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# On the combinatorics of iterated stochastic integrals

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**Keywords:** Semimartingale; iterated integrals; power jump processes; Itô's formula; stochastic exponential; chaotic representation.

**Abstract.** This paper derives several identities for the iterated integrals of a general semimartingale. They involve powers, brackets, exponential and the stochastic exponential. Their form and derivations are combinatorial. The formulae simplify for continuous or finite-variation semimartingales, especially for counting processes. The results are motivated by chaotic representation of martingales, and a simple such application is given.

## 1. INTRODUCTION AND THE MAIN RESULTS

We derive several identities involving the iterated integrals  $X^{(n)}$  of a general semimartingale  $X$  with  $X_0 = 0$ , defined inductively by  $X^{(0)} := 1$  and  $X^{(n)} = \int X_-^{(n-1)} dX$ . Thus,

$$X^{(1)} = X, \quad X^{(2)} = \int X_- dX = \int \int^- dX dX, \quad X^{(3)} = \int \int^- \int^- dX dX dX, \quad \text{etc.}$$

Our main result states that the series  $\sum_{n=0}^{\infty} X^{(n)}$  is absolutely convergent and converges to the Doléans-Dade stochastic exponential  $\mathcal{E}(X)$  of  $X$ :<sup>1</sup>

$$(1.1) \quad \mathcal{E}(X) = \sum_{n=0}^{\infty} X^{(n)}.$$

We derive the formula (1.2) below for  $X^{(n)}$  and (1.3) for the powers  $X^n$ . For a semimartingale  $X$  which is sum of its jumps, we show the alternative simpler formula (1.4) below and apply it to a counting process  $N$  to arrive at the identities (1.5) and (1.6). We derive several related identities and discuss an application to martingale representation.

Eq. (1.1) and the formula for  $X^{(n)}$  are well known when  $X$  is a *continuous* semimartingale, e.g., Revuz and Yor [6] (p. 142, 143). In this case, one simply computes

$$\mathcal{E}(X) = e^{X - [X]/2} = \sum_{i=0}^{\infty} \frac{X^i}{i!} \sum_{j=0}^{\infty} \frac{(-1)^j [X]^j}{2^j j!} = \sum_{n=0}^{\infty} I_n(X),$$

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This paper expands a 2005 version titled, "Various identities for iterated integrals of a semimartingale".

<sup>1</sup>See, e.g., Protter [5] for the definition and properties of  $\mathcal{E}(X)$  and other background assumed here.

where<sup>2</sup>

$$I_n(X) := \sum_{i,j \geq 0; i+2j=n} \frac{(-1)^j}{i!j!2^j} X^i [X]^j.$$

Eq. (1.1) now follows for the continuous case once one shows  $X^{(n)} = I_n(X)$ . Revuz and Yor [6] show this by applying the stochastic dominated convergence theorem while using  $\mathcal{E}(\lambda X) = \sum_{n=0}^{\infty} \lambda^n I_n(X)$  and  $d\mathcal{E}(\lambda X) = \lambda \mathcal{E}(\lambda X) dX$ . We prove it by induction using the recursion below which specializes to  $nX^{(n)} = X X^{(n-1)} - [X] X^{(n-2)}$  for the continuous case.

For a Brownian motion  $X$ , the formula  $X^{(n)} = I_n(X)$  specializes to that in Itô [1].

For a general semimartingale  $X$  with  $X_0 = 0$ , the definition of  $I_n(X)$  involves additionally the “power jump processes”  $X^{[n]}$ . This notion has been utilized in Naulart and Schoutens [3], Jamshidian [2], and Yan et.al. [7] in connection with chaotic representation of martingales. One defines  $X^{[n]}$  inductively by  $X^{[1]} = X$  and  $X^{[n]} = [X^{[n-1]}, X]$ . Thus,

$$X^{[2]} := [X] = [X]^c + \sum_{s \leq \cdot} (\Delta X_s)^2, \quad X_t^{[n]} := \sum_{s \leq t} (\Delta X_s)^n \text{ for } n \geq 3.$$

To derive the formula for  $X^{(n)}$  for a general semimartingale, we first establish the recursion

$$nX^{(n)} = \sum_{i=1}^n (-1)^{i-1} X^{[i]} X^{(n-i)}.$$

We then substitute by induction for each term in the recursion. The result is

$$(1.2) \quad X^{(n)} = \sum_{i_1, \dots, i_n \geq 0; i_1 + 2i_2 + \dots + ni_n = n} \frac{(-1)^{i_2 + i_4 + \dots + i_{2\lfloor n/2 \rfloor}}}{i_1! \dots i_n! 2^{i_2} \dots n^{i_n}} X^{i_1} (X^{[2]})^{i_2} \dots (X^{[n]})^{i_n} =: I_n(X).$$

Note, the sum has finitely many terms, and this definition of  $I_n(X)$  simplifies in continuous case to the earlier definition for a continuous  $X$ , since then  $X^{[k]} = 0$  for  $k \geq 3$ .

To prove (1.1) for a general semimartingale, we first show that if  $|\Delta X| < 1$  then

$$\mathcal{E}(X) = \exp\left(\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} X^{[k]}\right),$$

the series being absolutely convergent when  $|\Delta X| < 1$ . Hence, writing this exponential of a sum as a product of exponentials and rearranging terms we get,

$$\mathcal{E}(X) = \prod_{k=1}^{\infty} e^{(-1)^{k-1} X^{[k]}/k} = \prod_{k=1}^{\infty} \sum_{i=0}^{\infty} \frac{(-1)^{i(k-1)}}{k^i i!} (X^{[k]})^i = \sum_{n=0}^{\infty} I_n(X).$$

<sup>2</sup>The coefficients appearing in  $n!I_n(X)$  are those of the Hermite polynomial of degree  $n$  because

$$(-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2} = n! \sum_{i,j \geq 0; i+2j=n} \frac{(-1)^j}{i!j!2^j} x^i.$$

Since by (1.2),  $X^{(n)} = I_n(X)$ , this proves (1.1) for the case  $|\Delta X| < 1$ . The general case now follows easily from this and the finite variation case (see (1.4) below), by observing that if (1.1) holds for two processes  $X$  and  $Y$  then it holds for  $X + Y$  provided  $[X, Y] = 0$ .

We obtain an expansion similar to (1.2) for the powers  $X^n$ :

$$(1.3) \quad X^n = \sum_{i_1, \dots, i_n \geq 0; i_1 + 2i_2 + \dots + ni_n = n} \frac{(-1)^{n-i_1} n!}{i_2! \dots i_n! 2^{i_2} \dots n^{i_n}} X^{(i_1)} (X^{[2]})^{i_2} \dots (X^{[n]})^{i_n}.$$

When  $X$  is continuous, this simplifies to

$$X^n = \sum_{i, j \geq 0; i + 2j = n} \frac{n!}{j! 2^j} X^{(i)} [X]^j.$$

When  $X$  equals the sum of its jumps, we prove (1.1) directly by first showing that

$$(1.4) \quad X_t^{(n)} = \sum_{s_1 < \dots < s_n \leq t} \Delta X_{s_1} \dots \Delta X_{s_n}. \quad (\text{provided } X_t = \sum_{s \leq t} \Delta X_s)$$

An interesting case is a ‘‘counting process’’, i.e., a semimartingale  $N$  with  $N_0 = 0$  satisfying  $[N] = N$  (equivalently,  $N$  equals the sum of its jumps, all of which equal 1), e.g., a Poisson process or more generally a Cox process. Eq. (1.4) then simplifies to

$$(1.5) \quad N^{(n)} = 1_{N \geq n} \binom{N}{n}.$$

Inverting this yields an expression for  $N^n$  in terms of the Stirling numbers  $c_{n,i}$ :

$$(1.6) \quad N^n = \sum_{i=1}^n c_{n,i} N^{(i)}, \quad c_{n,i} := \sum_{j=0}^i (-1)^{i-j} \binom{i}{j} j^n.$$

Iterated integrals and Eq. (1.1) have well-known applications to the chaotic representation of martingales in a Brownian filtration; see e.g., Oertel [4] and the references there.<sup>3</sup> Different but related chaotic expansions of the powers  $X^n$  have been used in [3], [2] and [7] to exhibit chaotic representation of martingales under a filtration generated by Lévy (and more general) processes. Here, we illustrate this connection by applying (1.6) to a Cox process. For example, for a Poisson process  $N$  with intensity  $\lambda$ , we get for  $T > 0$ ,

$$(1.7) \quad N_T^n = \sum_{i=0}^n a_{n,i,T} (N - \lambda T)_T^{(i)},$$

where  $a_{n,i,T}$  are constants and given by

$$a_{n,i,T} := \sum_{k=0}^{n-i} \sum_{j=1}^{k+i} (-1)^{k+i-j} \frac{(k+i)! j^n}{(k+i-j)! j! k!} \lambda^k T^k.$$

<sup>3</sup>I wish to thank Frank Oertel for bringing to my attention his paper [4], where I encountered Eq. (1.1) for the first time (for the case of a continuous semimartingale with a deterministic quadratic variation).

## 2. THE IDENTITIES FOR A GENERAL SEMIMARTINGALE

In this section we derive the formula (1.2) for  $X^{(n)}$  and (1.3) for  $X^n$  for a general semimartingale  $X$  with  $X_0 = 0$ , and prove (1.1) for the case  $|\Delta X| < 1$ . The proof of (1.1) for the general case is completed using two results from the next section.

It is instructive to first derive these results for the simplest possible case because the general case uses essentially the same idea. Suppose  $X$  is continuous and of finite-variation with  $X_0 = 0$ . Then  $X^n = n \int X^{n-1} dX$ . Substituting  $X^{n-1} = (n-1) \int X^{n-2} dX$  gives  $X^n = n(n-1) \int \int X^{n-2} dX dX$ . Continuing in this manner, we see that  $X^n = n! X^{(n)}$ . This implies  $e^X = \sum_{n=0}^{\infty} X^{(n)}$ . But in this case,  $e^X$  also equals  $\mathcal{E}(X)$ .

There is a simple intuition behind Eq. (1.1). Since by definition  $X^{(0)} = 1$  and  $dX^{(n)} := X_-^{(n-1)} dX$ , heuristically (but far from rigorously) it is tempting to argue,

$$d \sum_{n=0}^{\infty} X^{(n)} = \sum_{n=1}^{\infty} dX^{(n)} = \sum_{n=1}^{\infty} X_-^{(n-1)} dX = \sum_{n=0}^{\infty} X_-^{(n)} dX.$$

This and uniqueness of solution of SDE  $d\mathcal{E}(X) = \mathcal{E}(X)_- dX$  indicate  $\mathcal{E}(X) = \sum_{n=0}^{\infty} X^{(n)}$ .

**2.1. The recursion formula.** The formula (1.2) for  $X^{(n)}$  uses the following recursion.

**Proposition 2.1.** *Let  $X$  be a semimartingale with  $X_0 = 0$ . Then for any  $n \in \mathbb{N}$  we have,*

$$(2.1) \quad nX^{(n)} = \sum_{i=1}^n (-1)^{i-1} X^{[i]} X^{(n-i)}.$$

*Proof.* By Itô's product rule on  $X^{[i]} X^{(n-i)}$ , and using  $[X^{[i]}, X^{(n-i)}] = \int X_-^{(n-i-1)} dX^{[i+1]}$ ,

$$X^{[i]} X^{(n-i)} = \int X_-^{(n-i)} dX^{[i]} + \int X_-^{(n-i-1)} dX^{[i+1]} + \int X_-^{[i]} dX^{(n-i)}.$$

Multiplying by  $(-1)^{i-1}$ , summing to  $n-1$ , and shifting the index of the second sum,

$$\begin{aligned} & \sum_{i=1}^{n-1} (-1)^{i-1} X^{[i]} X^{(n-i)} = \\ & \sum_{i=1}^{n-1} (-1)^{i-1} \int X_-^{(n-i)} dX^{[i]} + \sum_{j=2}^n (-1)^j \int X_-^{(n-j)} dX^{[j]} + \sum_{i=1}^{n-1} (-1)^{i-1} \int X_-^{[i]} dX^{(n-i)} \\ & = X^{(n)} + (-1)^n X^{[n]} + \sum_{i=1}^{n-1} (-1)^{i-1} \int X_-^{[i]} X_-^{(n-1-i)} dX, \end{aligned}$$

where for the last equality, we telescoped the first two sums to get some cancellations and we substituted  $dX^{(n-i)} = X_-^{(n-1-i)} dX$  in the third sum. Therefore, taking the second term to the left side and applying induction to the third term, we have

$$\sum_{i=1}^n (-1)^{i-1} X^{[i]} X^{(n-i)} = X^{(n)} + (n-1) \int X_-^{(n-1)} dX = nX^{(n)}.$$

The proof by induction is complete.  $\square$

If  $X$  is continuous, then  $X^{[k]} = 0$  for  $k \geq 3$ , so we obtain

**Corollary 2.2.** *Let  $X$  be a continuous semimartingale with  $X_0 = 0$ . Then*

$$(2.2) \quad nX^{(n)} = XX^{(n-1)} - [X]X^{(n-2)}.$$

**2.2. Iterated integrals.** Eq. (1.2) for  $X^{(n)}$  follows simply by substituting via induction for  $X^{(k)}$  in the right-hand side of the recursion (2.1), followed by index manipulation. Since the continuous case is more straightforward, for pedagogical reasons we do it first.

**Proposition 2.3.** *Let  $X$  be a continuous semimartingale with  $X_0 = 0$ . Then*

$$(2.3) \quad X^{(n)} = \sum_{i,j \geq 0, i+2j=n} \frac{(-1)^j}{i!j!2^j} X^i [X]^j.$$

*Proof.* Substituting in (2.2) from induction, followed by index manipulations,

$$\begin{aligned} nX^{(n)} &= \sum_{i',j \geq 0, i'+2j=n-1} \frac{(-1)^j}{i'!j!2^j} X^{i'+1} [X]^j - \sum_{i,j' \geq 0, i+2j'=n-2} \frac{(-1)^{j'}}{i!j'!2^{j'}} X^i [X]^{j'+1} \\ &= \sum_{i-1,j \geq 0, i+2j=n} \frac{i(-1)^j}{i!j!2^j} X^i [X]^j + \sum_{i,j-1 \geq 0, i+2j=n} \frac{2j(-1)^j}{i!j!2^j} X^i [X]^j \\ &= \sum_{i,j \geq 0, i+2j=n} \frac{i(-1)^j}{i!j!2^j} X^i [X]^j + \sum_{i,j \geq 0, i+2j=n} \frac{2j(-1)^j}{i!j!2^j} X^i [X]^j \\ &= \sum_{i,j \geq 0, i+2j=n} \frac{(i+2j)(-1)^j}{i!j!2^j} X^i [X]^j = n \sum_{i,j \geq 0, i+2j=n} \frac{(-1)^j}{i!j!2^j} X^i [X]^j. \end{aligned}$$

The proof by induction is complete.  $\square$

A similar argument, but based on (2.1) and with multi-indices, yields the general result:

**Theorem 2.4.** *Let  $X$  be a semimartingale with  $X_0 = 0$ . Then for any  $n \in \mathbb{N}$  we have,*

$$(2.4) \quad X^{(n)} = \sum_{i_1, \dots, i_n \geq 0; i_1+2i_2+\dots+ni_n=n} \frac{(-1)^{i_2+i_4+\dots+i_{2\lfloor n/2 \rfloor}}}{i_1! \dots i_n! 2^{i_2} \dots n^{i_n}} X^{i_1} (X^{[2]})^{i_2} \dots (X^{[n]})^{i_n}.$$

*Proof.* First we note that the sign can be alternatively written as

$$(2.5) \quad (-1)^{i_2+i_4+\dots+i_{2\lfloor n/2 \rfloor}} = (-1)^{i_2+2i_3+\dots+(n-1)i_n}.$$

Also, for simplicity, let us denote  $X_i := X^{[i]}$ . Then, by induction, for all  $m < n$ , we have

$$X^{(m)} = \sum_{i_1+2i_2+\dots+mi_m=m; i_1, \dots, i_m \geq 0} \frac{(-1)^{i_2+2i_3+\dots+(m-1)i_m}}{i_1! \dots i_m! 2^{i_2} \dots m^{i_m}} X_1^{i_1} \dots X_m^{i_m}.$$

Now, because of the constraint  $i_1 + 2i_2 + \cdots + mi_m = m$  in the sum, we can also write the sum over multi-indices  $i_1, \dots, i_n \geq 0$  subject to  $i_1 + 2i_2 + \cdots + ni_n = m$ , which of course implies  $i_j = 0$  for  $j > m$  (and so  $X_j^{i_j} = 1$  and  $(-1)^{(j-1)i_j} = 1$  for  $j > m$ ). Thus,

$$X^{(m)} = \sum_{i_1+2i_2+\cdots+ni_n=m; i_1, \dots, i_n \geq 0} \frac{(-1)^{i_2+2i_3+\cdots+(n-1)i_n}}{i_1! \cdots i_m! 2^{i_2} \cdots n^{i_n}} X_1^{i_1} \cdots X_n^{i_n}.$$

Substituting these in the right hand side of the recursion formula (2.1) we get,

$$\begin{aligned} nX^{(n)} &= \sum_{m=1}^n (-1)^{m-1} X_m \sum_{i_1+2i_2+\cdots+ni_n=n-m; i_1, \dots, i_n \geq 0} \frac{(-1)^{i_2+2i_3+\cdots+(n-1)i_n}}{i_1! \cdots i_n! 2^{i_2} \cdots n^{i_n}} X_1^{i_1} \cdots X_n^{i_n} \\ &= \sum_{m=1}^n \sum_{i_1+2i_2+\cdots+m(i_n+1)+\cdots+ni_n=n; i_1, \dots, i_n \geq 0} \frac{(-1)^{i_2+2i_3+\cdots+(m-1)(i_m+1)+\cdots+(n-1)i_n}}{i_1! \cdots i_n! 2^{i_2} \cdots n^{i_n}} X_1^{i_1} \cdots X_m^{i_m+1} \cdots X_n^{i_n} \end{aligned}$$

For the  $m$ -th summand, we change the index  $i_m$  by setting  $j_m = i_m + 1$ . In the  $m$ -th summand  $X_m^{i_m+1} = X_m^{j_m}$  appears. There, we also substitute  $1/i_m! = j_m/j_m!$  and  $1/m^{i_m} = m/j_m^{j_m}$ . The sign becomes  $(-1)^{i_2+2i_3+\cdots+(m-1)j_m+\cdots+(n-1)i_n}$ . In the  $m$ -summand  $j_m \geq 1$ , but we can run the sum from  $j_m = 0$  because the factor  $j_m/j_m!$  vanishes when  $j_m = 0$ . After these substitutions, we replace the symbol  $j_m$  with  $i_m$  in the  $m$ -th summand. We obtain,

$$\begin{aligned} nX^{(n)} &= \sum_{m=1}^n \sum_{i_1+2i_2+\cdots+ni_n=n; i_1, \dots, i_n \geq 0} m i_m \frac{(-1)^{i_2+2i_3+\cdots+(n-1)i_n}}{i_1! \cdots i_n! 2^{i_2} \cdots n^{i_n}} X_1^{i_1} \cdots X_n^{i_n} \\ &= n \sum_{i_1+2i_2+\cdots+ni_n=n; i_1, \dots, i_n \geq 0} \frac{(-1)^{i_2+2i_3+\cdots+(n-1)i_n}}{i_1! \cdots i_n! 2^{i_2} \cdots n^{i_n}} X_1^{i_1} \cdots X_n^{i_n}, \end{aligned}$$

where the last equality follows because  $\sum_{m=1}^n m i_m = n$  due to the constraint  $i_1 + 2i_2 + \cdots + ni_n = n$  in the inner sum. In view of (2.5), the inductive proof is complete.  $\square$

For example, for  $n \leq 5$  we have,

$$\begin{aligned} 2X^{(2)} &= X^2 - [X]. \\ 6X^{(3)} &= X^3 - 3[X]X + 2X^{[3]}. \\ 4!X^{(4)} &= X^4 - 6[X]X^2 + 3[X]^2 + 8XX^{[3]} - 6X^{[4]}. \\ 5!X^{(5)} &= X^5 - 10X^3[X] + 20X^2X^{[3]} + 15X[X]^2 - 30XX^{[4]} - 20[X]X^{[3]} + 4!X^{[5]}. \end{aligned}$$

For  $n \geq 6$ , monomials involving three or more  $X^{[k]}$  also appear. For example,  $6!X^{[6]}$  contains the term  $-120X[X]X^{[3]}$ . Of course, they do not appear in the continuous case because  $X^{[k]} = 0$  for  $k \geq 3$ . So, for example, when  $X$  is continuous we have

$$6!X^{(6)} = X^6 - 15[X]X^4 + 45[X]^2X^2 - 15[X]^3.$$

**2.3. The stochastic exponential.** The us begin with the simpler continuous case. For positive real numbers  $x$  and  $y$ , we have,

$$e^{x+y} = e^x e^y = \sum_{i=0}^{\infty} \frac{x^i}{i!} \sum_{j=0}^{\infty} \frac{y^j}{j!} = \sum_{n=0}^{\infty} \sum_{i,j \geq 0; i+2j=n} \frac{x^i y^j}{i! j!}.$$

The rearrangement of the sums is justified because all the terms are positive and the series convergent. Thus by the triangle inequality the calculation is valid for all  $x$  and  $y$  as the series on the right is absolutely convergent. Replacing  $x$  by  $X$  and  $y$  by  $-[X]/2$ , and using the formula  $\mathcal{E}(X) = e^{X-[X]/2}$ , we thus obtain in view of Eq. (2.3) the following result.

**Proposition 2.5.** *Let  $X$  be a continuous semimartingale with  $X_0 = 0$ . Then  $\mathcal{E}(X) = \sum_{n=0}^{\infty} X^{(n)}$ , with the series absolutely convergent.*

Now, instead of  $x + y$ , consider an absolutely convergent series  $\sum_{k=1}^{\infty} x_k$ . Then similarly,

$$(2.6) \quad e^{\sum_{k=1}^{\infty} x_k} = \prod_{k=1}^{\infty} e^{x_k} = \prod_{k=1}^{\infty} \sum_{i=0}^{\infty} \frac{x_k^i}{i!} = \sum_{n=0}^{\infty} \left( \sum_{i_1, \dots, i_n \geq 0; i_1 + 2i_2 + \dots + ni_n = n} \frac{x_1^{i_1} \dots x_n^{i_n}}{i_1! \dots i_n!} \right).$$

Again, the rearrangement is justified for it holds for  $x_i \geq 0$  and since in general the series on the right is absolutely convergent, in fact absolutely bounded by  $e^{\sum_{i=1}^{\infty} |x_k|}$ . With this in mind, we next derive an expression for  $\mathcal{E}(X)$  in terms of the  $X^{[k]}$  (recall  $X^{[1]} := X$ ).

**Proposition 2.6.** *Let  $X$  be a semimartingale such that  $|\Delta X| < 1$ . Then we have,  $\sum_{k=1}^{\infty} |X^{[k]}/k| < \infty$ . Moreover,*

$$(2.7) \quad \mathcal{E}(X) = \exp\left(\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} X^{[k]}\right).$$

*Proof.* We utilize the well-known formula (e.g., [5]) that in general

$$\mathcal{E}(X) = e^{X-[X]^c/2} \prod_{s \leq \cdot} (1 + \Delta X_s) e^{-\Delta X_s} = e^{X-[X]^c/2 + \sum_{s \leq \cdot} (\log(1 + \Delta X_s) - \Delta X_s)}$$

(The infinite product and sum are absolutely convergent and of finite variation.) Hence,

$$\begin{aligned} \mathcal{E}(X) &= e^{X-[X]^c/2 + \sum_{s \leq \cdot} \sum_{k=2}^{\infty} (-1)^{k-1} (\Delta X_s)^k / k} \\ &= e^{X-[X]^c/2 + \sum_{k=2}^{\infty} (-1)^{k-1} \sum_{s \leq \cdot} (\Delta X_s)^k / k} = e^{\sum_{k=1}^{\infty} (-1)^{k-1} X^{[k]} / k}. \end{aligned}$$

Above, for the first equality we used the expansion  $\log(1 + x) - x = \sum_{k=2}^{\infty} (-1)^{k-1} x^k / k$ , which is absolutely convergent for  $|x| < 1$ . For the second equality, we interchanged the sums over  $s$  and  $k$ , which is possible since  $\sum_{s \leq \cdot} \sum_{k=2}^{\infty} |\Delta X_s|^k / k = H(x) * \mu < \infty$ , where  $H(x) = 1_{|x| < 1} (\log(1 + |x|) - |x|)$  and  $\mu$  and is the random measure associated  $X$ . We also have  $\sum_{k=1}^{\infty} |X^{[k]}|/k = |X| + [X]^c/2 + H(x) * \mu < \infty$ . The proof is complete.  $\square$

We are now ready for the main step in the proof of Eq. (1.1).



**Lemma 2.7.** *Let  $X$  be a semimartingale with  $X_0 = 0$  such that  $|\Delta X| < 1$ . Then,  $\mathcal{E}(X) = \sum_{n=0}^{\infty} X^{(n)}$ , with the series absolutely convergent.*

*Proof.* Apply Eq. (2.6) with  $x_k = (-1)^{k-1} X^{[k]}/k$ . Since  $\sum_1^{\infty} x_k$  is then absolutely convergent by Prop. 2.6, we have by (2.6) (using also (2.5)) the absolutely convergent series

$$\mathcal{E}(X) = \sum_{n=0}^{\infty} \left( \sum_{i_1, \dots, i_n \geq 0; i_1 + 2i_2 + \dots + ni_n = n} \frac{(-1)^{i_2 + i_4 + \dots + i_{2[n/2]}}}{i_1! \dots i_n! 2^{i_2} \dots n^{i_n}} X^{i_1} (X^{[2]})^{i_2} \dots (X^{[n]})^{i_n} \right).$$

The desired result thus follows by Theorem 2.4.  $\square$

We now prove (1.1) in general, using two independent results from the next section.

**Theorem 2.8.** *Let  $X$  be a semimartingale with  $X_0 = 0$ . Then,*

$$(2.8) \quad \mathcal{E}(X) = \sum_{n=0}^{\infty} X^{(n)},$$

*with the series absolutely convergent.*

*Proof.* Define the semimartingale  $Z$  by  $Z_t = \sum_{s \leq t} 1_{|\Delta X_s| \geq 1} \Delta X_s$ . Set  $Y := X - Z$ . By Lemma 2.7, the Theorem holds for  $Y$ . And by Proposition 3.1 below the Theorem holds for  $Z$ . Since  $[Y, Z] = 0$ , it follows from Lemma 3.3 that the Theorem holds for  $Y + Z = X$ .  $\square$

**2.4. The powers.** Eq. (2.4) for  $X^{(n)}$  can be “inverted” by to yield a formula for  $X^n$ :

**Theorem 2.9.** *Let  $X$  be a semimartingale with  $X_0 = 0$ . Then for  $n \in \mathbb{N}$ , we have*

$$(2.9) \quad X^n = \sum_{i_1, \dots, i_n \geq 0; i_1 + 2i_2 + \dots + ni_n = n} \frac{(-1)^{n-i_1} n!}{i_2! \dots i_n! 2^{i_2} \dots n^{i_n}} X^{(i_1)} (X^{[2]})^{i_2} \dots (X^{[n]})^{i_n}.$$

*Proof.* The result follows from Eq. (2.4) simply by induction. But, let us give a more natural derivation for the case  $|\Delta X| < 1$ , using Eq. (2.7) and (2.8). Applied to  $\lambda X$ ,  $|\lambda| < 1$ , these equations imply (using  $(\lambda X)^{(n)} = \lambda^n X^{(n)}$ ,  $(\lambda X)^{[n]} = \lambda^n X^{[n]}$ ),

$$\sum_{j=0}^{\infty} \lambda^j X^{(j)} = \mathcal{E}(\lambda X) = e^{\sum_{k=1}^{\infty} (-1)^{k-1} \lambda^k X^{[k]}/k}.$$

Hence,

$$\begin{aligned} \sum_{n=0}^{\infty} \lambda^n \frac{X^n}{n!} &= e^{\lambda X} = \left( \sum_{j=0}^{\infty} \lambda^j X^{(j)} \right) e^{\sum_{k=2}^{\infty} (-1)^k \lambda^k X^{[k]}/k} \\ &= \left( \sum_{j=0}^{\infty} \lambda^j X^{(j)} \right) \prod_{k=2}^{\infty} \sum_{i=0}^{\infty} \frac{(-1)^{ki}}{k^i} \lambda^{ki} (X^{[k]})^i \\ &= \sum_{n=0}^{\infty} \left( \lambda^n \sum_{i_1, \dots, i_n \geq 0; i_1 + 2i_2 + \dots + ni_n = n} \frac{(-1)^{2i_2 + \dots + ni_n}}{i_2! \dots i_n! 2^{i_2} \dots n^{i_n}} X^{(i_1)} (X^{[2]})^{i_2} \dots (X^{[n]})^{i_n} \right). \end{aligned}$$

Eq. (2.9) now follows by setting the coefficients of  $\lambda^n$  on the two sides equal and noting that  $(-1)^{2i_2 + \dots + ni_n} = (-1)^{n-i_1}$  due to the constraint in the inner sum.  $\square$

Let us also give a direct inductive proof of (2.9) for the continuous case which uses a recursive relation similar to that Sections 2.1 and 2.1. By Itô's formula,

$$X^n = n \int X^{n-1} dX + \frac{1}{2}n(n-1) \int X^{n-2} d[X].$$

Hence, substituting for  $X^{n-1}$  and  $X^{n-2}$  by induction, we get,

$$\begin{aligned} X^n &= \sum_{i,j \geq 0, i+2j=n-1} \frac{n!}{j!2^j} \int [X]^j X^{(i)} dX + \sum_{i,j \geq 0, i+2j=n-2} \frac{n!}{j!2^{j+1}} \int X^{(i)} [X]^j d[X] \\ &= \sum_{i,j \geq 0, i+2j=n-1} \frac{n!}{j!2^j} \int [X]^j dX^{(i+1)} + \sum_{i,j \geq 0, i+2j=n-2} \frac{n!}{(j+1)!2^{j+1}} \int X^{(i)} d[X]^{j+1} \\ &= \sum_{i,j \geq 0, i+2j=n} \frac{n!}{j!2^j} \int [X]^j dX^{(i)} + \sum_{i,j \geq 0, i+2j=n} \frac{n!}{j!2^j} \int X^{(i)} d[X]^j \\ &= \sum_{i,j \geq 0, i+2j=n} \frac{n!}{j!2^j} X^{(i)} [X]^j. \end{aligned}$$

Above, in the last equality we integrated by parts, and in the third equality shifted by 1 the dummy index  $i$  (resp.  $j$ ) of the first (resp. second) sum. For example, we have

$$\begin{aligned} X^2 &= 2X^{(2)} + [X]. \\ X^3 &= 6X^{(3)} + 3[X]X. \\ X^4 &= 24X^{(4)} + 12[X]X^{(2)} + 3[X]^2. \\ X^5 &= 120X^{(5)} + 60[X]X^{(3)} + 15[X]^2X. \\ X^6 &= 720X^{(6)} + 360[X]X^{(4)} + 90[X]^2X^{(2)} + 15[X]^3. \end{aligned}$$

For a general semimartingale  $X$  with  $X_0 = 0$ , one can give a similar (albeit more complex) inductive proof based on the recursion  $X^n = \sum_{i=1}^n \binom{n}{i} \int X_-^{n-i} dX^{[i]}$  from [2].

### 3. ITERATED INTEGRALS OF FINITE-VARIATION PROCESSES

**3.1. Sum of jump processes.** The following was used in the proof of Theorem 2.8.

**Proposition 3.1.** *Let  $X$  be a finite variation semimartingale with  $X_0 = 0$  which is the sum of its jumps, i.e.,  $X_t = \sum_{s \leq t} \Delta X_s$ . Then*

$$(3.1) \quad X_t^{(n)} = \sum_{s_1 < \dots < s_n \leq t} \Delta X_{s_1} \cdots \Delta X_{s_n}.$$

Moreover we have,

$$(3.2) \quad \mathcal{E}(X) = \prod_{s \leq \cdot} (1 + \Delta X_s) = \sum_{n=0}^{\infty} X^{(n)},$$

with the sum absolutely convergent - in fact,  $\sum_{n=0}^{\infty} |X^{(n)}| \leq \exp(\sum_{s \leq \cdot} |\Delta X_s|) < \infty$ .

*Proof.* Since  $X$  is the sum of its jump, so it  $X^{(n)}$  by induction. Moreover, since  $dX^{(n)} = X_-^{(n-1)} dX$ , we have  $\Delta X^{(n)} = X_-^{(n-1)} \Delta X$ . Hence, using induction,

$$\begin{aligned} X_t^{(n)} &= \sum_{s \leq t} \Delta X_s^{(n)} = \sum_{s \leq t} X_{s-}^{(n-1)} \Delta X_s \\ &= \sum_{s \leq t} \left( \sum_{s_1 < \dots < s_{n-1} \leq s-} \Delta X_{s_1} \cdots \Delta X_{s_{n-1}} \right) \Delta X_s \\ &= \sum_{s_1 < \dots < s_{n-1} < s \leq t} \Delta X_{s_1} \cdots \Delta X_{s_{n-1}} \Delta X_s. \end{aligned}$$

This proves (3.1). Permuting  $s_1 < \dots < s_n$  and using the commutativity of product, we get from (3.1) also a sum over distinct jumps (below  $s_1 \neq \dots \neq s_n$  means the  $s_i$  are distinct):

$$X_t^{(n)} = \frac{1}{n!} \sum_{s_1 \neq \dots \neq s_n \leq t} \Delta X_{s_1} \cdots \Delta X_{s_n}.$$

Hence,

$$\begin{aligned} |X_t^{(n)}| &\leq \frac{1}{n!} \sum_{s_1 \neq \dots \neq s_n \leq t} |\Delta X_{s_1}| \cdots |\Delta X_{s_n}| \\ &\leq \frac{1}{n!} \sum_{s_1, \dots, s_n \leq t} |\Delta X_{s_1}| \cdots |\Delta X_{s_n}| \\ &= \frac{1}{n!} \prod_{i=1}^n \sum_{s_i=1}^{\infty} |\Delta X_{s_i}| = \frac{1}{n!} \left( \sum_{s \leq t} |\Delta X_s| \right)^n. \end{aligned}$$

Therefore,

$$\sum_{n=0}^{\infty} |X^{(n)}| \leq \sum_{n=0}^{\infty} \frac{1}{n!} \left( \sum_{s \leq \cdot} |\Delta X_s| \right)^n = \exp \left( \sum_{s \leq \cdot} |\Delta X_s| \right),$$

which is finite since  $\sum_{s \leq t} |\Delta X_s| < \infty$  a.s., as  $X$  is of finite variation. We further have,

$$\begin{aligned} \mathcal{E}(X)_t &= \prod_{s \leq t} (1 + \Delta X_s) \\ &= 1 + \sum_{n=1}^{\infty} \sum_{s_1 < \dots < s_n \leq t} \Delta X_{s_1} \cdots \Delta X_{s_n} = \sum_{n=0}^{\infty} X_t^{(n)}, \end{aligned}$$

where the first equality is standard and last equality follows from (3.1).  $\square$

Comparing Eq. (3.1) with (1.2) yields the following the purely combinatorial identity:

$$\sum_{1 \leq j_1 < \dots < j_n \leq m} \prod_{k=1}^n x_{j_k} = \sum_{i_1, \dots, i_n \geq 0; i_1 + 2i_2 + \dots + ni_n = n} \frac{(-1)^{i_2 + i_4 + \dots + i_{2[n/2]}}}{i_1! \cdots i_n! 2^{i_2} \cdots n^{i_n}} \prod_{k=1}^n \left( \sum_{j=1}^m x_j^k \right)^{i_k},$$

for  $m \geq n$  and real  $x_1, \dots, x_m$ . When  $n = m$ , this is a ‘‘polarization’’ identity for  $x_1 \cdots x_n$ .

**3.2. Iterated integrals of a sum.** The following result will be useful.

**Proposition 3.2.** *Let  $X$  and  $Y$  be semimartingales satisfying  $X_0 = Y_0 = 0$  and  $[X, Y] = 0$ . Then for  $n \in \mathbb{N}$  we have,*

$$(3.3) \quad (X + Y)^{(n)} = \sum_{i=0}^n X^{(i)} Y^{(n-i)}.$$

*Proof.* We employ induction. Using the definition of iterated integral, induction, the definition again, some index manipulations, and integration by parts using  $[X, Y] = 0$ ,

$$\begin{aligned} (X + Y)^{(n)} &= \int (X + Y)_-^{(n-1)} dX + \int (X + Y)_-^{(n-1)} dY \\ &= \sum_{i=0}^{n-1} \int X_-^{(i)} Y_-^{(n-1-i)} dX + \sum_{i=0}^{n-1} \int X_-^{(i)} Y_-^{(n-1-i)} dY \\ &= \sum_{i=0}^{n-1} \int Y_-^{(n-1-i)} dX^{(i+1)} + \sum_{i=0}^{n-1} \int X_-^{(i)} dY^{(n-i)} \\ &= \sum_{i=1}^n \int Y_-^{(n-i)} dX^{(i)} + \sum_{i=0}^{n-1} \int X_-^{(i)} dY^{(n-i)} \\ &= X^{(n)} + \sum_{i=1}^{n-1} \int Y_-^{(n-i)} dX^{(i)} + \sum_{i=1}^{n-1} \int X_-^{(i)} dY^{(n-i)} + Y^{(n)} \\ &= X^{(n)} + \sum_{i=1}^{n-1} X^{(i)} Y^{(n-i)} + Y^{(n)} = \sum_{i=0}^n X^{(i)} Y^{(n-i)}. \end{aligned}$$

This completes the inductive proof.  $\square$

The following consequence of Proposition 3.2 was used in the proof of Theorem 2.8.

**Lemma 3.3.** *Let  $X$  and  $Y$  be semimartingales satisfying  $X_0 = Y_0 = 0$  and  $[X, Y] = 0$ . Suppose  $\mathcal{E}(X) = \sum_{n=0}^{\infty} X^{(n)}$  and  $\mathcal{E}(Y) = \sum_{n=0}^{\infty} Y^{(n)}$  with both sums absolutely convergent. Then,  $\mathcal{E}(X + Y) = \sum_{n=0}^{\infty} (X + Y)^{(n)}$ , with the sum absolutely convergent.*

*Proof.* Using  $[X, Y] = 0$  and the assumption on  $\mathcal{E}(X)$  and  $\mathcal{E}(Y)$ , we have

$$\mathcal{E}(X + Y) = \mathcal{E}(X)\mathcal{E}(Y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} X^{(n)} Y^{(m)}$$

with the double sum absolutely convergent. Hence, we can rearrange the double summation to get,

$$\mathcal{E}(X + Y) = \sum_{n=0}^{\infty} \sum_{i=0}^n X^{(i)} Y^{(n-i)} = \sum_{n=0}^{\infty} (X + Y)^{(n)},$$

with the last equality following from Proposition 3.2.  $\square$

Lemma 3.3 and Proposition 3.1 provide a direct proof of Eq. (1.1) for a finite-variation semimartingale  $X$  without the use of Section 2. This is because  $X = Y + Z$ , where  $Y := \sum_{s \leq \cdot} \Delta X_s$  and  $Z := X - Y$  is a *continuous* finite-variation process, and we know both  $Y$  and  $Z$  satisfy (1.1). Moreover, applying Propositions 3.1 and 3.2 we get,

$$X_t^{(n)} = \sum_{i=0}^n \frac{1}{(n-i)!} \left( \sum_{s_1 < \dots < s_n \leq t} \Delta X_{s_1} \cdots \Delta X_{s_n} \right) (X_t - \sum_{s \leq t} \Delta X_s)^{n-i}.$$

Proposition 3.2 also applies to the continuous-discontinuous decomposition of a semimartingale because they have zero covariation.

It is possible to derive a formula for  $(X + Y)^{(n)}$  in general, without the assumption  $[X, Y] = 0$ . Since we will not need this, we content ourselves with the continuous case.

**Proposition 3.4.** *Let  $X$  and  $Y$  be continuous semimartingales with  $X_0 = Y_0 = 0$ . Then,*

$$(3.4) \quad (X + Y)^{(n)} = \sum_{i, j, k \geq 0; i+j+2k=n} \frac{(-1)^k}{k!} X^{(i)} Y^{(j)} [X, Y]^k.$$

*Proof.* Since  $X$  and  $Y$  are continuous, we have for any real number  $\lambda$ ,

$$\mathcal{E}(\lambda(X + Y)) = \mathcal{E}(\lambda X) \mathcal{E}(\lambda Y) e^{-\lambda^2 [X, Y]}.$$

Hence by Proposition 2.5 (applied thrice) we have,

$$\begin{aligned} \sum_{n=0}^{\infty} \lambda^n (X + Y)^{(n)} &= \sum_{i=0}^{\infty} \lambda^i X^{(i)} \sum_{j=0}^{\infty} \lambda^j Y^{(j)} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \lambda^{2k} [X, Y]^k \\ &= \sum_{n=0}^{\infty} \lambda^n \sum_{i, j, k \geq 0; i+j+2k=n} \frac{(-1)^k}{k!} X^{(i)} Y^{(j)} [X, Y]^k. \end{aligned}$$

The desired result follows by comparing the coefficients of  $\lambda^n$  on both sides.  $\square$

#### 4. THE CASE OF A COUNTING PROCESS

We call a semimartingale  $N$  with  $N_0 = 0$  a *counting process* if  $[N] = N$ , or equivalently,  $N$  is the sum of its jumps all which equal 1, implying  $N$  is piecewise constant, increasing, and integer valued. Examples are Poisson processes, or more generally, Cox processes. Another example is a finite sum of the indicator processes of independent stopping times.

**Proposition 4.1.** *Let  $N$  be a counting process. Then for  $\lambda, a \in \mathbb{R}$  and  $n \in \mathbb{N}$ , we have:*

$$(4.1) \quad (1 + \lambda)^N = \sum_{n=0}^{\infty} \lambda^n N^{(n)};$$

$$(4.2) \quad e^{aN} = \sum_{n=0}^{\infty} (e^a - 1)^n N^{(n)};$$

$$(4.3) \quad N^{(n)} = 1_{N \geq n} \binom{N}{n};$$

$$(4.4) \quad N^n = \sum_{i=1}^n c_{n,i} N^{(i)},$$

where

$$c_{n,i} := \sum_{j=0}^i (-1)^{i-j} \binom{i}{j} j^n, \quad n, i = 0, 1, 2, \dots \quad (c_{0,0} := 1)$$

*Proof.* Proposition 3.1 applied to  $X = \lambda N$  yields (using jumps of  $N$  equal 1),

$$\sum_{n=0}^{\infty} \lambda^n N^{(n)} = \mathcal{E}(\lambda N) = \prod_{s \leq \cdot} (1 + \lambda \Delta N_s) = (1 + \lambda)^N.$$

Eq. (4.1) follows. As for (4.2), set  $\lambda = (e^a - 1)$ . Then  $(1 + \lambda)^N = e^{aN}$ . So, (4.2) follows from (4.1). Next, we have  $(1 + \lambda)^N = \sum_{n=0}^N \binom{N}{n} \lambda^n$ . Thus (4.3) follows from (4.1) by comparing the coefficients  $\lambda^n$  on both sides. Finally, to show (4.4), we note that by Eq. (4.2),

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{a^n}{n!} N^n &= e^{aN} = \sum_{i=0}^{\infty} (e^a - 1)^i N^{(i)} = \sum_{i=0}^{\infty} \sum_{j=0}^i (-1)^j \binom{i}{j} e^{ja} N^{(i)} \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^i \sum_{n=0}^{\infty} (-1)^j \binom{i}{j} j^n \frac{a^n}{n!} N^{(i)} = \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} c_{n,i} \frac{a^n}{n!} N^{(i)}. \end{aligned}$$

Eq. (4.4) follows by comparing the coefficients of  $a^n$  and using  $c_{n,0} = 0 = c_{n,i}$  for  $i > n$ .  $\square$

The numbers  $c_{n,i}/i!$  are the Stirling numbers of the second kind, i.e., the number of partitions of  $\{1, \dots, n\}$  into  $i$  subsets.<sup>4</sup> (One has  $c_{n,0} = 0$ ,  $c_{n,n} = n!$ ,  $c_{n,i} = 0$  for  $1 \leq n < i$ .)

Another way to see (4.3) is that by Proposition 4.1,  $N_t^{(n)} = \sum_{1 \leq i_1 < \dots < i_n \leq N_t} \Delta N_{T_{i_1}} \cdots \Delta N_{T_{i_n}}$  where  $(T_i)_{i=1}^{N_t}$  are the jump times of  $N$  on  $[0, t]$ . If  $N_t < n$ , then the sum is taken over the empty set and is zero. Otherwise, since  $\Delta N_{T_i} = 1$ , the sum counts the multi-indices  $1 \leq i_1 < \dots < i_n \leq N_t$ , i.e., the number of subsets of  $\{1, \dots, N_t\}$  with  $n$  elements.

**4.1. Alternative derivations.** It is instructive to give alternative derivations of Eq. (4.4). One derivation uses the following identity from [2] for a semimartingale  $X$  with  $X_0 = 0$ :

$$X^n = \sum_{p=1}^n \sum_{i_1, \dots, i_p \in \mathbb{N}; i_1 + \dots + i_p = n} \frac{n!}{i_1! \cdots i_p!} \int \int^- \cdots \int^- dX^{[i_1]} \cdots dX^{[i_{p-1}]} dX^{[i_p]}.$$

Since  $N^{[i]} = N$  for all  $i$ , the iterated integral above is just  $N^{(p)}$  here. Therefore,

$$N^n = \sum_{p=1}^n \sum_{i_1, \dots, i_p \in \mathbb{N}; i_1 + \dots + i_p = n} \frac{n!}{i_1! \cdots i_p!} N^{(p)} = \sum_{p=1}^n c_{n,p} N^{(p)},$$

<sup>4</sup>We refer to the online-encyclopedia Wikipedia [7] for the basic properties of Stirling numbers.

as desired, where we used the readily verified identity,  $c_{n,p} = \sum_{i_1, \dots, i_p \in \mathbb{N}; i_1 + \dots + i_p = n} \frac{n!}{i_1! \dots i_p!}$ .

Another proof applies the recursion  $N^n = \sum_{i=0}^{n-1} \binom{n}{i} \int X_-^i dX_-^{[n-i]}$  from [2]. It follows,

$$N^n = \sum_{i=0}^{n-1} \binom{n}{i} \int N_-^i dN.$$

Hence, substituting on the right hand side for  $N_-^i$  and using induction (and  $c_{i,0} = 0$ ),

$$\begin{aligned} N^n &= \sum_{i=0}^{n-1} \binom{n}{i} \int \left( \sum_{j=0}^i c_{i,j} N^{(j)} \right)_- dN \\ &= \sum_{i=0}^{n-1} \binom{n}{i} \sum_{j=0}^i c_{i,j} N^{(j+1)} = \sum_{j=0}^{n-1} N^{(j+1)} \sum_{i=j}^{n-1} \binom{n}{i} c_{i,j} \\ &= \sum_{j=1}^n N^{(j)} \sum_{i=j-1}^{n-1} \binom{n}{i} c_{i,j-1} = \sum_{j=1}^n N^{(j)} c_{n,j}, \end{aligned}$$

where, in the last equality we used the easily verified fact that  $\sum_{i=j-1}^{n-1} \binom{n}{i} c_{i,j-1} = c_{n,j}$ .

**4.2. Martingale representation of the powers.** Let  $\Lambda$  be the (necessarily increasing) compensator of  $N$ . For example, if  $N$  is Poisson process of intensity  $\lambda$ , then  $\Lambda_t = \lambda t$ .

Set  $M := N - \Lambda$ . So,  $M$  is a local martingale.

**Proposition 4.2.** *With notation as above, assume  $\Lambda$  is continuous. Then for any  $n \in \mathbb{N}$ ,*

$$(4.5) \quad N^n = \sum_{i=0}^n A^{n,i} M^{(i)},$$

where

$$A_t^{n,i} := \sum_{k=0}^{n-i} \sum_{j=1}^{k+i} (-1)^{k+i-j} \frac{(k+i)! j^n}{(k+i-j)! j! k!} \Lambda_t^k.$$

*Proof.* Since  $\Lambda$  is continuous and of finite variation,  $\Lambda^{(n)} = \Lambda^n/n!$ . Hence by Prop. 3.2,

$$N^{(j)} = \sum_{i=0}^j M^{(i)} \Lambda^{(j-i)} = \sum_{i=0}^j M^{(i)} \frac{\Lambda^{j-i}}{(j-i)!}.$$

Therefore by Eq. (4.4) (and using  $c_{n,0} = 0$ ),

$$\begin{aligned} N^n &= \sum_{j=0}^n c_{n,j} N^{(j)} = \sum_{j=0}^n \sum_{i=0}^j c_{n,j} M^{(i)} \frac{\Lambda^{j-i}}{(j-i)!} \\ &= \sum_{i=0}^n \left( \sum_{j=i}^n c_{n,j} \frac{\Lambda^{j-i}}{(j-i)!} \right) M^{(i)} = \sum_{i=0}^n \left( \sum_{k=0}^{n-i} c_{n,k+i} \frac{\Lambda^k}{k!} \right) M^{(i)}. \end{aligned}$$

Substituting the expression for  $c_{n,k+i}$  completes the proof.  $\square$

When  $\Lambda$  is deterministic, then so are  $A^{n,i}$ . Thus, for  $T > 0$ , Eq. (4.5) furnishes an explicit martingale representation of  $N_T^n$  in terms of the martingales  $M^{(n)}$ . In particular,

$$\mathbb{E} N_T^n = A_T^{n,0} := \sum_{k=1}^n \sum_{j=1}^k \frac{(-1)^{k-j} j^n}{(k-j)! j!} \Lambda_T^k.$$

(Also, from (4.2) and the formula  $\mathbb{E} e^{aN_T} = e^{(e^a-1)\Lambda_T}$ , one easily gets,  $\mathbb{E} N_T^{(n)} = \Lambda_T^n/n!$ .)

Next suppose  $N$  is a Cox process, that is  $\Lambda$  is adapted to the subfiltration  $(\mathcal{F}_t^W)$  generated by a (vector) Brownian motion  $W$ . Assume also  $\Lambda_T^k$  is integrable for all  $k$ . Then, as  $A_T^{n,i}$  is  $\mathcal{F}_T^W$ -measurable, it has a representation  $A_T^{n,i} = M_T^{n,i}$ , where  $M^{n,i} = \mathbb{E}(A_T^{n,i}) + \int H^{n,i} dW$  for some predictable (vector) process  $H^{n,i}$  (depending on  $T$ ). Hence, by (4.5) we have,

$$N_T^n = \sum_{i=0}^n M_T^{n,i} M_T^{(i)} = M_T^{n,0} + \sum_{i=1}^n \left( \int_0^T M_t^{n,i} dM_t^{(i)} + \int_0^T M_{t-}^{(i)} dM_t^{n,i} \right),$$

where we integrated by parts using  $[M^{(i)}, M^{n,i}] = 0$ . So, all  $N_T^n$  can be represented in terms of the martingales  $M := N - \Lambda$  and  $W$ . When the law of  $N_T$  is exponentially decreasing, random variables of the form  $f(N_T)$  can be approximated in mean-square by a polynomial in  $N_T$  (e.g., [2]). Thus, such random variables admit a chaotic representation as described.

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