Is Prelec's function discontinuous at p = 1? (for the Einhorn Award of SJDM)

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(for the Einhorn Award of SJDM)

A possibility of the existence of a discontinuity of Prelec's (probability weighting) function at the probability $p = 1$ is discussed. This possibility is supported by the purely mathematical theorems and the "certain–uncertain" inconsistency of the random–lottery incentive experiments. The results of the well-known experiment support it as well.

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Contents

Introduction

The purpose of the paper

The purpose of the present working paper is to discuss a possibility of the existence of a discontinuity of Prelec's (probability weighting) function $W(p)$ at the probability $p = 1$.

This purpose is new and, in a sense, contradicts the accepted view. So, in the first stage of its study, the general methods of the work are mainly qualitative.

The data used are the well-established experimental results of other researchers. The techniques of analysis of the data are also mainly qualitative.

Therefore, the conclusions of the paper are qualitative as well.

Notation

There are a number of theories concerned with one or another concept of utility. They include, e.g., Bernoullian expected utility, von Neumann–Morgenstern expected utility, subjective expected utility, subjectively weighted utility theories (see, e.g., a review by Schoemaker, 1982); prospect theory (see Kahneman and Tversky, 1979) and cumulative prospect theory (see Tversky and Kahneman, 1992) or, in other terminology, original prospect theory and prospect theory; the salience theory of choice under risk (see Bordalo, Gennaioli, Shleifer, 2012); expected uncertain utility theory (Gul and Pesendorfer, 2014); etc.

In the present paper these theories are referred to as

utility and prospect theories.

The paper deals with the probability weighting function $W(p)$ necessarily and widely used in prospect theories. Here, it will be usually referred to as Prelec's weighting function (see Prelec, 1998) or, for short,

Prelec's function.

An invitation

6 May 2015, I received a letter from the Society for Judgment and Decision Making:

"The Society for Judgment and Decision Making is inviting submissions for the Hillel Einhorn New Investigator Award. The purpose of this award is to encourage outstanding work by new researchers. Individuals are eligible if they have not yet completed their Ph.D. or if they have completed their Ph.D. within the last five years (on or after July 1, 2010). To be considered for the award, please submit a journal-style manuscript on any topic related to judgment and decision making. …"

I have not yet completed my Ph.D. So, I may use this invitation to summarize and generalize the results of several my papers.

A propos, my first refereed article in an international journal was published in 2012. To 6 May 2015 I have published three refereed articles in international journals, three reports on international foreign conferences and a number of working papers.

The SJDM Einhorn Submission page in the Internet http://www.sjdm.org/awards/einhorn.upload.html determines the following types of the Status of Paper:

"Published or Published Online
Submitted or Under Review
Working Paper
Unpublished Manuscript
Other"

So, my manuscript is presented in the form of this working paper.

In this paper, I briefly review, summarize and generalize the considerations and results of my published articles, reports, articles under review, articles prepared for submission and working papers.

1. The “Luce problem” and “Luce question”

An essential part of problems of utility and prospect theories consists in the problems that are connected with a probability weighting (see, e.g., Tversky and Wakker, 1995). A probability weighting means that subjects treat the probability p by a probability weighting function $W(p)$ which is not equal to p. Prelec’s weighting function (Prelec, 1998) is one of the most popular probability weighting functions.

1.1. State of the art

During many years, in a lot of works, at least the vast majority of authors assumed by default that Prelec’s weighting function $W(p)$ is equal to 1 at $p = 1$. One may agree that the assumption $W(1) = 1$ is, indeed, quite evident and natural.

For example, we see in Wakker (1994), page 9: “DEFINITION 1. Rank-dependent utility, (RDU) holds if there exist a strictly increasing continuous utility function $U'' \ [0,M] \rightarrow N$ and a strictly increasing probability transformation $\varphi:[0,1] \rightarrow [0,1]$ with $\varphi(0) = 0$ and $\varphi(1) = 1$,”

We see in Prelec (1998) that Prelec’s formula “$w(p) = exp\{-\{-ln\ p\}^{\alpha}\},\ 0<a<1$” in itself assumes only $w(1) = 1$. ibid, page 515: “unique, nondecreasing weighting functions, satisfying $w(0) = 0, w(1) = 1$”

Note, there is no assumption of $W(1) \neq 1$ in these works.

1.2. In spite of the accepted view.
The “Luce problem” and “Luce question”
1.2.1. In spite of the accepted view. Two articles

In spite of the accepted view, R. Duncan Luce with Ragnar Steingrimsson and János Aczél had discovered a problem of a general mathematical and scientific nature. The essence of the problem was: “whether a well-known object is actually what it seems to the vast majority of people?”

In 2007, R. Duncan Luce with Ragnar Steingrimsson and János Aczél published two articles: Steingrimsson and Luce (2007) and Aczél and Luce (2007).

The first article was essentially devoted to the analysis of weighting functions with $W(1) = 1$ and without $W(1) = 1$. Two subchapters of the article are devoted to “function with $W(1) = 1$” and two subchapters of the article are devoted to “function without $W(1) = 1$.” Moreover, even the title of the second article contains the item “without assuming $W(1) = 1$.”

1.2.2. The "Luce problem" and "Luce question"

The problem of Steingrimsson and Luce (2007) and Aczél and Luce (2007) was: a special analysis of Prelec's function at $p = 1$ and $p \approx 1$.

Prelec's function has been much analyzed in the middle of the probability scale, but an analysis at $p = 1$ or at $p \approx 1$ is still an undeservedly too rare event.

One can name this problem after R. Duncan Luce as the "Luce problem," or after Steingrimsson, Luce and Aczél as the "SLA problem," etc. Here, this problem is referred to as the "Luce problem."

The question of Steingrimsson and Luce (2007) and Aczél and Luce (2007) was: whether Prelec's weighting function is actually equal to 1 at $p = 1$?

Here, this question is referred to also as the "Luce question."

1.2.3. An undeserved underestimation

The above two articles are well cited. As of 7 April 2015, Steingrimsson and Luce (2007) was cited by 23 and Aczél and Luce (2007) by 8 articles.

Nevertheless, the "Luce problem" and the "Luce question" are still undeservedly underestimated.

For example, we see in Diecidue, Schmidt, and Zank (2009), page 1105: "the weighting function $w: [0, 1] \rightarrow [0, 1]$ is strictly increasing and continuous with $w(0) = 0$ and $w(1) = 1$."

We see in Chechile and Barch (2013), page 16: "Assumption 2. If $p = 1$, then $w(p) = 1$"

We do not see an assumption like "If $p = 1$, then $w(p) \neq 1$" in these works.

We see also too few investigations of $W(p)$ at $p \approx 1$, even at $p > 0.9$.

However, we may note that the problem and question were considered by Luce with his co-authors not once. They were considered twice (and in various memberships of the co-authors). Therefore, one can conclude, in particular, the following:

1) The problem and question were not accidental.

2) The question was not a purely quantitative one. That is, the question was not: "whether Prelec's function $W(p)$ is a bit more or less than 1 at $p = 1$."

1.3. A modification of the "Luce question" and a possible discontinuity of Prelec's function $W(p)$ at $p = 1$

1.3.1. A modification of the "Luce question"

There is a deal of evidence for the existence of a difference between subjects' treatment of the probabilities of uncertain (probable) and certain outcomes (see, e.g., Kahneman and Tversky, 1979; Halevy, 2008). Therefore, in the general case, one should distinguish between the values of the probability weighting function $W(p)$ of a certain outcome and the limit of the probability weighting function $W(p)$ of uncertain outcomes as the probability of those uncertain outcomes tends to 1.

Let us specify a value $W_{Certain}$ of the probability weighting function $W(p)$ for a certain outcome. At that, $W_{Certain}$ may be assumed to be equal to 1. Otherwise, other values of $W(p)$ may be normalized by $W_{Certain}$.

Let us also specify a value $W(1)$ as the limit of the probability weighting function $W(p)$ for a probable (uncertain) outcome as p tends to 1

$$W(1) \equiv \lim_{p \to 1} W(p) \ .$$

If $W_{Impossible}$ is defined for the impossible case (for $p = 0$), then, similar to Aczél and Luce (2007), one can write

$$W(p) = \begin{cases} W_{\mathrm{Im}possible} & p = 0 \\ W(p) & p \in]0,1[\\ W_{Certain} & p = 1 \end{cases} \quad .$$

So, one may modify the "Luce question" whether $W(1) = 1$ into the modified "Luce question"

$$W_{Certain} - W(1) = ?$$

1.3.2. A possible discontinuity of Prelec's function at $p = 1$ and the crucial importance of the "Luce problem"

The question of the continuity of Prelec's function has been already considered mainly among other questions (see, e.g., Wakker, 1994, (see also Masson, 1974)). Let us highlight it and make a special consideration of it.

The question $W_{Certain} - W(1) = ?$ or whether $W(1) = W_{Certain}$ is the question whether $W(p)$ is continuous at $p = 1$.

If $W(1) = W_{Certain}$ then $W(p)$ is continuous (at $p = 1$). This is usually assumed by default. Nevertheless, this has not been proven for the general case. The answer

$$W_{Certain} - W(1) \neq 0$$

or

$$W(1) \neq W_{Certain}$$

to the modified "Luce question" means that $W(p)$ has a discontinuity at $p=1$.

A discontinuity is not a quantitative but a qualitative, moreover, a topological feature. Therefore, the possible discontinuity of Prelec's function can qualitatively change prospect theories, at least in their mathematical aspects.

So, the "Luce problem" can be of crucial importance for prospect theories.

This section has reviewed and generalized my elder works, e.g., Harin (2014).

2. Illustrative examples

Kahneman and Thaler (2006) pointed out that the problems of the utility and prospect theories have still not been adequately overcome.

One possible way to explain these problems has been widely discussed, e.g., in Schoemaker and Hershey (1992), Butler and Loomes (2007). Its essence consists in a proper attention to the widespread noise, imprecision, and other reasons that may cause scattering of real data.

The essential feature of these problems is their intense manifestation near the borders of the scale of probability (see, e.g., Tversky and Wakker, 1995).

The above mentioned Steingrimsson and Luce (2007) and Aczél and Luce (2007) have opened one more way which consists in paying proper attention to borders, boundaries and interfaces.

A purely mathematical investigation (see, e.g., Harin 2012b) has synthesized these two different ways. That is, it considers the dispersion of the data (or the influence of this dispersion) near the borders of the probability scale.

Purely mathematical theorems prove that the probability p cannot attain 1 under the condition of a non-zero dispersion of the data.

A mechanical analogy of vibrations near a rigid wall:

Suppose an electro-drill or any similar device, e.g., sewing-machine, vibrosieve, machine-gun, electric hammer etc. which (when working) can vibrate quickly. Presume the device has rigid flank sides and vibrates with a non-zero amplitude of, say, 1 mm.

Can we approach a flank side of the NON-WORKING drill (or of the device) to a rigid wall or ledge tightly? Yes. Surely.

And now turn the drill on. What will be the distance from the rigid wall to the working drill? Vibrations will repulse, shift the drill from the wall.

Can we approach a flank side of the WORKING drill to a rigid wall or ledge tightly? No.

The mean distance from the drill to the wall will be about a half of the non-zero amplitude of vibrations, that is about 0.5 mm (if we do not apply an essential force to specially press the drill to the wall).

This section illustrates the essence of the theorems.

Its first subsection presents a simple pictorial two-point example of restrictions on the mean in the presence of the non-zero dispersion.

Its second subsection presents a simple pictorial example of a restriction on the probability for a classical aiming firing at a target.

Its third subsection presents an example of the normal distribution.

This section has reviewed and generalized my elder works, e.g., Harin (2012a).

2.1. An illustrative example of restrictions on the mean

Let us consider briefly an illustrative example of restrictions on the mean.

2.1.1. Two points

Let us suppose given an interval $[A, B]$ (see Figure 1). Let us suppose that two points are determined on this interval: a left point x_{Left} and a right point x_{Right} : $x_{Left} < x_{Right}$. The coordinates of the middle, mean point may be calculated as $M=(x_{Left}+x_{Right})/2$.

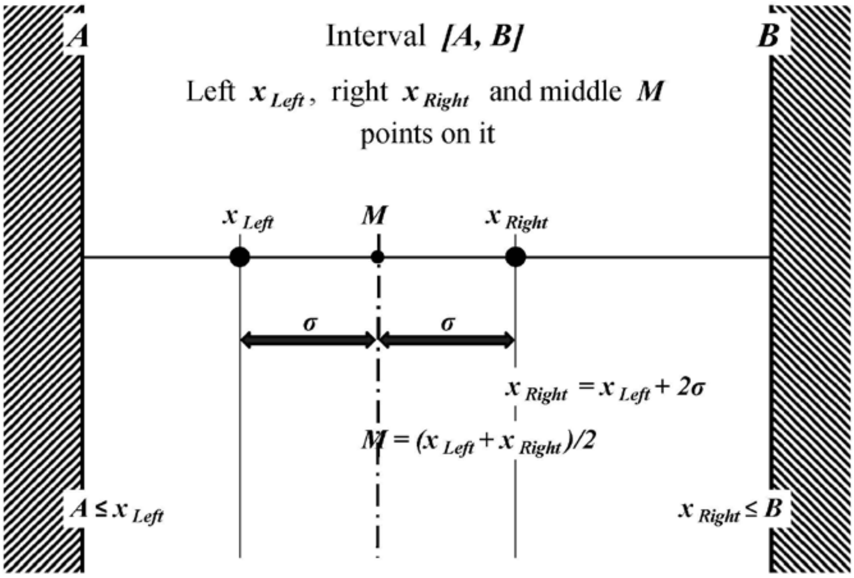

Figure 1. An interval $[A, B]$. Left x_{Left}, right x_{Right} and middle, mean M points on it

Let us suppose that $x_{Right}-x_{Left} \geq 2\sigma = 2Const_{\sigma} > 0$. So, of course, $x_{Right} \geq x_{Left}+2\sigma$ and $x_{Left} \leq x_{Right}-2\sigma$. For the sake of simplicity, Figures 1-3 represent the case of the equality $x_{Right}-x_{Left}=2\sigma$ and also, of course, $x_{Right}=x_{Left}+2\sigma$ and $x_{Left}=x_{Right}-2\sigma$ and $M-x_{Left}=x_{Right}-M=\sigma=Const_{\sigma}>0$.

So, $M=x_{Left}+\sigma>x_{Left}$ and $M=x_{Right}-\sigma<x_{Right}$.

Suppose further that $x_{Left} \geq A$ and $x_{Right} \leq B$.

One can easily see that two types of zones for M can exist in the interval:

1) The mean point M can be located only in the zone which will be referred to as "allowed" (see Figure 2).

2) The mean point M cannot be located in the zones which will be referred to as "forbidden" (see Figure 3).

2.1.2. Allowed zone

Due to the conditions of the example, the left point x_{Left} may not be located further left than the left border of the interval $x_{Left} \geq A$ and the right point x_{Right} may not be located further right than the right border of the interval $x_{Right} \leq B$.

For M, we have $M = x_{Left} + \sigma \geq A + \sigma > A$ and $M = x_{Right} - \sigma \leq B - \sigma < B$ (see Figure 2).

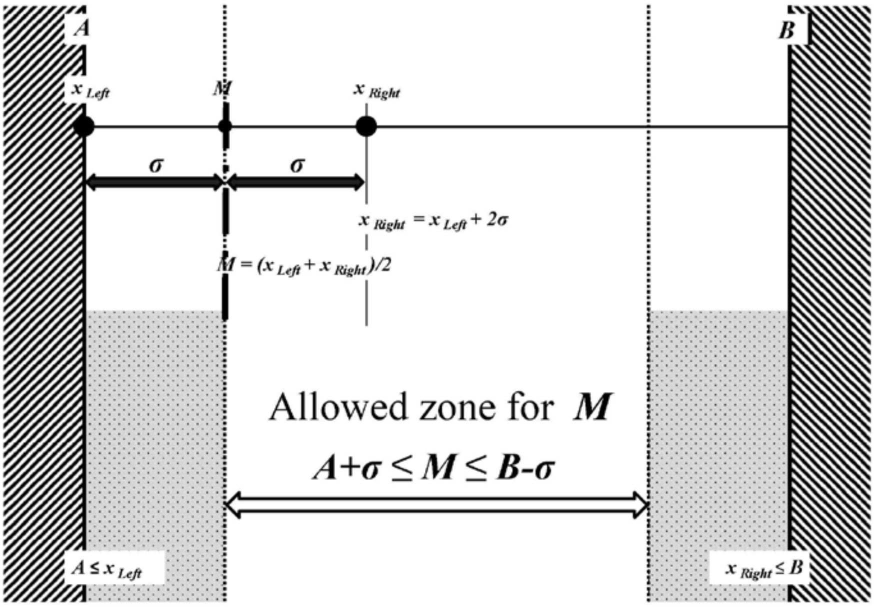

Figure 2. The allowed zone for M

The width of the allowed zone for M is equal to

$$B - \sigma - (A + \sigma) = (B - A) - 2\sigma \ .$$

It is less than the width $(B\text{-}A)$ of the total interval $[A, B]$ by 2σ. Also, the allowed zone is a proper subset of the total interval.

If the distance 2σ between the left x_{Left} and right x_{Right} points is non-zero, then the difference between the width of the allowed zone and the width of the interval is non-zero also. If this distance is greater than 2σ, then the difference is greater than 2σ also.

So, the mean point M can be located only in the allowed zone of the interval.

2.1.3. Restrictions, forbidden zones

Let us define the term “restriction” for the purposes of this paper:

Definition. A **restriction** (more exactly, a **restriction on the mean**) signifies the impossibility for the mean to be located closer to a border of the interval than some fixed distance. In other words, a restriction implies here a forbidden zone for the mean near a border of the interval.

The value of a restriction or the width of a forbidden zone signifies the minimal possible distance between the mean and a border of the interval. For brevity, the term “the value of a restriction” may be shortened to “restriction”.

If $A \le x_{Left}$, $x_{Right} \le B$ and $x_{Right}-x_{Left}=2\sigma$, then restrictions, forbidden zones with the width of one sigma σ exist between the mean point and the borders of the interval (see Figure 3). So there are two forbidden zones, located near the borders of the interval. The mean point M cannot be located in these forbidden zones.

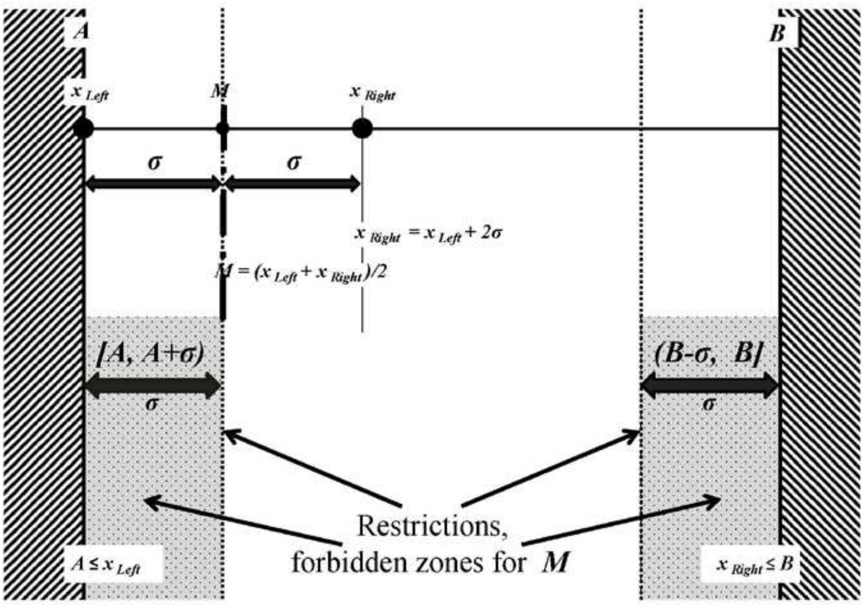

Figure 3. The forbidden zones, restrictions on M

The restrictions, the forbidden zones are shown by two dotted lines and by painting in the bottom part of Figure 3.

As one can easily see, restrictions on the mean (or forbidden zones) exist between the allowed zone of the mean M and the borders A and B of the interval $[A; B]$. The value of the restriction, or, equivalently, the width of the forbidden zone, is equal to σ.

So, the restrictions of the value σ on the mean point M exist near the borders of the interval in the presence of a non-zero dispersion.

2.2. An illustrative example of restrictions on the probability

Let us consider an illustrative example of restrictions on the probability.

2.2.1. A classical round target

Consider a classical example: an aiming firing at a target. Suppose a classical round target (Figure 4) of the diameter *2L*.

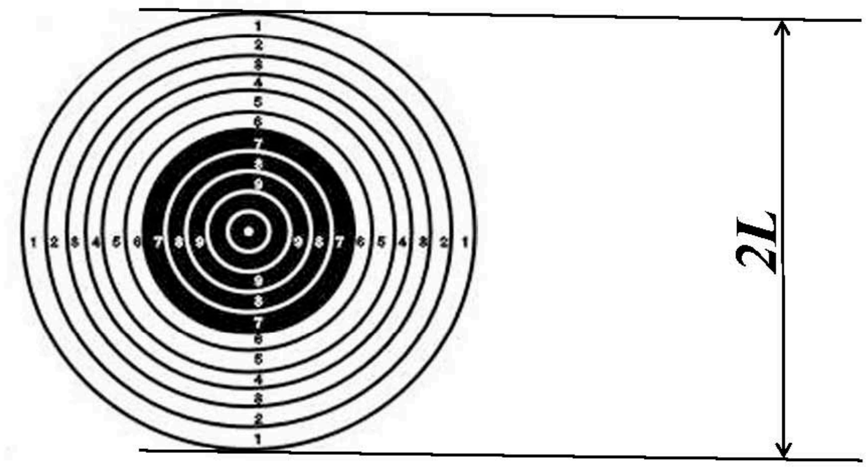

Figure 4. A target for firing

Suppose Mr. Somebody performs an aiming firing by batches of pellets or small shots at a target.

2.2.2. Two types of dispersion

For the obviousness suppose (Figure 5) the dispersion of pellets hits is uniformly distributed in a zone of the diameter 2σ (See an example of the normal distribution below).

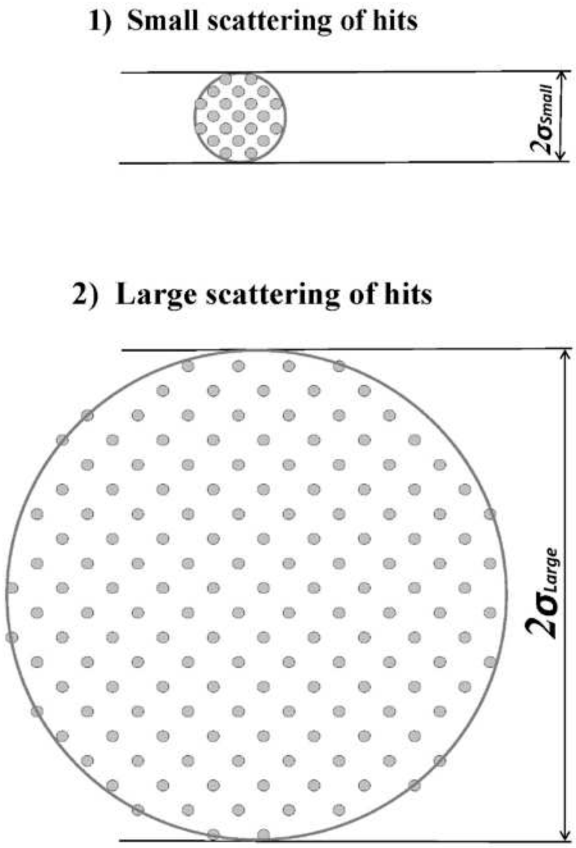

Figure 5. Dispersion of hits is uniformly distributed in a zone of the diameter 2σ

Notes about this figure:
Note 1: This is only a simplified example (See an example of the normal distribution below).
Note 2: The case 1) represents the case of small diameter $2\sigma_{Small}$ of the zone of dispersion of pellets hits.
The case 2) represents the case of large diameter $2\sigma_{Large}$ of the zone of dispersion of pellets hits.

Suppose the point of aiming may be varied between the center of the target and a point which is outside the target.

2.2.3. Small dispersion

The case, when the diameter $2\sigma_{Small}$ of the zone of dispersion of hits is considerably less than the diameter $2L$ of the target, is drawn on the figure 6.

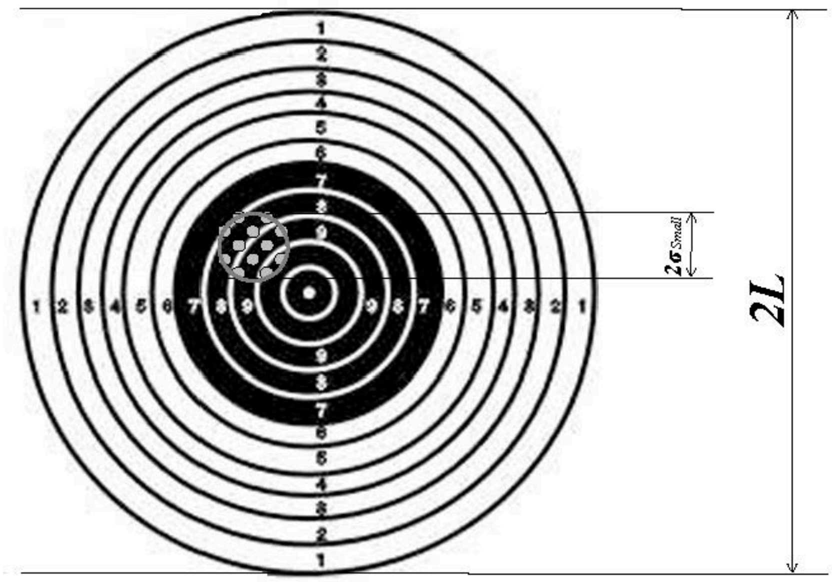

Figure 6. Firing for the small dispersion of hits

Note: The diameter $2\sigma_{Small}$ of the zone of dispersion of hits is considerably less than the diameter $2L$ of a target.

At the condition of the small dispersion of hits, the maximum possible probability of hit in the target can be equal to *1* (can reach the boundary of the probability scale).

When the point of aiming is varied between the center of the target and a point which is outside the target, the probability of hit in the target is varied from *1* to *0*. There are no restrictions in the probability scale.

2.2.4. Large dispersion

The case, when the diameter $2\sigma_{Large}$ of the zone of dispersion of hits is considerably more than the diameter $2L$ of the target, is drawn on the figure 7.

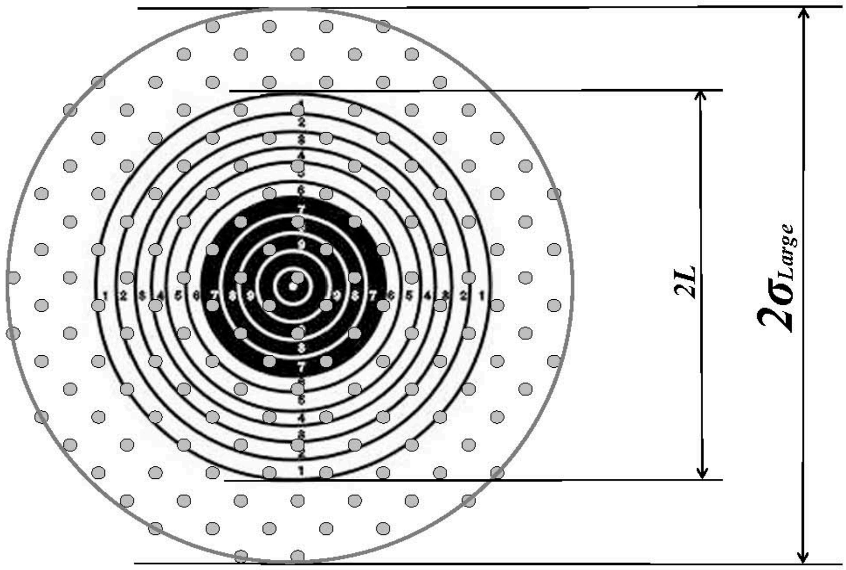

Figure 7. Firing for the large dispersion of hits

Note: The diameter $2\sigma_{Large}$ of the zone of dispersion of hits is considerably more than the diameter $2L$ of the target.

2.2.5. Restriction on the probability

At the condition of the large dispersion of hits (exactly speaking at the condition the diameter $2\sigma_{Large}$ of the zone of dispersion of hits is more than the diameter $2L$ of a target), the maximum possible probability of hit in the target cannot be equal to *1*.

So, the situation for the probability for this case is drawn on the figure 8.

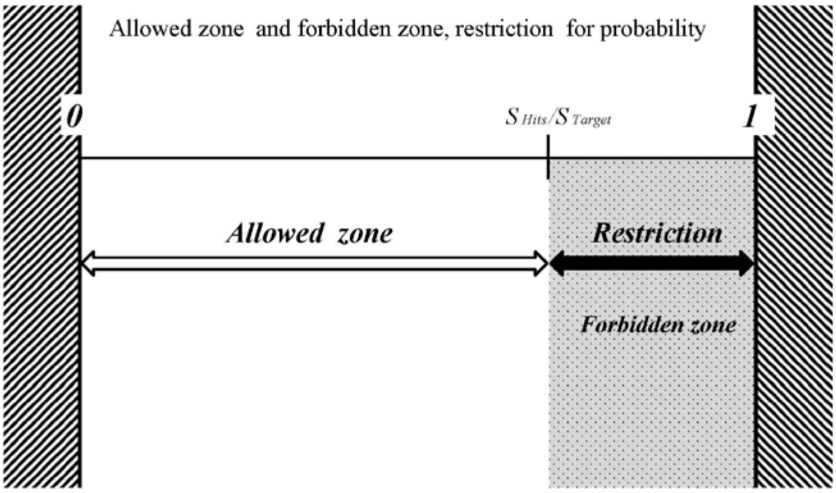

Figure 8. Restriction on the probability: Allowed zone and forbidden zone

Note: See the example of two restrictions for two boundaries below.

The value $P_{AllowedMax}$ of the maximal allowed probability of the allowed zone $[0,\ P_{AllowedMax}]$ may be estimated as the ratio of the mean number of the hits in the target to the total number of the hits. In this particular case, when the distribution of hits is supposed to be uniform, this ratio equals to the ratio of the area of hits scattering to the area of the target

$$P_{AllowedMax} = S_{Target} / S_{HitsLarge} = \pi L^2 / \pi\sigma^2{}_{Large} = L^2 / \sigma^2{}_{Large} \ .$$

If $L<\sigma_{Large}$, then $P_{AllowedMax}<1$. In this particular case, the probabilities of hit in the target, that are larger than $P_{AllowedMax}$, are impossible. The allowed probabilities of hit in the target belong to the allowed zone $[0,\ P_{AllowedMax}]$. The value of the restriction $R_{Restriction}$ may be estimated as the difference between unit and the maximal allowed probability $P_{AllowedMax}$ of hit in the target

$$R_{Restriction} = 1 - P_{AllowedMax} > 0 \ ,$$

and, if $L<\sigma_{Large}$, then $R_{Restriction}$ is a positive nonzero quantity. At the conditions of the figure 6, it is evident the probability $P_{AllowedMax}$ can not be more, then *0.5-0.7* (*50%-70%*) and the restriction $R_{Restriction}$ is as more as *0.3-0.5* (*30%-50%*).

2.3. An example of the normal distribution and of two restrictions for two boundaries

Let us consider concisely an example of the normal distribution and of two restrictions for two boundaries.

Conditions

Let us consider firing at a target in the one-dimensional approach. Let the dimension of the target be equal to $2L>0$ and the scatter of hits, when aim is precise, obeys the normal law with the dispersion σ^2. Then (see, e.g., Abramowitz and Stegun, 1972) the maximal probability P_{in_Max} of hitting the target and the minimal probability $P_{out\_min}=1-P_{in\_Max}$ of missing it are equal to:

Results

For $\sigma=0$:

$P_{in\_Max}=1$ and $P_{out\_min}=0$.

That is, there are no restrictions in the probability scale for hits and misses, that is $r_{expect}=1-P_{in\_Max}=P_{out\_min}=0$.

For $L=3\sigma$:

$0\le P_{in}\le P_{in\_Max}=0{,}997<1$ and $0<0{,}003=P_{out\_min}\le P_{out}\le 1$.

For this case, the restrictions r_{expect} in the probability scale for hits and misses are equal to $r_{expect}=0{,}003>0$.

For $L=2\sigma$:

$0\le P_{in}\le P_{in\_Max}=0{,}95<1$ and $0<0{,}05=P_{out\_min}\le P_{out}\le 1$.

For this case, the restrictions r_{expect} in the probability scale for hits and misses are equal to $r_{expect}=0{,}05>0$.

For $L=\sigma$:

$0\le P_{in}\le P_{in\_Max}=0{,}68<1$ and $0<0{,}32=P_{out\_min}\le P_{out}\le 1$.

For this case, the restrictions r_{expect} in the probability scale for hits and misses are equal to $r_{expect}=0{,}32>0$.

Deductions

For zero $\sigma=0$ there are no restrictions ($r_{expect}=0$).

For non-zero $\sigma>0$: The non-zero restriction $r_{expect}>0$ appears between the zone of possible values of the probability of hitting $0\le P_{in}\le P_{in\_Max}=1-r_{expect}<1$ and 1. The same non-zero restriction $r_{expect}>0$ appears between the zone of possible values of the probability of missing $0<r_{expect}=P_{out\_min}\le P_{out}\le 1$ and 0.

3. Theorems

3.1. A short review of an existence theorem for restrictions on the mean

Let us consider briefly (see, e.g., Harin, 2012b) existence theorems, from restrictions on the mean to restrictions on the probability.

3.1.1. Preliminary notes

Definition 3.1.1. Let us suppose given:

a) an interval $X=[A, B]$ satisfying
$$0 < Const_{Min.AB} \le (B-A) \le Const_{Max.AB} \ ,$$
b) a set of points $\{x_k\} : A \le x_k \le B,\ k=1, 2, \ldots K : 2 \le K \le \infty$,
c) a function f_K (a set of values $\{f_K(x_k)\}$), defined on $\{x_k\}$, satisfying
$$0 \le f_K(x_k)$$
and
$$\sum_{k=1}^{K} f_K(x_k) = W_K \ ,$$
where W_K (the total weight of f_K) is a constant satisfying
$$0 < W_K \ .$$

Without loss of generality, f_K may be and is normalized so that $W_K=1$. Under this condition, this function is referred to as a unitary function.

Let us define an analog of a moment.

Definition 3.1.2. An analog of the moment of n^{th} order of the function f_K relative to a point x_0 is the expression
$$E(X-X_0)^n \equiv \frac{1}{W_K}\sum_{k=1}^{K}(x_k-x_0)^n f_K(x_k) = \sum_{k=1}^{K}(x_k-x_0)^n f_K(x_k) \ .$$
From now on, for brevity, I refer to this analog of the moment of n^{th} order as simply the moment of n^{th} order.

3.1.2. Maximality

One may prove (see, e.g., Harin, 2015), that a function, which attains the maximal possible central moment, is concentrated at the borders of the interval. At that, the moduli of the central moments of such a function are not greater than the estimate
$$Max(|E(X-M)^n|) \le (M-A)^n \frac{B-M}{B-A} + (B-M)^n \frac{M-A}{B-A} \qquad (3.1).$$

3.1.3. Lemma about the tendency to zero for central moments

Lemma. If, for the nonnegative function f_K defined in Sub-subsection 3.1.1, $M \equiv E(X)$ tends to A or to B, then, for $n : 2 \le n < \infty$, $E(X-M)^n$ tends to zero.

Proof 1. For $M \to A$, the estimate (3.1) gives

$$|E(X-M)^n| \le (M-A)^n \frac{B-M}{B-A} + (B-M)^n \frac{M-A}{B-A} <$$
$$< [(B-A)^{n-1} + (B-A)^{n-1}] \frac{(M-A)(B-M)}{B-A} \le$$
$$\le 2(B-A)^{n-1}(M-A) \xrightarrow[M \to A]{} 0 \ .$$

This rough estimate is already sufficient for the purpose of this paper. But a more precise estimate may be obtained.

Proof 2. Let us transform

$$[(M-A)^{n-1} + (B-M)^{n-1}] \frac{(M-A)(B-M)}{B-A} =$$
$$= \left[\left(\frac{M-A}{B-A} \right)^{n-1} + \left(\frac{B-M}{B-A} \right)^{n-1} \right] (B-A)^{n-1} \frac{(M-A)(B-M)}{B-A} \ .$$

Let us consider the terms $(M-A)/(B-A)$ and $(B-M)/(B-A)$. Keeping in mind that $A \le M \le B$ we obtain $0 \le (M-A)/(B-A) \le 1$ and $0 \le (B-M)/(B-A) \le 1$. For $n \ge 2$ we have

$$\left(\frac{M-A}{B-A} \right)^{n-1} + \left(\frac{B-M}{B-A} \right)^{n-1} \le$$
$$\le \frac{M-A}{B-A} + \frac{B-M}{B-A} = \frac{B-A}{B-A} \equiv 1 \ .$$

So,

$$\left[\left(\frac{M-A}{B-A} \right)^{n-1} + \left(\frac{B-M}{B-A} \right)^{n-1} \right] (B-A)^{n-1} \frac{(M-A)(B-M)}{B-A} \le$$
$$\le (B-A)^{n-1} \frac{(M-A)(B-M)}{B-A} \le (B-A)^{n-1}(M-A) \ .$$

So,

$$|E(X-M)^n| \le (B-A)^{n-1}(M-A) \xrightarrow[M \to A]{} 0 \ .$$

For $M \to B$, the proof is similar and gives

$$|E(X-M)^n| \le (B-A)^{n-1}(B-M) \xrightarrow[M \to B]{} 0 \ .$$

So, if $(B-A)$ and n are finite and $M \to A$ or $M \to B$, then $E(X-M)^n \to 0$.
The lemma has been proved.

3.1.4. Existence theorem for restrictions on the mean

Definition. A "**non-zero restriction on the mean** r_{Mean}" (or, simply, a "non-zero restriction") signifies the impossibility for the mean to be located closer to a border of the interval than some non-zero distance.

In other words, a non-zero restriction designates the existence of a non-zero distance from a border of the interval. Within this distance, it is impossible for the mean to be located.

This restriction may be denoted also as a "forbidden zone" for the mean near a border of the interval.

The "restriction" for one border and the "restriction" for another border constitute the "restrictions" for the borders.

The value of a non-zero restriction (or the width of a non-zero "forbidden zone") signifies the minimal possible distance between the mean and a border of the interval. For brevity, the term "the value of a restriction" may be shortened to "the restriction."

Definition. At the beginning, let us define a "non-zero restriction on the dispersion $r^2_{Dispersion.2} \equiv r^2_{Disp.2} = \sigma^2_{Min}$" to be the minimal value of the analog of the dispersion $E(X-M)^2$ satisfying $E(X-M)^n \geq r^2_{Disp.2} > 0$.

Let us define analogously a general "**non-zero restriction on the n^{th} order central moment $|r^n_{Disp.n}|$**" to be the minimal absolute value of the analog of the n^{th} order central moment $E(X-M)^n$ satisfying $|E(X-M)^n| \geq |r^n_{Disp.n}| > 0$.

Theorem. If, for a nonnegative function f_K as in Sub-subsection 3.1.1, such that its mean $M \equiv E(X)$ and its analog of the n^{th} : $2 \leq n < \infty$, order central moment $E(X-M)^n$ exist, there exists a non-zero restriction on this analog of the n^{th} order central moment $|r^n_{Disp.n}| = Const_{Disp.n} > 0 : |E(X-M)^n| \geq |r^n_{Disp.n}|$, then a non-zero restriction on the mean $r_{Mean} = Const_{Mean} > 0$ exists and

$$A < (A + r_{Mean}) \leq M \equiv E(X) \leq (B - r_{Mean}) < B \ .$$

Proof. From the conditions of the theorem and from Lemma (3.1.3), for $M \to A$, we have

$$0 < | r^n{}_{Disp.n} | \leq | E(X-M)^n | \leq (B-A)^{n-1}(M-A) \ .$$

and

$$0 < \frac{| r^n{}_{Disp.n} |}{(B-A)^{n-1}} \leq (M-A) \ .$$

So,

$$(M-A) \geq r_{Mean} \equiv \frac{| r^n{}_{Disp.n} |}{(B-A)^{n-1}} > 0 \ .$$

For $M \to B$, the proof is similar and gives

$$(B-M) \geq r_{Mean} \equiv \frac{|r^n{}_{Disp.n}|}{(B-A)^{n-1}} > 0 \ .$$

These results may be rewritten as

$$A < \left(A + \frac{|r^n{}_{Disp.n}|}{(B-A)^{n-1}} \right) \leq M \leq \left(B - \frac{|r^n{}_{Disp.n}|}{(B-A)^{n-1}} \right) < B \ .$$

So, as long as $(B\text{-}A)$ and n are finite and $|r^n{}_{Disp.n}|=Const_{Disp.n}>0$, then $r_{Mean}=Const_{Mean}>0$ and $A<(A+r_{Mean})\leq M\leq(B\text{-}r_{Mean})<B$, which proves the theorem.

Remark 1

For $n=2$ the analog of the central moment is the analog of the dispersion, and r_{Mean} at A may be rewritten for the minimum σ_{Min} of the analog of the standard deviation σ, i.e., $\sigma\geq\sigma_{Min}\equiv r_{Disp.2}>0$, as

$$(M-A) \geq r_{Mean} \equiv \frac{r^2{}_{Disp.2}}{(B-A)} \equiv \frac{\sigma^2{}_{Min}}{(B-A)} > 0 \ .$$

The value of the restriction r_{Mean} at B may be also rewritten for the minimum σ_{Min} of the analog of the standard deviation σ as

$$(B-M) \geq r_{Mean} \equiv \frac{r^2{}_{Disp.2}}{(B-A)} \equiv \frac{\sigma^2{}_{Min}}{(B-A)} > 0 \ .$$

Remark 2

The estimates of the theorem are rather reliable ones, especially for $\frac{\sigma_{Min}}{B-A} \to 0$. They are, in a sense, as reliable as the Chebyshev inequality. Preliminary calculations (see, e.g., Harin, 2009) which were performed for real cases such as the normal, uniform and exponential distributions with the minimal values $\sigma^2{}_{Min}$ of the analog of the dispersion, gave much stronger restrictions r_{Mean} on the mean of the function (for $\frac{\sigma_{Min}}{B-A} \to 0$, for the unitary interval $[A, B] : (B\text{-}A)=1$) which are not worse than

$$r_{Mean} \geq \frac{\sigma_{Min}}{3} \ .$$

So, the inequalities $A<(A+r_{Mean})\leq M\leq(B\text{-}r_{Mean})<B$ for these cases may be rewritten as

$$A < \left(A + \frac{\sigma_{Min}}{3} \right) \leq M \leq \left(B - \frac{\sigma_{Min}}{3} \right) < B \ .$$

3.2. A short review of an existence theorem for restrictions on the probability

3.2.1. Lemma for the probability estimation

Definition. For a series of tests of number K, including $K\to\infty$, let the density f of a probability estimation, frequency F : $F\equiv M\equiv E(X)$, have the characteristics of f_K; in particular f is defined for $[0, 1]$ and $Const_f=1$.

Lemma. If f is defined as in Sub-subsection 3.1.1, and either $E[X]\to 0$ or $E[X]\to 1$, then, for $1<n<\infty$, we have $|E(X-M)^n|\to 0$.

Proof. As long as the conditions of this lemma satisfy the conditions of the lemma 3.1.3, then the statement of this lemma is as true as the statement of the lemma 3.1.3.

3.2.2. Theorem for the probability estimation

Theorem. If a probability estimation, frequency F_K, and $\{x_k\}$ are defined as in subsection 3.1.1, such that $M\equiv E[X]\equiv F_K$, there are n : $1<n<\infty$, and $r_{Dispers}>0$: $E[(X-M)^n]\geq r_{Dispers}>0$, then, for the probability estimation, frequency $F_K\equiv M\equiv E[X]$, a restriction r_{mean} exists such as $0<r_{Mean}\leq F_K\leq(1-r_{Mean})<1$.

Proof. As long as the conditions of this theorem satisfy the conditions of the theorem 3.1.4, then the statement of this theorem is as true as the statement of the theorem 3.1.4.

3.2.3. Theorem for the probability

Theorem. If, for the probability scale [$0;\ 1$], a probability P and the probability estimation, frequency F_K, for a series of tests of number K : $K>>1$, are determined such that when the number K of tests tends to infinity, the frequency F_K tends at that to the probability P, that is

$$P = \lim_{K\to\infty} F_K \ ,$$

non-zero restrictions r_{mean} : $0<r_{Mean}\leq F_K\leq(1-r_{Mean})<1$ exist between the zone of the possible values of the frequency and every boundary of the probability scale, then the same non-zero restrictions r_{Mean} exist between the zone of the possible values of the probability P and every boundary of the probability scale.

Proof. Consider the left boundary 0 of the probability scale [$0;\ 1$]. The frequency F_K is not less than r_{Mean}:

$$F_K \geq r_{mean} \ .$$

Hence, we obtain for P:

$$P = \lim_{K\to\infty} F_K \geq \lim_{K\to\infty} r_{mean} = r_{mean} \ .$$

For the right boundary 1 of the probability scale the proof is similar to that above. So, the theorem has been proved. For $r_{Mean}=\sigma^2\geq\sigma^2_{Min}>0$, one can write

$$0<\sigma^2_{Min}\leq p\leq\left(1-\sigma^2_{Min}\right)<1 \ .$$

or, at $\sigma^2_{Min}\to 0$, taking into account Remark 2,

$$0<\frac{\sigma_{Min}}{3}\leq p\leq\left(1-\frac{\sigma_{Min}}{3}\right)<1 \ .$$

4. A “certain–uncertain” inconsistency of the random–lottery incentive system

A natural question is bound to arise: Why was not this discontinuity discovered in numerous experiments?

This question is answered by Harin (2014). Here is a very brief review of this answer:

The random incentive procedure is usually referred to as the random–lottery incentive system (or the random lottery incentive system or random incentive system (RIS), etc.).

The random–lottery incentive system is the prevailing experimental procedure employed in the utility and prospect theories (see, e.g., Starmer and Sugden, 1991, Starmer, 2000, Andreoni and Sprenger, 2012, Baltussen et al., 2012, etc.).

4.1. Random (uncertain) incentives

Let us analyze one usual feature of experiments in utility and prospect theories. Let us consider some typical descriptions of the utility experiments. One can see in the literature (the **boldface** is my own):

Loewenstein and Thaler (1989), page 188: “The students … were told that the experimenter would select and implement one of their choices **at random**.”

Baltussen et al. (2012), page 424: “In the WRIS treatment, subjects play the game ten times, one of which for real payment. In the BRIS treatment, subjects play the game only once with a **one-in-ten** chance of real payment.” Page 425: “In both RIS treatments, a **ten-sided die** was thrown individually by each subject to determine her payment.”

Other sources such as Kahneman, Knetsch and Thaler (1991), Vossler, Doyon and Rondeau (2012), etc. give similar descriptions.

Such a procedure can be seen not only in the utility and prospect theories but also in other fields of the economics, see, e.g., Larkin and Leider (2012), page 193: “Subjects made fifteen choices between a lottery and a fixed payment. … Subjects were paid for one **randomly** selected decision.”

So, subjects are stimulated by random incentives. This is a well-known feature of the experiments, including in the field of utility and prospect theories.

4.2. Uncertain incentives and certain outcomes

Let us consider this feature more closely. One can see a detail in the literature (the **boldface** and <u>**underlining**</u> is my own):

Starmer and Sugden (1991), page 974: “subjects in groups B and C knew that they were taking part in a **random**–lottery experiment in which questions 21 and 22 had equal chances of being for real.” and “One problem, which we shall call P', required a choice between two lotteries R' (for "riskier") and S' (for “<u>**safer**</u>”). R' gave a 0.2 chance of winning £10.00 and a 0.75 chance of winning £7.00 (with the residual 0.05 chance of winning nothing); S' gave £7.00 <u>**for sure**</u>.”

Andreoni and Sprenger (2012), page 3365: "One choice for each subject was selected for payment by drawing a numbered card **at random**. Subjects were told to treat each decision as if it were to determine their payments." and page 3366: "Section I provided a testable hypothesis for behavior across **certain** and uncertain intertemporal settings."

Other sources such as Holt and Laury (2002), Harrison et al. (2005), Abdellaoui et al. (2011), etc. give the same detail.

So, the random incentive procedures are used not only in the uncertain but in the certain situations too. Let us consider this detail more closely.

4.3. The "certain–uncertain" inconsistency

So, a well-known feature of the experiments, including in the field of utility and prospect theories, is that subjects are stimulated by random incentives. For the purposes of this article, let us call this process as the stimulation by random incentives.

First, let us note that the stimulation (incentive) by a random payment selected from two or more alternatives may be called a random, uncertain stimulation. One may refer to it also as a stimulation by an uncertain incentive.

Further, let us consider a stimulation by this uncertain incentive for uncertain and certain choices.

Suppose, that subjects choose an uncertain choice, that is, a choice whose probability is strictly less than 1 (and strictly more than 0). In this case, the choice and the incentive are of the same type.

Suppose, that the subjects choose a certain choice, that is, a choice whose probability is strictly equal to 1. In this case, the choice and the incentive are of the essentially different types. The choice is certain but the incentive is uncertain. Moreover, this uncertain incentive can call into question the certain outcome.

Therefore, there is an evident inconsistency between the certain type of the choice and the uncertain type of the incentive.

Therefore, the correctness of the use of uncertain incentives for certain outcomes cannot be unquestionable. One may call this problem the "certain–uncertain" inconsistency.

This inconsistency is evident but I have found no mention about it in the literature: see, e.g., Andreoni and Sprenger (2012); Vossler, Doyon and Rondeau (2012); Baltussen et al. (2012); and also the "New Economics Papers. Utility Models & Prospect Theory" at http://econpapers.repec.org/scripts/nep.pf?list=nepupt for the period 2005–2014. The inconsistency was revealed in the report Harin (2014).

5. Experimental evidence

Conditions

One can see the following in the description of the well-known experiment of Starmer and Sugden (1991):

Page 974: “For groups A and D, this page began with an underlined text stating that question 22 would be played for real. For groups B and C, the corresponding text stated that one of the two questions would be played for real and that which question was to played out would be decided at the end of the experiment in the following way. The subject would roll a six-sided die. If the number on the die was 1, 2, or 3, then question 21 would be played; if the number was 4, 5, or 6, question 22 would be played.”

“One problem, which we shall call P', required a choice between two lotteries R' (for "riskier") and S' (for "safer"). R' gave a 0.2 chance of winning £10.00 and a 0.75 chance of winning £7.00 (with the residual 0.05 chance of winning nothing); S' gave £7.00 for sure.”

Results

So, in the R'-S' problem, R' gives $£10.00*0.2+£7.00*0.75 = £7.25$. S' gives $£7.00*1 = £7.00$. Here $R' = £7.25>S' = £7.00$.

Let us consider the results from table 2 on Page 976, those are of interest here (the **boldface** is my own):

- Group = B, Incentive = **Random lottery**, R':S' = **19:21**
- Group = C, Incentive = **Random lottery**, R':S' = **22:18**
- Group = D, Incentive = **P' real**, R':S' = **13:27**

So, the results for **P' real** incentive (**13:27**) differ evidently and essentially from the results for random lottery incentive (19:21 and 22:18).

Let us evaluate the percentage of the subjects choosing the uncertain outcome and the direction of the modification of $W(p)$. The total number of the subjects in each group is equal to $40=19+21=22+18=13+27$. So, the percentage is equal to $19/40=48\%$, $22/40=55\%$ and $13/40=33\%$. One may see that the modification of $W(p)$ by the random lottery incentives is directed from $13/40=33\%$ to $19/40=48\%$ and $22/40=55\%$. That is it is directed from 0 to 1.

Deductions

One can easily see that the experiment shows that the random lottery incentives can essentially modify subjects’ choices in comparison with the real incentives, when these choices include certain outcomes and the probability ($0.2 + 0.75 = 0.95 \sim 1$) of the uncertain choices is near the border of the probability scale.

The modification of $W(p)$ by the random lottery incentives is directed from 0 to 1. Therefore, the real unbiased probability weighting function $W(p)$ is located farther from 1 and nearer to 0 (at $p \sim 1$) than the function biased by the random lottery incentives.

6. A discontinuity of Prelec's function at the probability $p = 1$

So, this working paper allows to formulate:

1) The modified "Luce question" states that a possibility of the existence of a discontinuity of Prelec's function at the probability $p = 1$ should be considered.

2) The purely mathematical theorems prove that this discontinuity must take place in the presence of a non-zero dispersion of data.

3) The "certain–uncertain" inconsistency of the prevailing experimental procedure explains why this discontinuity has not still been detected.

4) The well-known experiment of Starmer and Sugden (1991) shows that this discontinuity can take place and that the random lottery incentives can hide it.

7. Possible consequences of the discontinuity

A discontinuity is not a quantitative but a qualitative, moreover, a topological feature. Therefore, the possible discontinuity of Prelec's function can qualitatively change prospect theories, at least in their mathematical aspects.

It may be supposed that such basic and useful tools as the random incentive systems, the overwhelming majority of the data already obtained by means of them, and the deductions from the data may and should continue to be used.

Apparently, the farther from $p = 1$ the less relevant is a possible discontinuity at $p = 1$ and the smaller can be corrections of the data and deductions. Note, that the experiments (see, e.g., Cubitt, Starmer and Sugden, 1998; Beattie and Loomes, 1997) at the probabilities that are less than 0.9 are not so sensitive to the "certain–uncertain" inconsistency as that of Starmer and Sugden (1991).

The following may be supposed:

In the narrow middle of the probability scale (where the probability weighting function intercepts the line $W(p) = p$) and in the obvious cases, the data and deductions may be used "as is".

In the wide middle of the probability scale, the data and deductions may be the same or slightly corrected. This may be true when the probability p is located sufficiently far from $p = 1\text{-}r_{mean}$ (see, e.g., Subsections 3.1 and 3.2 and Harin, 2012b).

When the probability tends to the restriction $p \rightarrow 1\text{-}r_{mean}$, the data should be used with non-linear corrections and the deductions should be recalculated by non-linear functions.

At the probabilities that are in the forbidden zone $1\text{-}r_{mean} \le p \le 1$, a new approach may be needed to make the deductions correct.

Conclusions

So, this working paper reviews, summarizes and generalizes my elder works.

It states that a discontinuity of Prelec's function can take place at the probability $p = 1$.

In particular, this working paper formulates:

1) The modified "Luce question" states that a possibility of the existence of a discontinuity of Prelec's function at the probability $p = 1$ should be considered.

2) The purely mathematical theorems prove that this discontinuity must take place in the presence of a non-zero dispersion of data.

3) The "certain–uncertain" inconsistency of the prevailing experimental procedure explains why this discontinuity has not still been detected.

4) The well-known experiment of Starmer and Sugden (1991) shows that this discontinuity can take place and that the random lottery incentives can hide it.

A discontinuity is not a quantitative but a qualitative, moreover, a topological feature. Therefore, the possible discontinuity of Prelec's function can qualitatively change prospect theories, at least in their mathematical aspects.

So, one can conclude:

There is a need of investigations of the "Luce problem" of a special analysis of Prelec's function at $p = 1$ and $p \approx 1$.

There is a need of investigations of the "Luce question" whether Prelec's weighting function is actually equal to 1 at $p = 1$?

There is a need of an independent analysis of the "certain–uncertain" inconsistency of the random–lottery incentive experiments.

There is a need of an independent investigations of the modified "Luce question" whether Prelec's weighting function has a discontinuity at $p=1$.

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