Nondegenerate Intervals of No-Trade Prices for Risk-averse Traders

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ABSTRACT. According to the local risk-neutrality theorem an agent who has the opportunity to invest in an uncertain asset does not buy it or sell it short iff its expected value is equal to its price, independently of the agent's attitude towards risk. Contrary to that it is shown that, in the context of expected utility theory with differentiable vNM utility function, but without the assumption of stochastic constant returns to scale, nondegenerate intervals of no-trade prices may exist. With a quasiconcave expected utility function they do if, and only if, the agent is risk averse of order one.

KEY WORDS: Portfolio choice, Risk aversion of order one, No-trade prices

1. INTRODUCTION

This paper is concerned with the existence of nondegenerate intervals of no-trade prices. This is a problem because, according to the well-known local risk-neutrality theorem (Arrow 1965), an agent who starts from a position of certain wealth will not trade in a given risky asset if, and only if, the expected (present) value of the asset coincides with its price. For any price less than the expected value the agent will buy some amount of the asset whereas for any higher price he will wish to sell it short. This holds in the absence of transactions costs whenever it is possible to buy small quantities of the asset, independently of the agent's attitude towards risk.[1] Observed investment behavior, on the other hand, suggests that there is typically an interval of prices within which the agent neither buys nor sells the asset short.

This fact has recently aroused increasing interest as is documented by a number of works seeking to explain inertia in investor's behavior in terms of their preferences and/or perceptions of risk and uncertainty. One possibility is to assume that the agent's behavior is representable by uncertainty aversion and nonadditive (subjective) probabilities.[2] This has been shown by Dow and da Costa Werlang

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(1992) building on an axiomatic approach of nonadditive probability measures due to Schmeidler (1982, 1989), Gilboa (1987) and Gilboa and Schmeidler (1989). Dow and da Costa Werlang's work can be seen as a generalization of the local risk-neutrality theorem.

Another possibility, as shown in this paper, is obtained by means of risk aversion of order one. This concept, introduced independently and in an essentially equivalent fashion by Montesano (1988) and by Segal and Spivak (1990), is defined by the condition that the risk premium $\pi(s)$ that the agent is willing to pay to avoid the lottery sX, where X is a given random variable and s a real number, is a kinked function at $s = 0$. As the above authors show, in the context of expected utility theory or nonexpected utility theory with a Fréchet differentiable functional and local utility functions (see Machina 1982), this cannot be obtained with differentiable von Neumann–Morgenstern utility functions; however, anticipated utility theory (i.e. expected utility with rank dependent probabilities), first presented by Quiggin (1982), is also compatible with first order risk aversion if the utility function is differentiable.

That risk aversion of order one yields intervals of no-trade prices will follow in a natural way from the formalization presented here. In fact it will be shown that, in the context of expected utility theory, first-order risk aversion occurs if, and only if, the expected utility function is kinked at the no-trade point. If it is also quasiconcave, this in turn is equivalent to the existence of nongenerate intervals of no-trade prices.

More precisely, in the present paper's model I have retained the assumptions of additive probability as well as expected utility (with rank *in*dependent probabilities and) with a differentiable von Neumann–Morgenstern utility function. To have this be compatible with first order risk aversion I give up what Arrow (1965) calls *stochastic constant returns to scale* (and which Montesano (1988) and Segal and Spivak (1990) have implicitly adopted), by which is meant that the distribution of the asset's rate of return is independent of the amount invested. Doing this permits the function which associates with each amount invested the variance of the agent's subjective probability distribution over the asset's random return to have a kink at zero. For a risk neutral agent this is of no importance but if he is risk averse, the kink in the variance function is equivalent to a kink in his expected utility function and consequently to inertia in his investment behavior. Finally, it turns out that being risk averse and having a kinked variance function is the same thing as being risk averse of order one.

The remainder of the paper is organized as follows. In Section 2, I introduce the model for the case in which the probability distributions have finite supports and present the fundamental technical results. Section 3 contains their application to the existence of non-degenerate intervals of no-trade prices. How this is linked to risk aversion of order one is shown in Section 4. Section 5 discusses the relation between the existence of intervals of no-trade prices and convergence concepts of probability measures. In Section 6 I extend the analysis to the case in which the distributions are given by means of density functions. Section 7 contains concluding remarks.

2. THE MODEL AND TECHNICAL RESULTS

Assume that an agent can trade an asset the uncertain return of which depends on the quantity s he buys ($s > 0$) or sells short ($s < 0$). Thus for each s the asset can be seen as a lottery $X(s)$. Its 'objective' distribution is unknown to the agent but he has some beliefs about it, captured by a subjective probability distribution $\mu(s) \in \Delta(\mathbb{R})$, where $\Delta(\mathbb{R})$ is the set of probability measures over $\mathbb{R}$. The distributions $\mu(s)$ are assumed to satisfy the following conditions.

(A1) There is a number $n \in \mathbb{N}$ such that for all $s \in \mathbb{R}$ the distribution $\mu(s)$ is of the form

$$\mu(s) = [(x_1(s), q_1(s)), \ldots, (x_n(s), q_n(s))],$$

with $x_i(\cdot)$ continuously differentiable and $q_i(\cdot)$ nonnegative, continuous, differentiable for $s \neq 0$, and such that there exist $\lim_{s\to 0^+} q_i'(s) =: q_i^+$ and $\lim_{s\to 0^-} q_i'(s) =: q_i^-$, for all $i = 1, \ldots, n$; moreover, $\sum_i q_i(s) = 1$ for all s.

(A2) $\mu(0) = \delta(0)$, where $\delta(x) \in \Delta(\mathbb{R})$ is the degenerate distribution concentrated in $x \in \mathbb{R}$.

(A3) The expected value function $s \mapsto \bar{x}(s) = \int x \, \mathrm{d}[\mu(s)](x)$ is continuously differentiable.

(A1) means that the support of $\mu(s)$ is finite. This is assumed here for transparency of the argument only; the extension to the continuum case is dealt with in Section 6. $q_i(s)$ is the probability the agent attaches to the outcome $x_i(s)$. Of course, since n is fixed, not all $q_i(s)$ need to be positive at every s.

(A2) expresses that the agent always has the option not to trade, in which case he is sure that his wealth is not influenced by the asset under consideration. (A3) is a regularity assumption on the agent's expectations.

Example 1. Assume $\mu(s)$ is such that, for any $s \in \mathbb{R}$, $x_i(s) = sx_i(1)$ and $q_i(s) = q_i(1)$ for all $i = 1, \ldots, n$. It is trivial that (A1)–(A3) are fulfilled. Moreover, this is the case of stochastic constant returns to scale invoked by Arrow (1965). Examples that satisfy the above axioms but where constant returns do not hold will be given at the end of this section.

From (A1) it is immediate that the function $\mu : \mathbb{R} \to \Delta(\mathbb{R})$ is weakly continuous.[3] In spite of this, and of the fact that $\mu(0)$ is concentrated in 0, $x_i(0)$ may differ from 0. However, in that event (A2) implies that it must have probability zero: in fact, assume $x_i(0) \neq 0$. If $q_i(0) \neq 0$, then $\mu(0) \neq \delta(0)$, in contradiction to (A2).

For future reference define $I = \{1, \ldots, n\}$ and $J = \{i \in I | x_i(0) = 0\}$. Then by the above argument $\sum_{i \in J} q_i(0) = 1$.

Denote by W the initial (certain) wealth of the agent and assume that the price of one unit of the asset, $X(1)$, is p. Then, for each s, $X(s)$ implies a lottery (the agent's final wealth) $Z(s|p) = W - ps + X(s)$. If the agent sells short s units of the asset, s is signed negatively and $X(s)$ indicates the random amount of money the agent will have to pay to the buyer of the lottery. Thus, from the agent's point of view, for $s < 0$, $X(s)$ will have negative outcomes.

Regarding the agent's preferences, I assume

(A4) The agent's preferences over lotteries $Z(s|p)$ can be represented by an expected utility function

$$U(s|p) = \int u(W - ps + x)\, \mathrm{d}[\mu(s)](x)$$

where u is a von Neumann–Morgenstern utility function which is strictly increasing and real analytic.

Assumptions (A1) and (A4) imply

$$U(s|p) = \sum_{i \in I} u(W - ps + x_i(s))q_i(s)$$

which is what the agent aims to maximize. By (A1) $U(s|p)$ is differentiable in s for $s \neq 0$. Its derivative (with respect to s) is

$$\begin{aligned} U'(s|p) = \sum_i \{&u'(W - ps + x_i(s))(x_i'(s) - p)q_i(s) \\ &+ u(W - ps + x_i(s))q_i'(s)\} \end{aligned} \tag{1}$$

I now consider $U'^{+}(p) := \lim_{s \to 0^+} U'(s|p)$ and $U'^{-}(p) := \lim_{s \to 0^-} U'(s|p)$. The following lemma presents the key technical insight for all the results to follow.

LEMMA 1.

$$U'^{+}(p) = \frac{\mathrm{d}}{\mathrm{d}s}[u(W - ps + \bar{x}(s))]_{s=0} + A^{+} \quad \text{and}$$

$$U'^{-}(p) = \frac{\mathrm{d}}{\mathrm{d}s}[u(W - ps + \bar{x}(s))]_{s=0} + A^{-},$$

where

$$A^{+} = \sum_{i \notin J} \left[\sum_{n=2}^{\infty} u^{(n)}(W) \frac{(x_i(0))^n}{n!} \right] q_i'^{+} \quad \text{and}$$

$$A^{-} = \sum_{i \notin J} \left[\sum_{n=2}^{\infty} u^{(n)}(W) \frac{(x_i(0))^n}{n!} \right] q_i'^{-}.$$

Proof. Differentiating $\bar{x}(s) = \sum_i x_i(s)q_i(s)$ yields

$$\sum_i x_i'(s)q_i(s) = \bar{x}'(s) - \sum_i x_i(s)q_i'(s) \quad \text{for all} \quad s \neq 0.$$

Taking limits as $s \to 0^+$,

$$\sum_{i \in J} x_i'(0)q_i(0) = \bar{x}'(0) - \sum_{i \in I} x_i(0)q_i'^{+} \tag{2}$$

by the definition of J and (A1).

From (1) and the fact that $q_i(0) = 0$ for $x_i(0) \neq 0$ it follows that

$$U^{'+}(p) = \sum_{i \in J} u'(W + x_i(0))(x_i'(0) - p)q_i(0) +$$

$$+ \sum_{i \in I} u(W + x_i(0))q_i^{'+}$$

$$= u'(W) \sum_{i \in J} (x_i'(0) - p)q_i(0) + \sum_{i \in I} u(W + x_i(0))q_i^{'+}$$

$$= - u'(W)p + u'(W) \sum_{i \in J} x_i'(0)q_i(0)$$

$$+ \sum_{i \in I} u(W + x_i(0))q_i^{'+}$$

which, using (2), becomes

$$U^{'+}(p) = - u'(W)p + u'(W)\bar{x}'(0) +$$

$$+ \sum_{i \in I} [u(W + x_i(0)) - u'(W)x_i(0)]q_i^{'+}$$

$$= \frac{\mathrm{d}}{\mathrm{d}s}[u(W - ps + \bar{x}(s))]_{s=0} + A^+$$

with $A^+ := \sum_{i \in I} [u(W + x_i(0)) - u'(W)x_i(0)]q_i^{'+}$. A^+ can be written, using the fact that u is analytic,

$$A^+ = \sum_{i \in I} \left[u(W) + \sum_{n=2}^{\infty} u^{(n)}(W) \frac{(x_i(0))^n}{n!} \right] q_i^{'+}$$

$$= u(W) \sum_{i \in I} q_i^{'+} + \sum_{i \in J} \left[\sum_{n=2}^{\infty} u^{(n)}(W) \frac{(x_i(0))^n}{n!} \right] q_i^{'+} +$$

$$+ \sum_{i \notin J} \left[\sum_{n=2}^{\infty} u^{(n)}(W) \frac{(x_i(0))^n}{n!} \right] q_i^{'+}.$$

The first term in this sum is zero because $\sum_{i \in I} q_i(s) = 1$ for all s implies $\sum_{i \in I} q_i^{'+} = 0$, the second because for $i \in J$ $x_i(0) = 0$. This proves the claim for $U^{'+}(p)$. The proof for $U^{'-}(p)$ is analogous. □

In the above lemma $u(W - ps + \bar{x}(s))$ is the agent's utility from choosing quantity s in case the outcome of the lottery happens to be

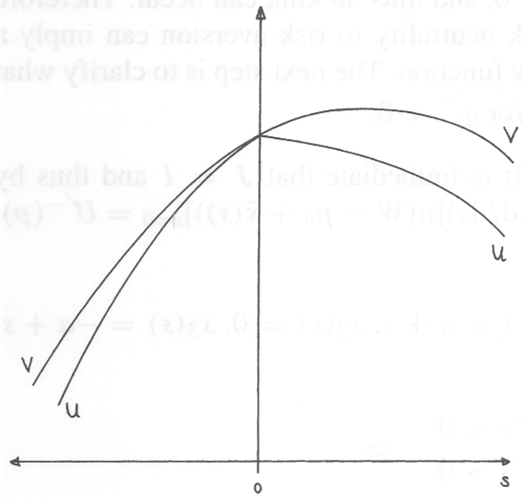

Figure 1.

equal to its expected value. It is also equal to the agent's expected utility in case he is risk neutral. Lemma 1 asserts that the derivative of that function at $s = 0$ differs from the right-hand-side derivative of the expected utility function under risk aversion at $s = 0$, $U^{'+}(p)$, if and only if A^+ differs from zero. Regarding the sign of A^+, we infer from $q_i(0) = 0$ for $i \notin J$ and $q_i(s) \geqslant 0$ for all $i \in I$ that $q_i^{'+} \geqslant 0$ for all $i \notin J$. On the other hand $\sum_{n=2}^{\infty} u^{(n)}(W)(x_i(0))^n/n! \leqslant 0$ iff u is concave and $q_i^{'+} > 0$ for at least one $i \notin J$. A symmetric reasoning applies to A^-. Thus we have proved:

PROPOSITION 1. $U^{'+}(p) \leqslant (\mathrm{d}/\mathrm{d}s)[u(W - ps + \bar{x}(s))]_{s=0} \leqslant U^{'-}(p)$. *The inequalities are strict if and only if u is strictly concave and* $q_i^{'+} > 0$ *and* $q_j^{'-} < 0$ *for at least one* $i \notin J$ *and one* $j \notin J$, *respectively.* □

For an interpretation of these results consider Figure 1, which shows the functions $s \mapsto V(s|p) := u(W - ps + \bar{x}(s))$ and $s \mapsto U(s|p)$. Since there is no uncertainty at $s = 0$, both functions coincide at that point. Due to (A3) and (A4) $V(\cdot|p)$ is differentiable everywhere, but if the conditions for strict inequalities in Proposition 1 are fulfilled, $U(\cdot|p)$ has a kink at $s = 0$. This fact will be decisive for obtaining a non-degenerate interval of prices p within which the agent will not want to trade. Observe that if the agent is risk neutral, $A^{+} = A^{-} = 0$, and thus no kink can occur. Therefore the mere switch from risk neutrality to risk aversion can imply a kink in the expected utility function. The next step is to clarify what it means that $q_i^{'+} > 0$ and/or $q_i^{'-} < 0$.

Example 1. [continued] It is immediate that $J = I$ and thus by Proposition 1 $U^{'+}(p) = (\mathrm{d}/\mathrm{d}s)[u(W - ps + \bar{x}(s))]_{s=0} = U^{'-}(p)$. Hence no kinks occur.

Example 2. Assume $x_1(s) = a + s$, $x_2(s) = 0$, $x_3(s) = -a + s$, $a \geqslant 0$, and

$$q_1 = \begin{cases} 0 & \text{if } s \leqslant 0 \\ s/(a+s) & \text{if } s > 0 \end{cases} \quad \Rightarrow$$

$$q_1' = \begin{cases} 0 & \text{if } s < 0 \\ a/(a+s)^2 & \text{if } s > 0 \end{cases}$$

$$q_2 = \begin{cases} 1 - [s/(-a+s)] & \text{if } s < 0 \\ 1 & \text{if } s = 0 \\ 1 - [s/(a+s)] & \text{if } s > 0 \end{cases} \quad \Rightarrow$$

$$q_2' = \begin{cases} a/(-a+s)^2 & \text{if } s < 0 \\ 1 - [a/(a+s)]^2 & \text{if } s > 0 \end{cases}$$

$$q_3 = \begin{cases} s/(-a+s) & \text{if } s < 0 \\ 0 & \text{if } s \geqslant 0 \end{cases} \quad \Rightarrow$$

$$q_3' = \begin{cases} -a/(-a+s)^2 & \text{if } s < 0 \\ 0 & \text{if } s > 0 \end{cases}.$$

One can easily verify that the functions x_i and q_i satisfy (A1)–(A3). In particular $\bar{x}(s) = s$ for all s. If $a = 0$, $X(s) = s$ with probability one, so assume $a > 0$. Then $J = \{2\}$, $q_1^{'+} = 1/a$, $q_3^{'+} = 0$, $q_1^{'-} = 0$ and $q_3^{'-} = -1/a$. Thus

$$A^{+} = \left[\sum_{n=2}^{\infty} u^{(n)}(W)\frac{a^n}{n!}\right](1/a) \leqslant 0$$

$$A^{-} = \left[\sum_{n=2}^{\infty} u^{(n)}(W)\frac{(-a)^n}{n!}\right](-1/a) \geqslant 0.$$

$A^{+} < 0$ and $A^{-} > 0$ iff u is strictly concave.

Example 3. $x_i(s)$, $i = 1, 2, 3$, as in Example 2, but in addition there is $x_4(s) = \bar{x}(s) = s$. I use script letters $\mathscr{q}_i$, $\mathcal{A}^{+}$ and $\mathcal{A}^{-}$ to distinguish the current variables from the corresponding ones of Example 2. Assuming $\mathscr{q}_4 \equiv q \in [0, 1]$ one obtains $\mathscr{q}_i(s) = q_i(s)(1 - q)$, $i = 1, 2, 3$. Consequently $\mathscr{q}_i'(s) = q_i'(s)(1 - q)$ for $i = 1, 2, 3$ and $\mathscr{q}_4' = 0$. This implies $\mathcal{A}^{+} = A^{+}(1 - q)$ and $\mathcal{A}^{-} = A^{-}(1 - q)$, and therefore it is again the case that risk aversion leads to a kink in the expected utility function at $s = 0$ provided the firm is risk averse (and $q < 1$).

Example 4. The only modification relative to Example 3 is that I now allow $\mathscr{q}_4 = q$ to be a function of s, $q(s)$. If $q(\cdot)$ is differentiable, then

$$\mathscr{q}_1' = \begin{cases} [a/(a+s)^2][1 - q(s)] - [s/(a+s)]q'(s) & \text{if } s > 0 \\ 0 & \text{if } s < 0 \end{cases}$$

$$\mathscr{q}_3' = \begin{cases} 0 & \text{if } s > 0 \\ [-a/(a+s)^2][1 - q(s)] - [s/(-a+s)]q'(s) & \text{if } s < 0 \end{cases}$$

Therefore $\mathscr{q}_1'^{+} = (1/a)[1 - q(0)]$ and $\mathscr{q}_3'^{-} = (-1/a)[1 - q(0)]$. In order that these numbers be zero if is necessary and sufficient that $q(0) = 1$. It is only in that case that the expected utility function does not have a kink.

3. EXISTENCE OF NONDEGENERATE INTERVALS OF NO-TRADE PRICES

I now discuss the relevance of the above results for the existence of nondegenerate intervals of no-trade prices. From Lemma 1,

$$\begin{aligned} U'^{+}(p) = 0 &\Leftrightarrow u'(W)(\bar{x}'(0) - p) + A^{+} = 0 \\ &\Leftrightarrow p = A^{+}/u'(W) + \bar{x}'(0) =: \underline{p}' \end{aligned}$$

and $U'^{+}(p) < 0$ for all $p > \underline{p}'$ since A^{+} does not depend on p. Equally,

$$U'^{-}(p) = 0 \Leftrightarrow p = A^{-}/u'(W) + \bar{x}'(0) =: \bar{p}'$$

and $U'^{-}(p) > 0$ for all $p < \bar{p}'$. Since $A^{+} \leqslant 0 \leqslant A^{-}$, $\underline{p}' \leqslant \bar{p}'$. The interval $[\underline{p}', \bar{p}']$ is nondegenerate if and only if $A^{+} < A^{-}$. For a risk averse agent this occurs if and only if there is at least one $i \in I \backslash J$ such that $q_i'^{+} > 0$ or $q_i'^{-} < 0$.

For every p between $\underline{p}'$ and $\bar{p}'$, $s = 0$ is a local maximizer of $U(s|p)$. If it is also a global maximizer, then the agent does not want to trade. A sufficient condition to ensure this is that the expected utility function $U(s|p)$ be quasiconcave in s.[4] If this is true for all p in a neighborhood $\mathbf{N}$ of $p = \bar{x}'(0)$, then there exists a nondegenerate interval $[\underline{p}, \bar{p}] \subseteq [\underline{p}', \bar{p}'] \cap \mathbf{N}$ of no-trade prices. We have thus proved:

THEOREM 1. *Assume* (A1)–(A4) *and that the expected utility function $U(s|p)$ is quasiconcave in s for all p in a neighborhood of $p = \bar{x}'(0)$. Then there exists a nondegenerate interval of prices $[\underline{p}, \bar{p}]$ for which the agent does not trade if and only if the agent is risk averse and there is at least one $i \in I \backslash J$ such that $q_i'^{+} > 0$ or $q_i'^{-} < 0$.* □

Note that assumptions (A1)–(A4) of Theorem 1 are satisfied in the above Examples 2 to 4 (in Example 4 if $q(0) < 1$). The existence of nondegenerate intervals of no-trade prices thus hinges on the behavior of $U(s|p)$ 'far away' from $s = 0$. In Example 6 I shall provide a complete example where $U(s|p)$ is quasiconcave and thus no-trade prices exist.

Before coming to this point it is worth while to shed further light on the intuitive significance of assumptions (A1)–(A4). I do this by looking at the variance of the lottery $X(s)$ implied by the subjective distribution $\mu(s)$.

Using (A1),

$$\operatorname{var} X(s) = \sum_i x_i(s)^2 q_i(s) - \bar{x}(s)^2.$$

The derivative of $\mathrm{var}X(s)$ for $s \neq 0$ is

$$\frac{\mathrm{d}\ \mathrm{var}X(s)}{\mathrm{d}s} = \sum_i [2x_i(s)x_i'(s)q_i(s) + x_i(s)^2 q_i'(s)] - 2\bar{x}(s)\bar{x}'(s). \tag{3}$$

Consider the limits as $s \to 0^+$ and $s \to 0^-$, that is $\lim_{s\to 0^+} \mathrm{d}\ \mathrm{var}X(s)/\mathrm{d}s =: \rho^+$ and $\lim_{s\to 0^-} \mathrm{d}\ \mathrm{var}X(s)/\mathrm{d}s =: \rho^-$, respectively. Since $\mathrm{var}X(0) = 0$, it is clear that $\rho^+ \geqslant 0$ and $\rho^- \leqslant 0$. What is not trivial is the following:

LEMMA 2. *$\rho^+ > 0$ if and only if $q_i'^+ > 0$ for at least one $i \notin J$; $\rho^- < 0$ if and only if $q_i'^- < 0$ for at least one $i \notin J$.*

Proof. From (A2) and the fact that $q_i(0) = 0$ for $x_i(0) \neq 0$ follows $\bar{x}(0) = 0$. Then (3) implies

$$\begin{aligned}\rho^+ &= \sum_{i\in I} [2x_i(0)x_i'(0)q_i(0) + x_i(0)^2 q_i'^+] \\ &= \sum_{i\in I} x_i(0)^2 q_i'^+ = \sum_{i\notin J} x_i(0)^2 q_i'^+ .\end{aligned}$$

Since for $i \notin J$ $x_i(0) \neq 0$, this proves the claim for ρ^+. The proof for ρ^- is symmetric. □

As to the interpretation of this result, note that $\rho^+ > 0$ or $\rho^- < 0$ mean that the variance function $s \mapsto \mathrm{var}X(s)$ is kinked at $s = 0$. By Proposition 1 and Lemma 2 , this kink is transferred to the expected utility function $U(s|p)$, but only if the agent is risk averse. To understand this intuitively consider the special case in which the vNM utility function u and/or the distributions $\mu(s)$ are such that expected utility depends on the expected value of $X(s)$ and its variance only, i.e. $U(s|p) = T(\bar{x}(s), \mathrm{var}X(s))$. If T is differentiable in its two arguments $\bar{x}$ and var, then it is clear that $U(\cdot|p)$ can have a kink only if $\mathrm{var}X(\cdot)$ has, since $\bar{x}(\cdot)$ is by assumption differentiable.

The above discussion is summarized in the following restatement of Theorem 1:

THEOREM 2. *Assume* (A1)–(A4) *and that the expected utility function $U(s|p)$ is quasiconcave in s for all p in a neighborhood of*

$p = \bar{x}'(0)$. *There exists a nondegenerate interval of prices* $[\underline{p}, \bar{p}]$ *for which the agent does not trade if and only if the agent is risk averse and the variance function* $s \mapsto \text{var}X(s)$ *is kinked at* $s = 0$. □

Example 5. We take the functions $x_i(\cdot)$ and $\varphi_i(\cdot)$, $i = 1, \ldots, 4$, as specified in Examples 2–4. For $s > 0$ one obtains

$$\begin{aligned} \text{var}X(s) &= (a+s)s(1-q) + s^2q - s^2 \\ &= as(1-q). \end{aligned}$$

Allowing for $q = q(s)$, this yields

$$\frac{\mathrm{d}\,\text{var}X(s)}{\mathrm{d}s} = a[1-q(s)] - asq'(s)$$

and therefore

$$\rho^+ = a[1-q(0)].$$

This expression is positive iff $q(0) < 1$. Comparison with the conclusions in Example 4 reveals that the variance function has a kink iff the expected utility function with risk aversion has a kink, as in fact predicted earlier.

4. RISK AVERSION OF ORDER ONE

As asserted in the Introduction, risk aversion of order one in the sense of Montesano (1988) or Segal and Spivak (1990) gives rise to intervals of no-trade prices. To see this, consider the certainty equivalent of the lottery $W - ps + X(s)$, $\Gamma(s|p)$, defined by

$$u(\Gamma(s|p)) = \int u(W - ps + x)\,\mathrm{d}[\mu(s)](x) = U(s|p).$$

Then the corresponding risk premium $\pi(s|p)$ is given by

$$\begin{aligned} \pi(s|p) &= E[W - ps + X(s)] - \Gamma(s|p) \\ &= W - ps + \bar{x}(s) - \Gamma(s|p). \end{aligned}$$

Note that $\pi(0|p) = 0$ since $\bar{x}(0) = 0$ and $\Gamma(0|p) = W$. Adapting Segal and Spivak's definition (p. 113) to the present framework one obtains:

DEFINITION 1. *The agent is risk averse at W of order 1 if* $\lim_{s\to 0^+} \pi'(s|p) > 0$ *or* $\lim_{s\to 0^-} \pi'(s|p) < 0$. *He is risk averse of order 2 if* $\pi'(0|p) = 0$, *but* $\lim_{s\to 0^+} \partial^2\pi/\partial s^2(s|p) > 0$ *or* $\lim_{s\to 0^-} \partial^2\pi/\partial s^2 (s|p) > 0$.

□ Since u is differentiable by assumption, having a kinked $U(\cdot|p)$ is equivalent to having a kinked $\Gamma(\cdot|p)$ which in turn is equivalent to having a kinked $\pi(\cdot|p)$, since $\bar{x}(\cdot)$ is differentiable. Applying Proposition 1 and Theorem 1, the claim follows. Note that, since I have not assumed stochastic constant returns to scale, first order risk aversion is possible although u is differentiable, in distinction to what Montesano (1988) and Segal and Spivak (1990) have established for the case in which stochastic constant returns to scale hold.

5. INTERVALS OF NO-TRADE PRICES AND CONVERGENCE CONCEPTS OF PROBABILITY MEASURES

It has already been observed above that weak continuity of the subjective distributions $\mu(s)$ is compatible with kinked expected utility functions. In decision theory under uncertainty, weak convergence is the standard assumption, and so on conventional decision theoretic grounds one cannot rule out the existence of intervals of no-trade prices. In order to do so, one needs a stronger concept of convergence. Gale (1979) has shown that such concept can be obtained by adding to weak convergence the concept of closed convergence. The latter results from the topology of closed convergence (see Hildenbrnad, 1974). Intuitively it means that, in order that a sequence of measures converge, their supports must converge (which is not required by weak convergence). In our context, this translates into the additional assumption (A5):

(A5) Let supp[μ] denote the support of the probability measure μ. Then, as $s \to 0$, supp[$\mu(s)$] $\to \{0\}$ with respect to the topology of closed convergence.

In terms of assumption (A1), (A5) implies that $x_i(0) = \lim_{s\to 0} x_i(s) = 0$ for all $i \in I$. In other words, $J = I$. But then there is no $i \notin J$, and therefore the conditions in Proposition 1 for the emergence of a kink in the expected utility function can never be fulfilled. Thus:

PROPOSITION 2. *Under the additional assumption of closed convergence* (A5), *nondegenerate intervals of no-trade prices do not exist.* □

Thus closed convergence is sufficient to eliminate kinks but it is not necessary. This is clear from Example 4 in which, for $q(\cdot)$ such that $q(0) = 1$, kinks do not occur although $\lim_{s\to 0^+} \text{supp}[\mu(s)] = \{0, a\}$, $\lim_{s\to 0^-} \text{supp}[\mu(s)] = \{-a, 0\}$ and $\text{supp}[\mu(0)] = \{0\}$.

Example 1. [continued] It is immediate that $\lim_{s\to 0} \text{supp}[\mu(s)] = \{0\}$ and thus, by Proposition 2, nondegenerate intervals of no-trade prices do not exist.

6. THE CASE OF DISTRIBUTIONS GIVEN BY DENSITY FUNCTIONS

I now consider the case in which the decision maker believes that the lottery $X(s)$ has a distribution given by means of a density function $x \mapsto f(x, s)$. The following result obtains.

THEOREM 3. *Assume that for $s \neq 0$ the subjective distribution $\mu(s)$ of $X(s)$ can be represented by means of a density function $f(x, s)$ continuous in x and differentiable in s. Assume moreover that* (A2)–(A4) *hold, that $\mu(s) \to \mu(0)$ weakly for $s \to 0$ and that the expected utility function*

$$U(s|p) = \int u(W - ps + x) f(x, s)\, \mathrm{d}x$$

is quasiconcave in s for all p in a neighborhood of $p = \bar{x}'(0)$. Then there exists a nondegenerate interval of no-trade prices $[\underline{p}, \bar{p}]$ if, and only if, the decision maker is risk averse and the function $s \mapsto \text{var}X(s)$ is kinked at $s = 0$.

Proof. Since the variance of $X(s)$ is

$$\text{var}X(s) = \int x^2 f(x, s)\, \mathrm{d}x - \bar{x}(s)^2,$$

its derivative, for $s \neq 0$, is

$$\frac{\mathrm{d}\operatorname{var} X(s)}{\mathrm{d}s} = \int x^2 \frac{\partial f}{\partial s}(x, s)\,\mathrm{d}x - 2\bar{x}(s)\bar{x}'(s).$$

For $s \to 0$, $\bar{x}(s)\bar{x}'(s) \to 0$ by (A3) and since $\mu(s) \to \delta(0)$ weakly, so

$$\rho^+ - \rho^- = \int x^2 \left[\frac{\partial f^+}{\partial s}(x, 0) - \frac{\partial f^-}{\partial s}(x, 0)\right] \mathrm{d}x \tag{4}$$

where

$$\frac{\partial f^+}{\partial s}(x, 0) = \lim_{s \to 0+} \frac{\partial f}{\partial s}(x, s),$$
$$\frac{\partial f^-}{\partial s}(x, 0) = \lim_{s \to 0-} \frac{\partial f}{\partial s}(x, s).$$

Since $\mu(s) \to \delta(0)$ weakly and f is continuous in x, $f(x, s) \to 0$ for $s \to 0$ whenever $x \neq 0$. Therefore, and since $f(x, s) \geqslant 0$ for all x and s, $(\partial f^+/\partial s)(x, 0) \geqslant 0$ and $(\partial f^-/\partial s)(x, 0) \leqslant 0$ for $x \neq 0$ which obviously implies

$$\frac{\partial f^+}{\partial s}(x, 0) - \frac{\partial f^-}{\partial s}(x, 0) \geqslant 0 \quad \text{for all} \quad x \neq 0.$$

Next consider the expected utility function

$$U(s|p) = \int u(W - ps + x) f(x, s)\,\mathrm{d}x.$$

Its derivative, for $s \neq 0$, is

$$U'(s|p) = \int u'(W - ps + x)(-p) f(x, s)\,\mathrm{d}x + \int u(W - ps + x) \frac{\partial f}{\partial s}(x, s)\,\mathrm{d}x.$$

Since $\mu(s) \to \mu(0)$ weakly, for $s \to 0$ the first integral tends to $-pu'(W)$. Therefore

$$U'^-(p) = -pu'(W) + \int u(W + x) \frac{\partial f^-}{\partial s}(x, 0)\,\mathrm{d}x,$$

$$U'^+(p) = -pu'(W) + \int u(W + x) \frac{\partial f^+}{\partial s}(x, 0)\,\mathrm{d}x.$$

The function $V(s|p) = u(W - ps + \bar{x}(s))$ gives rise to

$$\begin{aligned}
V'(0|p) &= u'(W)[\bar{x}'(0) - p] \\
&= u'(W)\left[\int x \frac{\partial f^-}{\partial s}(x,0)\,\mathrm{d}x - p\right] \\
&= -pu'(W) + \int u'(W)x\frac{\partial f^-}{\partial s}(x,0)\,\mathrm{d}x \\
&= -pu'(W) + \int u'(W)x\frac{\partial f^+}{\partial s}(x,0)\,\mathrm{d}x
\end{aligned}$$

where the last equality holds because $V(\cdot|P)$ is by assumption continuously differentiable. From this it follows that:

$$\begin{aligned}
&[U'^-(p) - V'(0|p)] - [U'^+(p) - V'(0|p)] \\
&= \int [u(W+x) - u'(W)x]\frac{\partial f^-}{\partial s}(x,0)\,\mathrm{d}x \\
&\quad - \int [u(W+x) - u'(W)x]\frac{\partial f^+}{\partial s}(x,0)\,\mathrm{d}x \\
&= \int \left[u(W) + \sum_{n=2}^{\infty} u^{(n)}(W)\frac{x^n}{n!}\right] \\
&\quad \times \left[\frac{\partial f^-}{\partial s}(x,0) - \frac{\partial f^+}{\partial s}(x,0)\right]\mathrm{d}x
\end{aligned}$$

which yields

$$\begin{aligned}
U'^-(p) - U'^+(p) = \int &\left[\sum_{n=2}^{\infty} u^{(n)}(W)\frac{x^n}{n!}\right] \\
&\times \left[\frac{\partial f^-}{\partial s}(x,0) - \frac{\partial f^+}{\partial s}(x,0)\right]\mathrm{d}x
\end{aligned} \tag{5}$$

since $\int f(x,s)\,\mathrm{d}x = 1$ for all s implies

$$u(W)\int \frac{\partial f^-}{\partial s}(x,0)\,\mathrm{d}x = 0 = u(W)\int \frac{\partial f^+}{\partial s}(x,0)\,\mathrm{d}x.$$

Comparing (4) and (5) one sees that, for a risk averse agent, the integral both in (4) and in (5) is nil iff $[(\partial f^+/\partial s)(x,0) -$

$(\partial f^-/\partial s)(x,0)]$ is zero for almost all $x \neq 0$. Therefore the expected utility function $U(s|p)$ is kinked at $s = 0$ iff the same holds for the variance function $\mathrm{var}X(s)$. Since a kinked and, for all p in a neighborhood of $\bar{x}'(0)$, quasiconcave $U(\cdot|p)$ implies a nondegenerate interval of no-trade prices, the assertion follows. □

Example 6. Assume $X(0) = 0$ with probability one and $X(s)$, $s \neq 0$, is believed to be normally distributed with expectation $\bar{x}(s)$ and variance $\sigma^2(s)$. Then $f(x,s) = g(x; \bar{x}(s), \sigma^2(s))$, with $g(x; \bar{x}, \sigma^2) = [2\pi\sigma^2]^{-1/2} e^{-(x-\bar{x})^2/2\sigma^2}$. $\mu(s) \to \mu(0)$ weakly for $s \to 0$ whereas $\mathrm{supp}[\mu(s)] = \mathbb{R} \neq \{0\} = \mathrm{supp}[\mu(0)]$ for all $s \neq 0$.

Assume moreover $u(z) = -e^{-z}$. Then, using the well-known result that $\int e^{tx} g(x; \nu, \sigma^2)\, \mathrm{d}x = e^{t\nu + t^2\sigma^2/2}$,[5] expected utility becomes

$$U(s|p) = -e^{-(W-ps)} \int e^{-x} g(x; \bar{x}(s), \sigma^2(s))\, \mathrm{d}x$$
$$= -e^{-(W-ps)} e^{-\bar{x}(s)+\sigma^2(s)/2} = -e^{-\tau(s|p)}$$

where

$$\begin{aligned}\tau(s|p) &= W - ps + \bar{x}(s) - \sigma^2(s)/2 \\ &= W - ps + EX(s) - \tfrac{1}{2}\mathrm{var}X(s) \\ &= EZ(s|p) - \tfrac{1}{2}\mathrm{var}X(s).\end{aligned}$$

On the other hand, from the definition of the certainty equivalent $\Gamma(s|p)$ it follows that $U(s|p) = -e^{-\Gamma(s|p)}$. Thus $\Gamma = \tau$ and, since $\mathrm{var}Z(s|p) = \mathrm{var}X(s)$,

$$\Gamma(s|p) = EZ(s|p) - \tfrac{1}{2}\mathrm{var}Z(s|p).$$

Moreover, the risk premium $\pi(s|p)$ is given by

$$\pi(s|p) = EZ(s|p) - \Gamma(s|p) = \tfrac{1}{2}\mathrm{var}X(s)$$

from which it is evident that the agent is risk averse of order one iff $\mathrm{var}X(\cdot)$ is kinked at $s = 0$.

To be still more specific, assume $\bar{x}(s) = s$ and $\sigma^2(s) = |s|^{2\gamma} e^{2(1-\gamma)|s|}$, where γ is a parameter in [0,1]. Then

$$|\mathrm{d}\,\mathrm{var}X(s)/\mathrm{d}s| = 2|s|^{2\gamma-1} e^{2(1-\gamma)|s|}[\gamma + (1-\gamma)|s|]$$

tends to zero for $s \to 0$ iff $\gamma > \frac{1}{2}$. For $\gamma \leqslant \frac{1}{2}$, therefore, according to Theorem 3, there exist nondegenerate intervals of no-trade prices, provided $U(s|p)$ is quasiconcave in s in a neighborhood of $p = \bar{x}'(0) = 1$.

To see this, note that, since $U(s|p)$ is strictly increasing in $\Gamma(s|p)$, to maximize U is equivalent to maximizing Γ which is now

$$\Gamma(s|p) = W + (1-p)s - |s|^{2\gamma} e^{2(1-\gamma)|s|}/2.$$

From this follows

$$\Gamma'(s|p) = 1 - p - s^{2\gamma-1} e^{2(1-\gamma)s}[\gamma + (1-\gamma)s], \quad \text{when } s > 0,$$

$$\Gamma'(s|p) = 1 - p - (-s)^{2\gamma-1} e^{-2(1-\gamma)s}[-\gamma + (1-\gamma)s], \quad \text{when } s < 0.$$

In particular $\Gamma'(s|1)$ is always strictly negative when $s > 0$ and strictly positive when $s < 0$, whatever is the value of γ. By continuity of $\Gamma'(s|p)$ in p this fact extends to a neighborhood of $p = 1$. Thus Γ is quasiconcave in s for all p in a neighborhood of 1 and, since quasiconcavity persists under positive monotone transformations, the same holds for U.

Let us now consider three cases in detail: $\gamma = 1$, $\gamma = \frac{1}{2}$ and $\gamma = 0$. For $\gamma = 1$ the above formulae yield $\Gamma'(s|p) = 1 - p - s$. Defining s^* by $\Gamma'(s^*|p) = 0$, one obtains $s^* = 0$ iff $p = 1$. Thus the only no-trade price is $p = 1$. (An analogous reasoning holds for any $\gamma > \frac{1}{2}$).

Next consider the case $\gamma = \frac{1}{2}$. Then, for $s > 0$

$$\Gamma'(s|p) = 1 - p - e^{s}(s+1)/2$$

and therefore $\Gamma'^{+}(0|p) = \frac{1}{2} - p \leqslant 0$ for $p \geqslant \frac{1}{2}$. For $s < 0$ one calculates

$$\Gamma'(s|p) = 1 - p - e^{-s}(s-1)/2$$

which yields $\Gamma'^{-}(0|p) = \frac{3}{2} - p \geqslant 0$ for $p \leqslant \frac{3}{2}$. Since $\Gamma''(s|p)$ is negative for any $s \neq 0$ and any p, $\Gamma(\cdot|p)$ is concave. Thus any price in the interval $[\frac{1}{2}, \frac{3}{2}]$ is a no-trade price.

Finally, for $\gamma = 0$ follows $\Gamma'^{+}(0) = -p \leqslant 0$ for $p \geqslant 0$ and $\Gamma'^{-}(0) = 2 - p \geqslant 0$ for $p \leqslant 2$. Since $\Gamma(\cdot|p)$ is concave for all p, the corresponding interval of no-trade prices is [0,2].

Note that in the present example there are stochastic constant returns to scale iff $\gamma = 1$, since only then $\mathrm{var}X(s) = \sigma^2(s) = s^2 = s^2\mathrm{var}X(1) = \mathrm{var}[sX(1)]$. The agent is risk averse of order one for $\gamma \in [0, \frac{1}{2}]$ and risk averse of order two for $\gamma \in (\frac{1}{2}, 1]$. In particular, this confirms the fact that, with a differentiable vNM utility function, first order risk aversion cannot occur when returns to scale are constant.

The example illustrates that a continuous variation in a single parameter can imply a qualitative change in the way the agent behaves. This is not exactly what intuition might have suggested, and it indicates that the origins of inertia in investment behavior may be quite intricate.

7. CONCLUDING REMARKS

In this paper it has been shown how abandoning the assumption of stochastic constant returns to scale can result in an invalidation of the local risk-neutrality theorem and, as a consequence, in the emergence of intervals of no-trade prices. The fact that observed behavior often does not conform with the theorem's prediction indicates that stochastic constant returns to scale are not necessarily taken as granted by economic agents. This may be due, for example, to the fact that fees vary with the amount of assets subscribed, to the presence of a big trader whose actions influence the stock price, or simply to particular subjective beliefs, in line with the view expressed in Arrow (1965, p. 13): "... choice is subjective; in choice among actions, both the values and the beliefs of the economic agent are relevant to explaining his choice, regardless of how these might differ from values or beliefs 'objectively' given in some sense."

In any case, the purpose of the present investigation was to find out how far one can get in the explanation of inertia in traders' behavior without giving up conventional assumptions like additive probabilities, differentiable von Neumann–Morgenstern utility function and expected utility. The main result is that, under quasi-concavity of the expected utility function, beliefs imply intervals of no-trade prices if, and only if, the corresponding perceived variance function of the random return of the asset under consideration is kinked at the no-trade point. This is equivalent to the presence of risk aversion of order one and it is perfectly compatible with the expected value of the asset being a smooth function of the quantity transacted, even and in particular at the no-trade point.

It would be possible to generalize this result to the case where the decision maker does not have the option of a certain final wealth but where that would be random, too, even if the agent did not trade. In that case the variance of final wealth would be positive at the no-trade point, but to obtain an interval of no-trade prices it would still be necessary and sufficient that the variance function of the random return of the asset under consideration be kinked.

NOTES

1. In case there are many risky assets, the unique no-trade price will not necessarily be the expected value of the asset under consideration.
2. I am grateful to Aldo Montesano for having pointed out to me the connection of the present paper's approach with the literature on nonadditive probabilities and on attitude towards risk of order one.
3. For any continuous and bounded function $f : \mathbb{R} \to \mathbb{R}$ and any sequence $\{s_n\} \to s$ continuity of $x_i(\cdot)$ and $q_i(\cdot)$ implies $\lim_{n\to\infty} \sum_i f(x_i(s_n))q_i(s_n) = \sum_i f(x_i(s))q_i(s)$, that is $\mu(s_n) \to \mu(s)$, by definition of weak convergence.
4. A function f is quasiconcave if the set $\{s | f(s) \geqslant \alpha\}$ is convex for any $\alpha \in \mathbb{R}$. In the present situation, where $f(s) = U(s|p)$ and $s = 0$ is a local maximizer which is locally unique, this implies that it is also a global one. In fact, if it were not, $\{s|U(s|p) \geqslant U(0|p)\}$ would not be convex.
5. See, e.g., Meyer (1970).

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