Shipping the Good Apples Out: Another Proof with A Graphical Representation

Saito, Tetsuya

SUNY at Buffalo

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An Expository Note on Alchian-Allen Theorem When Sub-Utility Functions are Homogeneous of Degree $n > 0$*

Tetsuya Saito

Department of Economics, SUNY at Buffalo

415 Fronczak Hall, Buffalo, NY 14260

tsaito5@buffalo.edu

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Abstract

This expository note shows Alchian and Allen's conjecture–consumers purchase fine quality relatively more than coarse one– is true under some specific conditions about homogeneity, inner solution and substitutability while allowing the influence of the income effect. In the proof, to be an exposition, I emphasize graphical representations of Alchian-Allen Theorem than algebra.

JEL Classifications: D01; F10

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1 Introduction

Alchian and Allen (1967) proposes consumers purchase superior quality (higher price) relatively more than inferior one (lower price) when a fixed transaction (transportation) fee is uniformly imposed. If the is no other goods, assuming two qualities substitute each other, their statement is plausible if these two qualities are close substitute because the percentage increase of the price of the superior one is less than that of the inferior one, and then consumers will purchase the

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superior one relatively more than before. If two qualities are not close substitute, we can easily find the conjecture does not hold—for example, consider a partial equilibrium model such that the fine quality is elastic but the coarse one is inelastic. With this regard, if two qualities are not close substitutes for each other—for example, luxury cars and low-end cars—they cannot be regarded as close commodities. Within the original statement, Alchian and Allen considers an example of apples sold locally and shipped to New York City proposing good apples in the distant marketplace more than the local market because of consumers' behavior we will discuss in this study. In this sense, we can justify the close substitutability assumption to support Alchian and Allen's argument.

Within a more general framework, applying Hicksian demand functions in order to eliminate income effects and introducing other goods than x, Borcherding and Silberberg (1978) have shown the conjecture holds and Bauman (2004) extended the proof with multiple Hicksian composite goods. In those models, demand elasticities are constrained by budget constrains as such known as the Third Law of Hicks (Hicks 1946). Yet, it is ambiguous when there is some income effects. Gould and Segall (1969) shows Alchian and Allen's conjecture is ambiguous and empirical works are necessary, and also Umbeck (1980) criticizes the specification of Borcherding and Silberberg that ignores income effects. In applications, it is also difficult to interpret Hicksian demand functions, especially in empirical studies. This expository note shows Alchian and Allen's conjecture holds under some conditions about homogeneity, inner solution and substitutability when there is a possibility that income effect may exist.

2 Proof of the Theorem

Suppose there are two goods x and y, and x has two alternative qualities, fine and coarse, respectively denoted as x_1 and x_2. I assume y is the numèraire of this model. The difference in the quality of x is characterized by prices. Let p_1 and p_2 be respective prices of x_1 and x_2; thence, $p_1 > p_2 > 0$ applies. Let $V = V(x_1, x_2, y)$ be the utility function of a consumer. Let x_1, x_2 and y are substitutes of each other, then, without loss of the generality, we can modify the definition of the utility function as $V[U(x_1, x_2), y]$, where $X = U(x_1, x_2)$ is the sub-utility function. Properties of V and U are then summarized as follows.

Assumption 1 *V possesses following properties; $\partial V/\partial x_1 > 0$, $\partial V/\partial x_2 > 0$ and $\partial V/\partial y > 0$; $\partial^2 V/\partial {x_1}^2 < 0$, $\partial^2 V/\partial {x_2}^2 < 0$ and $\partial^2 V/\partial y^2 < 0$; and $\partial^2 V/\partial y \partial x_1 > 0$ and $\partial^2 V/\partial y \partial x_2 > 0$.*

Assumption 2 *U possesses following properties; $\partial U/\partial x_1 > 0$ and $\partial U/\partial x_2 > 0$; $\partial^2 U/\partial {x_1}^2 < 0$ and $\partial^2 U/\partial {x_2}^2 < 0$; and $\partial^2 V/\partial x_1 \partial x_2 > 0$.*

Consider the following optimization problem of the consumer regarding the sub-utility level X as a good:

$$\text{Maximize} \quad V(X, y) \quad \text{subject to} \quad PX + y \leqslant m, \tag{1}$$

where $P > 0$ is the price index of x and $m > 0$ is the income respectively in terms of the price of y. Assume no corner solution. Then, this problem gives an optimum value of X and y as functions of P and m, as such, $X^* = X^*(P, m)$ and $y^* = y^*(P, m)$. Let X^* be given by Problem (1) to consider the next problem:

$$\text{Minimize} \quad p_1 x_1 + p_2 x_2 \quad \text{subject to} \quad U(x_1, x_2) \geqslant X^*, \tag{2}$$

which determines consumptions of x_1 and x_2 with respect to X^* and prices.

Lemma 1 *Suppose U is homogeneous of degree $n > 0$ and the constraint of Problem 2 is not slack. Then, at the optimum, Problem (1) is identical to the next problem if it has an inner solution:*

$$\text{Maximize} \quad V(x_1, x_2, y) \quad \text{subject to} \quad p_1 x_1 + p_2 x_2 + y \leqslant m. \tag{3}$$

Proof: Let $\lambda \geqslant 0$ be the Lagrange multiplier for Problem (3). From the first order condition for an inner solution, at the optimum, we have

$$p_1 x_1 + p_2 x_2 = \frac{1}{\lambda} \cdot \left(x_1 \cdot \frac{\partial U}{\partial x_1} + x_2 \cdot \frac{\partial U}{\partial x_2} \right) \equiv \frac{n}{\lambda} \cdot U(x_1, x_2), \tag{4}$$

where the last equivalence follows from Euler's Homogeneous Function Theorem because of the assumption such that U is homogeneous of degree $n > 0$. Applying $P = n/\lambda$ and rewriting V with the sub-utility function, we can see Problem (1) and Problem (3) are identical at the optimum. Note, in order to guarantee $P \in (0, \infty)$, we require $n \in (0, \infty)$ and $\lambda \in (0, \infty)$. ■

Theorem 1 (Alchian-Allen Theorem) *Assume Assumption 1 and Assumption 2. Suppose there is no corner solution. Then the consumer raises the consumption on the fine quality relative to that of the coarse one if the sub-utility function is homogeneous of degree $n > 0$.*

Proof: Applying Lemma 1, we can work on the two-step optimization process represented by Problem 1 and following Problem 2 if the sub-utility function is homogeneous of degree $n > 0$. Let $T > 0$ be a fixed transaction cost. Then, by the assumption on prices, we have

$$\frac{p_1}{p_2} > \frac{p_1 + T}{p_2 + T}, \tag{5}$$

hence the relative price of the fine quality declines in terms of the coarse one when the fixed transaction cost is uniformly imposed. In addition, I also note the slope of the iso-utility locus of V on y-X plane is represented by

$$\frac{dX}{dy} = -\frac{\partial V/\partial y}{\partial V/\partial X}, \tag{6}$$

which follows from

$$dX = \frac{\partial U}{\partial x_1} \cdot dx_1 + \frac{\partial U}{\partial x_2} \cdot dx_2 \tag{7}$$

and

$$dV = \frac{\partial V}{\partial X}\frac{\partial U}{\partial x_1} \cdot dx_1 + \frac{\partial V}{\partial X}\frac{\partial U}{\partial x_2} \cdot dx_2 + \frac{\partial V}{\partial y} \cdot dy = 0. \tag{8}$$

Then we can also see the slope of iso-utility locus of V on y-X plane increases as y increases by Assumption 1.

Based on the above arguments, we can analyze the influence of the fixed transportation cost using Figure 1 and Figure 2 (Assumption 2 gives the shape of the iso-utility locus of U on x_1-x_2 plane). Suppose A in Figure 1 is the initial consumption point. If a fixed transaction cost is imposed, the income in terms of P declines because λ decreases as consumer prices of x_1 and x_2 rises—in particular, λ is not altered by marginal utilities and that is determined after the change in consumer prices, hence, only consumer prices matter at this point. Then this change is depicted by (i) in the figure. Accordingly the consumer brings the utility level in accordance with (ii), thence, the optimum point moves from A to B in the figure.

On x_1-x_2 plane (Figure 2), the decline in X is depicted by the downward movement of the sub-utility function denoted by (iii) on the figure because $n > 0$. Suppose C is the initial consumption point corresponding to A. If the fixed transaction cost is imposed, the iso-utility local of the sub utility function shift downward and the consumer minimizes the expenditure at E because the slope of the sub-utility function must be the relative price of fine and coarse qualities. In this sense, the decline in the relative price of the fine quality in terms of the coarse one is depicted by (iv) in the figure. Accordingly the consumption ration of the fine quality to the coarse one increases as depicted by (v) in the figure because the assumption of the homogeneity on the sub-utility function guarantees the slope of OD, is larger than that of OE, where D represents the consumption point under the same price levels with the utility level with the fixed transportation cost. Hence we can see the fixed transaction cost raises the relative consumption of the fine quality to the coarse one under Assumption 1 and Assumption 2, and additional assumptions such that there is no corner solution and the sub-utility function is homogeneous of degree $n > 0$. ■

This proof can be easily extended to multiple Hicksian composite goods case such as studied by Bauman (2004) if the sub-utility function for those Hicksian composite goods is homogeneous of degree $n > 0$ because Lemma 1 is also applicable to the other goods as well. In this case, we need substitutabilities among x_1, x_2 and the other goods while there is no substitutability restriction within the other goods.

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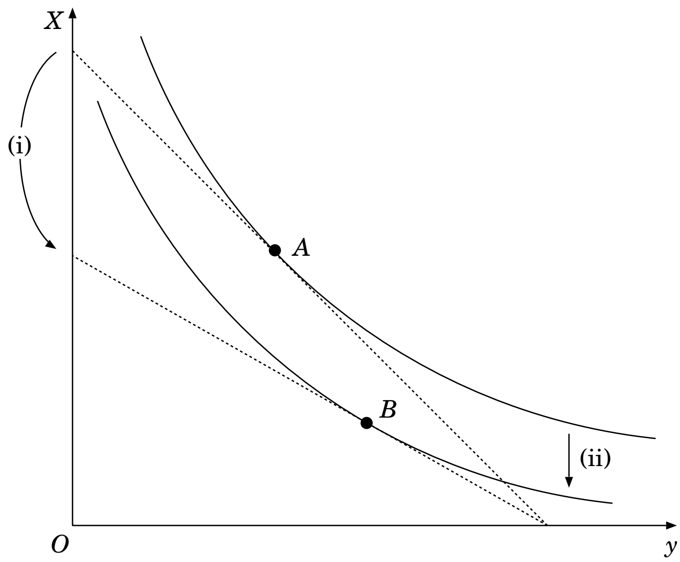

Figure 1: Changes in consumptions on X and y

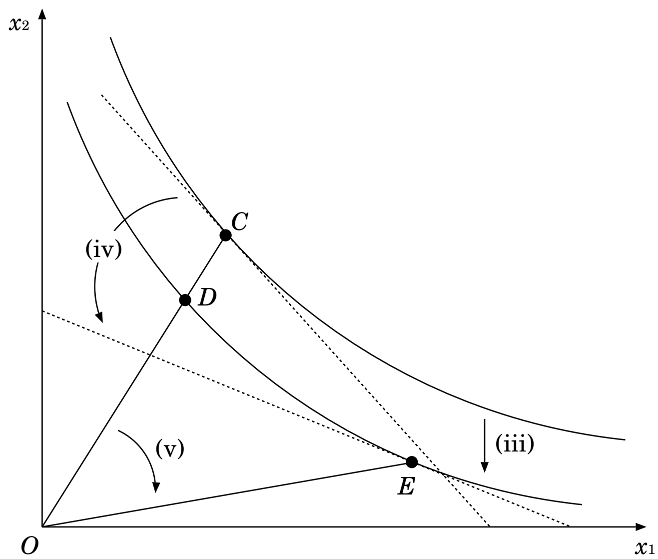

Figure 2: Changes in consumptions on x_1 and x_2