^{m} z_i$ has a variance $m\sigma^2$. As m increases the dispersion becomes greater and the probability that a value of $\sum z_i$ will remain within the limits logB and logA tends to zero. The mean $\bar{z}$ tends to a normal distribution with mean σ^2/m and therefore the probability that it falls between (logB)/m and (logA)/m tends to zero.

Consider a sample for which L_m lies between A and B for the first n-1 trials and then $L_m \geq A$ at the nth trial, so we accept H_1 (and reject H_0). By definition, the probability of getting such a sample is at least A times as large under H_1 as under H_0. The probability of accepting H_1 when H_0 is true is α and that of accepting H_1 when H_1 is true is $1-\beta$. Therefore:

$$1-\beta \geq A\alpha$$

or

$$A \leq \frac{1-\beta}{\alpha}. \tag{3.4.1}$$

Similarly, we see that when we accept H_0

$$\beta \leq B(1-\alpha)$$

or

$$B \geq \frac{\beta}{1-\alpha}. \tag{3.4.2}$$

Wald (1947) showed that for all practical purposes the above inequalities hold as equalities. Thus:

$$A = \frac{1-\beta}{\alpha} \text{ and } B = \frac{\beta}{1-\alpha}.$$

Suppose that $a = \frac{1-\beta}{\alpha}$ and $b = \frac{\beta}{1-\alpha}$ and that the true errors of first and second kind for the limits a and b are α' and β'. Then, from (3.4.1):

$$\frac{\alpha'}{1-\beta'} \leq \frac{1}{a} = \frac{\alpha}{1-\beta}$$

and from (3.4.2)

$$\frac{\beta'}{1-\alpha'} \leq b = \frac{\beta}{1-\alpha}.$$

Therefore

$$\alpha' \leq \frac{\alpha(1-\beta')}{1-\beta} \leq \frac{\alpha}{1-\beta} \tag{3.4.3}$$

$$\beta' \leq \frac{\beta(1-\alpha')}{1-\alpha} \leq \frac{\beta}{1-\alpha} \tag{3.4.4}$$

Furthermore

$$\alpha'(1-\beta)+\beta'(1-\alpha) \leq \alpha(1-\beta')+\beta(1-\alpha')$$

or

$$\alpha'+\beta' \leq \alpha+\beta . \tag{3.4.5}$$

In practice α and β are small. From (3.4.3) and (3.4.4) we see that the amount that α' can exceed α or β' exceed β is negligible. In addition, from relation (3.4.5) we see that either $\alpha' \leq \alpha$ or $\beta' \leq \beta$. Therefore the use of α and β in place of A and B can only increase one of the errors and only by a very small amount.

3.4.1. Application of the SPRT to the CUSUM for the lognormal distribution

Based on the theory of Wald we will derive a CUSUM scheme for the parameter μ of the lognormal distribution. Define L_0 and L_1 to be the likelihood functions of the random sample under H_0 and H_1, respectively, and let the likelihood ratio L_1/L_0 be denoted by Λ. That is:

$$L_1 = \left[\prod_{i=1}^{n} X_i\left(\sigma\sqrt{2\pi}\right)^n\right]^{-1} \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(\ln X_i-\mu_1\right)^2\right]$$

$$L_0 = \left[\prod_{i=1}^{n} X_i\left(\sigma\sqrt{2\pi}\right)^n\right]^{-1} \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(\ln X_i-\mu_0\right)^2\right]$$

and Λ is

$$\Lambda = \exp\left\{-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n}\left(\ln X_i-\mu_1\right)^2-\sum_{i=1}^{n}\left(\ln X_i-\mu_0\right)^2\right]\right\}.$$

We assumed that σ is known which is reasonable if control charts have been used to monitor the process for a period. Define A and B as follows:

$$A = \frac{1-\beta}{\alpha} \text{ and } B = \frac{\beta}{1-\alpha}.$$

We accept the lot if $\Lambda \leq B$. We reject the lot if $\Lambda \geq A$. If $B<\Lambda<A$ we continue sampling. After some calculations, we conclude that

(a) Accept the lot if $Y \geq Y_1 = nS + h_1$

(b) Reject the lot if $Y \leq Y_2 = nS + h_2$

(c) Continue to sample if $Y_2 < Y < Y_1$

where $Y = \sum_{i=1}^{n} \ln X_i$, $S = \frac{\mu_0 + \mu_1}{2}$, $h_1 = \frac{\sigma^2 \ln\left(\frac{\beta}{1-\alpha}\right)}{\mu_1 - \mu_0}$ and $h_2 = \frac{\sigma^2 \ln\left(\frac{1-\beta}{\alpha}\right)}{\mu_1 - \mu_0}$.

In the above three cases h_1 and h_2 are parallel lines referred as acceptance and rejection lines respectively. The results were developed for the case of $\mu_1 > \mu_0$ meaning that μ_1 is an upper specification limit. In the same way, we can derive results for the case $\mu_1 < \mu_0$.

4. ARL performance

Control charts are usually evaluated using the average run length (ARL). ARL is the average number of samples until signal. A computer program was written for the computation of the ARL based on the theory of Brook and Evans (1972). In this program the values of k and h are provided and we receive the value of the ARL. In the following several results of the program are presented and also the effect of the decision interval h is illustrated. The symbols μ and σ are the parameters of the lognormal distribution.

μ=1		σ=0,6
K	h	ARL
11	5	632.6
11	4	449.8
11	3	316.1
10.5	5	532.3
10.5	4	376.6
10.5	3	263.1
μ=0		σ=1
K	h	ARL
11	5	358.5
11	4	294.8
11	3	240.2

5. Conclusion

In this paper, a new CUSUM control chart for the lognormal distribution is presented. The close relationship of the CUSUM and the SPRT is also shown in detail. Various versions of the new chart for monitoring σ are illustrated. Finally, a computer program for comparing different control charts using average run length (ARL) is provided.

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