CES Function, Generalized Mean and Human Poverty Index: Exploring Some Links

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Abstract

The Sennian capability approach has facilitated to capture poverty in its multi-dimensional incidence and thus to raise a new aggregate poverty index – the UNDP's Human Poverty Index (HPI). The UNDP has found power mean of order $\alpha > 1$ as possessing some of the most desirable properties in describing the distribution of deprivation dimensions and hence as the most appropriate aggregate index of multi-dimensional deprivation. The UNDP elevates power mean of order $\alpha > 1$ (PM) in comparison with arithmetic mean (AM) commonly used for averaging, leaving out others. It would hence be worthwhile to look into the links among the means, both the known and the potential ones, and their strengths and weaknesses in terms of their properties in comparison with each other. The present paper is a preliminary attempt at this. We find that the means we commonly use, the AM, the geometric mean (GM) and the harmonic mean (HM), along with the PM, are special cases of the CES function. We acknowledge the possibility of an inverse CES function, and hence, that of an inverse power mean (IPM) also. Among these means, the AM is an average, typical of *all* the components, but its infinite elasticity of substitution renders it less desirable. To the extent that we need an average typical of the components, we seek for one that is closer to the AM, so that this second best choice will have the minimum deviations next to the AM. And we find this basic criterion is satisfied by the IPM only. Hence, while the PM captures the multi-dimensional deprivation, its inverse, the IPM, seems to offer a multi-dimensional development index.

JEL Classification: C43; I32.

Key Words: Generalised mean, CES function, Human Poverty Index, Deprivation, development.

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1. Introduction

Poverty – A Multi-dimensional Concept

Poverty was traditionally measured uni-dimensionally in terms of inadequate income. Following Amartya Sen, however, it has come to be recognised as multidimensional in terms of deprivation of capability to fulfil essential functions in human life. These functions concern not only the possibility to adequately feed and clothe oneself, and to have a shelter, but also the possibility to avoid preventable diseases and to have a long and healthy life, to have an education, and to join society through work, political participation and social relations (Sen 1999). It goes without saying that reaching these goals depends on the available economic resources, as well as on: who we are, the characteristics we possess, and the economic, social and cultural environment we are in. In other words, affluence (or poverty, for that matter) is not describable in terms of possession of economic resources alone; it also depends on what an individual *can do* and *be*, on the set of doings and beings, or simply on functionings. This in turn suggests that assessing the quality of life amounts to evaluating these functionings and capability to function. In this sense, poverty is seen as a capability deprivation, as a condition where the people are deprived of the capability for certain vital functionings. These diverse functionings make poverty multi-dimensional.

Asserting that "*Poverty must be addressed in all its dimensions, not income alone*" (UNDP 1997: 5), the eighth Human Development Report, 1997, introduced the human poverty index in addition to the human and gender development indices. Whilst the Human Development Index (HDI) measures the progress of the country in achieving development, the Human Poverty Index (HPI) "*combines basic dimensions of poverty* *and reveals interesting contrasts with income poverty*" (*ibid.*).[1] The HPI focuses on deprivation in three essential elements of human life already reflected in the HDI – longevity, knowledge and a decent standard of living. Deprivation in the first dimension relates to survival, that is, the vulnerability to death at a relatively early age, and that in the second dimension relates to exclusion from the world of knowledge in terms of education. Deprivation in the third dimension relates to the lack of access to overall economic provisioning. Two HPI indices are there in currency: HPI-1 is a measure of absolute poverty in Less Developed Countries and HPI-2 is a measure of relative poverty in More Developed Countries.

A major problem involved in HPI methodology relates to weighting and aggregation. This problem arises on account of the possibility for overlapping of the poverty dimensions. Suppose for a particular country, we find 30 percent of the people in each of the (say) 3 fields of deprivations, that is $X_1 = X_2 = X_3 = 30$ percent. Three possibilities are here in this distribution. 1) Non-disjointness: the *same* 30 percent of the people suffer from all the three deprivations together, so that only 30 percent in the country are affected by poverty, but they are deprived in respect of all the three fields. 2) Disjointness: the 30 percent is entirely *different* in each field, so that a total of 90 percent of the population are in poverty, but each 30 percent group is deprived in respect of only one field. 3) Overlapping: a combination of the two extremes. "However, when it comes to constructing an index, it is not easy to decide whether 30 percent of people with inadequacies of all three types represents larger social poverty than 90 percent of people having one deficiency each. It is a matter of the importance to be given to depth vis-à-vis breadth. For the purpose of the HPI, the two cases have been treated as equivalent, so that in some sense depth and width have been equally considered." (UNDP 1997: 20). This assumption facilitates averaging of all the dimensions to represent aggregate poverty. But in averaging, there comes the problem of choice of an average and of weighting.

[1] Rightly, "by combining in a single poverty index the concerns that often get pushed aside when the focus is on income alone, the HPI makes a useful addition to the measures of poverty" (UNDP 1997: 5).

The UNDP has found power mean of order $\alpha > 1$ as possessing some of the most desirable properties in describing the distribution of deprivation dimensions (UNDP 1997: 117-121), and hence as the most appropriate aggregate index of multi-dimensional deprivation. UNDP has selected power mean (PM) of order 3 for some obvious reasons, detailed in the Technical Note 1 in UNDP (1997: 117-121), over the usual, everyday-means of arithmetic (AM) and geometric mean (GM). Remember AM is the only summary measure typical of a set of data, in the sense that it is the coordinate of the centre of gravity, balancing the values on either side of it, and the sum of squared deviations of the values from it is the minimum – the minimum variance property. It is this representativeness that makes it a good measure of averaging. Similarly, GM is theoretically considered the best average in the construction of index numbers as well as for ratios and percentages (in which units empirical deprivation measures are usually expressed) as also for their rate of changes. The other less used mean, harmonic mean (HM), is useful in cases where the values of a variable are compared with a constant (or unit) quantity of another variable, for example, distance covered within certain time and quantity purchased or sold per unit price.

The power mean of order α is usually considered the generalised mean[2], and is given for, say, three dimensions, X_1, X_2, and X_3 by: $X(\alpha) = [w_1 X_1^{\alpha} + w_2 X_2^{\alpha} + w_3 X_3^{\alpha}]^{1/\alpha}$, where $w_1 + w_2 + w_3 = 1$ and α may be any real number except zero. The arithmetic, quadratic and harmonic means are its special cases when, respectively, $\alpha = 1$, $\alpha = 2$, and $\alpha = -1$. The UNDP (*ibid.*) elevates power mean of order $\alpha > 1$ in comparison with arithmetic mean, leaving out others. It would hence be worthwhile to look into the links among the means, both the known and the potential ones, and their strengths and weaknesses in terms of their properties in comparison with each other. The present paper is a preliminary attempt at this. We take in this paper the power mean of order α

[2] See, for example, Abramowitz and Stegun (1965: 10).

> 1 (this special case is hereafter called PM) as one of an array of special cases, distributed in the continuum of the domain of a function, more popularly known in economics as the Constant Elasticity of Substitution (CES) function. This array of special cases of means includes the 4 known means: PM, arithmetic mean (AM), geometric mean (GM), and harmonic mean (HM) as well as 3 potential means which we call CES proper mean (CPM), inverse CES mean (ICM) and inverse power mean (IPM). It should, however, be noted that the power mean (function) in its entire domain ($-\infty$ to $+\infty$) can still be regarded as a generalised mean with all the seven means we consider here being its special cases. In this sense power mean and CES mean are substitutable.

What follows is divided into two sections: the next section reexamines the means: we find that the means we commonly use, the AM, the GM and the HM, along with the PM, are special cases of the CES function. We acknowledge the possibility of an inverse CES mean, and hence, that of an inverse power mean (IPM) also. Next we analyse the links among these means and attempts to bring out their properties on a comparative plane. In the last section, we sum up our results where we argue that the inverse power mean (of order 2) can be a good second best alternative to the arithmetic mean, useful as a representative average to index development.

2. The Means Reexamined

CES Function as Generalised Mean

Needless to say, averages are functions, but all functions are not averages. The two well-known functions, Cobb-Douglas (CD) and Constant Elasticity of Substitution (CES), commonly used in economic analysis, *are* averages. While the CD function generalises geometric mean (GM), the CES function does generalise all the means as well as the CD function.

Given two variable inputs (or dimensions, as in this case), X_1 and X_2, the CD function is: $AX_1^{a}X_2^{1-a}$, $0 < a < 1$, and A is a constant. Taking A =1, and a = 0.5, we have the geometric mean (GM) of the two dimensions: $(X_1X_2)^{1/2}$. Similarly, for the two inputs, the CES function is given by: $A(w_1X_1^{-\rho} + w_2X_2^{-\rho})^{-1/\rho}$, $w_1 + w_2 = 1$. Taking $A = 1$, this function represents a generalised mean of wider range, which we call the CES mean (or CM): $X(\rho) = (w_1X_1^{-\rho} + w_2X_2^{-\rho})^{-1/\rho}$, $w_1 + w_2 = 1$. It can be seen that the power mean of order $\alpha > 1$ (PM) is a special case of it (with $\rho = -\alpha$): $(w_1X_1^{\alpha} + w_2X_2^{\alpha})^{1/\alpha}$. We have also the arithmetic mean (AM: $(w_1X_1 + w_2X_2)$, with $\rho = -1$) and the harmonic mean (HM: $(0.5X_1^{-1} + 0.5X_2^{-1})^{-1}$, with $\rho = 1$) as its special cases. Again, it can be seen, using *L'Hopital*'s rule, that, with $w_1 = w_2 = 0.5$, and $\rho = 0$, the GM also is a special case of the CM.[3]

Still another condition also generates the three means directly from the CES mean. Using proportionate weights, $w_i = X_i / \sum_i X_i$, we can see that the CM yields AM when $\rho = 1$, GM, when $\rho = 2$, and (approximately) HM when $\rho = 3$ (see Table 1).

Now defining $\rho > 1$, $X(\rho)$ may be taken as a 'CES proper mean' (CPM) and hence we can also have an 'inverse CES mean' (ICM) of order γ with $\rho = 1/\gamma$; that is, $X(1/\gamma)$ = $\left(w_1X_1^{-1/\gamma} + w_2X_2^{-1/\gamma}\right)^{-\gamma}$, for $\gamma > 1$, (or $0 < \rho < 1$) and hence an 'inverse power mean' (IPM) of order β with $\rho = -1/\beta$; that is $X(-1/\beta)$ = $\left(w_1X_1^{1/\beta} + w_2X_2^{1/\beta}\right)^{\beta}$, for $\beta > 1$, (or $-1 < \rho < 0$). Note that for $0 < \gamma < 1$, the former is identical with the CPM (with $\rho > 1$), and for $0 < \beta < 1$, the latter is identical with the power mean (with $\alpha > 1$). Thus we can think of seven different means as special cases of the CES function (or CES mean) as

[3] The proofs are very simple and hence are not reported; however, they will be available from the author upon request.

the parameter ρ varies in its domain: PM, AM, IPM, GM, ICM, HM and CPM, as given in the following figure (Fig. 1), which shows the distribution of the means in the continuum of the domain of the CES function from – ∞ to + ∞. (also see Table 1).

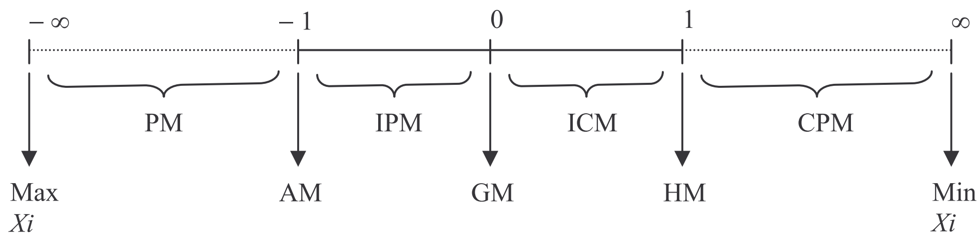

Fig. 1: Distribution of the means in the domain of the CES function as ρ varies from – ∞ to + ∞.

Links among the Means

The seven means, distributed in the continuum of the number line, are expected to exhibit some regular pattern of links among them. Though the following we prove for a two variable case with an order of 2 for the general means of PM, IPM, ICM and CPM, the same is true in general

1. It is easy to see that we have the following order of relationship among them, as ρ varies from – ∞ to + ∞. This is true in general for the means (irrespective of the number of variables).

$$\text{Max } \{X_i\} \geq \text{PM} \geq \text{AM} \geq \text{IPM} \geq \text{GM} \geq \text{ICM} \geq \text{HM} \geq \text{CPM} \geq \min \{X_i\}.$$

2. It follows from the above that the sum of the deviations of the components from the PM is negative (remember that from the AM is zero), and that from each of the other means is positive. This also is a general result.

3. Given the above relationship among the means, we can also find that the AM lies much farther from the PM than from the IPM:

In general we find that among these 7 means, as special cases of the CES function distributed in the continuum of its domain, PM and CPM mark respectively the upper and lower extremes and lie farther from their immediate neighbour means. Moreover, the means are by no means symmetric about GM, which is though obtained as the generalised mean (CM) approaches its centre of gravity of its domain, that is, zero.

Properties of the Means

We consider the following properties (Table 2).

1. Additivity: By this property we mean that a mean exists even if one or more, but not all, components (or dimensions, X_i) are zero. This is possible only if the components are expressed in the mean in a *meaningfully* additive way, such as in arithmetic mean, power mean and inverse power mean. The CES proper and the inverse CES means along with the harmonic mean, though additive, are not defined when any component, X_i, is zero. So is the geometric mean. Additivity need not be a necessary property of an average, since in most cases the zero value may be just ignored as a non-entity; however, where absences are involved in multi-dimensional contexts, additivity is an essential property and a mean devoid of it is meaningless, unless the components have a *combined existence*, as in a production function.[4] As long as the dimensions (as in the case of deprivation, say) are considered independent in averaging, only the arithmetic mean, power mean and inverse power mean come out qualified here.

[4] For example, in a production function with two factors, X_1 and X_2, we have the following *combined existence* condition: $f(X_1, 0) = f(0, X_2) = 0$.

2. Additional Weight: The three means, AM, GM and HM have no provision for any *additional* weights to the components. It should, however, be noted that GM gives less weight to large components and more to small ones than does the AM; this leaves the GM never larger than the AM. Since the components are taken in terms of their reciprocals in the HM, the component with the least absolute value will have the maximum impact on the mean value, and this leaves the HM never greater than the GM. This is so in the case of the CES proper and the inverse CES means also though with additional weights from power term. This weighting has much relevance in the marginalist approach in the demand and production analysis of economics, where a scarce variable has higher importance, and these means are thus useful for indexing such economic variables. On the other hand, the components are expressed in positive power terms in the PM and its inverse (IPM) and this results in greater contribution to the mean from larger components, and in the case of deprivation, this is a desirable property. Because the usual assumption here is that as the deprivation in a certain field increases, the weight to be put on removing deprivation in that dimension should also increase. That is, that particular deprivation field should have higher additional weight. This is achieved through $\alpha > 1$ in PM and $\beta > 1$ in IPM. However, it should be noted that the weight structure differs in these cases, with $\alpha > 1$ for PM, ($\alpha = 1$ for AM) and $0 < \alpha < 1$ (equivalent to $\beta > 1$) for IPM.[5]

3. Slope (first derivative): It goes without saying that the mean should vary directly with the component, that it is monotonic increasing in each X_i. That is, its first derivative be positive. And it is so in all the cases we consider here.

4. Curvature (second derivative): While the first derivative of a function tells us about its rate of change, or the slope of its graph, the second derivative indicates its

[5] Such weight structure explains why we have PM ≥ AM ≥ IPM ≥ GM ≥ ICM ≥ HM ≥ CPM.

curvature, which in turn describes concavity/convexity of the function. If the second derivative of a function is negative (positive) for all its argument, then that (primitive) function must be a strictly concave (convex) function. AM is a linear function, and is hence out of consideration here. And we find that all other functions except the PM are concave; there is a 'law of diminishing returns' at work in these cases. It should be noted that the IPM is concave subject to the permissible domain of $\beta > 1$ only; as already mentioned, for $0 < \beta < 1$, it is identical with the PM (with $\alpha > 1$), which is convex. Thus the function is increasing at an increasing rate ('increasing returns') only in the case of PM (for $\alpha > 1$). UNDP (1997: 213) prescribes that a poverty index be convex with respect to each of its deprivation dimensions, X_i; "This is equivalent to saying that [the poverty index] decreases with reductions in [X_i], and at a diminishing rate." (*ibid.*) In this light, it is as well reasonable to assume that a development index exhibit 'diminishing returns'; it rises with increases in a development indicator but at a diminishing rate, reflecting the tension and friction that set in as more and more development accumulates. Hence all the 5 concave functions (IPM, GM, ICM, HM and CPM) are possible candidates for a development index. However, the 'marginalist' weighting of all but the first of these means makes them less desirable. IPM, on the other hand, attaches only fractional weights compared with PM (and AM).

5. Linear Homogeneity: A function is homogeneous of degree one (linear), if a certain proportionate change in its arguments alters the function also by the same proportion. For example, if all the X_is are increased by 10 percent, then the function also increases by 10 percent. This is a desirable property of an average, and all the means we consider here are linearly homogeneous.

6. Elasticity of Substitution: The elasticity of substitution (denoted by σ) is the rate at which substitution between, say, X_1 and X_2 along an iso-$X(\alpha)$ curve, keeping X_3 constant, takes place, and is defined as the percentage change in (X_1/X_2) for a unit

percentage change in the slope of the tangent along this curve, projected onto X_1-X_2 space at the given value of X_3 (UNDP 1997: 121). For the PM of order α, it is given by $\sigma = 1/(\alpha - 1)$. The elasticity of substitution (σ) between any two X_is, given the other dimension(s), is infinite for AM (with $\alpha = 1$). In such perfect substitutability case, that is, with AM, the impact on the mean of a unit change in any of the dimensions is the same (since there is no additional weight attached to the dimension), regardless of the level of deprivation implied in them. However, this goes against the usual assumption that as the deprivation in a certain field increases, the weight to be put on removing deprivation in that dimension should also increase. That is, that particular deprivation field should have higher additional weight. This necessitates $\alpha > 1$, as for the PM. However, as the order α increases infinitely from unity, the elasticity of substitution for the PM decreases monotonically from infinity to zero. that is, in zero substitutability case, the aggregate index tends to the maximum of the given dimensions, leaving the changes in the other dimensions with no effect at all on the index; poverty in effect just becomes uni-dimensional. Hence this case also is untenable. Hence the selection by the UNDP of $\alpha = 3$, where we have a reasonable intermediate case of $\sigma = 1/2$, and this puts greater weight on that dimension with higher deprivation.. For the CD function and hence GM, it is unity, and for HM, one-half. For the CES proper mean, it lies between one-half and zero, as ρ moves from unity to infinity. In the case of the inverse CES mean, for which $\sigma = \frac{\gamma}{\gamma + 1} = \frac{1}{\rho + 1}$, as for the CES one, the opposite movement holds: as γ increases from unity, the elasticity measure rises from one-half to unity. For the IPM, with $\beta > 1$, implying $\rho < 1$, we have $\sigma = \frac{\beta}{\beta - 1} = \frac{1}{1 - \rho}$, and as β increases from unity, the elasticity falls from infinity to unity.

To recap: the minimum to maximum values of the elasticity of substitution in the case of CPM: zero to one-half; HM: one-half; ICM: one-half to unity; GM: unity; IPM: unity to infinity; AM: infinity; and PM: zero to infinity.

Now for $\alpha = 3$, the UNDP's choice, the PM has a limited substitution elasticity of one-half. This choice implies an absolute value of $\rho = 3$, and for the CES function, then, the elasticity is 1/4, and for the inverse CES function, it is 3/4 for $\gamma = 3$. In the case of the IPM for $\beta = 3$, the elasticity of substitution is 1.5.

7. Impact Multiplier: In the case of power mean, for any α, the relative impact[6] on the aggregate index of a unit change in X_1 compared with that in X_2 is given by $(X_1 / X_2)^{\alpha-1}$. For arithmetic mean, when $\alpha = 1$, the relative impact is unity: impact is the same for any dimension, whatever be the level of deprivation. On the other hand, as α tends to infinity (for the PM) the impact from a unit increase in X_1 becomes infinitely larger, so that it becomes the only determinant of the index. For GM, it is always given by the ratio of the two dimensions, and for HM, by the square of the ratio. In the case of IPM, the weight of the relative impact, given by the reciprocal of the elasticity of substitution, is always less than unity, giving the lightest impact among the means, and in ICM, greater than unity. The greatest impact is experienced in the case of CPM. An example will illustrate the points clearly.[7] Let X_1 = 50 percent and X_2 = 25 percent (with X_3 = 40 percent, being kept constant). In the case of PM, for any α, the relative impact on the mean of a unit change in X_1 compared with that in X_2 is then given by $2^{\alpha-1}$. For AM, it is unity,

[6] Note that the impact of any dimension X_i on the power mean is given by its first derivative with respect to that dimension. The above result is obtained by putting i = 1 and 2, and taking the ratio of the two derivatives (assuming equal weights).

[7] Also see UNDP (1997: 121).

GM: 2, HM: 4, IPM: $2^{(\beta-1)/\beta}$; ICM: $2^{(\gamma+1)/\gamma}$; and for CPM: $2^{\rho+1}$. For an absolute order of 3 (as used by the UNDP for α), the relative impact for PM is 4; for IPM: 0.67; ICM: 1.33; and for CPM: 16.

3. To Sum Up

The power mean as an aggregate index of multi-dimensional deprivation has thus a number of desirable properties, but it too has its own limitations, just as the other means do.[8] For example, the PM fails as a representative average. Remember AM is the only summary measure representative of a set of data, with the least variance property. PM, being much farther on the upper end from AM, is thus entirely atypical of its components in the above sense. Similarly, GM is theoretically considered the best average in the construction of index numbers as well as for ratios and percentages. Again, where scarce factors command higher marginal importance, HM along with CPM and ICM, the last two with additional weights, deserves consideration. However, none goes unblemished; all these have their own limitations too. AM is a linear function, incapable of further differentiation and has an infinite elasticity of substitution that render it less desirable, and the others violate the additivity condition.

IPM, on the other hand, has some additional qualities. We have seen that AM is much closer to IPM than to PM. To the extent that we need an average typical of the components in the face of an unacceptable AM, we may seek for one that is closer to the AM, so that this second best choice will have the minimum deviations next to the AM, but is free from its unacceptable blemishes. And we find this basic criterion is satisfied by the IPM only.

[8] The originators themselves admit, this index also is not without failings. "Like all measures, the HPI has weaknesses – in data and in concept. Like all measures, it cannot capture the totality of human poverty." (*ibid.*).

Given that the IPM, as compared with the PM, is typical, in a second best sense of being closer to the AM, of all the components considered, we now turn to its other properties. Along with the PM, it also enjoys additional weight on each component, the larger component contributing a greater effect to the mean, desirable for a deprivation index. With an order of 2, the PM has an elasticity of substitution of 1 (unity) and the IPM, 2, much limited compared with the infinity of the AM. It should be noted that as the order falls, both the PM and the IPM approach the AM, the latter much closer than the former; in both the cases, 2 is the minimum possible order that keeps the measures closer to the AM with a limited substitution elasticity. The IPM, however, differs from the PM in respect of convexity and impact multiplier. But as we have already noted earlier, the 'diminishing returns' property of the IPM appears appropriate for a development index. Thus while the PM captures the multi-dimensional deprivation, its inverse, the IPM seems to offer a multi-dimensional development index.

In Table 3, we compare these 7 means and their ranks in respect of the reported deprivation in three dimensions of longevity, knowledge and economic provisioning of 78 countries, as given in UNDP (1997). An order of 2 is taken for the general means of PM, IPM, ICM and CPM. Note that the ranks of the countries must remain the same, irrespective of the order of α in the case of the power mean, though the ranks of some of the countries given in Table 3 are different from those in UNDP (1997). This is just because we use the data as given in the UNDP Report, which may be different from the actual fractionally enough to distort some ranks.

We find that the mean values are distributed in a descending order from PM to CPM and AM is much closer to IPM than to PM, and HM much closer to ICM than to CPM. The rank of the countries by AM is equal in 20 cases to that by PM and in 28 cases to that by IPM. Number of cases of rank equality among the different means are as follows: (i) PM with AM: 20; with IPM: 14; with GM: 10; with ICM: 9 with HM: 5; and with CPM: 5; (ii) AM with IPM: 28; with GM: 20; with ICM: 17; with HM: 12; and with CPM: 5; (iii) GM with IPM: 29; with ICM: 34; and (iv) HM with ICM: 32; with CPM: 23.

The proximity of IPM to AM compared with that of PM to AM is further evident from the following: that is, the sum of the squares of the deviations between the ranks by PM and by AM is 520, whereas that by AM and by IPM is only 188. The same for the other paired consecutive means are: (i) IPM–GM:188.5; (ii) GM–ICM: 156; (iii) ICM–HM: 110; and (iv) HM–CPM: 227.5. It is interesting to compare this result with our conclusion in Appendix 5 that (PM – AM) > (AM – IPM) = (IPM – GM) > (GM – ICM) > (ICM – HM) < (HM – CPM).

Table 1: The Special Cases of the Generalised (CES) Mean: $X(\rho) = \left(\sum_i^n w_i X_i^{-\rho} \right)^{-1/\rho}$

1.	$\lim_{\rho \to \infty} X(\rho) = \min \{X_i\}$, $i = 1, 2, \ldots, n$.
2.	$X(\rho)$ = CES Proper Mean, $\rho > 1$.
3.	$X(1)$ = Harmonic Mean ($\rho = 1$).
4.	$X(1/\gamma)$ = Inverse CES Mean, $\gamma > 1$. ($0 < \rho < 1$)
5.	$\lim_{\rho \to 0} X(\rho)$ = Geometric Mean
6.	$X(-1/\beta)$ = Inverse Power Mean, $\beta > 1$. ($-1 < \rho < 0$)
7.	$X(-1)$ = Arithmetic Mean ($\rho = -1$).
8.	$X(-\alpha)$ = Power Mean, $\alpha > 1$. ($\rho < -1$)
9.	$\lim_{\rho \to -\infty} X(\rho) = \max \{X_i\}$, $i = 1, 2, \ldots, n$.
	With Proportionate Weights: $w_i = X_i / \Sigma X_i$
10.	$X(1)$ = Arithmetic Mean
11.	$X(2)$ = Geometric Mean
12.	$X(3) \approx$ Harmonic Mean

Note: $\sum_i w_i = 1$.

Table 2: Properties of the Means

	When any Xi = 0,	Additional Weight of each Xi	Slope	Curvature	Degree of Homogeneity	Elasticity of Substitution	Impact Multiplier
AM	$\neq 0$	No	Positive	Zero	Linear	Infinity	1
GM	= 0	No	Positive	Concave	Linear	Unity	(X_1/X_2)
HM	Not defined	No	Positive	Concave	Linear	1/2	$(X_1/X_2)^2$
PM of order $\alpha > 1$	$\neq 0$	Maximum with the maximum Xi	Positive	Convex	Linear	$\frac{1}{\alpha - 1}$	$(X_1 / X_2)^{\alpha - 1}$
IPM of order $\beta > 1$	$\neq 0$	Maximum with the maximum Xi	Positive	Concave	Linear	$\frac{\beta}{\beta - 1}$	$(X_1 / X_2)^{(\beta - 1)/\beta}$
CPM of order $\rho > 1$	Not defined	Maximum with the minimum Xi	Positive	Concave	Linear	$\frac{1}{\rho + 1}$	$(X_1 / X_2)^{\rho + 1}$
ICM of order $\gamma > 1$	Not defined	Maximum with the minimum Xi	Positive	Concave	Linear	$\frac{\gamma}{\gamma + 1}$	$(X_1 / X_2)^{(\gamma + 1)/\gamma}$

Table 3: Comparison of the Different Averages of Deprivation and Their Ranks of 78 Countries, 1997.

	Deprivation in			PM	AM	IPM	GM	ICM	HM	CPM	Ranks						
	Longevity	Knowledge	Eco. Prov								PM	AM	IPM	GM	ICM	HM	CPM
Trinidad & Tobago	5.4	2.1	3	3.77	3.50	3.37	3.24	3.12	3.02	2.84	1	1	1	1	1	1	1
Cuba	6.2	4.6	4	5.02	4.93	4.89	4.85	4.81	4.77	4.70	2	2	2	2	2	2	5
Chile	4.6	5	6	5.23	5.20	5.18	5.17	5.15	5.14	5.11	3	3	3	3	4	5	6
Singapore	3.2	9	5	6.22	5.73	5.48	5.24	5.02	4.81	4.47	4	4	4	4	3	3	3
Costa Rica	4.1	5.3	9	6.48	6.13	5.97	5.80	5.65	5.52	5.28	5	5	5	5	5	7	7
Colombia	6.3	8.9	14	10.25	9.73	9.48	9.22	8.98	8.76	8.36	6	9	10	11	11	12	12
Jordan	9.2	14.5	5	10.33	9.57	9.15	8.74	8.33	7.94	7.28	7	7	8	9	9	10	11
Uruguay	5.4	2.9	17	10.43	8.43	7.38	6.43	5.67	5.09	4.38	8	6	6	6	6	4	2
Panama	6.2	9.5	15	10.86	10.23	9.91	9.60	9.29	9.00	8.50	9	11	12	12	12	13	13
Mexico	8.3	10.8	13	10.87	10.70	10.61	10.52	10.43	10.34	10.17	10	14	14	14	15	16	17
Thailand	8.9	6.5	16	11.22	10.47	10.10	9.75	9.42	9.13	8.64	11	13	13	13	13	14	14
Jamaica	4.3	15.6	11	11.30	10.30	9.69	9.04	8.37	7.74	6.72	12	12	11	10	10	8	9
Mauritius	6.2	17.6	6	11.32	9.93	9.27	8.68	8.19	7.80	7.25	13	10	9	8	8	9	10
United Arab emirates	3.6	21.4	4	12.74	9.67	8.07	6.75	5.82	5.22	4.60	14	8	7	7	7	6	4
Ecuador	9.9	10.4	20	14.21	13.43	13.06	12.72	12.41	12.14	11.69	15	15	16	17	20	21	21
Mongolia	16	17.8	12	15.46	15.27	15.17	15.06	14.96	14.85	14.63	16	18	20	22	24	25	28
Philippines	12.8	5.6	24	16.03	14.13	13.06	11.98	10.96	10.05	8.69	17	16	16	16	16	15	15
China	9.1	19.1	20	16.81	16.07	15.63	15.15	14.65	14.13	13.16	18	20	22	23	22	23	23
Zimbabwe	18.4	15.3	18	17.29	17.23	17.21	17.18	17.15	17.12	17.06	19	23	26	28	30	32	32
Libyan Arab Jamahiriya	16.2	25	4	17.35	15.07	13.51	11.74	10.00	8.53	6.65	20	17	17	15	14	11	8
Dominican Rep	10.2	18.5	22	17.61	16.90	16.50	16.07	15.63	15.19	14.33	21	22	24	25	25	26	25
Sri Lanka	7.9	9.9	29	18.27	15.60	14.29	13.14	12.19	11.45	10.46	22	19	18	18	17	19	20
Syrian Arab Rep	10.3	30.2	12	19.68	17.50	16.45	15.51	14.71	14.05	13.11	23	25	23	24	23	22	22
Indonesia	14.8	16.8	27	20.25	19.53	19.19	18.86	18.56	18.28	17.79	24	30	31	32	33	33	34
Paraguay	9.2	8.1	33	20.32	16.77	15.01	13.50	12.31	11.43	10.36	25	21	19	19	19	18	18.5

	Deprivation in			PM	AM	IPM	GM	ICM	HM	CPM	Ranks						
	Longevity	Knowledge	Eco. Prov								PM	AM	IPM	GM	ICM	HM	CPM
Iran, Islamic Rep of	11.7	31.4	13	20.75	18.70	17.72	16.84	16.08	15.44	14.52	26	27	27	26	27	27	26
Peru	13.4	11.7	32	21.14	19.03	18.03	17.12	16.33	15.68	14.72	27	28	28	27	28	28	29
Honduras	10.8	28	21	21.15	19.93	19.24	18.52	17.78	17.05	15.74	28	31	32	31	31	31	31
Botswana	15.9	31.3	11	21.24	19.40	18.49	17.62	16.84	16.15	15.05	29	29	29	30	29	29	30
Tunisia	10.5	34.8	7	21.37	17.43	15.43	13.68	12.28	11.24	9.95	30	24	21	20	18	17	16
Bolivia	19.6	17.5	28	22.17	21.70	21.47	21.26	21.05	20.85	20.49	31	33	33	35	35	36	38
Viet Nam	12.1	7	37	22.84	18.70	16.56	14.63	13.07	11.88	10.36	32	27	25	21	21	20	18.5
Algeria	10.6	40.6	12	25.20	21.07	19.04	17.29	15.88	14.83	13.50	33	32	30	29	26	24	24
Kenya	22.3	23	31	25.74	25.43	25.29	25.15	25.01	24.88	24.64	34	38	38	38	39	40	41
Nicaragua	13.6	34.7	25	25.91	24.43	23.61	22.77	21.91	21.08	19.57	35	35	36	37	37	37	37
El Salvador	11.7	29.1	34	26.71	24.93	23.83	22.62	21.35	20.10	17.91	36	36	37	36	36	35	35
Lesotho	23.9	29.5	28	27.24	27.13	27.08	27.03	26.97	26.92	26.81	37	39	39	40	42	42	44
Iraq	15.4	43.2	14	27.69	24.20	22.53	21.04	19.80	18.81	17.45	38	34	34	34	34	34	33
Congo	22.1	26.1	36	28.67	28.07	27.77	27.49	27.21	26.94	26.45	39	41	41	42	43	43	42
Myanmar	25.6	17.3	41	29.64	27.97	27.12	26.28	25.49	24.74	23.44	40	40	40	39	40	39	40
Egypt	16.6	49.5	10	30.69	25.37	22.63	20.18	18.17	16.63	14.62	41	37	35	33	32	30	27
Cameroon	25.4	37.9	28	30.91	30.43	30.20	29.98	29.77	29.57	29.19	42	42	44	44	45	45	45
Papua New Guinea	28.6	28.8	37	31.71	31.47	31.35	31.24	31.13	31.02	30.82	43	44	46	46	46	47	52
Ghana	24.9	36.6	34	32.23	31.83	31.62	31.41	31.19	30.96	30.50	44	45	47	47	47	46	49
Guatemala	14.5	44.3	35	33.65	31.27	29.81	28.22	26.58	24.98	22.21	45	43	42	43	41	41	39
Zambia	35.1	23.4	42	34.37	33.50	33.03	32.55	32.06	31.57	30.60	46	48	48	48	48	48	50
India	19.4	48.8	29	34.64	32.40	31.27	30.17	29.13	28.16	26.52	47	46	45	45	44	44	43
Rwanda	42.1	40.8	28	37.51	36.97	36.67	36.37	36.05	35.72	35.06	48	50	51	53	55	57	57
Morocco	12.3	57.9	28	37.81	32.73	29.91	27.12	24.55	22.34	19.15	49	47	43	41	38	38	36
Togo	28.4	49.6	33	38.10	37.00	36.47	35.96	35.47	35.02	34.20	50	51	50	51	52	55	56
Zaire	30	23.6	55	38.65	36.20	35.01	33.89	32.87	31.95	30.44	51	49	49	49	49	49	48
Tanzania, U Rep of	30.6	33.2	50	38.90	37.93	37.47	37.03	36.62	36.23	35.54	52	52	52	55	56	58	58

	Deprivation in			PM	AM	IPM	GM	ICM	HM	CPM	Ranks						
	Longevity	Knowledge	Eco. Prov								PM	AM	IPM	GM	ICM	HM	CPM
Lao People's Dem Rep	32.7	44.2	42	39.95	39.63	39.47	39.30	39.13	38.95	38.60	53	55	57	58	60	62	64
Sudan	25.2	55.2	35	40.44	38.47	37.48	36.51	35.60	34.73	33.22	54	53	53	54	54	54	55
Uganda	39	38.9	45	41.07	40.97	40.92	40.87	40.82	40.78	40.69	55	58	61	62	66	66	70
Nigeria	33.8	44.4	45	41.39	41.07	40.90	40.72	40.54	40.36	39.98	56	59	60	61	64	65	67
Central African Rep	35.4	42.8	46	41.64	41.40	41.28	41.15	41.03	40.90	40.64	57	61	62	64	67	68	69
Namibia	21.1	60	37	42.48	39.37	37.71	36.05	34.44	32.94	30.36	58	54	55	52	51	51	47
Guinea-Bissau	43.2	46.1	41	43.48	43.43	43.41	43.38	43.36	43.33	43.28	59	65	69	70	70	71	73
Pakistan	22.6	62.9	36	43.83	40.50	38.79	37.13	35.56	34.12	31.72	60	57	56	56	53	52	53
Cote d'Ivoire	23.1	60.6	40	43.99	41.23	39.75	38.26	36.79	35.38	32.90	61	60	58	57	58	56	54
Senegal	25.3	67.9	26	44.45	39.73	37.49	35.48	33.76	32.36	30.34	62	56	54	50	50	50	46
Mauritania	31.7	63.1	31	44.53	41.93	40.71	39.58	38.56	37.66	36.22	63	62	59	60	59	59	59
Bhutan	33.2	58.9	38	44.78	43.37	42.69	42.04	41.43	40.86	39.86	64	64	66	68	68	67	66
Haiti	27.1	55.9	47	44.97	43.33	42.42	41.45	40.45	39.44	37.49	65	63	64	66	63	63	62
Malawi	38.3	44.2	53	45.57	45.17	44.97	44.77	44.57	44.38	44.00	66	69	71	71	72	72	74
Yemen	25.6	58.9	47	45.95	43.83	42.64	41.38	40.09	38.80	36.38	67	67	65	65	62	61	60
Bangladesh	26.4	62.7	42	46.16	43.70	42.41	41.12	39.85	38.64	36.47	68	66	63	63	61	60	61
Burundi	33.8	65.4	33	46.58	44.07	42.88	41.78	40.79	39.90	38.47	69	68	68	67	65	64	63
Guinea	41.3	65.2	30	47.81	45.50	44.35	43.23	42.16	41.16	39.40	70	70	70	69	69	69	65
Madagascar	32.1	54.2	56	48.66	47.43	46.74	46.01	45.25	44.47	42.90	71	72	72	72	73	73	71
Mozambique	43.8	60.5	42	49.47	48.77	48.43	48.10	47.79	47.49	46.94	72	73	75	75	75	75	75
Cambodia	31.9	65	50	50.80	48.97	47.98	46.98	45.96	44.96	43.04	73	74	74	74	74	74	72
Mali	28.4	70.7	49	52.30	49.37	47.78	46.16	44.56	43.00	40.20	74	75	73	73	71	70	68
Burkina Faso	36.1	81.3	21	52.77	46.13	42.72	39.50	36.65	34.24	30.68	75	71	67	59	57	53	51
Ethiopia	35.7	65.5	59	54.91	53.40	52.56	51.67	50.75	49.81	47.95	76	76	76	76	76	76	76
Sierra Leone	52.1	69.7	52	58.53	57.93	57.65	57.37	57.10	56.85	56.37	77	77	77	78	78	78	78
Nigeria	43.2	86.9	50	63.03	60.03	58.60	57.26	56.01	54.89	52.99	78	78	78	77	77	77	77

Source: The first three columns are from UNDP (1997).
Note: Eco. Prov. = Economic Provisioning; PM = Power Mean of order 2; AM = Arithmetic Mean; IPM = Inverse Power Mean of order 2; GM = Geometric Mean; ICM = Inverse CES Mean of order 2; HM = Harmonic Mean; and CPM = CES Proper Mean of order 2.

(These appendices are not meant for publication with the main text)

APPENDIX 1

Here we prove that the CES mean (CM) becomes the GM as $\rho \to 0$, when the weights (w_i) are equal, (just as the CES function becomes the Cobb-Douglas function for $\rho = 0$ (with unitary elasticity of substitution), by virtue of *L'Hopital*'s rule).

Consider the CM with two components of equal weights: $X(\rho) = (0.5X_1^{-\rho} + 0.5X_2^{-\rho})^{-1/\rho}$. The *L'Hopital*'s rule states that if $\lim_{y \to a} f(y) = 0$ and $\lim_{y \to a} g(y) = 0$, then $\lim_{y \to a} \frac{f(y)}{g(y)} = \lim_{y \to a} \frac{f'(y)}{g'(y)}$ (see Apostol 1967: 292 – 295).

Taking the natural logarithm of the $X(\rho)$, given above, we have

$$\ln X(\rho) = \frac{-\ln(0.5X_1^{-\rho} + 0.5X_2^{-\rho})}{\rho} = \frac{f(\rho)}{g(\rho)}.$$

Evidently, as $\rho \to 0$, we have $f(\rho) \to 0$, and $g(\rho) \to 0$. Taking the derivatives of the denominator and the numerator,

$$g'(\rho) = 1, \text{ and } \ f'(\rho) = \frac{0.5X_1^{-\rho} \ln X_1 + 0.5X_2^{-\rho} \ln X_2}{0.5X_1^{-\rho} + 0.5X_2^{-\rho}},$$

which converges to $0.5 \ln X_1 + 0.5 \ln X_2$, as $\rho \to 0$. Hence, the limiting case is

$\ln X(\rho) = 0.5 \ln X_1 + 0.5 \ln X_2$, or

CM $= X(\rho) = (X_1 X_2)^{1/2} =$ GM.

APPENDIX 2

Here we prove that PM ≥ AM ≥ IPM ≥ GM ≥ ICM ≥ HM ≥ CPM.

Let a and b be two positive numbers such that $a \neq b$. We consider power mean (PM), inverse power mean (IPM), inverse CES mean (ICM) and CES proper mean (CPM) all of order 2 only.

We have AM = $\frac{a+b}{2}$, GM = $\sqrt{ab}$, HM = $\left(\frac{a^{-1}+b^{-1}}{2}\right)^{-1} = \frac{2ab}{a+b}$,

CES = $\left(\frac{a^{-2}+b^{-2}}{2}\right)^{-1/2} = \frac{\sqrt{2}ab}{(a^2+b^2)^{1/2}}$, ICES = $\left(\frac{a^{-1/2}+b^{-1/2}}{2}\right)^{-2} = \frac{4ab}{\left(\sqrt{a}+\sqrt{b}\right)^2}$,

PM = $\left(\frac{a^2+b^2}{2}\right)^{1/2}$, and IPM = $\left(\frac{a^{1/2}+b^{1/2}}{2}\right)^2$

1. First we prove PM > AM.

Suppose it is true, that is, PM > AM. Then, $\left(\frac{a^2+b^2}{2}\right)^{1/2} > \frac{a+b}{2}$. Or

$$\left(\frac{a^2+b^2}{2}\right) > \left(\frac{a+b}{2}\right)^2 = \frac{a^2+b^2+2ab}{4}.$$

Bringing the RHS terms to the LHS, we get

$\frac{a^2+b^2-2ab}{4} = \left(\frac{a-b}{2}\right)^2 > 0$. Since the square of any real quantity is always positive, $\left(\frac{a-b}{2}\right)^2 > 0$ is true, and hence PM > AM also is true.

2. Now we proceed to prove AM > IPM.

We start with the assumption that AM > IPM, so that $\frac{a+b}{2} > \left(\frac{a^{1/2}+b^{1/2}}{2}\right)^2$. Or

$$\frac{a+b}{2} > \left(\frac{a^{1/2}+b^{1/2}}{2}\right)^2 = \left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2.$$

That is, $\frac{a+b}{2} > \frac{a+b+2\sqrt{ab}}{4}$.

Bringing the RHS terms to the LHS, we get

$$\frac{a+b-2\sqrt{ab}}{4} = \left(\frac{\sqrt{a}-\sqrt{b}}{2}\right)^2 > 0.$$

Since the square of any real quantity is always positive, $\left(\frac{\sqrt{a}-\sqrt{b}}{2}\right)^2 > 0$ is true, and hence AM > IPM also is true.

3. We now prove IPM > GM.

Starting with the assumption IPM > GM, we have $\left(\frac{a^{1/2}+b^{1/2}}{2}\right)^2 = \left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2 >$ $\sqrt{ab}$.

That is, $\frac{a+b+2\sqrt{ab}}{4} - \sqrt{ab} > 0$. Or,

$$\frac{a+b-2\sqrt{ab}}{4} = \left(\frac{\sqrt{a}-\sqrt{b}}{2}\right)^2 > 0.$$

Since the square of any real quantity is always positive, $\left(\frac{\sqrt{a}-\sqrt{b}}{2}\right)^2 > 0$ is true, and hence IPM > GM also is true.

4. Next we prove GM > ICM.

Proceeding with the assumption that GM > ICM, we have $\sqrt{ab}$ > $\left(\frac{a^{-1/2}+b^{-1/2}}{2}\right)^{-2} = \frac{4ab}{\left(\sqrt{a}+\sqrt{b}\right)^2}$. Or

$a+b+2\sqrt{ab} > 4\sqrt{ab}$, which gives

$a+b-2\sqrt{ab} > 0$. Or $(\sqrt{a}-\sqrt{b})^2 > 0$.

Since the square of any real quantity is always positive, $\left(\sqrt{a}-\sqrt{b}\right)^2 > 0$ is true, and hence GM > ICM also is true.

5. Now we prove ICM > HM.

Let us assume that ICM > HM, then we have $\left(\frac{a^{-1/2}+b^{-1/2}}{2}\right)^{-2} = \frac{4ab}{\left(\sqrt{a}+\sqrt{b}\right)^2}$ > $\left(\frac{a^{-1}+b^{-1}}{2}\right)^{-1} = \frac{2ab}{a+b}$. Or

$a+b > \frac{1}{2}\left(\sqrt{a}+\sqrt{b}\right)^2$, which gives

$a+b-2\sqrt{ab} > 0$. Or $(\sqrt{a}-\sqrt{b})^2 > 0$.

Since the square of any real quantity is always positive, $\left(\sqrt{a}-\sqrt{b}\right)^2 > 0$ is true, and hence ICM > HM also is true.

6. Finally we prove HM > CPM.

Starting with the assumption that HM > CPM, we have $\left(\dfrac{a^{-1}+b^{-1}}{2}\right)^{-1} = \dfrac{2ab}{a+b} >$

$\left(\dfrac{a^{-2}+b^{-2}}{2}\right)^{-1/2} = \dfrac{\sqrt{2}ab}{(a^2+b^2)^{1/2}}$. Or

$2(a^2+b^2) > (a+b)^2$, which gives

$a^2+b^2-2ab > 0$. Or $(a-b)^2 > 0$.

Since the square of any real quantity is always positive, $(a-b)^2 > 0$ is true, and hence HM > CPM also is true.

From the above it follows that PM > AM > IPM > GM > ICM > HM > CPM.
If a and b are equal, then PM = AM = IPM = GM = ICM = HM = CPM.
Thus we have PM ≥ AM ≥ IPM ≥ GM ≥ ICM ≥ HM ≥ CPM.
Since a mean lies between the maximum and the minimum of the components, it follows that max $\{X_i\}$ is greater than PM and min $\{X_i\}$ is lower than CPM.

APPENDIX 3

Here we show that the sum of the deviations of the components from the AM is zero, that from the PM is negative and that from each of all other means, positive:

1. Sum of the deviations of the components from the AM is zero:

Given the components X_i, i = 1, …, n, we have the AM: $\overline{X} = \frac{1}{n}\sum_1^n X_i$. Now let $\sum_1^n (X_i - \alpha) = 0$, where α is a constant. Then, $\sum_1^n X_i = n\alpha$. Therefore, α must be the AM.

2. Sum of the deviations of the components from the PM is negative:

Suppose $\sum_1^n (X_i - a) = \sum_1^n X_i - na < 0$, where a is a constant. This then gives AM (or $\overline{X}$) < a. Since in our case AM < PM, a must be the PM here.

3. Sum of the deviations of the components from all the means other than AM and PM is positive:

Let $\sum_1^n (X_i - b) = \sum_1^n X_i - nb > 0$, where b is a constant. This then gives AM (or $\overline{X}$) > b. Since we have AM > IPM and all other means below that, b must be any of the means other than AM and PM.

APPENDIX 4.

Let a and b be two positive numbers such that $a \neq b$. We consider CES proper mean (CPM), inverse CES mean (ICM), power mean (PM), and inverse power mean (IPM) of order 2.

We have AM = $\dfrac{a+b}{2}$, GM = $\sqrt{ab}$, HM = $\left(\dfrac{a^{-1}+b^{-1}}{2}\right)^{-1} = \dfrac{2ab}{a+b}$,

CPM = $\left(\dfrac{a^{-2}+b^{-2}}{2}\right)^{-1/2} = \dfrac{\sqrt{2}ab}{(a^2+b^2)^{1/2}}$, ICM = $\left(\dfrac{a^{-1/2}+b^{-1/2}}{2}\right)^{-2} = \dfrac{4ab}{\left(\sqrt{a}+\sqrt{b}\right)^2}$,

PM = $\left(\dfrac{a^2+b^2}{2}\right)^{1/2}$, and IPM = $\left(\dfrac{a^{1/2}+b^{1/2}}{2}\right)^2$

Appendix 4(a)

1. First we show that the absolute deviation of the AM from the PM is greater than that from the IPM. This results from the following proofs: (i) the deviation between the AM and the GM is twice that between the AM and the IPM, that is, AM – GM = 2(AM – IPM); and (ii) the deviation between the squared PM and the squared GM is twice that between the squared PM and the squared AM, that is, $PM^2 - GM^2 = 2(PM^2 - AM^2)$.

i) The deviation between the AM and the GM is twice that between the AM and the IPM, that is, AM – GM = 2(AM – IPM).

Suppose AM – GM = m, a constant. That is,

$$\frac{a+b}{2} - \sqrt{ab} = \frac{a+b-2\sqrt{ab}}{2} = \frac{\left(\sqrt{a}-\sqrt{b}\right)^2}{2} = m. \qquad \ldots(1)$$

And let AM – IPM = q, a constant. That is,

$$\frac{a+b}{2} - \left(\frac{a^{1/2}+b^{1/2}}{2}\right)^2 = \frac{a+b}{2} - \frac{a+b+2\sqrt{ab}}{4} = \left(\frac{\sqrt{a}-\sqrt{b}}{2}\right)^2 = q. \qquad \ldots(2)$$

From (1) and (2), we have $m = 2q$. That is,

AM – GM = 2(AM – IPM), which also implies that

AM – IPM = IPM – GM, which in turn also implies that

AM – GM = 2(IPM – GM).

That is, the absolute deviation of the IPM from the AM is as much as that from the GM. This, together with the relationship that AM > IPM > GM, then suggests that the three measures are equi-distant! This also suggests that IPM = (AM + GM)/2; that is, IPM is the arithmetic mean of AM and GM.

ii) Now we prove that the deviation between the squared PM and the squared GM is twice that between the squared PM and the squared AM, that is, $PM^2 - GM^2 = 2(PM^2 - AM^2)$.

Let $PM^2 – GM^2 = k$, a constant. This then gives

$$\frac{a^2+b^2}{2} - ab = k \text{ . Or, } \frac{(a-b)^2}{2} = k \text{ .} \qquad \text{….(3)}$$

Also assume $PM^2 – AM^2 = r$, a constant. Thus we have

$$\frac{a^2+b^2}{2} - \left(\frac{a+b}{2}\right)^2 = r \text{ , which gives } \left(\frac{a-b}{2}\right)^2 = r \text{ .} \qquad \text{….(4)}$$

From (3) and (4), we have $k = 2r$. That is,

$PM^2 – GM^2 = 2(PM^2 – AM^2)$, which also implies that

$PM^2 – AM^2 = AM^2 – GM^2$, which in turn also implies that

$PM^2 – GM^2 = 2(AM^2 – GM^2)$.

That is, the absolute deviation of the squared AM from the squared PM is as much as that from the squared GM. That is, (given PM > AM > GM), the squared values of these three measures are equi-distant! This together with the relationship that PM > AM > IPM > GM and the above proof (in 1) that AM, IPM, and GM are equi-distant then suggests that the PM is much farther from the AM than the IPM.

Also note from the above that $AM = \{(PM^2 + GM^2)/2\}^{1/2}$. That is, AM is the quadratic mean (power mean of order 2) of PM and GM.

Appendix 4(b)

For the particular case of two variables (or dimensions), we have a number of interesting results, as reported below:

1. By definition, we have
   - i) $AM = GM^2/HM$;
   - ii) $PM = GM^2/CPM$; and
   - iii) $IPM = GM^2/ICM$.

Then it follows obviously that

i) PM/AM = HM/CPM;

ii) AM/IPM = ICM/HM;

iii) PM/IPM = ICM/CPM;

iv) PM/GM = GM/CPM;

v) AM/GM = GM/HM; and

vi) IPM/GM = GM/ICM.

The last three suggests that GM is the geometric mean of i) PM and CPM; ii) AM and HM; and iii) IPM and ICM.

2. AM – GM ≥ GM – HM.

Proof:

We have AM – GM = $\frac{a+b}{2} - \sqrt{ab} = \frac{\left(\sqrt{a}-\sqrt{b}\right)^2}{2}$ and

GM – HM = $\sqrt{ab} - \frac{2ab}{a+b} = \frac{\sqrt{ab}}{a+b}\left(\sqrt{a}-\sqrt{b}\right)^2$.

Hence AM – GM = (GM – HM) $\frac{a+b}{2\sqrt{ab}}$.

Now $(\sqrt{a}-\sqrt{b})^2 \geq 0$ implies $\frac{(a+b)}{2\sqrt{ab}} \geq 1$.

Therefore the result: AM – GM ≥ GM – HM.

3. IPM – GM ≥ GM – ICM

We have IPM – GM = $\left(\frac{a^{1/2}+b^{1/2}}{2}\right)^2 - \sqrt{ab} = \left(\frac{\sqrt{a}-\sqrt{b}}{2}\right)^2$, and

GM – ICM = $\sqrt{ab} - \frac{4ab}{(\sqrt{a}+\sqrt{b})^2} = \sqrt{ab}\,\frac{\left(\sqrt{a}-\sqrt{b}\right)^2}{\left(\sqrt{a}+\sqrt{b}\right)^2}$.

Therefore IPM – GM = (GM – ICM) $\frac{\left(\sqrt{a}+\sqrt{b}\right)^2}{4\sqrt{ab}}$.

Now $\frac{\left(\sqrt{a}+\sqrt{b}\right)^2}{4\sqrt{ab}} \geq 1$ (since, $(\sqrt{a}-\sqrt{b})^2 \geq 0$ implies $\frac{(a+b)}{2\sqrt{ab}} \geq 1$; and $\frac{(a+b)}{4\sqrt{ab}} \geq 1/2$).

Hence the result: IPM – GM ≥ GM – ICM.

4. GM – ICM ≥ ICM – HM.

We have GM – ICM = $\sqrt{ab} - \frac{4ab}{(\sqrt{a}+\sqrt{b})^2} = \sqrt{ab}\frac{\left(\sqrt{a}-\sqrt{b}\right)^2}{\left(\sqrt{a}+\sqrt{b}\right)^2}$ and

ICM – HM = $\frac{4ab}{(\sqrt{a}+\sqrt{b})^2} - \frac{2ab}{(a+b)} = \frac{2ab}{(a+b)}\frac{\left(\sqrt{a}-\sqrt{b}\right)^2}{\left(\sqrt{a}+\sqrt{b}\right)^2}$.

Therefore GM – ICM = (ICM – HM) $\frac{a+b}{2\sqrt{ab}}$.

Since $\frac{a+b}{2\sqrt{ab}} \geq 1$, the result follows: GM – ICM ≥ ICM – HM.

5. ICM – HM ≤ HM^2 – CPM^2.

We have ICM – HM = $\frac{2ab}{(a+b)}\frac{\left(\sqrt{a}-\sqrt{b}\right)^2}{\left(\sqrt{a}+\sqrt{b}\right)^2}$, and

HM^2 – CPM^2 = $\frac{4a^2b^2}{(a+b)^2} - \frac{2a^2b^2}{a^2+b^2} = \frac{2a^2b^2}{a^2+b^2}\frac{(a-b)^2}{(a+b)^2}$.

Comparing the two, we have the above result.

6. PM^2 – AM^2 ≥ AM – IPM.

We have seen above that $\mathrm{PM}^2 - \mathrm{AM}^2 = \left(\frac{a-b}{2}\right)^2$, and

$\mathrm{AM} - \mathrm{IPM} = \left(\frac{\sqrt{a}-\sqrt{b}}{2}\right)^2$.

Since $(a-b)^2 \geq \left(\sqrt{a}-\sqrt{b}\right)^2$, we have the above result.

7. $\mathrm{PM}^2 - \mathrm{GM}^2 \geq \mathrm{GM}^2 - \mathrm{CPM}^2$.

We have from (1) above, $\mathrm{PM}^2 - \mathrm{GM}^2 = \frac{(a-b)^2}{2}$, and

$\mathrm{GM}^2 - \mathrm{CPM}^2 = ab - \frac{2a^2b^2}{(a^2+b^2)} = \frac{ab}{(a^2+b^2)}(a-b)^2$.

Therefore, $\mathrm{PM}^2 - \mathrm{GM}^2 = (\mathrm{GM}^2 - \mathrm{CPM}^2)\frac{a^2+b^2}{2ab}$.

Since $(a-b)^2 \geq 0 \Rightarrow \frac{a^2+b^2}{2ab} \geq 1$, we get the result: $\mathrm{PM}^2 - \mathrm{GM}^2 \geq \mathrm{GM}^2 - \mathrm{CPM}^2$.

8. Finally, $\mathrm{GM}^2 - \mathrm{CPM}^2 \leq 2(\mathrm{GM}^2 - \mathrm{HM}^2)$.

From the above, we have $\mathrm{GM}^2 - \mathrm{CPM}^2 = \frac{ab}{(a^2+b^2)}(a-b)^2$, and

$\mathrm{GM}^2 - \mathrm{HM}^2 = ab - \frac{4a^2b^2}{(a+b)^2} = ab\frac{(a-b)^2}{(a+b)^2}$.

Therefore, $\mathrm{GM}^2 - \mathrm{CPM}^2 = (\mathrm{GM}^2 - \mathrm{HM}^2)\frac{(a+b)^2}{a^2+b^2} = (\mathrm{GM}^2 - \mathrm{HM}^2)\left(1 + \frac{2ab}{a^2+b^2}\right)$.

Since $(a-b)^2 \geq 0 \Rightarrow \frac{2ab}{a^2+b^2} \leq 1$, the result follows: $\mathrm{GM}^2 - \mathrm{CPM}^2 \leq 2(\mathrm{GM}^2 - \mathrm{HM}^2)$,

which in turn also implies that $\mathrm{HM}^2 - \mathrm{CPM}^2 \leq \mathrm{GM}^2 - \mathrm{HM}^2$.

(a) All the above results together give the following relationship:

$(PM^2 – AM^2) = (AM^2 – GM^2) \geq (AM – IPM) = (IPM – GM) \geq (GM – ICM) \geq$

$\geq (ICM – HM) \leq (HM^2 – CPM^2) \leq (GM^2 – HM^2)$.

From this we have the following conclusion:

$(PM – AM) \geq (AM – IPM) = (IPM – GM) \geq (GM – ICM) \geq$

$\geq (ICM – HM) \leq (HM – CPM)$.

(b) We also have $PM^2 – GM^2 \geq GM^2 – CPM^2$,

So that we may conclude: $PM – GM \geq GM – CPM$.

From this it follows that in the case of only two variables, with an order of 2 for the general means,

i) AM, IPM, and GM are equidistant;

ii) AM is farther from PM than from IPM;

iii) GM is farther from IPM than from ICM;

iv) ICM is farther from GM than from HM;

v) HM is farther from CPM than from ICM; and

vi) GM is farther from PM than from CPM.

For example, consider the mean values and successive differences in the following Table for $a = 35$ and $b = 60$; the parameter values of ρ of the CES function correspond to the special means:

Means	PM	AM	IPM	GM	ICM	HM	CPM
ρ	–2	–1	–0.5	0	0.5	1	2
Values	49.12	47.5	46.663	45.826	45.004	44.211	42.755
Difference		1.62	0.84	0.84	0.82	0.79	1.46

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