Risky Swaps.

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Abstract.

In [10] we presented a reduced form of risky bond pricing. At the default date a bond seller fail to continue fulfill his obligation and the price of the bond sharply drops down. If the face value of the defaulted bond for no-default scenarios is $1 then the bond price just after default is called its recovery rate (RR). Rating agencies and theoretical models are trying to predict RR for companies or sovereign countries. The main theoretical problem with a risky bond or with general debt problems is presenting the price given the RR.

The problem of a credit default swap (CDS) pricing is somewhat an adjacent problem. Recall that the corporate bond price is inversely depends on interest rate. The credit risk on a debt investment is related to the loss if default occurs. There exist a possibility for a risky bond buyer to reduce his credit risk. This can be achieved by buying a protection from a protection seller. The bondholder would pay a fixed premium up to maturity or default, which one comes first. In exchange if default comes before maturity the protection buyer will receive the difference between the initially set face value of the bond and RR. This difference is called 'loss given default'. This contract represents CDS. The counterparty that pays a fixed premium is called CDS buyer or protection buyer and the opposite party is the CDS seller. Note that in contrast to corporate bond CDS contract does not assume that buyer of the CDS is a holder of the underlying bond. Note that underlying to the swap can be any asset. It is called the reference asset or reference entity. Thus CDS is a credit instrument that separates credit risk from corresponding underlying entity.

Thus the formal type of the CDS can be described as follows. The buyer of the credit swap pays fixed rate or coupon until maturity or default if it occurs sooner than maturity. In case of default protection buyer delivers cash or default asset in exchange of the face value of the defaulted debt. These are known as cash or physical settlements correspondingly.

Introduction.

The option valuation benchmark was developed by Black, Scholes, and Merton in 70s [2, 14]. It uses the present value neutralized reduction of the underlying security for the instrument pricing. It was highlighted in [5-8] that the underlying logic of the benchmark approach in many respects contradicts the common sense of the pricing definition. Indeed either the Black Scholes option pricing equation or developed later the binomial scheme does not depend on a real return of the underlying security. Therefore these approaches suggest the same price for the option written on securities having equal risk characteristic (volatility) and different expected rates of return regardless whether it positive or negative. Thus the benchmark price is the same for the options that promised positive payoff at maturity with probability as closed to 1 as we wish and with probability 0 as closed as we wish [5-7].

Recall that this incorrect pricing is constructed on the base of the idea known as 'self-financing' or 'no-arbitrage'. Briefly the self-financing pricing scheme can be outlined as follows. The price of a financial instrument can be found using standard transactions. First borrow funds from the bank at the risk free interest rate. That is going short in bonds and then investing these funds in the instrument. The rule of pricing the instrument is that: the price of the new instrument should have a total balance equal to 0 at any time in the future. Realization of such idea in stochastic setting can be achieved by putting the portfolio changes over the time equal to 0. This strategy represents 'no arbitrage'.

This approach has explicit drawbacks. If the instrument is risk free then the self-financing strategy makes sense. In a stochastic setting self-financing scenario automatically implies deterministic interpretation of the market prices. Indeed borrowing and investing funds in a risky market contain a risk to lose investment. In another words the risk of investing in stochastic market could be described as getting a return bellow than it was initially planned.

In theory one can assume that underlying security distribution and its parameters are known. Fixing the distribution might lead to an opportunity to define a generalization of the classical arbitrage. The stochastic arbitrage is one when an investor could hope to receive a statistical advantage. Nevertheless in finance it is difficult to exercise statistical arbitrage.

Indeed theoretical distributions used for models in finance are implied and therefore statistical arbitrage for implied distributions is also implied. From mathematical point of view arbitrage is a necessary condition of a correct pricing. That is if pricing is correct then arbitrage could not exist but there is no arbitrage say between stochastic stock price and deterministic bond price. Now let us formally express our remarks.

European call option price is a solution of the Black Scholes equation

$$\frac{\partial c}{\partial t} + r x \frac{\partial c}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 c}{\partial x^2} - r c = 0 , \qquad t \in [\,0 , T\,) , \ x > 0 \qquad (BSE)$$

$$c\,(T , x) = \max\,(\,x - K , 0\,)$$

Here risk free interest rate r, volatility coefficient σ, and strike price K are assumed to be given constants. The unique solution of the problem (BSE) admits probabilistic representation in the form

$$c(t,x) = \exp - r(T - t)\, E \max\,(S_r(T) - K , 0)$$

where the random process $S_r(v) = S_r(v; t, x),\ S_r(t) = x,\ t \le v \le T$ is a solution of the stochastic differential equation

$$d S_r(t) = r S_r(t)\, d t + \sigma\, S_r(t)\, d w(t)$$

on a complete probability space $\{ \Omega , F , P \}$ and $w (t)$ is a Wiener process on this space. Assume that option underlying security is governed by the equation

$$d S_{\mu} (t) = \mu S_{\mu} (t) d t + \sigma S_{\mu} (t) d w (t)$$

and $\mu \neq r$. The first remark to the Black Scholes pricing is that the solution of the problem (BSE) could not be called price. Indeed the notion of the price is commonly associated with the present value PV of the future cash flow. If the cash flow strictly excesses other then the spot price of the first cash flow should be greater than the spot price of the second. Otherwise there exists an arbitrage. This principle can be expressed formally as the **Equal Investment Principle** (EIP).

Two investment opportunities are called equal at the moment of time t if they promise equal instantaneous rate of return. Two investments are equal on $[t , T]$ if they are equal at any moment of time over this time interval.

Thus equal investments are generated by the equal cash flows and this implies equal spot prices. Assume that $0 < r < \mu$. Then as it follows from comparison theorem for one dimensional stochastic differential equations(SDE) $S_{r} (v ; t , x) < S_{\mu} (v ; t , x), \ t \leq v \leq T$ with probability 1 and therefore the cash flow generated by the real μ-security is greater than the cash flow generated by the r-security. Hence the present values of two cash flows are also not equal. Remarkably two investments could have equal present values but they could be not equal in according to EIP. This remark shows that the benchmark present value principle is incomplete. The EIP was used earlier in [7, 9, 10] for fixed income and option valuations.

Looking at the equation (BSE) one can probably note that underlying securities having equal risk characteristic σ and promising arbitrary expected return $(- \infty , + \infty)$ are priced by the same amount. As far as the future cash flows of two processes having opposite signs of the expected return satisfy inequality $S_{-\mu} (v ; t , x) < S_{\mu} (v ; t , x), \ t \leq v < + \infty$ then the present values of the correspondent cash flows over $[t , T]$ are different. That is

$$PV_{t}^{T} S_{-\mu} (\cdot ; t , x) < PV_{t}^{T} S_{\mu} (\cdot ; t , x)$$

That implies that the Black Scholes equation solution could not be called the price defined based on the present value principle. Thus though this function does not present arbitrage in usually accepted sense nevertheless this function also can not be associated with the price notion and therefore cannot be used as the option price.

There exist other aspect of the derivatives pricing we wish to highlight here. For the recent generation of the derivatives called credit derivatives the risk neutral setting is stems from the (BSE) interpretation. Note that some researchers are also used risk neutral setting for interest rate derivatives pricing. This problem relates to the probability theory and actually does not relate to finance. This is so-called risk neutral pricing. Let us briefly outline the essence of the problem. Usually researchers referred to as risk neutral world when they want to emphasize that in this world real μ-security would transform into neutralized r-security. Mathematical techniques behind this transformation are known as Girsanov theorem. Application this theorem to the random processes S_{μ} , S_{r} states that measures m_{μ} , m_{r} corresponding these processes are absolutely continuous to each other and in particular

$$\frac{d\, m_\mu}{d\, m_r}(\omega) = \exp\{\lambda\, [\, w(T) - w(t)\,] - \frac{1}{2}\lambda^2\,(T - t)\}$$

where

$$\lambda = \frac{\mu - r}{\sigma}$$

Denote measure Q on the same measurable space $\{\Omega, F\}$ with the help of the equality

$$Q(A) = \int_A \frac{d\, m_\mu}{d\, m_r}(\omega)\, P(d\omega)$$

Then the random process S_μ is the solution of the equation

$$d\,S(t) = r\,S(t)\,d\,t + \sigma\, S(t)\,d\,w_Q(t)$$

with the Wiener process $w_Q(t) = w(t) + \lambda\, t$ on the probability space $\{\Omega, F, Q\}$. One could note that the measure Q depends on parameter λ and therefore it depends on parameter μ. Moreover calculation of the expectation of a functional given on risk neutral world $\{\Omega, F, Q\}$ automatically converts it to the correspondent functional with respect to the measure P according to the formula

$$E_Q f(S_r(*; t, x)) = E f(S_\mu(*; t, x))$$

It is common to state that the solution of the problem (BSE) can be represented as expected value of the functional over the risk neutral process S_r on the risk neutral world $\{\Omega, F, Q\}$. Note that functional is a mathematical term that covers variety payoff classes used in derivative contracts. Taking into account the change of variable represented in above equality one can see that risk neutral world does not perform the transformation of the real word security price S_μ into neutralized price S_r on risk neutral world. That means that commonly stated the risk neutral setting fails to perform the task of the real world transformation of the parabolic equation with the real return to the parabolic equation having risk free coefficient at the first order derivatives with respect to price variable.

On the other hand it is easy to apply the process S_r determined on the original probability space in order to present the solution of the parabolic equation (BSE) in the form represented above.

For credit derivatives modeling it is not a mistake starting on original probability space to make Girsanov transformation and arrive at the risk neutral world. To follow this way one should assume a certain number N of security instruments that compose a security market. Note that this number N as well as a particular choice of these securities is a subjective heuristically notion. Choosing the market representation one could assume different stochastic dynamics on the risk neutral world. Nevertheless the pricing and accurate calculation of the risk characteristics connected to these notions would display their dependence on security market parameters. That in turn is irrelevant and incorrect. As well as it is difficult to reveal details of underlying calculations it looks like that primary software simply omitted Girsanov density and used so-called neutralized security that is of course mathematically incorrect though can present close quantitative or qualitative results if the risk free and real returns are closed to each other.

CDS

A credit default swap is an over-the-counter (OTC) bilateral financial instrument used for hedging default risk of a risky debt instrument or a basket of risky debt instruments.
Default risk is a risk when one or two counterparties fail to make a scheduled payment.
CDS one of the most popular class of credit derivatives that allows trading of the counterparty risk from one to another without changing ownership of an underlying instrument.
A CDS is one of the most liquid instrument types in which one party called protection buyer seeks for the credit protection on risky debt such as corporate bonds or loans. These are referred to as to reference entity. On the opposite side of the agreement is another party that agrees to pay in the event of default of the reference entity. This counterparty called a protection seller. The protection buyer pays to protection seller a fee in the form of periodic fixed payments usually paid semiannually or quarterly. At the event of default the fixed payments stop and the protection seller pays to the protection buyer the predetermined amount. There are two common ways of the CDS settlement. With the physical settlement the protection buyer delivers the defaulted reference security or its equivalent to the protection seller and in exchange receives the par value of the reference security. Other way is the cash settlement. In this case protection buyer receives the difference between par and recovery value of the security. This amount is called loss given default. Actually the physical settlement is more common way of delivery. The most popular notional value is 5 or 10 million dollars with 5 years until expiration. Most of the CDS contracts are single name having a single corporate bond underlying. A portfolio names swap is written on a basket of bonds. For such contracts most popular is the first-to-default swap. It terminates when the first credit event happens or at the contract expiration. Other type of the swap written on a basket of the N bonds is n-th to default, $n \leq N$. This is a CDS contract that pays buyer of protection the difference between its face value and recovery rate at the n-th default event among the reference pool. One probably noted that the introduced CDS contract is more likely to the insurance contract.

There is a relationship between risky bond evaluation and CDS pricing. In a risky coupon bond valuation the face value and coupon payments can be interpreted as given parameters of the exposure for CDS seller. The variable risky bond price dynamics then is subject to study. The CDS valuation is the problem of finding CDS spread that the constant rates. Thus the CDS valuation is somewhat adjacent counterpart to the risky bond pricing.

Let us now outline CDS theoretical framework. We consider two approaches to the CDS valuation. The first valuation approach is a standard and based on comparison present values of the two counterparties involved in CDS. The main distinction of our approach is that we use stochastic setting in contrast to commonly accepted methods dealing with expected values of the cash flows to counterparties. Then we also consider another approach that uses the option pricing. Note that option-pricing method used in [5,7] in any respect does not relate to the Black Scholes benchmark derivative pricing [2,14].

Assume first for simplicity that default of the corporate coupon bond occurs only at a specified series dates t_k, $k = 1, 2 \ldots N$. These dates can or cannot be the coupon payment dates. Let $\tau(\omega)$ denote a random time of default and D_k denote the default event at the date

t_k defined on a probability space $\{ \Omega , F , P \}$. Thus $D_k = \{ \omega : \tau (\omega) = t_k \}$, $k = 0, 1, ..., N$. Let T be a maturity of the bond. Note that it is also possible to assume that $t_N < T$ though for simplicity we put $t_N = T$. If a scenario $\omega \in D_k$ then a protection buyer pays a fixed periodic premium \$q at the dates $t_1 , \ldots , t_{k-1}$. We assume that the premium q does not be paid at the date of default. On the opposite side of the swap contract the protection seller would compensate the losses of the protection buyer at the date of default. This compensation value is an amount 'loss given default'. There are several reasonable possibilities to define the value of 'loss given default' (LGD). For instance it can be defined as

$$[F - \Delta_k] \chi\{ \tau (\omega) = t_k \}$$
$$[F - R_q (t_k - 0, T ; \omega)] \chi\{ \tau (\omega) = t_k \}$$
$$[F B(t_k, T) - R_q (t_k , T ; \omega)] \chi\{ \tau (\omega) = t_k \}$$

Here $R_q (t , T ; \omega)$ is the value of the risky coupon bearing bond at the date t and maturity T. The first equality admits that the value of the risky bond at the default event be equal to its recovery rate. Therefore LGD is defined as the promised face value minus recovery. In the second equation the LGD is the differential between bond's face value and its value just before default. The last value corresponds to the case when default debt in either form delivered to the protection seller in exchange to the risk free bond with the equal face value and maturities. Note that this case also covers the 'cheapest-to-delivery' delivery option as far as $B(t_k, T)$ could be issued at arbitrary moment in the past. Indeed let $B(t_k, T ; t)$ denote the risk free bond price at the date t_k issued at t with maturity T. Putting

$$B(t_k, T) = \min\{ t, t \le t_k : B(t_k, T ; t)\}$$

we note that the third equality can be interpreted as the 'cheapest-to-delivery' settlement.

The present value of the future payment is considered by definition as the spot value of the future payment. This reduction is usually presented in the form in which the future payment is multiplied by the zero coupon risk-free bond having maturity at the specified date in the future. Thus the discounted value is the seller's spot price. The series of the future payments received over a particular time period, as coupon payments will accumulate by the bond buyer at the maturity or at the default date, which one comes up earlier. Thus this other present value can be determined as a present value of the total balance at the date that is the minimum between maturity or default dates. This present value relates to the bond buyer spot price and is by construction a random variable. On the other hand the bond buyer can buy the bond and go short on the same amount equal to discounted coupon future payments. In this case that looks like somewhat unrealistic one need to use the benchmark present value for the bond buyer's valuation. We consider such case as unrealistic because in a two party deal the security seller is a party looking for financing while security buyer represents an investor. Therefore in case when the investor buying a security goes short getting cash is a mixed strategy combining investor's and financier's strategy. It might be reasonable way but it should be more accurately specified. In this paper we consider market deals parties behavior in their common sense. Security buyers are investors interested in getting maximum return on their investments while security sellers are looking for financing.

The CDS value by definition is the value of a periodic payment q for which the balance of the cash flows to counterparties is equal to 0. This value q can be also interpreted as a coupon or premium but it is commonly called also CDS spread. Hence the CDS spread is a fixed rate q during the lifetime of the CDS contract for which the cash flow to either counterparty is 0. The present value of the cash flow over the lifetime CDS to the protection seller is

$$\sum_{k=1}^{N} B(t, t_k)[q \sum_{j=0}^{k-1} B^{-1}(t_j, t_k) - (F - R(t_k, T; \omega))] \chi(\tau(\omega) = t_k) + \tag{1}$$
$$+ B(t, T)\, q [\sum_{k=1}^{N} B^{-1}(t_k, T)] \chi(\tau(\omega) > T) = 0$$

The solution of the equation (1) is a random variable $q = q_b(\omega)$ equal to

$$q_b = \frac{\sum_{k=1}^{N} B(t, t_k)[F - R(t_k, T; \omega)] \chi(\tau(\omega) = t_k)}{\sum_{k=1}^{N} [\chi(\tau(\omega) = t_k) B(t, t_k) \sum_{j=1}^{k-1} B^{-1}(t_j, t_k) + \chi(\tau(\omega) > T) B(t, T) B^{-1}(t_k, T)]} = \tag{2}$$
$$= \sum_{k=1}^{N} \frac{[F - R(t_k, T; \omega)]}{\sum_{j=1}^{k-1} B^{-1}(t_j, t_k)} \chi(\tau(\omega) = t_k)$$

The value q_b is the exact solution of the CDS pricing problem. Note that from (2) follows in particular that for a scenario ω associated with 0-default event the value of the premium is 0, as it should be expected. There is no need to buy protection for no default over the lifetime of the risky bond scenarios. Then for $\omega \in D_k = \{ \tau(\omega) = t_k \}$ there is only one term in denominator and numerator that is not equal to 0 and in this case

$$q_b(\omega)\, \chi(\tau(\omega) = t_k) = \frac{[F - R(t_k, T; \omega)]}{\sum_{j=1}^{k-1} B^{-1}(t_j, t_k)}$$

and $q_b(\omega) = 0$ for $\omega \in \{ \tau(\omega) > T \}$. Summing up these equalities we arrive at the equality (2).

Remark. The benchmark CDS valuation models [4, 11] reduce the CDS spread notion to the break-even value that makes the expected value of the cash flows to counterparties be equal. It is easy to check that the expected value of the exact solution of a linear algebraic equation with random coefficients does not coincide with the solution of the problem in which real random cash flows are replaced by the expectations of these cash flows. Besides that dealing with expectation of the cash flows one would loss the market risk exposure implied by the random coefficients. One can also note that protection buyer and protection seller have asymmetric exposure to the credit risk. Indeed for no default scenarios protection buyer paid periodic premium over the lifetime of the CDS whereas protection seller does not pay anything to the protection buyer. The present value of these cash flows is presented below by the first term of the equality (3). If default occurred at the date t_k then the protection buyer would receive from protection seller 'loss given default' amount equal to $(F - R (t_k , T ; \omega))$ in exchange for the fixed periodic payments q protection buyer has paid on dates t_j, $j < k$. For instance the present value to the protection buyer is equal to 0 if

$$\sum_{k=1}^{N} [\, B(t, t_k)\,(F - R(t_k, T; \omega)) - q_s \sum_{j=1}^{k-1} B(t, t_j)\,]\, \chi(\tau(\omega) = t_k) -$$
$$- q_s \sum_{k=1}^{N} B(t, t_k)\, \chi(\tau(\omega) > T) = 0 \tag{3}$$

The periodic coupon implied by equation (3) is

$$q_s(\omega) = \frac{\sum_{k=1}^{N} B(t, t_k)[\, F - R(t_k, T; \omega)\,]\, \chi(\tau(\omega) = t_k)}{\sum_{k=1}^{N} [\{ B(t, t_k) \sum_{j=1}^{k-1} B(t, t_j)\, \chi(\tau(\omega) = t_k)\} + \chi(\tau(\omega) > T)\,]} =$$
$$= \sum_{k=1}^{N} \frac{[\, F - R(t_k, T; \omega)\,]}{\sum_{j=1}^{k-1} B(t, t_j)}\, \chi(\tau(\omega) = t_k) \tag{4}$$

The risk analysis of the CDS contract can be established as follows. Recall that the market risks of the protection seller and protection buyer are different and can be determined using formulas (2, 4). Counterparty risk depends on default time distribution. There are variety assumptions regarding default time distributions might be reasonable to apply, see for example [10]. In addition given D_k the protection buyer risk at t depends on future rates over periods $[t_j, t_k]$, $1 \le j < k \le N$ in formula (4). In these formulas we ignored protection seller's risk of default on claim amount at the date of default. As far as the seller's and the buyer's valuation formulas are different then it makes sense to present these risk formulas separately. When two counterparties come to the agreement regarding premium value then this value should be applied to estimate the market risk for either party of the contract. Thus a settlement market price μ_q on CDS contract implies the risk for both counterparties. Let μ_q denote a market spread. Then market risk can be described as follows. The protection buyer's risk is a probability that the market spread value μ_q exceeds $q_b(\omega)$, i.e. $P\{ q_b(\omega) < \mu_q \}$. This probability represents the measure of the chance that the market price exceeds the exact contract price. Recall that for each scenario $\omega \in D_k$ the value $R(t_k, T; \omega)$ is defined by its recovery rate [10]. This recovery rate can be found based on formula (4). If it is reasonable to use binomial distribution as an approximation of the time of default we can see that

$$P\{\tau = t_j\} = (1-p)\,p^{j-1}\ , \qquad P\{\tau > t_j\} = 1 - p^j$$

Here the value 1 - p denotes the probability of default. Using these formulas one can easy calculate any statistical characteristics of the swap.

Some CDS variations.

Though CDS contract is the most popular credit derivative instrument there are several important variations of the standard CDS on the credit market. Let us first consider a **constant maturity default swap** (CMDS) contract. Goldman Sachs first introduced this contract in1997. A CMDS contract is almost identical to the standard CDS. The primary difference relates to the premium leg of the contract. Recall that for the standard CDS premium leg is a specified constant coupon paid periodically to the protection seller until earlier date between default or maturity. For a CMDS contract several parameters should be specified in advance. These are the length of the constant maturity, a sequence of reset dates when the previous market spread is replaced by the new on-the-run CDS market spread with the specified maturity, and a percentage factor that would be applied for. This percentage factor is also known as a participation rate. Thus given available information regarding CDS contracts the valuation of CMDS contract is the problem of calculation unknown percentage factor. First note that the protection leg payments is the same for the either type of the contract CDS or CMDS and equal to

$$\sum_{k=1}^{N} [\, F - R(t_k, T; \omega)]\, \chi(\tau(\omega) = t_k)$$

A modification presented for CMDS contracts can be studied as follows. Let L (s , t), s ≤ t be the Libor (Eurodollar) rate at date s with maturity t . It has been defined as the simple interest rate for Eurodollar deposit at s with maturity t. The cash flows from protection buyer to the protection seller for CDS and CMDS contracts are different and can be written in the forms

$$[\sum_{k=1}^{N} \chi(\tau(\omega) = t_k) \sum_{j=1}^{k-1} L^{-1}(t_j, t_k) + \chi(\tau(\omega) > T)\,]\, q_s$$

$$[\sum_{k=1}^{N} \chi(\tau(\omega) = t_k) \sum_{j=1}^{k-1} q_H(t_{j-1}) L^{-1}(t_j, t_k) + q_H(t_{N-1})\, \chi(\tau(\omega) > T)\,]\, p$$

correspondingly. Here p is unknown constant representing percentage factor and $q_H(t)$ is the spot CDS spread at the date t with maturity t + H. Protection seller receives payments on scheduled dates prior to default or maturity, which one comes first. Recall that Libor rate is used here as a discount factor. Taking into account that the protection leg is the same for either CDS or CMDS contracts we remark that the cash flows from protection buyers should be equal for these contracts too. Thus writing the equality in which the value of inflow is equal to the value of outflow from protection buyer to the protection seller at the date of default we see that

$$p(\omega) = q_s(\omega)\,[\,\sum_{k=1}^{N} \chi(\tau(\omega) = t_k)\,\frac{\sum_{j=1}^{k-1} L^{-1}(t_j, t_k)}{\sum_{j=1}^{k-1} q_H(t_{j-1}) L^{-1}(t_j, t_k)} + \frac{\chi(\tau(\omega) > T)}{q_H(t_{N-1})}\,] \tag{5}$$

Remark. It makes sense to highlight a technical problem that relates to the derivatives valuation. This is a common for finance valuation practice of the replacement stochastic cash flows by its expectations. This reduction cannot be always accepted without critical remarks. Sometimes for a particular problem it might make sense. On the other hand it can have no sense. Let us consider illustrative example. Let y be an unknown parameter and the present value of cash flows to and from an investor can be modeled $[\, w^2(t) + 1\,]\, y$ and $5\, w^2(t) + 3$. If we consider the equation generated by expected cash flows we arrive at the solution

$$<< y >> = \frac{5t + 3}{t + 1}$$

On the other hand the exact solution of the problem is

$$y = \frac{5w^2(t) + 3}{w^2(t) + 1}$$

and its expectation does not coincide with <<y>>. This highlight the point that replacement stochastic flows by its expectations can lead to the crude problem solution. Thus in the case when a pricing problem admits an exact solution advanced reduction of the problem to the expected flows might be even mathematically incorrect. We also illustrate this point of view bellow.

The **equity default swap** (**EDS**) was launched by JP Morgan Chase in 2003 though the first equity swap was when Amoco Pension exchanged fixed rate on Japanese stock index excised in 1990. As a CDS benchmark EDS contract exchanges variable rate on a constant rate until maturity or a credit event which one comes first. A stock or a basket of stocks could be a provider of the variable rate. For example a stock basket can be referred to a traded index or a composed virtual index. Thus EDS references on equity market rather than to credit market.

Remark. First let us make a comment related to the equity swap (ES) a contract pricing with zero chance of default. Such contracts first started to trade at the late 80's. The pricing models of the ES are well known [3]. The problem is: given stochastic an equity price $S(t, \omega)$ to derive a fixed rate R of the swap contract. The benchmark formula that was developed using self-financing and no arbitrage general principles is quite simple. Following [3] the ES spread value is equal to

$$R = \frac{1 - B(0, t+n)}{\sum_{i=1}^{n} B(0, t+i)}$$

Here B(0 , T) is a Treasury bond price at the time 0 and maturity T, and B(T , T) = $1. Though in [3] it was remarked that "surprisingly the level of stock is irrelevant in determining the value of swap" the correct conclusion was not provided probably based on widely prevalent over financial community faith in perfection of the valuation methods. These methods might be reasonable when the market is constituted by securities which prices are subject to self-financing no arbitrage valuation. In stochastic setting security price S (t) is a given random process which price does not governed by this rule it is impossible to expect that its derivative would be governed by self-financing and no arbitrage principle that relates to the default free government treasuries. In [3] it was also noted that the stock price S (t) does effect on swap pricing relates to the market practical activity but as we see that it does not appeared in the above theoretical formula. This is an example that shows when common sense follows behind the faith in the method.

From our point of view the formula provided above is incorrect. Indeed the introduced above formula suggests the same fixed rate spread value for equity swap on different stocks for which expected return over a specified period is equal for example to 13%, 5%, 0% or -10%. One also can remark that this formula does not depend on volatility of the stocks. The volatility of the stocks can be equal or not. It is obvious that pricing method that leads to the above formula for the fixed rate R is incorrect regardless of its popularity or simplicity.

Let us briefly outline other framework of the ES pricing. In contrast to self-financing method we assume that an investor has a needed volume of funds for investment. There are two investment opportunities investing in fixed or variable legs. Assume that market provides complete information to market participants. That is an investor is free to choose long or short base on return analysis. A simple approximate formula for the fixed rate R can be received if one equates the rate of returns for both sides of the risk free contract over the period [0 , T]. It leads to the equation

$$\$1(t_0) \prod_{j=1}^{N} \frac{S(t_{j+1})}{S(t_j)} = \$1(t_0)\,(1 + R)^N$$

From which it follows that

$$R = \left[\prod_{j=1}^{N} \frac{S(t_{j+1})}{S(t_j)} \right]^{\frac{1}{N}} - 1 = \left[\frac{S(t_N)}{S(t_0)} \right]^{\frac{1}{N}} - 1$$

In this derivation we did not take into account the equity swap the rule that states the only netted amount of the transactions change hands. In this case the option pricing method is a precise approach that provides a correct reduction of the cash flows to a spot price. Bellow we outline the application of the option pricing approach to the risky swap pricing.

Before writing general formulas let us consider a simple numeric example that illustrates typical equity swap transactions. Put the notional principal be equal to 10. Then set of transactions can be specified by the table

Dates	t_0	t_1	t_2	$t_3 = T$
Floating rate: S (t)	2	6	3	12
Fixed rate: R	2	2	2	2
Transactions value		10[(6/2-1)-2]=0	10[(3/6-1)-2= -25	10[(12/3-1)-2]=10

The calculations show the cash flows to the holder of the variable equity rate. The variable rate of return is exchanged for a fixed rate multiplied by the notional principal. At the date t_1 there is no cash changes hands. Then at t_2 amount 25 is gone from floating leg holder to the fixed rate holder and at the maturity T amount of 10 goes from fixed rate holder to the counterparty.

Now in general case let us assume that S (t) is a stochastic process. Denote 'A' a counterparty that receives a fixed rate and pays floating. Counterparty 'B' is an opposite leg: receives floating rate and pays fixed. Then the stochastic cash flows from counterparties can be represented in the form

$$I_{A \to B}(*, \omega) = \sum_{i=0}^{N-1} \chi_{i+1} \left[\frac{S(t_{i+1}, \omega) - S(t_i, \omega)}{S(t_i, \omega)} - R \right] \chi\left\{ \frac{S(t_{i+1}, \omega) - S(t_i, \omega)}{S(t_i, \omega)} > R \right\}$$

where $\chi_{i+1} = \chi\{ t = t_{i+1} \}$and the symbol '$*$' expresses a functional dependence on time. The cash flow to the A is

$$I_{B \to A}(*, \omega) = \sum_{i=0}^{N-1} \chi_{i+1} \left[R - \frac{S(t_{i+1}, \omega) - S(t_i, \omega)}{S(t_i, \omega)} \right] \chi\left\{ \frac{S(t_{i+1}, \omega) - S(t_i, \omega)}{S(t_i, \omega)} \leq R \right\}$$

The value of the swap is defined by the netted cash flows to and from counterparty B at the contract initiation. Thus the problem is to present date-t reduction of the cash flows $I_{B \to A}$ and $I_{A \to B}$. We use option pricing method introduced in [5]. This approach is consistent with the investment equality definition given above. Following [5] the call and put European option pricing equations are

$$\frac{S(T)}{S(t)} \chi\{S(T) > K\} = \frac{C(T, S(T))}{C(t, S(t))}$$

$$\frac{S(T)}{S(t)} \chi\{S(T) < K\} = \frac{P(T, S(T))}{P(t, S(t))}$$

Here K is a known strike price and European call and put payoffs at expiration date T are defined as

$$C(T, S(T)) = \max\{ S(T) - K, 0 \}$$

$$P(T, S(T)) = \max\{ K - S(T), 0 \}$$

correspondingly. This approach represents other option price definition and does not coincide with Black Scholes pricing in two major issues. First, this approach does not advice the same derivative price for two instruments having the same volatility and different expected rates of return within $[- \mu , \mu]$ where $\mu > 0$ is an arbitrary constant. The second is that it does not relevant to self-financing pricing. Investors have funds and they make a decision where to invest funds based on expectation of the future return regardless where the funds have been received. In stochastic market the decision could lead an investor either to profit or loss. Self-financing pricing approach neglects profit - loss and binds pricing with risk free rate only. The solutions of the equation are

$$C(t, S(t)) = \frac{S(t)}{S(T)} C(T, S(T)) \chi \{S(T) > K\}$$

$$P(t, S(t)) = \frac{S(t)}{S(T)} P(T, S(T)) \chi \{S(T) < K\} \tag{EO}$$

Let us consider payments $I_{B \to A}(* , \omega)$, $I_{A \to B}(* , \omega)$ generated by the exchange floating rate $S(t_{j+1}) / S(t_j)$, $j = 0, 1, \ldots, N - 1$ for fixed rate R. Thus using option pricing solution we enable to produce the value of the cash flows at $t = t_0$ as follows

$$I_{A \to B}(t, \omega) = \sum_{i=0}^{N-1} \frac{S(t_1, \omega) S(t_i, \omega)}{S(t_0, \omega) S(t_{i+1}, \omega)} \left[\frac{S(t_{i+1}, \omega)}{S(t_i, \omega)} - 1 - R \right] \chi \left\{ \frac{S(t_{i+1}, \omega)}{S(t_i, \omega)} > 1 + R \right\}$$

$$I_{B \to A}(t, \omega) = \sum_{i=0}^{N-1} \frac{S(t_1, \omega) S(t_i, \omega)}{S(t_0, \omega) S(t_{i+1}, \omega)} \left[1 + R - \frac{S(t_{i+1}, \omega)}{S(t_i, \omega)} \right] \chi \left\{ \frac{S(t_{i+1}, \omega)}{S(t_i, \omega)} < 1 + R \right\}$$

Note that scenarios in which indicator contains equality sign can be omitted as far as the corresponding term in the sum is equal to 0. The value R for which the right hand sides of the above formulas are equal represents a solution of the equity swap pricing problem. This solution of the problem can be written in a simple compact form. Indeed the equality of the two cash flows at t results

$$I_{B \to A}(t, \omega) = I_{A \to B}(t, \omega)$$

Using equality $\chi\{ Q > x \} = 1 - \chi\{ Q \le x \}$ we see that the fixed rate is a random variable equal to

$$R(\omega) = \frac{\sum_{i=0}^{N-1} [1 - \frac{S(t_i, \omega)}{S(t_{i+1}, \omega)}]}{\sum_{i=0}^{N-1} \frac{S(t_i, \omega)}{S(t_{i+1}, \omega)}} = [N^{-1} \sum_{i=0}^{N-1} \frac{S(t_i, \omega)}{S(t_{i+1}, \omega)}]^{-1} - 1 \qquad (6)$$

This is the definition of the fixed rate of the equity swap. On the other hand when counterparties agree about a particular price they are subject to risk. For example let counterparties agreed to apply a value <R> as a contractual fixed rate of the swap. For example this value can be associated with the expectation of the R (ω). The value of the risk is stipulated by a chance that the real world rate of return to counterparties for the chosen fixed rate value is bellow that implied by the exact value R (ω). Let <R> be the fixed rate for the equity swap. Then the swap value is then by definition equal to

$$I_{A\to B}(t) - I_{B\to A}(t) = \frac{S(t_1, \omega)}{S(t_0, \omega)} \sum_{i=0}^{N-1} [1 - (1 + R)\frac{S(t_{i+1}, \omega)}{S(t_i, \omega)}] \qquad (7)$$

Formulas (6) and (7) represent a solution of the equity swap pricing assuming 0 chance of default. Now let us consider the equity default swap (EDS). First we clarify the use of the terms. Recall steps of the bond valuations. We first perform risk free valuation formulas. Then we defined risky bond price [10] and then CDS contract represents a cost of the default protection of the bond. Here we just defined the price of the 0-default swap. Following bond's program the second step should be a pricing of a risky swap, i.e. the swap that admits default. The last step would be a pricing of the protection fixed rate. The latter step could be named a credit default swap over the underlying risky swap.

We begin with the pricing of the risky swap. The cash flows to counterparties are scheduled at the date t_j , j = 1, 2,…, N can be written in the form

$$I_{A\to B}(*, \tau(\omega)) = \sum_{i=1}^{N} \chi\{\tau(\omega) = t_i\} \sum_{j=0}^{i-2} \chi_j [\frac{S(t_{j+1}, \omega) - S(t_j, \omega)}{S(t_j, \omega)} - R_d] \times$$
$$\times \chi\{\frac{S(t_{j+1}, \omega) - S(t_j, \omega)}{S(t_j, \omega)} > R_d\} + I_{A\to B}(*, \omega)\, \chi\{\tau(\omega) > T\}\, \chi_N$$
$$I_{B\to A}(*, \tau(\omega)) = \sum_{i=1}^{N} \chi\{\tau(\omega) = t_i\} \sum_{j=0}^{i-2} \chi_j [R_d - \frac{S(t_{j+1}, \omega) - S(t_j, \omega)}{S(t_j, \omega)}] \times$$
$$\times \chi\{\frac{S(t_{j+1}, \omega) - S(t_j, \omega)}{S(t_j, \omega)} < R_d\} + I_{B\to A}(*, \omega)\, \chi_N$$

Here the symbol $\chi_j = \chi\{t = t_j\}$ denotes indicator of the scenario that transaction takes place at the date t_j . These formulas show that at the scheduled sequence of the reset dates counterparties exchange their rates. 'A' receives fixed and pays equity floating rates whereas

'B' receives floating and pays fixed rates. At the default there are no transactions. Note that it is possible to add to these formulas the term that covers recovery payments from defaulted side in exchange for full or a portion of payment from protection seller. In this case it would be a trilateral credit contract.

Applying the option pricing method used above for transactions for every scenario $\omega \in \{ \omega : \tau(\omega) = t_j \}$, j = 1, 2, …, N and then summing them up we arrive at the formulas for swap fixed rate and swap value

$$R_d(\omega) = \sum_{i=0}^{N-1} \chi\{\tau(\omega) = t_i\} \{ [\frac{1}{i-1} \sum_{j=0}^{i-2} \frac{S(t_j, \omega)}{S(t_{j+1}, \omega)}]^{-1} - 1 \} + \chi\{\tau(\omega) > T\} R(\omega) \tag{8}$$

$$I_{A \to B}(t, \tau(\omega) \wedge T) - I_{B \to A}(t, \tau(\omega) \wedge T) = \frac{S(t_1, \omega)}{S(t_0, \omega)} \sum_{i=0}^{N-1} \chi\{\tau(\omega) = t_i\} \times$$

$$\times \sum_{j=0}^{i-2} [1 - (1 + R) \frac{S(t_{j+1}, \omega)}{S(t_j, \omega)}] + [I_{A \to B}(t) - I_{B \to A}(t)] \chi\{\tau(\omega) > T\}$$

Here expressions $R(\omega)$ and $I_{B \to A}(t) - I_{A \to B}(t)$ are defined above in formulas (6), (7) for the risk free equity swap. Thus the stochastic spread $R - R_d$ is stipulated by the possibility of default.

In the structural approach default is defined as the first moment of time when company's stock value sinks bellow of a certain fraction q of the initial price. In this case default time would be defined as

$$\tau(\omega) = \min \{ t_i : S(t_i) / S(t_0) < q \}$$

Hence if default occurred during the lifetime of the equity swap then the rate q is a threshold which separates default from no default period. Given a distribution of the random process S (*) one can find an appropriate approximation of the default time distribution.

Now let us consider valuation of the premium which protection buyer should pay to protection seller in order to receive a complete compensation in the case of default. First we need to specify a reasonable value of claim at the default event. If q is the default barrier then a claim amount in the discrete time setting could be defined as

$$F [Q - S(\tau(\omega)) / S(t_0)] \tag{9}$$

where F is a notional principal and Q, Q > q is initially specified rate that can be either a constant such a potion of R or a specified function depending on t. For example Q can be represented by a particular index value.

Credit default swap on a risky equity swap is a value of a fixed rate premium that should be paid periodically by protection buyer to protection seller. Protection buyer can be a receiver of the floating rate in EDS or not. The pricing problem is to derive this premium in exchange for protection (9) delivered by protection seller at default. This problem is close to CDS valuation problem. Let R_B, R_A denote premium values from protection buyer and protection seller perspective correspondingly. When the credit event is coming the protection seller pays amount (9) to the protection buyer. The time of default can be written in the discrete time setting in the form

$$\tau(\omega) = \sum_{i=1}^{N} t_i \chi\{\tau(\omega) = t_i\} = \sum_{i=1}^{N} t_i \chi\{\frac{S(t_i)}{S(t_0)} \le q\} \prod_{j=1}^{i-1} \chi\{\frac{S(t_i)}{S(t_0)} > q\}$$

If $\omega \in \{\tau(\omega) = t_i\}$ then protection buyer makes $(i-1)$ regular payments FR_A on the scheduled reset dates. On the equity leg protection seller pays the amount (9) at the date $\tau(\omega)$ for any ω for which $\tau(\omega) < T$. If there is no credit event over the lifetime of the EDS then protection seller accumulates the scheduled payments until the swap expiration and does not pay default compensation. This reasoning leads to the equation

$$\sum_{i=1}^{N} \chi\{\tau(\omega) = t_i\} \sum_{j=1}^{i-1} FR_A B(t, t_j) + \chi\{\tau(\omega) > T\} \sum_{j=1}^{N} FR_A B(t, t_j) = \tag{10}$$

$$= \sum_{i=1}^{N} \chi\{\tau(\omega) = t_i\} F[Q - \frac{S(t_i)}{S(t_0)}] + 0\,\chi\{\tau(\omega) > T\}$$

From which follows that

$$R_A(\omega) = \frac{\sum_{i=1}^{N} \chi\{\tau(\omega) = t_i\} [Q - \frac{S(t_i)}{S(t_0)}]}{\sum_{i=1}^{N} \chi\{\tau(\omega) = t_i\} \sum_{j=1}^{i-1} B(t, t_j) + \chi\{\tau(\omega) > T\} \sum_{j=1}^{N} B(t, t_j)} =$$

$$= \sum_{i=1}^{N} \frac{[Q - \frac{S(t_i)}{S(t_0)}]}{\sum_{j=1}^{i-1} B(t, t_j)} \chi\{\tau(\omega) = t_i\}$$

On the other hand compare the payments from protection buyer perspective one can easy figure out that

$$R_B(\omega) = \frac{\sum_{i=1}^{N} \chi\{\tau(\omega) = t_i\} [Q - \frac{S(t_i)}{S(t_0)}]}{\sum_{i=1}^{N} \chi\{\tau(\omega) = t_i\} \sum_{j=1}^{i-1} B^{-1}(t_j, t_i) + \chi\{\tau(\omega) > T\} \sum_{j=1}^{N} B^{-1}(t_j, t_i)} =$$

$$= \sum_{i=1}^{N} \frac{[\, Q - \frac{S(t_i)}{S(t_0)} \,]}{\sum_{j=1}^{i-1} B^{-1}(t_j, t_i)} \chi \{ \tau(\omega) = t_i \}$$

Remark. In formula (10) if we dealt with expected values then the last term on the left-hand side would be non-zero term. Denote $<<R_A>>$ a seller premium when stochastic cash flows replaced by its expected values. Then premium is a deterministic constant equal to

$$<< R_A >> = \frac{\sum_{i=1}^{N} E\chi\{\tau = t_i\} [\, Q - \frac{S(t_i)}{S(t_0)} \,]}{\sum_{i=1}^{N} P\{\tau = t_i\} \sum_{j=1}^{i-1} B(t, t_j) + P\{\tau > T\} \sum_{j=1}^{N} B(t, t_j)}$$

That does not coincide with expected value of the expectation of the rate $R_A(\omega)$.

An **asset-swap** contract is a contract that transforms the difference between spot price and the face value of the risky bond in a series of future payments in which each payment represents periodically adjusted London Interbank Offered Rate (LIBOR) plus a constant spread. The asset-swap market is an important segment of the Credit Derivatives Market attached to LIBOR rate, which is usually interpreted as AA risky rate.

Let company X sells to a counterparty Y company's Z risky zero-coupon bond for par at date t and then enters to the interest rate swap IRS paying fixed rate \$c to the counterparty Y. Denote $< R(t, T) >$ the market price of the risky 0-coupon bond at t. If there is no default during the lifetime of the bond the company Y would receive the face value of the bond \$1 at the bond maturity T. On the other hand on the predetermined dates the IRS fixed payments of \$c would be paid by X to Y. In return Y pays to X variable LIBOR rate plus a spread s_X. Consider the case when credit events might occur only when underlying bond default. This approximation might be realistic if default risk of the company Z is more significant than the 'X-Y' counterparty risk. Nevertheless in general both sides of the contract can default on a particular transaction and it is possible to study this general case too. Bearing in mind described above structure the corresponding asset swap transactions can be formally determined as follows.

Let $R(t, T; \omega)$ denotes the risky 0 coupon bond price at t with maturity T. Company X sells risky bond for $< R(t, T) >$ to the company Y at the date t. The risk of the deal is getting lower return than implied by the price $< R(t, T) >$. This risk discussed in details in [10]. Thus the cash flow to the company Y is

$$\sum_{j=1}^{i-1} (c - L_j - s_X)\, \chi(c > L_j + s_X)\, \chi\{ t = t_j \}$$

for a scenario ω for which $\tau(\omega) = t_i$, $i = 1, 2, \dots, N$. If $\tau(\omega) > T$ then

$$1\,\chi\{t=T\}+\sum_{j=1}^{N}(c-L_j-s_X)\,\chi\,(\,c>L_j+s_X\,)\,\chi\{t=t_j\}$$

Note that the face value \$1 received by the company Y if there is no default of the bond until T can also be paid by the company Z though we study the case when it paid by X. The cash flow from the company Y to X could be presented as follows

$$<R(t,T)> \qquad \text{at the date } t,$$

$$\sum_{j=1}^{i-1}(\,L_j+s_X-c\,)\,\chi\,(\,c<L_j+s_X\,) \quad \text{at dates } t_j,\ j=1,\ldots,i-1 \quad \text{if } \tau=t_i,\ i=1,2,\ldots N$$

$$\sum_{j=1}^{N}(L_j+s_X-c\,)\,\chi\,(\,c<L_j+s_X\,) \qquad \text{at dates } t_j,\ j=1,\ldots,N \qquad \text{if } \tau>T.$$

Let us consider a more general when investor X pays a known recovery protection of 1 - Δ to Y at the default date. In order to avoid arbitrage opportunity the value of the cash flows to and from the counterparty Y should be equal and therefore

$$\chi\{\tau>T\}\,[\,1\,\chi\{\,t=T\}+\sum_{j=1}^{N}(\,\gamma-L_j-s_X\,)\,\chi\,(\,\gamma>L_j+s_X\,)\,\chi\{\,t=t_j\}\,]\,+$$

$$+\sum_{i=1}^{N}\chi\{\tau=t_i\}\,[\sum_{j=1}^{i-1}(\,\gamma-L_j-s_X\,)\,\chi\,(\,\gamma>L_j+s_X\,)\,\chi\{\,t=t_j\}+(\,1-\Delta\,)\chi\{\,t=t_i\}\,]=$$

$$=<R\,(t,T)>\chi\{\,t=t_0\}+\sum_{i=1}^{N}\chi\{\tau=t_i\}\sum_{j=1}^{i-1}(\,L_j+s_X-\gamma\,)\,\chi\,(\,\gamma<L_j+s_X\,)\,\chi\{\,t=t_j\}\,+$$

$$+\;\chi\{\tau>T\}\sum_{j=1}^{N}(\,L_j+s_X-\gamma\,)\,\chi\,(\,\gamma<L_j+s_X\,)\,\chi\{\,t=t_j\}$$

Here the left hand side of this equation represents cash flow paid to and right hand side paid by counterparty Y. There are two ways to discount future payments. One way is attached to US Treasury rate and other is attached to the LIBOR rate. There is also a possibility to use specified or a constant maturity samples. Let D (t , T) denote a discount factor over [t , T]. Then the spread value s_X from the counterparty Y perspective is equal to

$$
s_X(\omega) = \begin{cases} \dfrac{D(t,T) + \sum_{j=1}^{N} (c - L_j)\, D(t,t_j) - \langle R(t,T)\rangle}{\sum_{j=1}^{N} D(t,t_j)} \chi\{\tau > T\} \\[2ex] \dfrac{(1-\Delta) D(t,t_i) + \sum_{j=1}^{i-1} (c - L_j)\, D(t,t_j) - \langle R(t,T)\rangle}{\sum_{j=1}^{N} D(t,t_j)} \chi\{\tau = t_i\} \end{cases}
$$

$i = 2, 3, \ldots, N$. Note that market data of the spread implies risk for either counterparty. This risk value is expressed by probability of receiving return less than implied by the market data. In contrast to the spread formula presented above one can apply reduction of the cash flows to the future date $\tau \wedge T = \min\{\tau, T\}$ and then calculate its present value. Then

$$
\sum_{i=2}^{N} D(t,t_i)\,\chi(\tau = t_i) \sum_{j=1}^{i-1} D^{-1}(t_j, t_i)(c - L_j - s_X)\,\chi(c > L_j + s_X) +
$$

$$
+ \chi(\tau > T) D(t,T)\left[1 + \sum_{j=1}^{N} D^{-1}(t_j, T)(c - L_j - s_X)\chi(c > L_j + s_X)\right] =
$$

$$
= \sum_{i=2}^{N} D(t,t_i)\,\chi(\tau = t_i) \sum_{j=1}^{i-1} D^{-1}(t_j, t_i)(c - L_j - s_X)\,\chi(c > L_j + s_X) +
$$

$$
+ \chi(\tau > T) \sum_{j=1}^{N} D(t,T)\, D^{-1}(t_j, T)(c - L_j - s_X)\chi(c > L_j + s_X) -
$$

$$
- \sum_{j=2}^{N} \chi(\tau = t_i) \sum_{j=1}^{i-1} D(t,t_i)\, D^{-1}(t_j, t_i)(c - L_j - s_X) -
$$

$$
- \chi(\tau > T) \sum_{j=1}^{N} D(t,T)\, D^{-1}(t_j, T)(c - L_j - s_X)
$$

Hence

$$
\sum_{i=2}^{N} \chi(\tau = t_i) \sum_{j=1}^{i-1} D(t,t_i) D^{-1}(t_j, t_i)(c - L_j - s_X) +
$$

$$+ \chi(\tau > T) \sum_{j=1}^{N} D(t, T) D^{-1}(t_j, T)(c - L_j - s_X) = 0$$

and therefore

$$s_X(\omega) = \sum_{i=2}^{N} \chi(\tau = t_i) \frac{\sum_{j=1}^{i-1} D(t, t_i) D^{-1}(t_j, t_i)(c - L_j)}{\sum_{j=1}^{i-1} D(t, t_i) D^{-1}(t_j, t_i)} +$$

$$+ \chi(\tau > T) \frac{\sum_{j=1}^{N} D(t, t_j) D^{-1}(t_j, t_i)(c - L_j)}{\sum_{j=1}^{N} D(t, T) D^{-1}(t_j, T)}$$

Note that LIBOR rate used by floating leg of the interest rate swap could be expressed in different ways. For example at the date t_j the Libor rate $L_j = l(t_j; H)$ one can apply for the next period $[t_j, t_{j+1}]$ where $H \geq T$ is a fixed maturity. Libor rate could be assumed in this formula either stochastic or deterministic.

With a **total return swap** (TRS) two counterparties X and Y exchange their cash flows. The lifetime of the TRS contract is stipulated by a risky bond issued by a third party, company Z. Let us assume that bond might default only at the moments t_j, $j = 1, 2, \ldots N$. If there is no default during the lifetime of the TRS contract then the face value of the bond of \$F is paid by the company X to Y at the bond's maturity $t_N = T$. If default occurred at t_i then
X pays to Y at the dates t_j, $j = 1, 2, \ldots, i - 1$
*) $\langle R(t, T) \rangle$ bond price at the date t ;
*) a specified coupon payment \$c ;
*) $[R(t_j, T; \omega) - R(t_{j-1}, T; \omega)] \chi \{R(t_j, T; \omega) > R(t_{j-1}, T; \omega)\}$;
*) the recovery rate $1 - \Delta_i$ at default date t_i .
In return if default occurred at t_i company Y pays to X at the dates t_j, $j = 1, 2, \ldots, i - 1$
*) the Libor rate L_{j-1} specified by the previous time period plus spread s ;
*) $[R(t_{j-1}, T; \omega) - R(t_j, T; \omega)] \chi \{R(t_j, T; \omega) < R(t_{j-1}, T; \omega)\}$;
*) the par value of the bond \$F at T if there is no default.
First note that a reference obligation that can be any asset implies a risk. This is a risk that market price of the reference obligation does not coincide with the future cash flow implies by this price. Note that similar to asset swap underlying risk can be considered separately [10]. We begin with the standard PV pricing estimate used usually for construction an approximate solution of the valuation problems. Setting to be equal the present value of the cash flows to and from Y we arrive at the equality

$$< R(t,T) > - FD(t,T) + \sum_{i=1}^{N} \chi(\tau = t_i)\{ D(t,t_i)(1-\Delta_i) + \sum_{j=1}^{i-1} D(t,t_j)[c - L_j - s +$$

$$+ R(t_j, T; \omega) - R(t_{j-1}, T; \omega)] + \chi(\tau > T) \sum_{i=1}^{N} D(t,t_i)(c - L_i - s)]\} = 0 \tag{11}$$

Note that if default occurs at t_1 then by definition we assumed that Y paid the notional and received recovery rate and there is no other transaction for this scenario. From equation (11) it follows that TRS spread is equal to

$$s(\omega) = \frac{< R(t,T) > - FD(t,T)}{\sum_{i=1}^{N} \chi(\tau = t_i) \sum_{j=1}^{i-1} D(t,t_j) + \chi(\tau > T) \sum_{i=1}^{N} D(t,t_i)} +$$

$$+ \sum_{i=1}^{N} \frac{\chi(\tau = t_i)}{\sum_{j=1}^{i-1} D(t,t_j)} \sum_{j=1}^{i-1} D(t,t_j)[c - L_j + R(t_{j+1}, T; \omega) - R(t_j, T; \omega)] + \tag{12}$$

$$+ \chi(\tau > T) \sum_{i=1}^{N} \frac{D(t,t_i)}{\sum_{j=1}^{N} D(t,t_j)} (c - L_i)]$$

We arrive at the more accurate formula of the TRS if we replace the terms $D(t, t_j)$ on $D(t, T) D^{-1}(t_j, T)$ on the right hand side (12)

The next is a swap hybrid contract known as a **credit-linked note** (CLN). This is a funded type of credit derivatives, which synthetically combines two instruments a corporate bond and a standard CDS. Lifetime of the CLN issued by a company Y is defined by the lifetime of the corporate bond. This bond is assumed to be issued by a company Z. If there is no default of the company Z bond then the company Y makes a fixed periodic coupon payments to a CLN buyer at specified payment dates t_k, k = 1, …N and a principal CLN at the bond maturity T. If a credit event such as default or bankruptcy does occur before CLN maturity T then the CLN contract is terminated. The next coupon payments are not paid and CLN holders will receive value Δ on defaulted bond. Along with CLN the company Y enters into CDS deal. Though it can be done before or later the CLN issue date we suppose for simplicity that the initiation dates of CLN and CDS are the same. With CDS contract the company Y is a protection seller who will pays a ‘loss given default’ at the date of default in exchange for periodic premium until default and recovery rate at the default date. This recovery rate goes to CLN investors and CDS premium is passed on to the CLN investors to increase yield on Z company notes. The CLN holders deliver either the bond of the company Z or the amount representing the market value of the defaulted bond to the CLN issuer Y and in return Y pays loss given default to X. Recall that these types of settlements are known as physical and cash correspondingly.

Let $Q_{LN}(t, T; \omega)$ denotes the CLN price at the date t. Then using synthetic pricing one can split $Q_{LN}(t, T; \omega)$ price into two components. The PV of the one component is the cash flow from CLN issuer (Y) to the CLN buyers is

$$-\sum_{i=1}^{N} \chi(\tau = t_i)\,[\sum_{j=1}^{i-1} B(t, t_j)\, c_{LN} + \Delta B(t, t_i)] + \chi(\tau > T)\,[\,B(t, T) - \sum_{i=1}^{N} B(t, t_i)\, c_{LN}\,]$$

Other cash flow stems from CDS transactions. The CDS cash components for counterparty Y are recovery rate 1 – Δ paid by Y to the CDS protection buyer. In exchange Y receives defaulted bond or the equivalent cash Δ at the date of default and fixed rate coupon from CDS buyer. Thus the PV of the CDS transaction is

$$\sum_{i=1}^{N} \chi(\tau = t_i)\ \{\ [\,\Delta - (1 - \Delta)\,]\, B(t, t_i) + \sum_{j=1}^{i-1} B(t, t_j)\, s_{CDS}\ \}$$

$$+\ \chi(\tau > T) \sum_{i=1}^{N} B(t, t_i)\, s_{CDS}$$

If there is no default the CLN issuer would pay CLN coupon and receives CDS periodic payments s_{CDS}. Thus approximation of the CLN price represented by the PV reduction is

$$\sum_{i=1}^{N} \chi(\tau = t_i)\,\{\ \sum_{j=1}^{i-1} B(t, t_j)\,[\,s_{CDS} - c_{LN}\,] - (1 - \Delta)\, B(t, t_i)\ \} +$$

$$+\ \chi(\tau > T)\,\{\sum_{i=1}^{N} B(t, t_i)[\,s_{CDS} - c_{LN}\,] + B(t, T)\,\} = 0 \qquad (13)$$

From which it follows that

$$c_{LN} = s_{CDS} - \sum_{i=1}^{N} \chi(\tau = t_i)\ \frac{(1 - \Delta_i)\, B(t, t_i)}{\sum_{j=1}^{i-1} B(t, t_j)} + \chi(\tau > T)\ \frac{B(t, T)}{\sum_{i=1}^{N} B(t, t_i)}$$

This is representation of the CLN spread based on PV reduction. Admitting a particular default time distribution it is easy to present calculations of the expected value of the CLN premium as well as its risk characteristics. Taking into account (2, 4) we can perform more accurately representation of the CLN prices

$$\sum_{i=1}^{N} \chi(\tau = t_i)\{\sum_{j=1}^{i-1} [B(t, t_i)B^{-1}(t_j, t_i)\, s_{CDS} - B(t, t_j)\, c_{LN}] - (1 - \Delta_i)B(t, t_i)\} +$$

$$+ \chi(\tau > T)[\sum_{j=1}^{N} [B(t, T)B^{-1}(t_j, T)\, s_{CDS} - B(t, t_i)c_{LN}] = 0 \tag{13'}$$

Note that in this adjustment the payments made by the counterparty Y are discounted by the standard PV reduction while received payments first are summing up at the earliest of maturity or default time using future rates. Then this cumulative sum is discounted to the date t. The correspondent formula for CLN coupon can be written in the form

$$c_{LN} = \sum_{i=1}^{N} \chi(\tau = t_i) \frac{B(t, t_i)[\sum_{j=1}^{i-1} B^{-1}(t_j, t_i)s_{CDS} - (1 - \Delta_i)]}{\sum_{j=1}^{i-1} B(t, t_j)} +$$

$$+ \chi(\tau > T) \frac{B(t, T) \sum_{j=1}^{N} B^{-1}(t_j, T)s_{CDS}}{\sum_{i=1}^{N} B(t, t_i)}$$

Let us consider a CLN pricing bearing in mind joint effect of the counterparty risk. Based on a term of CDS contract and a value of the loss given default it is clear that the protection seller's creditworthiness should be taking into account for more accurate pricing. Thus assume that protection seller might also default on protection transaction. Denote $D_{ps}(\tau(\omega))$ a default event when protection seller fails on delivery recovery $\$(F - R(\tau(\omega), T; \omega))$ to the protection buyer at the date of default. Therefore if underlying security default at t_k, i.e. $\omega \in \{\tau(\omega) = t_k\}$there exist a chance $D_{ps}(t_k)$ that protection seller fails to fulfill CDS obligation. Subscript 'ps' here stands for the 'protection seller'. Thus

$$(F - R(t_k, T; \omega))\chi\{\tau(\omega) = t_k\} = \Delta_{ps}(F - R(t_k, T; \omega))\chi\{\tau(\omega) = t_k\}\chi(D_{ps}(t_k)) +$$

$$+ (F - R(t_k, T; \omega))\chi\{\tau(\omega) = t_k\}[1 - \chi(D_{ps}(t_k))]$$

Here Δ_{ps} denote a recovery rate on default of the transaction 'loss given default' from the protection seller to the protection buyer. Substitution of the right hand side of this equality in (2) and (4) leads to the formula refinement that takes into account protection seller's credit risk. In this case for a scenario $\omega \in \{\tau(\omega) = t_k\} \cap D_{ps}(t_k)$, $k = 1, 2, .., N$ protection buyer's loss is

$$(1 - \Delta_{ps})(F - R(t_k , T; \omega)) \chi\{ \tau(\omega) = t_k \} \chi(D_{ps} (t_k))$$

The corresponded value of the spread will be reduced as follows

$$q_b^{(c)} = \sum_{k=1}^{N} \frac{[F - R(t_k , T ; \omega)]}{\sum_{j=1}^{k-1} B^{-1}(t_j , t_k)} [1 - (1 - \Delta_{ps}) \chi(D_{ps} (t_k))] \chi\{ \tau(\omega) = t_k \} \tag{2'}$$

Thus the spread determined by the protection buyer's fixed rate regular payment will be reduced as far as there exist a chance that protection seller will not pay the protection payment in full. The correspondent seller exposure will also reduced and equal to

$$q_s^{(c)} = \sum_{k=1}^{N} \frac{[F - R(t_k , T ; \omega)]}{\sum_{j=1}^{k-1} B(t, t_j)} [1 - (1 - \Delta_{ps}) \chi(D_{ps} (t_k))] \chi\{ \tau(\omega) = t_k \} \tag{4'}$$

Note that for calculation of the mean or variance of the random variables (2′) and (4′) we need to know the join conditional distributions of the random vector $(R (t_k, T ; \omega) , D_{ps} (t_k))$ conditioning on $\{ \tau (\omega) = t_k \}$. The event $D_{ps} (t_k)$ can be assumed to be independent on the event $\{ \tau (\omega) = t_k \}$ at least for the first order approximation nevertheless the distribution $R (t_k , T ; \omega)$ in general should correlate with the event $\{ \tau (\omega) = t_k \}$. Therefore for an accurate modeling one need a realistic assumption regarding this conditional distribution.

Now let us consider the continuous time CDS contract. Assume that coupon is paid on the fixed dates t_k , k = 1, 2…N. Then the balance equation (3) should be adjusted taking into account continuous distribution of the default event. Bearing in mind accrual interest and a possibility to default of the protection seller on delivery 'loss given default' occurred at the date $t_{k, i}$ equality (3) can be rewritten in the form

$$\lim_{\varepsilon \downarrow 0} \sum_{k=1}^{N} \sum_{i=0}^{n} \chi\{ \tau(\omega) \in [t_{k,i} , t_{k,i+1}) \} \{ B(t , t_{k,i+1}) [F - R(t_{k,i+1} , T; \omega)] \times$$

$$\times [1 - (1 - \Delta_{ps}) \chi (D_{ss} (t_{k,i}))] - q_s [\sum_{j=0}^{k-1} B(t, t_j) + [R(t_{k,i} , t_{k+1} ; \omega) - R(t_k , t_{k+1} ; \omega)]\} -$$

$$- q_s \sum_{j=0}^{N} B(t, t_j) \chi(\tau(\omega) > T) = 0$$

Here $t_k = t_{k,0} < t_{k,1} < t_{k,2} < \ldots < t_{k,n} = t_{k+1}$ is a sub-partition of the interval $[t_k, t_{k+1}]$, and $t_{k,i+1} - t_{k,i} = \varepsilon$. Denote $i_d(s, t; \omega)$ the discount interest rate of the risky bond over the interval $[s, t]$. Then the term

$$q_s [R(t_{k,i}, t_{k+1}; \omega) - R(t_k, t_{k+1}; \omega)] =$$

$$q_s [i_d(t_{k,i}, t_{k+1}; \omega) - i_d(t_k, t_{k+1}; \omega)] \times (t_{k,i} - t_k) / 360$$

is the accrued interest unknown at the date t. Note that if default occurred exactly at t_k then from this formula follows that accrued interest will not be paid. If the chance that default occurs at initiation date $t = 0$ does not equal to 0 then the term equal to

$$(F - R(t, T))\, \chi\{ \tau(\omega) = 0 \}$$

must be added to the left hand side of the above formula. In this case it looks reasonable to pay the first coupon payment \$q at $t = 0$. In the limit it was assumed that protection settlement is occurred immediately after default although for other variations this formula can be easily adjusted. The spread formula can be received from the above equation. Indeed

$$q_s = \lim_{\varepsilon \downarrow 0} \frac{\sum_{k=1}^{N} \sum_{i=0}^{n} B(t, t_{k,i+1}) [F - R(t_{k,i+1}, T; \omega)]\, [1 - (1 - \Delta_{ps}) \chi(D_{ps}(t_{k,i}))] \times}{\sum_{k=1}^{N} \sum_{i=0}^{n} \chi\{\tau(\omega) \in [t_{k,i}, t_{k,i+1})\} \{ \sum_{j=0}^{k-1} B(t, t_j) + [R(t_{k,i}, t_{k+1}; \omega) -}$$

$$\frac{\times\, \chi\{ \tau(\omega) \in (t_{k,i}, t_{k,i+1}] \}}{- R(t_k, t_{k+1}; \omega)] \} + \sum_{j=0}^{N} B(t, t_j)\, \chi\{\tau(\omega) > T\}} =$$

$$= \frac{B(t, \tau(\omega)) [F - R(\tau(\omega), T; \omega)] [1 - (1 - \Delta_{ps}) \chi(D_{ps}(t_{k,i}))]\, \chi\{ \tau(\omega) \le T \}}{\sum_{k=1}^{N} \chi\{\tau(\omega) \in [t_k, t_{k+1})\} \{ [\sum_{j=0}^{k-1} B(t, t_j)] + R(\tau(\omega), t_{k+1}; \omega) - R(t_k, t_{k+1}; \omega) \}}$$

Here in denominator on the right hand side of the above equality the term that corresponds to no default scenario can be omitted as far as the value of the numerator for such scenarios is equal to 0. Indeed for no default scenarios it is not reasonable for the buyer to pay a protection.

Floating rate risky bond.

Let us briefly outline risk free contract called floating rate bond. Beside the popularity of this contract it is also important with its application to the valuation of the floating leg of the interest rate swap. The valuation method introduced below follows [9].

Consider an interval $[t, T]$ and let $t = t_0 < t_1 < \ldots < t_N = T$ be the interest rate reset dates and assume that the step $\varepsilon = t_{j+1} - t_j$ does not depend on j. Let $i(t_j, t_{j+1})$ be the floating rate at t_j which is applied over the period $[t_j, t_{j+1}]$. For writing simplicity assume that notional principal is \$1. Otherwise the values of transactions should be proportionally changed. The floating interest rates that would be applied for regular payments from the contract buyer to its seller are represented in the table bellow

Dates	t_0	t_1	t_2	…	$t_N = T$
Floating flow	-1	$i(t_0, t_0 + \varepsilon)$	$i(t_1, t_1 + \varepsilon)$	…	$1 + i(t_{N-1}, t_{N-1} + \varepsilon)$

Looking at this table one can see that one-dollar at date t_{N-1} is equal to

$$\$1(t_{N-1}) = \$[1 + i(t_{N-1}, T)](T)$$

Hence in particular

$$\$1(t_{N-1})\, i(t_{N-2}, t_{N-1}) + \$(T)[1 + i(t_{N-1}, T)] = \$1(t_{N-1})\,[1 + i(t_{N-2}, t_{N-1})]$$

Therefore the cumulative cash flow to the bond buyer over the time period $[t, T]$ can be calculated in the backward of time starting from the date T to t. It yields

$$\$(t_1)\, i(t_0, t_1) + \$(t_2)\, i(t_1, t_2) + \ldots + \$(T)[1 + i(t_{N-1}, T)] =$$

$$= \$(t_1)\, i(t_0, t_1) + \$(t_2)\, i(t_1, t_2) + \ldots + \$(t_{N-1})[1 + i(t_{N-2}, t_{N-1})] = \ldots$$

$$\ldots = \$(t_1)[1 + i(t_0, t_1)] = \$1(t)$$

These calculations prove that \$1 invested in the risk free security at the date t generates a floating rate cash flow. Thus from bond seller perspective the bond buyer paying \$1 at t will receive equivalent cash payments over the period $(t, T]$. The variability of the interest rate here does not affect this valuation. That is

Dates	t_0	t_1	t_2	…	$t_N = T$
Floating flow	1	$-i(t_0, t_0 + \varepsilon)$	$-i(t_1, t_1 + \varepsilon)$	…	$-[1 + i(t_{N-1}, t_{N-1} + \varepsilon)]$

This floating rate bond valuation has used for the present value reduction in order to justify the pricing model. Thus floating bond seller receives \$1 at the date $t_o = t$. Investing it and paying coupon $i(t_{k-1}, t_k)$ at t_k, $k = 1, 2, \ldots, N - 1$ and $1 + i(t_{N-1}, t_N)$ at the bond maturity T the amount of would exhausted the upfront funding of \$1. Note that this construction actually does not depend on T and ε and therefore can be applied for arbitrary T and ε. On the other hand bond buyer estimates the future value of the contract using formula

$$Fl(T) = \sum_{j=1}^{N} i(t_{j-1}, t_j)\, B^{-1}(t_j, T) + [1 + i(t_{N-1}, t_N)]$$

This formula presents the date-T value of the floating bond payments and the ε-compounding interest rate formula should be applied for the estimation of the bond value. To avoid arbitrage over [t , T] one should expect that the floating bond and the 0-coupon bond issued by the same Government should provide the same rate of return. That implies in particular that

$$Fl(T) / Fl(t) = 1 / B(t, T)$$

The solution of this equation is

$$Fl(t) = B(t, T)\,[\,\sum_{j=1}^{N} i(t_{j-1}, t_j)\, B^{-1}(t_j, T) + 1 + i(t_{N-1}, t_N)\,]$$

Rates $i(t_{j-1}, t_j)$ are unknown at the date t therefore it might make sense to interpret these unknowns as a sequence of the random variables.

Assume that underlying to the floating bond contract is a risky bond. Denote $FB_\lambda(t, T)$ $\lambda = \{ t_j ; j = 1, 2, \ldots, N)$ the cash flow generated by the sequence of payments $i(t_{j-1}, t_j)$ paid at t_j, $j = 1, 2, \ldots, N - 1$ and $1 + i(t_{N-1}, t_N)$ paid at T. Then by definition $\$1(t) = FB_\lambda(t, T)$ for any λ and T regardless whether the rates are random or deterministic. A seller of the risky floating bond would pay λ-reset floating interest rate payments until default or maturity which one comes first. Assume that at the default date the bond seller would pay a specified ratio $0 \le \Delta < 1$. In return a floating bond investor pays \$s at initiation of the contract. The floating bond valuation problem is to derive the value of the upfront premium \$s given the nonrandom recovery rate Δ and a distribution of the default time. Bond buyer pays upfront \$s and receives from the bond seller the cash flow equal to

$$\sum_{i=1}^{N} \chi(\tau = t_i)[\sum_{j=1}^{i-1} i(t_{j-1}, t_j) + \Delta] + \chi(\tau > T)\sum_{i=1}^{N} i(t_{i-1}, t_i) = \sum_{j=1}^{N} \chi(\tau = t_j)[\,FB_\lambda(t, t_j) -$$

$$- 1 + \Delta\,] + \chi(\tau > T)[\,FB_\lambda(t, T) - 1\,] = [\,FB_\lambda(t, \tau) - 1\,] - \Delta\,\chi(\tau < T)$$

Thus upfront premium value is

$$s(\omega) = [\,FB_\lambda(t, \tau) - 1\,] - \Delta\,\chi(\tau < T)$$

In a CDS contract written on a floating rate bond a protection buyer buys a protection that would cover possible default losses. Let recovery rate of the risky floating bond is Δ and default occurs at the date t_j then a protection seller should reimburse at the date t_j or at the next date the loss (1 - Δ). On the other hand the protection buyer would pay a fixed premium until earliest between the date of default or maturity T. Assume that date of default is t_k and Δ is the recovery rate. Then the loss $L_k(t, \lambda)$ occurred at the dates $t = t_j$, $j = k, \ldots, N$ is equal to

$$L_k(\lambda) = \$[\, i(t_{k-1}, t_k) - \Delta\,]\,\chi(t = t_k) + \sum_{j=k+1}^{N} \$ i(t_{j-1}, t_j)\chi(t = t_j) + \$1\chi(t = t_N) =$$

$$= (1 - \Delta)\,\$(t_{k-1})$$

Thus protection seller payment to protection buyer can be represented by the loss function L_k in the form of 'loss given default'

$$L(\lambda) = \sum_{k=1}^{N} L_k(\lambda)\chi(\tau = t_k) = \sum_{k=1}^{N} \chi(\tau = t_k)(1 - \Delta) = (1 - \Delta)\,\chi(\tau \le T))$$

On the other side of the contract cash flow to protection seller from protection buyer is

$$q\,[\sum_{i=1}^{N} (i - 1)\,\chi(\tau = t_i) + N\chi(\tau > T)\,]$$

Note that by definition we put $q(\omega) = 0$ for $\omega \in \{\omega : \tau(\omega) = t_1\}$. It follows from the fact that if default occurs immediately after contract initiation and therefore coupon will not be paid there is not reason to define coupon value. From last two equalities it follows that premium s is a random variable equal to

$$q(\omega) = \frac{(1 - \Delta)\,\chi(\tau \le T))}{\sum_{i=2}^{N} (i - 1)\,\chi(\tau = t_i) + N\chi(\tau > T)} = (1 - \Delta)\sum_{j=1}^{N-1} j^{-1}\,\chi(\tau = t_{j+1}) \qquad (14)$$

The term $N\,\chi(\tau > T)$ in denominator can be omitted as far as for such scenarios numerator is equal to 0. Note in particular that moments of the spread can be written in compact form

$$E s^{n}(\omega) = (1 - \Delta)^{n} \sum_{j=1}^{N-1} j^{-n}\, P(\tau = t_{j+1}) \qquad (15)$$

where n = 1, 2,… .

Remark. In contemporary credit derivatives research it is commonly accepted in floating leg calculation replacing random cash flow by its expectation. It seems important to note that the value of the spread received with such popular approach does not coincide with the first moment of the exact solution. Indeed the spread value that follows from equal expected cash flows to and from a counterparty of the CDS written on the risky floating bond contract is

$$\langle\langle q \rangle\rangle = \frac{(1 - \Delta)\,P(\tau \le T)}{\sum_{j=2}^{N-1} j\,P(\tau = t_{j+1}) + N P(\tau > T)}$$

The additional term $N P(\tau > T)$ can be either small or large making this approximation latter case crude and biased.

Counterparty Risk of the Interest Rate Swap.

Follow [9] let us first recall a valuation model of interest rate swap (IRS) with 0 chance of default. A standard IR swap is a two party contract. The counterparty A makes fixed semiannual or quarterly payments to counterparty B. The magnitude of each fixed payment is usually a pre-specified percent of the notional principal. In return, counterparty B pays floating rate payments to A. All payments are made in the same currency and only netted amount is paid.

Let $t = t_0 < t_1 < \ldots < t_N = T$ be reset dates, q and $l(*, *)$ denote a fixed and a floating (LIBOR) interest rates correspondingly, and \$1 is the notional principal. The fixed flow line in the table below represents the scheduled payments should be made by counterparty A to B and the floating line is the scheduled payments of the counterparty B to A.

Dates	t_0	t_1	t_2	…	$t_N = T$
Fixed flow	0	q	q	…	$1 + q$
Floating flow	0	$l(t_0, t_0 + \varepsilon)$	$l(t_1, t_1 + \varepsilon)$	…	$1 + l(t_{N-1}, t_{N-1} + \varepsilon)$

where $t_k + \varepsilon = t_{k+1}$, $k = 0, 1, \ldots N - 1$. Recall that only netted payments are paid. If the notional principal is \$F then all entries in lines should be multiplied by the F to present the real cash stream. Let us recall some important points of the swap valuation. The domestic risk free rate usually uses for calculations. If for a particular scenario $q > l(t_k, t_{k+1})$ then the payment of $q - l(t_k, t_{k+1})$ would made by A to B at t_{k+1}. This real world amount would be held until maturity is unknown at t. Thus cumulative future values at the IRS maturity paid to counterparty B and A are

$$\sum_{k=0}^{N-1} [\, q - l(t_k, t_{k+1}) \,]\, \chi(\, q > l(t_k, t_{k+1}))\, B^{-1}(t_{k+1}, T)$$

$$\sum_{k=0}^{N-1} [\, l(t_k, t_{k+1}) - q \,]\, \chi(\, q < l(t_k, t_{k+1}))\, B^{-1}(t_{k+1}, T) \qquad (16)$$

correspondingly. It also might have sense to replace the future value by the PV reduction. The PAR swap rate is by definition the rate q for which the present value of the all fixed side payments is equal to the present value of floating payments. In another words it is a value q for which the value of the swap at t is 0. Multiplying both expressions in (16) by the same factor $B(t, T)$ we arrive at

$$q = \frac{\sum_{k=0}^{N-1} l(t_k, t_{k+1}) B^{-1}(t_k, T)}{\sum_{k=0}^{N-1} B^{-1}(t_k, T)} \tag{17}$$

Definition. The value of the swap $s = s(u, T)$, is the difference between fixed and floating sides values at the date $u \in [t, T]$.

Though the present values (future) is still a benchmark concept in asset pricing it should be clear that it presents an approximation of the general Equal Investment Principal pricing concept formulated above. Moreover along with risk neutralization concept it defines option price incorrectly. It was demonstrated in details in [5-7]. Let us very briefly illustrate the primary deficiency of the Present Value pricing. Let $t = t_0 < t_1 < t_2 = T$ be the dates of trade. Assume for instance that at date t the risk free interest rate term structure is $i(t, t_1) = 4.1\%$, $i(t, T) = 4.2\%$ and $i(t_1, T) \neq i(t, T)$. In this case two counterparties are subject to the market risk. This risk is that the bond buyer loses expected return if it is occurred that bond is exercised at t_1. This simple example shows necessity of the stochastic modeling of the future interest rates.

Now we apply option-pricing approach for the risk free IRS valuation. We will employ the option's valuation method [5-7] for the swap valuation. Recall that the cash flow to the counterparties A and B can be represented in the forms

$$\begin{aligned} C_A(\lambda, t) &= \sum_{k=1}^{N} [\, l(t_k) - q\,]\, \chi\{l(t_k) > q\} \\ P_B(\lambda, t) &= \sum_{k=1}^{N} [\, q - l(t_k)\,]\, \chi\{l(t_k) < q\} \end{aligned} \tag{18}$$

where $t = t_0, l(t_k) = l(t_{k-1}, t_k), \lambda = \{t_k, k = 1, 2, \ldots, N\}$. We interpret the value $l(t_k)$ as a variable portion of the dollar value at t_k. Functions $C_A(*, t, T)$, $P_B(*, t, T)$ in (18) represent a discrete time cash flows in which each term on the right hand side are payoff of the call and put options correspondingly on variable interest $l(t)$ with the strike q. We apply the option pricing formulas follow [5-7]. Let us recall the option price definition. The European call and put option prices $C(t, x)$, $P(t, x)$ at date t on underlying security $S(u)$, $u \geq t$, and $S(t) = x$ with strike price K and maturity T defined above by (EO). This definition leads to

$$\begin{aligned} C_A^{(\lambda)}(t) &= \sum_{j=1}^{N} \frac{l(t_1)}{l(t_j)} [\, l(t_j) - q\,]\, \chi\{\, l(t_j) > q\,\} \\ P_A^{(\lambda)}(t) &= \sum_{j=1}^{N} \frac{l(t_1)}{l(t_j)} [\, q - l(t_j)\,]\, \chi\{\, l(t_j) < q\,\} \end{aligned} \tag{19}$$

These formulas present the values of derivatives instruments known as floorlets and caplets contracts. Hence the swap value s by definition is the difference

$$s = C_A^{(\lambda)}(t) - P_B^{(\lambda)}(t) = \sum_{j=1}^{N} \frac{l(t_1)}{l(t_j)} [\, l(t_j) - q \,] \tag{20}$$

Now the formula (17) can be interpreted as an approximation of the swap spread value presented by the equation (19). Indeed we see that left-hand side in (16), (17) contain all payments to and from party A. Additional difficulties in presenting statistical calculations are the floating forward future rates $l(t_{k+1}) = l(t_k, t_{k+1})$. Hypothetical log-normal implied distribution is commonly used for options valuation. As far as historical data is available the statistical test can provide quantitative likelihood of the implied distribution. If the swap value s is given then fixed leg rate q is equal to

$$q = \frac{N\, l(t_1) - s}{l(t_1) \sum_{j=1}^{N} l^{-1}(t_j)}$$

Putting in this formula s = 0 we obtain the value q that represents the fixed rate (CDS spread) that equates floating payments.

Now we consider the case when one or two counterparties of the IRS are subject to credit risk. Let us assume first that floating rate payer B is subject to default. Let the only dates t_j, j = 1, 2, …, N can be the dates of default. The party B might default before or after the date the underlying bond defaults. The default of the counterparty B means that B is fail to deliver the protection amount $[\, l(t_k) - q \,]$ to A. Assume that recovery rate of the party B implied by its rating is a constant $\Delta_B < 1$. If party B defaults at t_k it pays to counterparty A the fraction of the amount due $\Delta_B [\, l(t_k) - q \,]$. Thus the cash flow from B to A given that B defaults prior to A is

$$C_{B \to A|B}(t, \lambda) = \sum_{k=1}^{N} \{ [\, \chi(\tau(\omega) = t_k)\, \chi(\tau_B(\omega) \geq t_k) + \chi(\tau_B(\omega) = t_k)\, \chi(\tau(\omega) > t_k) \,] \times$$

$$\times [\, \sum_{j=1}^{k-1} [\, l(t_j) - q \,] + \chi\{\tau_B(\omega) = t_k\}\, \chi\{\tau(\omega) > t_k\}\, \Delta_B [\, l(t_k) - q \,] \,\}$$

Here $\tau(\omega)$ denotes the default time of the underlying security. Note this time of default $\tau(\omega)$ and $\tau_B(\omega)$ can be correlated. This cash flow above has two components. One when the default of underlying security is coming before the default of B. The second component of the cash flow corresponds to scenarios when B defaults earlier than underlying security. Our task is to present fixed rate of the swap. This value should be found at initiation date t. A corresponding reduction that could be used is the option valuation method. Given $\omega \in \{\, \omega : \tau(\omega) = t_k \,\}$ there are two mutually exclusive scenarios $\{ \omega : \tau_B(\omega) > t_k \}$ and $\{ \omega : \tau_B(\omega) = t_k \}$ for which we can apply formulas (19). Then

$$C_{B\to A|B}(t) = \sum_{k=1}^{N} \{ [\chi(\tau = t_k)\,\chi(\tau_B \geq t_k) + \chi(\tau_B = t_k)\,\chi(\tau > t_k)] \times$$

$$\times \sum_{j=1}^{k-1} \frac{1(t_1)}{1(t_j)} [1(t_j) - q]\,\chi(1(t_j) > q) + \tag{21}$$

$$+ \chi(\tau_B = t_k)\,\chi(\tau > t_k)\,\Delta_B \frac{1(t_1)}{1(t_k)}[1(t_k) - q]\,\chi(1(t_k) > q)\} +$$

$$+ \chi(\tau > T)\,\chi(\tau_B > T)\,C_A^{(\lambda)}(t)$$

where the value of $C_A^{(\lambda)}(t)$ is given by (19). Note that the cash flow from A to B at the date t given that B might default is

$$P_{A\to B|B}(t) = \sum_{k=1}^{N} \{ [\chi(\tau = t_k)\,\chi(\tau_B \geq t_k) + \chi(\tau_B = t_k)\,\chi(\tau > t_k)] \times$$

$$\times \sum_{j=1}^{k-1} \frac{1(t_1)}{1(t_j)} [q - 1(t_j)]\,\chi(q > 1(t_j)) +$$

$$+ \chi(\tau_B = t_k)\,\chi(\tau > t_k)\,\Delta_B \frac{1(t_1)}{1(t_k)}[q - 1(t_k)]\,\chi(q > 1(t_k))\}$$

The swap value at t is a random variable depending on a scenario and time t. This value is equal to

$$S_B(t) = C_{B\to A|B}(t) - P_{A\to B|B}(t)$$

The random value of $q = q(\omega)$ for which $S_B = 0$ is the default swap rate. The formula for the counterparty risky IRS swap rate q can be presented in analytic form. Indeed taking into account equality

$$(l_j - q)\,\chi(l_j > q) - (q - l_j)\,\chi(l_j < q) = (l_j - q)$$

one can easy figure out that the solution of the problem $C_{B\to A|B}(t) = P_{A\to B|B}(t)$ can be written in the form

$$q_B = \frac{\sum_{k=1}^{N} \{ [\chi(\tau_B > \tau = t_k) + \chi(\tau > \tau_B = t_k)](k-1) + \chi(\tau > \tau_B = t_k)\Delta_B \}}{\sum_{k=1}^{N} \{ [\chi(\tau_B > \tau = t_k) + \chi(\tau > \tau_B = t_k)] \sum_{j=1}^{k-1} l^{-1}(t_j) + \chi(\tau > \tau_B = t_k) l^{-1}(t_k)\Delta_B \}}$$

Assume now that party A is subject to default. In this case the only sign of the cash flow will be changed and all formulas remain the same as when only company B is subject to default. Also note that the recovery rates Δ, Δ_A, Δ_B can be also are assumed to be random and could correlate.

Now let us consider the case when two counterparties can default simultaneously. The swap value then is defined as

$$S_{AB} = \sum_{k=1}^{N} \{ \chi[\tau_A(\omega) = \tau_B(\omega) = t_k] \chi(\tau(\omega) > t_k) \sum_{j=1}^{k-1} \{ \frac{1(t_0)}{1(t_j)} [1(t_j) - q] +$$

$$+ \frac{1(t_0)}{1(t_k)} [\Delta_A 1(t_k) - \Delta_B q] \} + \tag{22}$$

$$+ \sum_{k=1}^{N} \chi\{ \min[\tau_A(\omega), \tau_B(\omega)] > t_k \} \chi(\tau(\omega) = t_k) \sum_{j=1}^{k-1} \frac{1(t_0)}{1(t_j)} [1(t_j) - q] \}$$

The swap rate is the value of $q_{AB}(\omega)$ for which $S_{AB}(\omega) = 0$. From (22) it follows that

$$q_{AB} = \frac{\sum_{k=1}^{N} \{ [\chi(\tau_A = \tau_B = t_k)\chi(\tau > t_k)](k - 1 + \Delta_A) + \chi(\tau = t_k)\chi(\tau_A \wedge \tau_B > t_k)(k-1) \}}{\sum_{k=1}^{N} \{ [\chi(\tau_A = \tau_B = t_k)\chi(\tau > t_k)] \sum_{j=1}^{k-1} l^{-1}(t_j) + l^{-1}(t_k)\Delta_B \} + \chi(\tau = t_k)\chi(\tau_A \wedge \tau_B > t_k) \sum_{j=1}^{k-1} l^{-1}(t_j)}$$

where $a \wedge b = \min(a, b)$. In general case when counterparties might default during the lifetime of the contract. In this case there are three mutually exclusive scenarios representing default

$$\alpha = \{ \tau_A(\omega) > \max[\tau(\omega), \tau_B(\omega)] \}, \quad \beta = \{ \tau_B(\omega) \geq \max[\tau(\omega), \tau_A(\omega)] \}$$

$$\gamma = \{ \tau(\omega) > \max[\tau_B(\omega), \tau_A(\omega)] \}.$$

For each of the possible scenarios one can apply a formula represented above and therefore in general case the value and the rate of the counterparty risky IRS is

$$S = S_A \chi(\alpha) + S_B \chi(\beta) + S_{AB} \chi(\gamma)$$

$$q = q_A \chi(\alpha) + q_B \chi(\beta) + q_{AB} \chi(\gamma)$$

Now all components of the counterparty risky interest rate swap value are presented.

Appendix. Remarks on Credit Swap Spread.

In this section we revise couple basic notions related to the Credit Swap Market. Let us define risky present value (RPV) that is an generalization of the risk-free present value (PV). Let t and T , $t < T$ be two moments of time. Let a risky instrument promised payment of $C at the date T. Then RPV (t) at the date t is by definition a random variable $CB(t, T)\chi\{\tau > T\}$ where $\chi\{\tau > T\}$ denotes the indicator of the event that there is no default over the period $[t, T]$, $B(t, T)$ is US Treasury bond price at t and maturity T, and $B(T, T) = \$1$.

It is a standard in Credit Risk papers (see for example [7]) to define PRV as $C B(t, T) P\{\tau > T\}$. This definition is expected value of the random variable $C B(t, T)\chi\{\tau > T\}$. It seems confusing to call 'risky' the expected value. Indeed for instance let us assume that Wiener process describes profit or losses of a portfolio over the time. Then the portfolio change is measured by its expectation then there is no change in the portfolio. Then it looks somewhat awkwardly to associate portfolio dynamics with the risk. It looks more common to use mathematical notions as their used in mathematics. In particular to interpret expected value as a characteristic of the random variable associated with the risk.

Definition [1]. The standard RPV of a promised payment by definition is the PV adjusted for probability to receive this payment. The RPV of a series of payment is a weighted sum of the individual payments. We will show bellow that given definition can be interpreted in different ways. Indeed, let λ denote a partition $t = t_0 < t_1 < \ldots < t_N = T$. Dates t_j are a sequence of the future dates of payment and C_j is a payment due at t_j, $j = 1, 2, \ldots N$. Let us consider expressions

$$\sum_{k=1}^{N} P\{\tau = t_k\} \sum_{j=1}^{k-1} C_j B(t, t_j)\,; \quad \sum_{j=1}^{N} P\{\tau = t_{j+1}\} C_j B(t, t_j)\,; \quad P\{\tau > T\} \sum_{j=1}^{N} C_j B(t, t_j)$$

Each of these expressions can be chosen for RPV definition. To see that one can note that in each expression the probability factor related to particular date implies that all preceding payments would be received. These three sums present different values and therefore the definition should be refined. On the other hand these values are expected values of the random variables that are indeed risky PVs in contrast to its expectations representing only a risk characteristic. Let us define as risky present value of the series of future payments $\{C_j\}$ as follows

$$\chi\{\tau(\omega) > T\} \sum_{j=1}^{N} C_j B(t, t_j)$$

Note that the stochastic definitions that correspond to other expected values above would be fail to present scenarios for which all scheduled payments will be received. Indeed if a scenario

$$\mathrm{RPV}(t,\lambda;\omega) = \sum_{j=1}^{N} \chi\{\tau(\omega) = t_{j+1}\}\, C_j\, B(t, t_j) \tag{23}$$

admits payment at a t_j then it is not true that this scenario admits payment at t_k for $k > j$. This is explicit advantage of this stochastic definition that makes construction more visible. Note in particular that for n = 1,2, … we have

$$E[\mathrm{RPV}(t,\lambda;\omega)]^n = \sum_{j=1}^{N} P\{\tau(\omega) = t_{j+1}\}\, [C_j\, B(t, t_j)]^n$$

One of the most important risk characteristics is the RPV01. Let L denotes a fixed leg coupon, i.e. $L = C_j$ for all j. Then the definition of the RPV01 is the risky present value of receiving coupon payments of 1bp = 0.01 L over the scheduled dates. That is

$$\mathrm{RPV01}(t,\lambda;\omega) = L\,\frac{1\mathrm{bp}}{4}\sum_{j=1}^{N} B(t, t_j)\,\chi\{\tau = t_{j+1}\} \tag{24}$$

It was assumed in above formula that payments made quarterly. From this definition follows that the value RPV01 depends on scenario ω. It is benchmark to present definition of RPV01 as the risky present value as expectation of the correspondent random variable. That is

$$\boldsymbol{\mathit{RPV01}}(t,T) = E\,\mathrm{RPV01}(t,T;\omega) = 0.25L\sum_{j=1}^{N} B(t, t_j)\,P\{\tau = t_{j+1}\}$$

Denote q (u , T ; ω) CDS-spread at the date u with maturity T. The mark-to-market CDS position over the interval [u , v] we define as

$$\mathrm{MTM}(u,v;T,\omega) = L\,[\,q(v,T;\omega) - q(u,T;\omega)\,]\,\mathrm{RPV01}(v,T;\omega) \tag{25}$$

In general the expected value of the market-to-market position does not coincide with the benchmark market-to-market as it usually implied. In order that these two expressions be equal it is sufficient that the expression in the brackets and random RPV01 value on the right hand side (25) were uncorrelated.

References.

1. Bansal G., and Skarabot J. US Fixed Income Strategy. Global Structured Credit Products. Citigroup Global Markets. April 20, 2006.
2. Black F., Scholes M., The Pricing of Options and Corporate Liabilities. The Journal of Political Economy (May' 1973).
3. Don Chance, Equity Swaps and Equity Investing, 07-2003
4. Duffie D., Credit Swap Valuation, Financial Analysts Journal, 1999, 73-87.
5. Gikhman I, On Black-Scholes Equation. ICFAI Jornal of Applied Finance, v.10, No.4, April 2004, p.47-74.
6. Gikhman I., Drawbacks in Options Definition http://papers.ssrn.com/sol3/papers.cfm?abstract_id=895679, 2006.
7. Gikhman I., Option valuation. http://papers.ssrn.com/sol3/papers.cfm?abstract_id=900111, 2007
8. Gikhman I., Some basic remarks on options valuation. Actes de la 3Xme Conférence Internationale de Finance - IFC 3, Formalisation et Modélisation, Hors Série N°3, IFC3 - 3-5 March, 2005, Hammamet, Tunisia, p.121-136.
9. Gikhman I., Fixed-income pricing. http://papers.ssrn.com/sol3/papers.cfm?abstract_id=958838, 2007.
10. Gikhman I., Corporate debt pricing 1. http://papers.ssrn.com/sol3/papers.cfm?abstract_id=
11. Hull J., and White A., Valuing credit default swaps I: No counterparty default risk, Working Paper, University of Toronto, 2000; Journal of Derivatives 8, Fall, 2000, 29-40.
12. Jarrow R. and Turnbull S., Pricing Options on Financial Securities Subject to Default Risk", Working Paper, Graduate School of Management, Cornell University, 1992
13. R. Jarrow R. and Turnbull S., Pricing Options on Financial Securities Subject to Credit Risk", Journal of Finance 50, 1995, 53-85.
14. Merton R., On the pricing of corporate debt: The risk structure of interest rates. 684-73, November 1973.(Presented at AFA Meetings NY December 1973.
15. The Lehman Brothers Guide to Exotic Credit Derivatives, 2003.
16. C. Pedersen, S. Sen, "Valuation of Constant Maturity Default Swaps", Quantitative Credit Research Quarterly, Lehman Brothers, June 2004.