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Salnykov, Mykhaylo and Zelenyuk, Valentin

Ukrainian Productivity and Efficiency Group, Simon Fraser University, Kyiv School of Economics

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SALNYKOV, M. and V. ZELENYUK

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ON COMENSURABILITY OF DIRECTIONAL DISTANCE FUNCTIONS

by Mykhaylo Salnykov

Department of Economics
Simon Fraser University
8888 University Drive,
Burnaby, BC, Canada
V5A 1S6
e-mail: salnykov@sfu.ca
tel.: +1 604 512 21 96
fax:+1 604 291 59 44

and

Valentin Zelenyuk

Institut de Statistique
Université Catholique de Louvain,
20 voie du Roman Pays, B 1348, Louvain-la-Neuve, Belgium
and UPEG at EROC-EERC-Kiev
e-mail: zelenyuk@stat.ucl.ac.be
tel.: +32 10 47 43 11
fax: +32 10 47 30 32

Abstract

Shephard's distance functions are widely used instruments for characterizing technology and for estimating efficiency in contemporary economic theory and practice. Recently, they have been generalized by the Luenberger shortage function, or Chambers-Chung-Färe directional distance function. In this study, we explore a very important property of an economic measure known as *commensurability* or independence of units of measurement up to scalar transformation. Our study discovers both negative and positive results for this property in the context of the directional distance function, which in turn helps us narrow down the most critical issue for this function in practice—the choice of direction of measurement.

Keywords:

Directional distance functions, commensurability, efficiency.

***JEL Code*:** D24, C43.

Introduction

Since Shephard [12], the production theory in neo-classical economics, especially duality issues, has been dominated by what his followers called the Shephard's distance functions (DFs). Recently, Luenberger [8, 9] and Chambers, Chung and Färe [2, 3] have introduced and explored what is now widely known as the directional distance function (DDF), or the shortage and benefit function in the terminology of Luenberger. The DDF was shown to be a generalization of DFs: it also gives complete characterization of a technology set, under quite weak regularity conditions, and has somewhat more powerful duality relationships (not only to revenue and cost, but also to profit function). One of the major critiques, however, that is often pitched into the DDF, at least from empirical researchers, is that it is often not clear what directional vector must be chosen for each particular empirical study.

Indeed, while researchers often argued whether input or output orientation must be chosen in a particular research involving Shephard's DFs, now, for DDF there is infinite (in fact, a continuum) of possibilities. A natural way to reduce the set of possibilities for the directional vector would be to postulate a list of desirable properties that the DDF must satisfy. This is in the fashion of axiomatic approach to efficiency analysis, where DF and DDFs are extensively used, proposed by Färe and Lovell [6] and elaborated by Bol [1], Russell [10, 11] and others to justify the use of some measures and warn about using others. In this paper we will focus on one of the most important properties for efficiency measure—commensurability—introduced by Russell [10], on the analogy of property introduced by Eichhorn and Voeller [5], as an independence of an efficiency measure from units of measurement (up to scalar transformation) of the data.

The rest of the paper is organized as follows. We first study what we call absolute-commensurability and ranking-commensurability properties for the DDF with the *unit* directional vector. To our surprise, we find that none of these properties is satisfied. We then examine whether such DDF can be 'de-commensurated' both ex *post* and ex *ante*. Finally, we find a particular type of DDF that does possess commensurability property.

I. Basic Definitions

Let $\mathbf{x} \in \Re_{+}^{N}$ denote a vector of inputs, while $\mathbf{y} \in \Re_{+}^{M}$ denote a vector of outputs and assume technology can be characterized by a *technology set* T, defined in general terms as

$$T \equiv \{(\mathbf{x}, \mathbf{y}) : \mathbf{x} \textit{ can produce } \mathbf{y}\}. \tag{1.1}$$

We assume T satisfies the standard regularity conditions of neo-classical production economics. In particular, we assume

A1. T is closed and non-empty.

A2. Inputs and outputs are freely disposable: $(\mathbf{x}, \mathbf{y}) \in T \Rightarrow (\mathbf{x}', \mathbf{y}') \in T$, $\forall \mathbf{x}' \geqq \mathbf{x},\ \mathbf{y}' \leqq \mathbf{y}$.

A3. There is no free lunch, *i.e.* $(\mathbf{0}_N, \mathbf{y}) \in T \Rightarrow \mathbf{y} = \mathbf{0}_M$.

A4. Doing nothing is possible, *i.e.* $(\mathbf{x}, \mathbf{0}_M) \in T$, $\forall \mathbf{x} \in \Re_{+}^{N}$.

A5. $P(\mathbf{x}) \equiv \{\mathbf{y} : (\mathbf{x}, \mathbf{y}) \in T\}$ is bounded for all $\mathbf{x} \in \Re_{+}^{N}$.

A6. Technology is productive, i.e. $P(\mathbf{x}) \neq \{\mathbf{0}_M\}$ for some $\mathbf{x} \in \Re_{+}^{N}$.

Given these regularity conditions and any (directional) vector $(-\mathbf{d}_\mathbf{x}, \mathbf{d}_\mathbf{y}) \in \Re_{-}^{N} \times \Re_{+}^{M}$, the directional distance function (DDF), defined as

$$\vec{D}(\mathbf{x}, \mathbf{y} \mid -\mathbf{d}_\mathbf{x}, \mathbf{d}_\mathbf{y}) \equiv \sup\{\theta \geq 0 : ((\mathbf{x} - \theta\, \mathbf{d}_\mathbf{x}), (\mathbf{y} + \theta\, \mathbf{d}_\mathbf{y})) \in T\}, \tag{1.2}$$

gives a complete characterization of technology set *T*. Luenberger [8, 9], Chambers, Chung and Färe [2, 3], and Färe and Grosskopf [7] have derived other properties of the DDF, but no one has addressed the *commensurability* property of DDF—the issue we address in the next section.

II. Commensurability of distance functions

The *commensurability axiom* has been introduced to efficiency analysis by Russell [10] in one of his works on axiomatics of efficiency measurement. Russell convincingly argued that commensurability (independence of units of measurement up to scalar transformation) is a very desirable property of any efficiency measure. Indeed, an efficiency measure not satisfying commensurability may cause different researchers, that use the same data and the same methodology, to arrive to different results just because one used, for example, kilograms and the other one used pounds to measure inputs or outputs. Formally, commensurability (adapted to the case of DDF, where some additional parameters *p*, such as directional vector, are set exogenously) can be defined as follows.

<u>Definition 1</u> *(Absolute Commensurability)*

Let $E(\mathbf{x},\mathbf{y},\mathbf{p}) \in \Re_+$ for $\forall \mathbf{x} \in \Re_+^N, \mathbf{y} \in \Re_+^M$ be an efficiency measure, where $\mathbf{p} \in \Re_+^Z$ is a Z-dimensional vector of exogenous parameters of the efficiency measure (*e.g.*, directional vector coordinates in the case of DDF). Let $\tilde{\mathbf{x}} = \Omega_{\mathbf{x}}\mathbf{x}$ and $\tilde{\mathbf{y}} = \Omega_{\mathbf{y}}\mathbf{y}$, where $\Omega_{\mathbf{x}}$ and $\Omega_{\mathbf{y}}$ are (any) diagonal matrices (further called 'commensuration' matrices) of dimensions *N*x*N* and *M*x*M*, respectively, with all diagonal elements being strictly positive constants. The efficiency measure $E(\mathbf{x},\mathbf{y},\mathbf{p})$ is said to be commensurable in inputs and outputs *if and only if*

$$E(\mathbf{x},\mathbf{y},\mathbf{p}\,|\,T) = E(\tilde{\mathbf{x}},\tilde{\mathbf{y}},\mathbf{p}\,|\,\tilde{T}), \qquad \forall \mathbf{x} \in \Re_{+}^{N}; \mathbf{y} \in \Re_{+}^{M} \tag{2.1}$$

where

$$\tilde{T} \equiv \{(\tilde{\mathbf{x}},\tilde{\mathbf{y}}) : (\mathbf{x},\mathbf{y}) \in T\} = \{(\tilde{\mathbf{x}},\tilde{\mathbf{y}}) : (\mathbf{\Omega}_{\mathbf{x}}^{-1}\tilde{\mathbf{x}}, \mathbf{\Omega}_{\mathbf{y}}^{-1}\tilde{\mathbf{y}}) \in T\}. \tag{2.2}$$

Intuitively, absolute commensurability shall be understood as a property of independence (of the score yielded by the efficiency measure) from the scale of any inputs and any outputs. For example, efficiency score obtained for any observation with the inputs expressed in tons and outputs in Watts should be *identical* to efficiency score of the same observation but when inputs are expressed in kilograms and outputs in tons of oil equivalent.

Consider first the DDF with a *unit* directional vector (henceforth UDDF), i.e.,

$$\vec{D}(\mathbf{x},\mathbf{y}\,|-\mathbf{1}_N,\mathbf{1}_M) \equiv \sup\{\theta \geq 0 : ((\mathbf{x}-\theta\mathbf{1}_N),(\mathbf{y}+\theta\mathbf{1}_M)) \in T\}. \tag{2.3}$$

This direction is one very common choice for the DDF that has been used in practice—perhaps due to its simplicity, normalizing nature and, as a consequence, convenience in explaining the results of measurement. Specifically, efficiency measure based on such direction gives *one* number indicating (regardless of units of measurement) how many units of each input must be deducted and how many units of each output must be added to any particular point in technology set to reach the (upper) frontier of this set. Despite such appealing nature, it turns out that UDDF is *not* absolute-commensurable, as we show in the next proposition.

Proposition 1. UDDF is not absolute-commensurable for all technologies.

□ Using (2.2), it follows that

$$\vec{D}(\tilde{\mathbf{x}},\tilde{\mathbf{y}}\,|-\mathbf{1}_N,\mathbf{1}_M) = \sup\{\theta \geq 0 : ((\mathbf{x}-\theta\boldsymbol{\Omega}_{\mathbf{x}}^{-1}\mathbf{1}_N),(\mathbf{y}+\theta\boldsymbol{\Omega}_{\mathbf{y}}^{-1}\mathbf{1}_M)) \in T\}. \tag{2.4}$$

Let us scale up all inputs and outputs by the same positive scalar, *i.e.*

$$\boldsymbol{\Omega}_{\mathbf{x}} = \gamma\mathbf{I}_N \qquad \gamma > 1 \quad \Rightarrow \quad \boldsymbol{\Omega}_{\mathbf{x}}^{-1} = \frac{1}{\gamma}\mathbf{I}_N \qquad \gamma > 1 \tag{2.5}$$

and

$$\boldsymbol{\Omega}_{\mathbf{y}} = \gamma\mathbf{I}_M \qquad \gamma > 1 \quad \Rightarrow \quad \boldsymbol{\Omega}_{\mathbf{y}}^{-1} = \frac{1}{\gamma}\mathbf{I}_M \qquad \gamma > 1 \tag{2.6}$$

Then (2.4) transforms into

$$\vec{D}(\tilde{\mathbf{x}},\tilde{\mathbf{y}}\,|-\mathbf{1}_N,\mathbf{1}_M) = \gamma\vec{D}(\mathbf{x},\mathbf{y}\,|-\mathbf{1}_N,\mathbf{1}_M) \neq \vec{D}(\mathbf{x},\mathbf{y}\,|-\mathbf{1}_N,\mathbf{1}_M). \tag{2.7}$$

Hence, $\exists\boldsymbol{\Omega}_{\mathbf{x}},\boldsymbol{\Omega}_{\mathbf{y}}$ such that $\vec{D}(\tilde{\mathbf{x}},\tilde{\mathbf{y}}\,|-\mathbf{1}_N,\mathbf{1}_M) \neq \vec{D}(\mathbf{x},\mathbf{y}\,|-\mathbf{1}_N,\mathbf{1}_M)$ and, thus, confirming that UDDF is not absolute-commensurable. ■

The practical implication of this result is that different researchers using the same data and methodology may arrive to different estimates—just because the researchers used different units of measurement. One may wonder whether the results would be the same at least qualitatively: i.e., if, under some units of measurement, firm A was more efficient than firm B then this ranking would, hopefully, remain the same under any other units of measurement (different by a scalar transformation). We thus call this concept *ranking-commensurability*, and formally define and apply it to UDDF below.

<u>Definition 2</u> *(Ranking-Commensurability)*

An efficiency measure $E(\mathbf{x},\mathbf{y},\mathbf{p})$ is said to be ranking-commensurable if and only if

$$E(\mathbf{x}_k, \mathbf{y}_k, \mathbf{p}) > E(\mathbf{x}_j, \mathbf{y}_j, \mathbf{p}) \Leftrightarrow E(\tilde{\mathbf{x}}_k, \tilde{\mathbf{y}}_k, \mathbf{p}) > E(\tilde{\mathbf{x}}_j, \tilde{\mathbf{y}}_j, \mathbf{p}) \tag{2.8}$$

$$\forall \mathbf{x}_k, \mathbf{x}_j \in \Re_+^N; \mathbf{y}_k, \mathbf{y}_j \in \Re_+^M,$$

where $\tilde{\mathbf{x}} = \mathbf{\Omega}_{\mathbf{x}}\mathbf{x}$ and $\tilde{\mathbf{y}} = \mathbf{\Omega}_{\mathbf{y}}\mathbf{y}$ are as in definition 1.

The intuition behind (2.8) is that if an efficiency measure is ranking-commensurable, then changing the units of measurement of any input or/and output by a scalar transformation should not affect the *ranking* of the efficiency scores, although may change the scores per se. It turns out that UDDF is also *not* ranking commensurable, as we show it in the next proposition.

Proposition 2. UDDF is not ranking-commensurable for all technologies.

□ To prove this statement, consider a simple single-input-single-output technology,

$$T = \{ (x, y): \ y \le 6 \text{ if } x \ge 1.5, \ y = 0 \text{ if } 0 \le x < 1.5\} \quad x, y \in \Re_+, \tag{2.9}$$

where the numbers are provided for the sake of illustration. Measuring DFU scores of two observations: *A* at $(x_A = 2, y_A = 0)$ and *B* at $(x_B = 3.5, y_B = 4.5)$, we would conclude that observation *A* is more efficient than *B*, since $\vec{D}(2,0 \mid -1,1) = \theta_A = 0.5$ and $\vec{D}(3.5,4.5 \mid -1,1) = \theta_B = 1.5$. Graphical illustration of this problem is provided in Figure 1.

Now, suppose we decide to transform inputs and outputs in such a way that we measure inputs in $\tilde{x} = 3x$ and outputs in $\tilde{y} = y/3$. Combining (2.3) and (2.9) implies

$$\tilde{T} = \{(\tilde{x}, \tilde{y}): \tilde{y} \le 2 \text{ if } \tilde{x} \ge 4.5, \tilde{y} = 0 \text{ if } 0 \le \tilde{x} < 4.5\} \qquad \tilde{x}, \tilde{y} \in \Re_+. \tag{2.10}$$

Therefore, changing the units of measurement of observations would transform observation A into $\tilde{A}(\tilde{x}_{\tilde{A}} = 6, \tilde{y}_{\tilde{A}} = 0)$ and observation B into $\tilde{B}(\tilde{x}_{\tilde{B}} = 10.5, \tilde{y}_{\tilde{B}} = 1.5)$ respectively. This would result in $\vec{D}(6,0\,|-1,1) = \theta_{\tilde{A}} = 1.5$ and $\vec{D}(10.5,1.5\,|-1,1) = \theta_{\tilde{B}} = 0.5$, *i.e.*, observation B is now concluded to be more efficient than A (graphical illustration is provided in Figure 2).

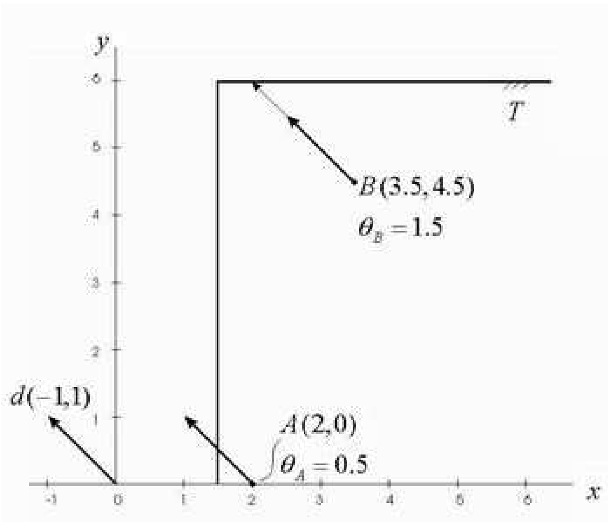

Figure 1. Graphical illustration of technology (2.9), observations A and B and their DFU scores

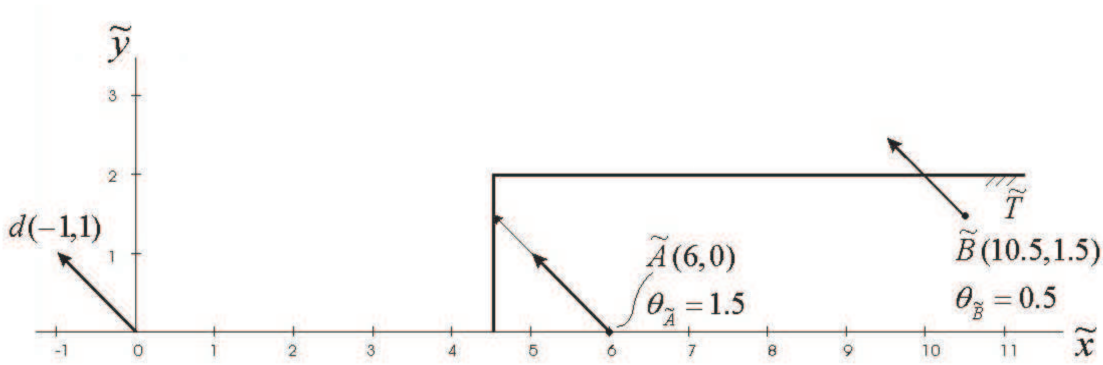

Figure 2. Graphical illustration of technology (2.10), observations $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ and their DFU scores

Therefore, the change in units of measurement have caused different ranking. Thus, UDDF measure is not ranking commensurable for all technologies. ■

The practical implication of this result is that even the *qualitative* difference in efficiency measurement with UDDF may occur just due to change in the units of measurement, which in turn may even lead to radically different policy implications. It must be clear that the type of technology we used in our argument was taken to be simple for illustration purposes, not some pathological one, and the same argument would hold for many other technologies, though one is enough to prove our claim. A natural question now is whether we could cure the situation with UDDF—with, for example, what we call here as *ex post* and *ex ante* 'de-commensurations'

Definition 3. *Ex post de-commensuration*

An efficiency measure $E(\mathbf{x}, \mathbf{y}, p)$ can be de-commensurable *ex post* if and only if

$$\exists G : \Re_{+}^{1} \rightarrow \Re_{+} : \quad G(E(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}, p)) = E(\mathbf{x}, \mathbf{y}, p), \ \forall T, \ \ \forall \mathbf{x} \in \Re_{+}^{N}; \mathbf{y} \in \Re_{+}^{M}, \qquad (2.11)$$

and where $\tilde{\mathbf{x}} = \mathbf{\Omega}_{\mathrm{x}}\mathbf{x}$ and $\tilde{\mathbf{y}} = \mathbf{\Omega}_{\mathrm{y}}\mathbf{y}$ are as in definition 1.

In words, this is a situation when it is possible to transform, *ex post*, i.e., after computation, a score from UDDF for 'commensurated' data in such a way that it becomes identical to the UDDF score for the original data. Note that here we require that de-commensurating transformation *G* works for all technologies. Therefore, transformation can depend on observed inputs and outputs as well as commensuration matrices, but shall be independent of the parameters of technology. We now will see that UDDF cannot be 'cured' by the *ex post* de-commensuration.

Proposition 3. UDDF is not de-commensurable *ex post*, independently of technology.

□ Consider a single-input-single-output CRS technology

$$T = \{(x, y) : \lambda x \geq y\} \qquad x, y \in \Re_+ . \tag{2.12}$$

Then,

$$\vec{D}(x, y \mid -1,1) \equiv \sup\{\theta \geq 0 : \lambda(x-\theta) \geq y+\theta\} = \frac{\lambda x - y}{\lambda + 1}. \tag{2.13}$$

Let $\Omega_x = \gamma > 1$ and $\Omega_y = 1$, thus $\tilde{x} = x$ and $\tilde{y} = \gamma\, y$. Combining (2.3) and (2.12) implies

$$\tilde{T} = \{(\tilde{x}, \tilde{y}) : \lambda\gamma\tilde{x} \geq \tilde{y}\}, \qquad x, y \in \Re_+ \tag{2.14}$$

and therefore,

$$\vec{D}(\tilde{x}, \tilde{y} \mid -1,1) = \frac{\lambda\gamma\, x - \gamma\, y}{\lambda\gamma + 1} = \frac{\lambda\, x - y}{\lambda + 1/\gamma}. \tag{2.15}$$

Finally, combining (2.13) and (2.15) implies

$$\vec{D}(x, y \mid -1,1) = \frac{\lambda + 1/\gamma}{\lambda + 1}\vec{D}(\tilde{x}, \tilde{y} \mid -1,1). \tag{2.16}$$

Thus, (2.16) proposes a (unique) way to transform (2.15) to obtain (2.13) under single-input-single-output CRS technology when output is scaled up. However, this transformation *depends* on parameter of the technology λ, while our definition required such independence (since true technology sets are typically unobserved in practice by researchers). Thus, (2.11) does not hold and therefore UDDF is not de-commensurable *ex post*. ■

In words, once UDDF was calculated for the commensurated data, there is no unique (independent of technology) function that would transform UDDF score into the one for non-commensurated original data.

What shall be clear from the intuition of the proofs is that not only the UDDF but also *any* DDF with a *fixed* directional vector will not satisfy even ranking-commensurability for all technologies. The intuition for this can be explained by the *additive* nature of such DDF, which becomes irreconcilable with the *multiplicative* nature of the commensurability property. This intuition encourages us to raise another natural question: Is it possible to cure UDDF measure in some way to ensure that an efficiency measure *after* commensuration is equal to its value *before* commensuration? One possible cure might be applied to the (fixed) directional vector. We call this concept *ex ante* 'de-commensuration,' defining it below.

Definition 4. *Ex ante de-commensuration*

An efficiency measure $E(\mathbf{x},\mathbf{y},\mathbf{p})$ is said to be de-commensurable *ex ante* if and only if

$$\exists F:\Re^{z}\to\Re^{z} \quad : \qquad E(\tilde{\mathbf{x}},\tilde{\mathbf{y}},F(\mathbf{p}))=E(\mathbf{x},\mathbf{y},\mathbf{p}),\quad \forall T,\quad \forall\,\mathbf{x}\in\Re_{+}^{N};\mathbf{y}\in\Re_{+}^{M}\,. \qquad (2.17)$$

and where $\tilde{\mathbf{x}}=\mathbf{\Omega}_{\mathrm{x}}\mathbf{x}$ and $\tilde{\mathbf{y}}=\mathbf{\Omega}_{\mathrm{y}}\mathbf{y}$ are as in definition 1.

In words, an efficiency measure is de-commensurable *ex ante* if and only if along with the change of units of measurement of inputs and outputs one shall also modify the parameter vector **p** (*e.g.*, directional vector in the case of DDF) to obtain the same numerical value of efficiency as for the observation in the original units of measurements. Trying this on, for our UDDF, we finally get a positive result: UDDF is de-commensurable *ex ante* if its directional vector is 'commensurated' along with the inputs and outputs by pre-multiplying it by the respective commensuration matrices. Formally, it can be easily shown that

$$\vec{D}\left(\tilde{\mathbf{x}},\tilde{\mathbf{y}}\,|\,\mathbf{\Omega}_{\mathrm{x}}(-\mathbf{1}_{N}),\mathbf{\Omega}_{\mathrm{y}}\mathbf{1}_{M}\right)=\vec{D}\left(\mathbf{x},\mathbf{y}\,|-\mathbf{1}_{N},\mathbf{1}_{M}\right)\,. \qquad (2.19)$$

It is worth noting, however, that once the directional vector of UDDF's has been pre-multiplied by the (non-identity) commensuration matrices, the DDF is no longer a UDDF. Therefore, once commensuration takes place and a researcher wants to compare her results to the results of the other study with the 'non-commensurated' data, she cannot use UDDF, but shall rather use directional distance function with a 'commensurated' directional vector. In other words, only the 'lucky first' gets to use DDF with unit directional vector and thus setting the standard for units of measurement towards which others have to comply by commensurating their unit directional vectors with the same matrices that relate their units of measurement to that of this standard. As standard scale is rather a question of tastes, traditions and practices, which may differ from school to school and from society to society, it might be difficult to achieve a consensus over this issue, as it is hard to convince the Europeans, for example, to switch to British weights and measures (e.g., pounds) or Americans to metric system (e.g., kilograms). A compromise might be to use a directional distance function with a directional vector that would 'commensurate' itself when the data is commensurated. In fact, the result (2.19) hints us on how to obtain a more general conclusion. Specifically, the necessary and sufficient condition for the DDF to yield the same values for commensurated and non-commensurated data is that the directional vector is commensurated along with the data. Formally,

$$\vec{D}(\mathbf{x},\mathbf{y} \mid -\mathbf{d}_{\mathrm{x}},\mathbf{d}_{\mathrm{y}}) = \vec{D}(\tilde{\mathbf{x}},\tilde{\mathbf{y}} \mid -\tilde{\mathbf{d}}_{\mathrm{x}},\tilde{\mathbf{d}}_{\mathrm{y}}) \qquad \forall(\mathbf{x},\mathbf{y}) \in T; (\tilde{\mathbf{x}},\tilde{\mathbf{y}}) \in \tilde{T} \tag{2.20}$$

$$\textit{if and only if } \; \tilde{\mathbf{d}}_{\mathrm{x}} = \Omega_{\mathrm{x}}\,\mathbf{d}_{\mathrm{x}} \;\; \tilde{\mathbf{d}}_{\mathrm{y}} = \Omega_{\mathrm{y}}\,\mathbf{d}_{\mathrm{y}} \,. \tag{2.21}$$

An interesting special (but still quite broad) case that ensures (2.20), which actually would not require commensuration of the directional vector explicitly, but would be obtained automatically, is when $(-\mathbf{d}_{\mathrm{x}},\mathbf{d}_{\mathrm{y}}) = (-\Xi_x \mathbf{x}, \Xi_y \mathbf{y})$, where Ξ_x and Ξ_y are any diagonal matrices of dimensions NxN and MxM, respectively, where elements are constants that, intuitively, assign weights of each input (output) relative to other inputs (outputs) in measuring the distance. (The diagonal elements of Ξ_x and Ξ_y can be zero or even negative). Special cases of this function have appeared before. For example, Chung, Färe and Groskopf [4] used it for

$$\Xi_x = \mathbf{0}_{N\times N}, \quad \Xi_y = \left(\begin{array}{c|c} \mathbf{I}_{G\times G} & \mathbf{0}_{G\times B} \\ \hline \mathbf{0}_{B\times G} & -\mathbf{I}_{B\times B} \end{array}\right),$$

where $\mathbf{I}_{G\times G}$ is a G by G identity matrix assigning equal weights to G good outputs, $\mathbf{I}_{B\times B}$ is a B by B identity matrix assigning equal weights to B bad outputs and the rest are zero matrices of dimensions indicated in subscripts. Also, Zelenyuk [13] used it when Ξ_x and Ξ_y were scalars multiplied by corresponding identity matrices. Even more special cases are the popular directional vectors $(-\mathbf{x}, \mathbf{y})$, as well as $(\mathbf{0}, \mathbf{y})$ and $(-\mathbf{x}, \mathbf{0})$ under which one-to-one closed-form relationship with the Shpehard's distance functions are known.

Concluding, it might be worth noting that the test for satisfying absolute (and ranking) commensurability helps us reducing the problem of choosing the direction of measurement for DDF considerably. In particular, it discourages from using DDF with any *fixed* vector, but the one satisfying (2.20)-(2.21). Moreover, since the DDF in (2.20)-(2.21) is a special case of the general DDF, it thus not only passes the tests for commensurability, but (under the regularity conditions) also satisfies all the general properties derived by Luenberger [8, 9], Chambers, Färe and Chung [2, 3], and Färe and Grosskopf [7]. Finally, a natural extension to our work would be an exploration of other properties for DDF that are generally desirable for theoretical or empirical characterizations of technologies and efficiency measurement.

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