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Abstract

This paper analyses the optimal saving behaviour of a risk-averse and prudent consumer who faces two sources of income risk: risk as described by a given probability distribution and risk in the distribution itself. The latter is captured by the randomness in the parameters underlying the probability distribution and is referred to as distributional risk. Stochastic volatility, which generally refers to the randomness in the variance, can be viewed as a form of distributional risk. Necessary and sufficient conditions by which an increase in distributional risk will induce the consumer to save more are derived under two specifications of preferences: expected utility preferences and Selden/Kreps-Porteus preferences. The connection (or lack of) between these conditions and stochastic volatility is addressed. The additional conditions under which a prudent consumer will save more under greater volatility risk are identified.

Keywords: Stochastic volatility, stochastic convexity, precautionary saving.

JEL classification: D81, D91, E21.

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1 Introduction

This paper analyses the optimal saving behaviour of a risk-averse and prudent consumer who receives a random income drawn from a mixture of probability distributions. A mixture model is used to capture two sources of income risk: risk as described by a given probability distribution and risk in the distribution itself. The latter is referred to as *distributional risk*. The consumer is said to display a precautionary saving motive against distributional risk if he saves more when there is an increase in such risk.[1] The main purpose of this study is to derive a set of conditions for this to occur under two specifications of preferences: expected utility preferences and Selden/Kreps-Porteus preferences.[2] Our interest in this topic is motivated by the recent development in the stochastic volatility literature. Stochastic volatility, which generally refers to the randomness in the variance of some exogenous variables, can be viewed as one form of distributional risk. There is now ample empirical evidence showing that the volatility of major economic variables are time-varying and stochastic.[3] These findings have inspired a surge of interest in understanding how volatility risk (also known as uncertainty risk) would affect individual choices and market outcomes.[4] In particular, several authors have suggested that an increase in volatility risk will induce consumers to save more out of precautionary motives.[5] The results in this paper can be used to shed light on the theoretical foundation of such claim.

To analyse the effects of distributional risk on consumption-saving decisions, we adopt a similar two-period framework as in Leland (1968), Sandmo (1970), Kimball (1990) and Kimball and Weil (2009). This model focuses on a single risk-averse consumer who faces income risk only in the second period and who can self-insure by holding a single risk-free asset. In the present study, the second-period income is assumed to be drawn from a mixture of a collection of probability distributions, denoted by $\mathcal{F} = \{F(\cdot \mid \theta), \ \theta \in \Theta\}$. Each member in this collection

[1] The consumer may also display a precautionary saving motive against the first source of income risk. These two types of precautionary saving motives are parallel but independent of one another.

[2] Although expected-utility preferences can be viewed as a special case of Selden/Kreps-Porteus preferences, the conditions for precautionary savings under these two types of preferences are rather different. This is true even in the absence of distributional risk [see Kimball (1990), Gollier (2001, Section 20.3) and Kimball and Weil (2009)]. Thus, in our main analysis we will deal with these two types of preferences separately.

[3] At the aggregate level, Stock and Watson (2002) and Sims and Zha (2006), among others, have documented the time-varying nature of business cycle volatility in postwar US economy. At the household level, Meghir and Pistaferri (2004), Storesletten *et al.* (2004) and Guvenen *et al.* (2014) have shown that the variance of US household earnings is fluctuating over time and correlated with macroeconomic conditions. In the asset pricing literature, Bansal and Yaron (2004) have provided evidence of stochastic volatility in US consumption. Other indicators of volatility risk have been discussed in Bloom (2014).

[4] See Fernández-Villaverde and Rubio-Ramírez (2013) and Bloom (2014) for concise reviews on models with stochastic volatility or uncertainty shocks. Following this literature, we will use the terms "risk" and "uncertainty" interchangeably to describe random events that can be quantified by a well-defined probability distribution.

[5] See, for instance, Bloom (2014, p.165) and Basu and Bundick (2014).

is indexed by a vector of parameters θ, which is itself a random variable. The randomness in θ is what we referred to as distributional risk. The extent of such risk is captured by another distribution function $G(\theta)$. The consumer is assumed to know both $\mathcal{F}$ and G so that there is no ambiguity regarding the probability distribution of the second-period income. Under this framework, changes in income risk can be due to changes in $F(\cdot \mid \theta)$ brought by changes in θ, or changes in the mixing distribution G. Our focus is on the latter. Specifically, we examine how an increase in the riskiness of G would affect individual savings. In light of the stochastic volatility literature mentioned earlier, such an increase can be the result of deteriorating macroeconomic conditions which lead to a more volatile prospect for individual consumers.

Our main findings can be summarised as follows: Firstly, we observe that the standard conditions for risk aversion and prudence are not directly applicable to distributional risk. To see this precisely, consider an expected-utility consumer with von Neumann-Morgenstern utility function $u(\cdot)$ and marginal utility $u'(\cdot)$.[6] For explanatory convenience, suppose θ is just a scalar. In the presence of distributional risk, the expected utility of future income is given by $\int E[u(y) \mid \theta]\, dG(\theta)$, where $E[\cdot \mid \theta]$ is the expectation formed under the distribution $F(y \mid \theta)$. Similarly, define $\int E[u'(y) \mid \theta]\, dG(\theta)$ as the expected marginal utility of future income. Using the standard textbook definition of risk aversion, the consumer is said to dislike distributional risk if and only if $E[u(y) \mid \theta]$ exhibits concavity in θ. Likewise, by the same argument as in Kimball (1990), the consumer is said to be prudent towards distributional risk if and only if $E[u'(y) \mid \theta]$ exhibits convexity in θ.[7] The problem is that the concavity of $u(\cdot)$ does not necessarily imply the concavity of $E[u(y) \mid \theta]$ under an arbitrary set of distribution functions $\mathcal{F}$.[8] In fact, it is easy to construct examples in which $u(\cdot)$ is globally concave but $E[u(y) \mid \theta]$ is globally *convex* in θ. Similarly, the convexity of $u'(\cdot)$ does not necessarily imply the convexity of $E[u'(y) \mid \theta]$ under an arbitrarily given $\mathcal{F}$. Thus, a convex marginal utility function alone is not enough to ensure the existence of precautionary savings against distributional risk.

The above discussion makes clear that some restrictions on $\mathcal{F}$ are necessary in order to establish an aversion towards distributional risk and a precautionary saving motive against this type of risk. The main contribution of this paper is to make clear what these restrictions are.

[6] The argument below also applies to Selden/Kreps-Porteus preferences, but in order to explain this precisely we need to introduce more notations. For this reason, we choose to defer this discussion until Section 3.2.

[7] More specifically, the consumer is said to be prudent towards distributional risk if an increase in the riskiness of θ will raise the expected marginal utility $\int E[u'(y) \mid \theta]\, dG(\theta)$. In the absence of distributional risk, Kimball (1990) shows that an expected-utility consumer is prudent if and only if the marginal utility function is convex.

[8] The integrability of $u(\cdot)$ under $F(\cdot \mid \theta)$ is one issue, but the main problem remains even if we focus on those $u(\cdot)$ and $F(\cdot \mid \theta)$ such that $E[u(y) \mid \theta]$ exists and is finite.

Specifically, we provide a necessary and sufficient condition under which the concavity of $u(\cdot)$ and the convexity of $u'(\cdot)$ are inherited by $E[u(y) \mid \theta]$ and $E[u'(y) \mid \theta]$, respectively. This single condition thus ensures that both the risk aversion and prudence properties are transferred from $u(\cdot)$ to $E[u(y) \mid \theta]$. We refer to this condition as *stochastic convexity.* With expected-utility preferences, an increase in distributional risk will lead a consumer with convex marginal utility to save more if and only if the stochastic convexity condition is satisfied. We also derive an analogous result for Selden/Kreps-Porteus preferences.

Equipped with these findings, we are now in position to comment on the existence of precautionary savings in stochastic volatility models. The implications of our results are clear: an increase in volatility risk can induce a risk-averse and prudent consumer to save more if and only if the stochastic convexity condition is satisfied. In Section 2.3, we show that while stochastic volatility can be viewed as a form of distributional risk, it does not imply (and is not implied by) stochastic convexity. Thus, stochastic convexity is an additional condition needed to ensure the existence of precautionary savings against volatility risk.

The rest of this paper is organised as follows: Section 2 introduces some basic concepts and results. Section 3 describes the model environment. Section 4 analyses the existence of precautionary savings under expected utility preferences and Selden/Kreps-Porteus preferences. Section 5 provides some concluding remarks.

2 Preliminaries

The purpose of this section is to introduce some basic concepts and results that are essential for our analysis. In Section 2.1, we present a basic framework for defining a mixture of probability distributions and make clear the meaning of distributional risk. In Section 2.2, we provide a formal definition of stochastic convexity and establish an important characterisation result which will be used throughout the paper. In Section 2.3, we discuss the connection between stochastic volatility and distributional risk.

2.1 Mixture of Probability Distributions

Let $\mathcal{F} = \{F(\cdot \mid \theta) : \theta \in \Theta\}$ be a collection of probability distributions defined on the support $S \subseteq \mathbb{R}$. Each member of this collection is indexed by a random vector θ drawn from a set $\Theta \subset \mathbb{R}^m$, for some m. The probability distribution of θ is denoted by $G : \Theta \rightarrow [0, 1]$. Let Y be a random variable with probability distribution given by

$$H(y) \equiv \Pr(Y \leq y) = \int_{\Theta} F(y \mid \theta)\, dG(\theta), \qquad \text{for all } y \in S. \tag{1}$$

In words, H is a mixture or weighted average of the distribution functions in $\mathcal{F}$, with weights assigned according to G.[9] This setup can be interpreted in a number of ways. For instance, one can view (Y, θ) as a vector of correlated random variables. In this case, θ represents a set of covariates of Y (or background risks) with marginal distribution G; $F(y \mid \theta)$ is the distribution of Y conditioned on θ and H is the marginal distribution of Y. In the multiple-prior models of ambiguity, the mixture equation in (1) is used to represent the subjective beliefs of a decision maker who is ambiguous about the true distribution of Y. In this context, the second-order distribution $G(\cdot)$ captures the degree of ambiguity, while $\mathcal{F} = \{F(\cdot \mid \theta) : \theta \in \Theta\}$ represents a set of plausible first-order distributions or priors.[10]

In the present study, Y is an exogenous random variable that directly affects the choices of a risk-averse consumer, whereas θ is a set of random parameters that will affect those choices *indirectly* through the distribution of Y. For this reason, we refer to the randomness in θ as *distributional risk*. The extent of such risk in the mixture H is captured by the distribution function G. There is no ambiguity regarding the probability distributions of θ and Y.

2.2 Stochastic Convexity

In order to define the concept of stochastic convexity, we need to introduce some additional notations. Let $\mathcal{C}(S)$ be the set of all real-valued, continuous functions defined on S that are integrable with respect to the probability distributions in $\mathcal{F}$. Define an operator Γ on $\mathcal{C}(S)$ according to

$$(\Gamma\psi)(\theta) \equiv \int_{S} \psi(y)\, dF(y \mid \theta) = E[\psi(y) \mid \theta], \qquad \text{for all } \theta \in \Theta. \tag{2}$$

[9] The above definition can be generalised in at least two ways. First, $\mathcal{F}$ can be an arbitrary collection of multivariate distributions. Second, Θ can be taken as an arbitrary index set. In other words, the distribution functions in $\mathcal{F}$ need not be parametric. For further details on this, see Teicher (1960).

[10] See Section 5 for yet another interpretation of the mixture model in (1).

For any given $\theta \in \Theta$, $(\Gamma\psi)(\theta)$ is the expectation of the function ψ under $F(\cdot \mid \theta)$. Our attention will be focused on those distributions $G(\cdot)$ that satisfy the following condition:

$$\int_{\Theta} E[\psi(y) \mid \theta]\, dG(\theta) < \infty, \qquad \text{for all } \psi \in \mathcal{C}(S). \tag{3}$$

This ensures that every function ψ in $\mathcal{C}(S)$ is integrable with respect to $H(\cdot)$. The set of all distribution functions $G(\cdot)$ that satisfy (3) is denoted by $\mathcal{L}(\Theta)$. Since $(\Gamma\psi)(\theta) < \infty$ for all $\psi \in \mathcal{C}(S)$ and for all $\theta \in \Theta$, the set $\mathcal{L}(\Theta)$ includes all Dirac distributions that assign unit probability to a single point in Θ.

Let $\mathcal{C}'(S)$ be an arbitrary subset of $\mathcal{C}(S)$. The operator Γ defined in (2) is said to be *stochastically convex with respect to* $\mathcal{C}'(S)$ if $\Gamma\psi$ is a convex function of θ for all $\psi \in \mathcal{C}'(S)$. For example, if $\mathcal{C}'(S)$ is the set of all increasing functions in $\mathcal{C}(S)$, then stochastic convexity means that Γ will map every increasing function of y in $\mathcal{C}(S)$ to a convex function of θ. This form of stochastic convexity has been discussed and analysed in Topkis (1998, Section 3.9.1). For our purposes here, the relevant form of stochastic convexity is the one with respect to all decreasing convex functions, so from this point onward $\mathcal{C}'(S)$ denotes the set of all decreasing convex functions in $\mathcal{C}(S)$.

In practice, it is difficult (if at all possible) to check the convexity of $\Gamma\psi$ for every function in $\mathcal{C}'(S)$. Thus, a more operational characterisation of stochastic convexity is called for. This is achieved in Theorem 1. For any $y \in S$, define an auxiliary function $\Phi(\cdot; y) : \Theta \to \mathbb{R}_+$ by

$$\Phi(\theta; y) \equiv \int_{\underline{y}}^{y} F(\omega \mid \theta)\, d\omega, \tag{4}$$

where $\underline{y}$ is the infimum of S.

Assumption A1 For any $y \in S$, the function $\Phi(\theta; y)$ defined in (4) is convex in θ.

Theorem 1 states that Assumption A1 is both necessary and sufficient for Γ to be stochastically convex with respect to $\mathcal{C}'(S)$.[11] Its corollary follows immediately from the fact that $-\psi$ is increasing concave whenever ψ is decreasing convex. Unless otherwise stated, all proofs can be found in the Appendix.

[11] It is also possible to derive a necessary and sufficient condition under which Γ exhibits stochastic convexity with respect to *all* convex functions in $\mathcal{C}(S)$. This condition, however, is stronger than Assumption A1 as it enforces stochastic convexity on a larger set of functions. In this study, we choose to exploit both the monotonicity and concavity/convexity of the von Neumann-Morgenstern utility function $u(\cdot)$ and its first derivative $u'(\cdot)$ so that stochastic convexity can be obtained under a weaker condition, which is Assumption A1.

Theorem 1 *Let $\mathcal{C}'(S)$ be the subset of $\mathcal{C}(S)$ consisting of decreasing convex functions. For any $\psi \in \mathcal{C}'(S)$, $\Gamma\psi$ is a convex function of θ if and only if Assumption A1 is satisfied.*

Corollary 2 *For any increasing concave function ψ in $\mathcal{C}(S)$, $\Gamma\psi$ is a concave function of θ if and only if Assumption A1 is satisfied.*

Examples of distribution functions that satisfy Assumption A1 can be easily constructed as follows: Let $F_1(\cdot)$ and $F_2(\cdot)$ be two distribution functions with support in S. For each $\theta \in \Theta$, define $F(\cdot \mid \theta)$ according to

$$F(y \mid \theta) \equiv p(\theta) F_1(y) + [1 - p(\theta)] F_2(y), \qquad \text{for all } y \in S, \tag{5}$$

where $p : \Theta \to [0, 1]$ is a weighting function. The function Φ defined in (4) then becomes

$$\Phi(\theta; y) = p(\theta) \int_{\underline{y}}^{y} [F_1(\omega) - F_2(\omega)]\, d\omega + \int_{\underline{y}}^{y} F_2(\omega)\, d\omega.$$

Suppose $F_2(\cdot)$ is a mean-preserving spread of $F_1(\cdot)$. Then for any $y \in S$, $\Phi(\theta; y)$ is convex in θ if and only if $p(\cdot)$ is a concave function.[12]

Alternatively, since convexity is preserved by integration, Assumption A1 is satisfied if $F(y \mid \theta)$ is convex in θ for all $y \in S$.

Stochastic Dominance and Stochastic Convexity

Before proceeding further, it is useful to discuss the differences between stochastic dominance and stochastic convexity. In the context of equation (2), stochastic dominance can be viewed as defining the monotonicity of $E[\psi(y) \mid \theta]$ in θ for a certain class of function ψ, whereas stochastic convexity defines the convexity of $E[\psi(y) \mid \theta]$ in θ for a certain class of ψ. When viewed in this light, it is clear that there is no direct connection between these two concepts. To give a concrete example, suppose $F(\cdot \mid \theta_1)$ first-order stochastically dominates $F(\cdot \mid \theta_2)$ for any $\theta_1 \geq \theta_2$ in Θ. In words, this means the likelihood of drawing a large value of Y is monotonically increasing in θ. It follows that the expected value of Y under $F(\cdot \mid \theta_1)$ is greater than that under $F(\cdot \mid \theta_2)$. Thus, in this setting distributional risk (i.e., the randomness in θ) implies the randomness in the first moment of Y. This type of stochastic dominance is also equivalent to the assertion that $E[\psi(y) \mid \theta]$ is increasing in θ for all increasing function ψ in $\mathcal{C}(S)$.

[12] This uses the fact that $F_2(\cdot)$ is a mean-preserving spread of $F_1(\cdot)$ if and only if $\int_{\underline{y}}^{y} [F_1(\omega) - F_2(\omega)]\, d\omega \leq 0$, for all $y \in S$.

The difference between this type of stochastic dominance and stochastic convexity can also be seen as follows: Recall that a necessary and sufficient condition for first-order stochastic dominance is that $F(y \mid \theta)$ is decreasing in θ for all $y \in S$. Clearly, this condition does not imply and is not implied by Assumption A1. This again confirms that there is no direct connection between the two. But they are not incompatible. In particular, it is possible to construct a collection $\mathcal{F}$ whereby $F(y \mid \theta)$ is decreasing in θ *and* $\Phi(\theta; y)$ is convex in θ for all $y \in S$.

Another example is the second-order stochastic dominance criterion. This type of ordering is closely related to the notion of stochastic variance, hence it will be discussed in the next section.

2.3 Stochastic Volatility and Distributional Risk

If the variance of Y is a function of θ, then stochastic volatility of Y can be generated by the randomness in θ. One way to achieve this is by imposing the following assumptions: (i) all distributions in $\mathcal{F}$ share the same mean, and (ii) $F(\cdot \mid \theta_1)$ second-order stochastically dominates $F(\cdot \mid \theta_2)$ whenever $\theta_1 \geq \theta_2$. These conditions are equivalent to the assertion that $E[\psi(y) \mid \theta]$ is increasing in θ for all concave functions ψ defined on S. This in turn implies that the variance under $F(\cdot \mid \theta_1)$ is lower than that under $F(\cdot \mid \theta_2)$. Hence, the randomness in θ will imply the randomness in the variance of Y.

How is this type of stochastic dominance related to stochastic convexity? Given the equal mean assumption, a necessary and sufficient condition for second-order stochastic dominance is that $\Phi(\theta; y)$ is decreasing in θ for all $y \in S$. In other words, this type of stochastic dominance is characterised by the monotonicity of $\Phi(\theta; y)$ in θ, whereas stochastic convexity is characterised by its convexity. Thus, there is no direct connection between the two but they are also not mutually exclusive. We illustrate this by means of two examples. First, consider the set of probability distributions $\{F(\cdot \mid \theta) : \theta \in \Theta\}$ defined by (5). Since $F_2(\cdot)$ is a mean-preserving spread of $F_1(\cdot)$, all the distributions in this collection will have the same mean but different variances. In particular, a random θ will generate a random variance of Y, regardless of the shape of $p(\cdot)$. But the condition of stochastic convexity is satisfied only when $p(\cdot)$ is concave.

In the second example, Y is generated by a simple linear model:

$$y = \mu + \sigma\varepsilon, \tag{6}$$

where μ is a deterministic constant, ε is a random variable with zero mean and unit variance, and σ is a positive random variable that captures stochastic volatility. The parameter θ is now a scalar which corresponds to the inverse of σ (i.e., the precision). Let $J(\varepsilon)$ be the distribution function of ε defined over the support $[\underline{\varepsilon}, \infty)$. The function $\Phi(\theta; y)$ is then given by[13]

$$\begin{aligned}
\Phi(\theta; y) &\equiv \int_{\underline{y}}^{y} F(\omega \mid \theta)\, d\omega = \int_{S} \max\{y - \omega, 0\}\, dF(\omega \mid \theta) \\
&= \frac{1}{\theta} \int_{\underline{\varepsilon}}^{\infty} \max\{\varepsilon - \xi, 0\}\, dJ(\xi) \\
&= \frac{1}{\theta} \int_{\underline{\varepsilon}}^{\varepsilon} J(\xi)\, d\xi,
\end{aligned}$$

which is strictly decreasing and strictly convex in θ for all ε in $[\underline{\varepsilon}, \infty)$. The linear model in (6) thus entails both stochastic volatility and stochastic convexity.

3 The Model

Consider a single risk-averse consumer who lives for two periods. The consumer is endowed with a known amount of wealth $z > 0$ in the first period, and faces a random income y in the second period. The consumer can self-insure by saving or borrowing at a single risk-free interest rate. An *ad hoc* borrowing constraint is in place to limit the amount of debt that the consumer can have. Our main focus is on the consumption-saving decision in the first period.

The novelty of this model lies in the introduction of distributional risk as defined in Section 2.1. Specifically, let Θ be the parameter space which is a convex subset of $\mathbb{R}^m$, and let $\mathcal{F} = \{F(\cdot \mid \theta),\ \theta \in \Theta\}$ be a collection of probability distributions with support $S = [\underline{y}, \infty)$, where $\underline{y} > 0$. The parameter θ is itself a random variable with distribution function $G : \Theta \to [0, 1]$ that satisfies (3). The unconditional probability distribution of y is then defined by (1). The consumer is assumed to have perfect knowledge regarding $\mathcal{F}$ and $G(\cdot)$ when he makes his choices in the first period. For ease of future reference, we will refer to y as the random income generated by $\{\mathcal{F}, G\}$.

3.1 Preferences

Two specifications of preferences are considered in this paper. The first one is the standard expected utility (EU) specification under which preferences are separable over time and across

[13] The second equality uses integration by parts. The third equality uses the equations: $y = \mu + \sigma\varepsilon$ and $\omega = \mu + \sigma\xi$, where ω and ξ are dummy variables of integration.

state of nature. The second one is the Selden/Kreps-Porteus (SKP) preferences which allow for a separation between attitude towards risk and attitude towards intertemporal substitution.

Let c_1 and c_2 denote consumption in the first and the second period, respectively. Under the EU specification, the consumer's preferences are represented by

$$E\left[U\left(c_1, c_2\right)\right] = u\left(c_1\right) + \beta E\left[u\left(c_2\right)\right],$$

where E is the expectation operator conditioned on the information available in the first period, $\beta \in (0, 1)$ is the subjective discount factor and $u(\cdot)$ is the von Neumann-Morgenstern (vNM) utility function. The vNM utility function is assumed to satisfy the following properties.

Assumption A2 The function $u : \mathbb{R}_+ \to \mathbb{R}$ is continuous, differentiable, increasing and concave. Both $u(\cdot)$ and $u'(\cdot)$ are integrable with respect to the distribution functions in $\mathcal{F}$.

Under the SKP specification, the consumer's preferences are given by

$$E\left[U\left(c_1, c_2\right)\right] = v\left(c_1\right) + \beta v\left(M\left(c_2\right)\right), \tag{7}$$

where $v(\cdot)$ is the period utility function for non-stochastic values, and $M(\cdot)$ is a certainty equivalent operator defined by

$$M\left(c_2\right) = \phi^{-1}\left\{E\left[\phi\left(c_2\right)\right]\right\}. \tag{8}$$

The function $\phi(\cdot)$ is the atemporal vNM utility function. As is well-known in this literature, the curvature of $v(\cdot)$ captures the consumer's willingness to smooth consumption over time, while the curvature of $\phi(\cdot)$ captures his attitude towards risk. These two functions are assumed to have the following properties:

Assumption A3 The function $v : \mathbb{R}_+ \to \mathbb{R}$ is continuous, differentiable, increasing and concave.

Assumption A4 The function $\phi : \mathbb{R}_+ \to \mathbb{R}$ is continuous, differentiable, *strictly* increasing and concave. Both $\phi(\cdot)$ and $\phi'(\cdot)$ are integrable with respect to the distributions in $\mathcal{F}$.

The properties in Assumptions A3 and A4 largely mirror those in Assumption A2. The function $\phi(\cdot)$ is required to be *strictly* increasing so that its inverse is a well-defined function. In what follows, we will refer to a consumer with an increasing concave vNM utility function as risk averse.

3.2 Attitude Towards Distributional Risk

Does a risk-averse consumer necessarily dislikes distributional risk? The answer to this question is (surprisingly) no.[14] In this subsection, we show that a risk-averse consumer (under both types of preferences) dislikes distributional risk if and only if Assumption A1 is satisfied. We also provide an example to show that in the violation of this assumption such a consumer will actually prefer *more* distributional risk to *less.*

To start, let $G'(\cdot)$ and $G''(\cdot)$ be two arbitrary distributions in $\mathcal{L}(\Theta)$. Suppose $G'(\cdot)$ is smaller than $G''(\cdot)$ under the multivariate convex order (denoted by $G' \leq_{cx} G''$), i.e.,

$$\int_{\Theta} \zeta(\theta)\, dG'(\theta) \leq \int_{\Theta} \zeta(\theta)\, dG''(\theta), \tag{9}$$

for any real-valued convex function ζ defined on Θ, provided the expectations exist. The above ordering can be viewed as a multivariate version of the standard second-order stochastic dominance criterion (the two coincides when θ is a scalar).[15] Let y' and y'' be the random incomes generated by $\{\mathcal{F}, G'\}$ and $\{\mathcal{F}, G''\}$, respectively. Then y'' is said to have a larger degree of distributional risk than y'.

A risk-averse EU consumer is said to dislike distributional risk if his expected utility under y' is greater than that under y'', i.e.,

$$\int_{\Theta}\int_{S} u(y')\, dF(y' \mid \theta)\, dG'(\theta) \geq \int_{\Theta}\int_{S} u(y'')\, dF(y'' \mid \theta)\, dG''(\theta). \tag{10}$$

Theorem 3 states that (10) is true if and only if Assumption A1 is satisfied. Intuitively, this result states that the only way to transfer the risk aversion property from $u(y)$ to $\widetilde{u}(\theta) \equiv E[u(y) \mid \theta]$ is by imposing the stochastic convexity condition. This result thus highlights the importance of Assumption A1 in characterising the consumer's attitude towards distributional risk.[16]

[14] Under the expected-utility hypothesis, the consumer is indifferent between a compound lottery form by $\{\mathcal{F}, G\}$ and another lottery with the same distribution as H (without compounding). This, however, does not imply that the consumer is indifferent between two different compound lotteries $\{\mathcal{F}, G'\}$ and $\{\mathcal{F}, G''\}$.

[15] For a textbook treatment of the multivariate convex order, see Shaked and Shanthikumar (2007, Chapter 7).

[16] Theorem 3 can also be interpreted as follows: Let H' and H'' be the compound distributions generated by $\{\mathcal{F}, G'\}$ and $\{\mathcal{F}, G''\}$, respectively. Suppose $G' \leq_{cx} G''$. Then H'' second-order stochastically dominates H' if and only if Assumption A1 is satisfied.

Theorem 3 *Let $u(\cdot)$ be an increasing concave function in $\mathcal{C}(S)$. Then any expected-utility consumer with utility function $u(\cdot)$ will prefer y' over y'' if and only if Assumption A1 is satisfied.*

As an illustration of this result, consider the collection of distribution functions $\mathcal{F}$ defined by (5). The expected utility under $\{\mathcal{F}, G\}$ can be expressed as

$$\begin{aligned}&\int_\Theta \int_Y u(y)\, dF(y \mid \theta)\, dG(\theta)\\ = \ &\left[\int_\Theta p(\theta)\, dG(\theta)\right]\left[\int_S u(y)\, dF_1(y) - \int_S u(y)\, dF_2(y)\right] + \int_S u(y)\, dF_2(y).\end{aligned}$$

In this example, the effect of distributional risk is entirely captured by the expected value of the weighting function. In particular if $p(\cdot)$ is a linear function, then

$$\int_\Theta p(\theta)\, dG'(\theta) = \int_\Theta p(\theta)\, dG''(\theta),$$

which means the consumer is neutral or indifferent towards distributional risk. If $p(\cdot)$ is convex, then we have

$$\int_\Theta p(\theta)\, dG'(\theta) \leq \int_\Theta p(\theta)\, dG''(\theta).$$

This, together with the assumption that $F_2(\cdot)$ is a mean-preserving spread of $F_1(\cdot)$, implies the following: (i) $\Phi(\theta; y)$ is a *concave* function in θ for any $y \in S$, and (ii) any risk-averse EU consumer is either indifferent or strictly prefers more distributional risk to less, i.e.,

$$\int_\Theta \int_S u(y')\, dF(y' \mid \theta)\, dG'(\theta) \leq \int_\Theta \int_S u(y'')\, dF(y'' \mid \theta)\, dG''(\theta).$$

Finally, the result in Theorem 3 can be easily extended to SKP preferences. Since $v(\cdot)$ is increasing, the consumer is averse to distributional risk if and only if the certainty equivalence under y' is greater than that under y'', i.e.,

$$\phi^{-1}\left[\int_\Theta \int_S \phi(y')\, dF(y' \mid \theta)\, dG'(\theta)\right] \geq \phi^{-1}\left[\int_\Theta \int_S \phi(y'')\, dF(y'' \mid \theta)\, dG''(\theta)\right].$$

Since $\phi^{-1}(\cdot)$ is also strictly increasing, this essential boils down to

$$\int_\Theta \int_S \phi(y')\, dF(y' \mid \theta)\, dG'(\theta) \geq \int_\Theta \int_S \phi(y'')\, dF(y'' \mid \theta)\, dG''(\theta). \tag{11}$$

It is immediate to see that (11) can be obtained by replacing u with ϕ in (10). This leads to the following slightly revised version of Theorem 3, which we state without proof.[17]

Theorem 4 *Let $\phi(\cdot)$ be a strictly increasing concave function in $\mathcal{C}(S)$. Suppose Assumption A1 is satisfied. Then any consumer with SKP preferences will prefer y' over y''.*

4 Precautionary Saving

We now explore the conditions under which an increase in distributional risk will promote savings under EU preferences and SKP preferences. Regardless of the preference specification, the consumer's first-period choices are subject to the sequential budget constraints: $c_1 + s = z$ and $c_2 = y + (1+r)s$, and an *ad hoc* borrowing constraint: $s \geq -\underline{b}$, where $r > 0$ is the risk-free interest rate, s denotes savings in the first period, and $\underline{b} > 0$ is the borrowing limit.

4.1 EU Preferences

Consider an EU consumer with utility function $u(\cdot)$ that satisfies Assumption A2. The consumer's problem in the first period is given by

$$\max_{s \in [-\underline{b}, z]} \left\{ u(z-s) + \beta \int_{\Theta} \int_{S} u[y + (1+r)s] \, dF(y \mid \theta) \, dG(\theta) \right\}. \tag{P1}$$

Since the objective function is continuous and the constraint set is compact, the above problem has at least one solution.[18] By the concavity of $u(\cdot)$, the Kuhn-Tucker first-order conditions are both necessary and sufficient to identify the solutions of (P1). To rule out the uninteresting case where first-period consumption is zero (i.e., $s = z$), the condition $\beta(1+r) < 1$ is imposed.[19] It follows that a feasible value s is optimal if and only if it satisfies the Euler equation

$$u'(z-s) \geq \beta(1+r) \int_{\Theta} \int_{S} u'[y + (1+r)s] \, dF(y \mid \theta) \, dG(\theta), \tag{12}$$

with equality holds if $s > -\underline{b}$. The left side of (12) captures the marginal cost of saving more, while the expression on the right is the discounted gain in expected future utility brought by an

[17] Note that we have lost the "only if" part in Theorem 4. This is because $\phi(\cdot)$ is required to be *strictly* increasing so that $\phi^{-1}(\cdot)$ is a well-defined function. Thus, in the "only if" part, starting from (2) we can only establish the stochastic convexity property of Γ with respect to all *strictly* increasing concave function in $\mathcal{C}(S)$, which is a smaller set than $\mathcal{C}'(S)$.

[18] Obviously one can characterise the solution of (P1) more sharply by imposing some stronger conditions on $u(\cdot)$, such as strict concavity and the Inada condition. By doing so, however, we will sacrifice the necessity of Assumption A1 in our main result.

[19] Since $u'(\cdot)$ is decreasing, we have $\int_{\Theta} \int_{S} u'[y + (1+r)z] \, dF(y \mid \theta) \, dG(\theta) \leq u'(0)$. This, together with $\beta(1+r) < 1$, implies that the marginal benefit of consuming more in the first period is strictly greater than the marginal cost of doing so. Hence, it is not optimal to have $c_1 = 0$.

increase in savings. An increase in distributional risk will induce the consumer to save more if and only if the marginal benefit of saving is higher under a riskier distribution of θ. Formally, let $G'(\cdot)$ and $G''(\cdot)$ be two distributions in $\mathcal{L}(\Theta)$ such that $G' \leq_{cx} G''$. Let y' and y'' be the random incomes generated by $\{\mathcal{F}, G'\}$ and $\{\mathcal{F}, G''\}$, respectively. Let s'' be *any* solution of (P1) under y''. Then an increase in distributional risk will lead the consumer to save more out of precautionary motives if and only if

$$\int_{\Theta}\int_{S} u'\left[y'' + (1+r)s''\right] dF\left(y'' \mid \theta\right) dG''(\theta) \geq \int_{\Theta}\int_{S} u'\left[y' + (1+r)s''\right] dF\left(y' \mid \theta\right) dG'(\theta). \tag{13}$$

In words, this means saving s'' under y'' will give a greater expected marginal benefit than saving the same amount under y'. This will then induce the consumer to save less under y', where the degree of distributional risk is lower.

As is well-known in the precautionary saving literature, in the absence of distributional risk, a precautionary motive of saving exists if and only if the EU consumer is prudent, i.e., the marginal utility function $u'(\cdot)$ is convex.[20] Thus, it seems natural to ask whether this type of consumer will save more when there is an increase in distributional risk. The answer is provided in Theorem 5, which states that condition (13) holds for any decreasing convex $u'(\cdot)$ if and only if the condition for stochastic convexity is satisfied.

Theorem 5 *Suppose Assumption A2 and $\beta(1+r) < 1$ are satisfied. Then any prudent expected-utility consumer will save more under y'' than under y' if and only if Assumption A1 is satisfied.*

The intuition of this result is straightforward. An increase in the dispersion of θ will increase the marginal benefit of saving if and only if the *expected* marginal utility $E\{u'[y + (1+r)s''] \mid \theta\}$ is convex in θ. On the other hand, the convexity of $u'(\cdot)$ will transpire into the convexity of $E\{u'[y + (1+r)s''] \mid \theta\}$ if and only if Assumption A1 is satisfied. Thus, the only way to transfer the prudence property from $u(y)$ to $\widetilde{u}(\theta) \equiv E[u(y) \mid \theta]$ is by imposing the stochastic convexity condition.

4.2 SKP Preferences

We now repeat the same exercise under SKP preferences. Consider a consumer with preferences defined by (7)-(8). Suppose Assumptions A3 and A4 are satisfied. Define the composite function

[20] If $u(\cdot)$ is thrice-differentiable, then this is equivalent to $u'''(c) \geq 0$ for all $c > 0$. Our main result, however, does not require $u(\cdot)$ to be thrice-differentiable.

$\psi(x) \equiv v\left[\phi^{-1}(x)\right]$. Since both $v(\cdot)$ and $\phi^{-1}(\cdot)$ are continuous, differentiable and increasing, so is $\psi(\cdot)$. The consumer's problem is now given by

$$\max_{s\in[-\underline{b},z]}\left\{v(z-s)+\beta\psi\left[\int_\Theta\int_S \phi[y+(1+r)s]\,dF(y\mid\theta)\,dG(\theta)\right]\right\}. \tag{P2}$$

Gollier (2001, Section 20.3) shows that, in the absence of distributional risk, precautionary savings under SKP preferences exist if two additional conditions are satisfied. The first one requires $\phi'(\cdot)$ to be a convex function, and the second one requires $\psi(\cdot)$ to be concave.[21] Here we will refer to a consumer with SKP preferences that satisfy these two additional conditions as a prudent SKP consumer. But it is important to note that there is more than one way to define prudence under SKP preferences.[22] We choose to use this set of conditions because it is a direct generalisation of the prudence condition for the EU specification. To see this, first note that the EU specification corresponds to the case when $\psi(x) \equiv v\left[\phi^{-1}(x)\right]$ is a linear (hence weakly concave) function. Once this is granted, the convexity of $\phi'(\cdot)$ is equivalent to the convexity of $u'(\cdot)$ in the EU model.

Our next lemma summarises the main properties of a solution of (P2) under these conditions.

Lemma 6 *Suppose Assumptions A3-A4 and $\beta(1+r)<1$ are satisfied. In addition, suppose $\phi'(\cdot)$ is convex and $\psi(\cdot)\equiv v\left[\phi^{-1}(\cdot)\right]$ is concave. Then a solution to (P2) exists. A feasible value s is optimal if and only if it satisfies*

$$\begin{aligned} & v'(z-s) \qquad (14)\\ \geq\ & \beta(1+r)\,\psi'\left[\int_\Theta\int_S\phi[y+(1+r)s]\,dF(y\mid\theta)\,dG(\theta)\right]\int_\Theta\int_S\phi'[y+(1+r)s]\,dF(y\mid\theta)\,dG(\theta), \end{aligned}$$

with equality holds if $s>-\underline{b}$.

Equation (14) is the counterpart of (11) under SKP preferences, and can be interpreted in the same way. Specifically, the left side of (14) captures the marginal cost of saving, while the right side captures the marginal benefit. By the same logic as in Section 4.1, an increase in distributional risk will induce the consumer to save more if and only if the marginal benefit of saving is increased. Under the same conditions in Lemma 6, this happens if and only if Assumption A1 is satisfied. This result is formally stated in Theorem 7. This, together with

[21] The second condition is equivalent to requiring that the period utility function $v(\cdot)$ be more concave than the atemporal vNM utility function $\phi(\cdot)$.

[22] See Kimball and Weil (2009) for an alternative set of sufficient conditions.

Theorem 5, demonstrates the importance of Assumption A1 in creating a precautionary saving motive against distributional risk.

Theorem 7 *Suppose Assumptions A3-A4 and $\beta(1+r) < 1$ are satisfied. Then any prudent SKP consumer will save more under y'' than under y' if and only if Assumption A1 is satisfied.*

5 Some Concluding Remarks

This paper introduces the concept of distributional risk into an otherwise standard model of precautionary saving. In a broader context, this paper is an effort to explore the effects and implications of distributional risk. We believe our results, especially Theorem 1, can find use in many different applications. Here we will provide two other interpretations of this result. Suppose θ is a set of covariates of Y as described in Section 2.1. Then Theorem 1 states that stochastic convexity is a necessary and sufficient condition under which an increase in background risk will increase the riskiness of Y. All other results in this paper can be rephrased accordingly. The mixture model in (1) can also be used to represent within-goup and between-group heterogeneity. Specifically, consider a population that is divided into different groups, each indexed by a value $\theta \in \Theta$. The function $F(y \mid \theta)$ then denotes the distribution of Y within group θ (within-group heterogeneity); $G(\cdot)$ denotes the distribution across groups (between-group heterogeneity) and H is the distribution of Y in the entire population. In this context, Theorem 1 states that stochastic convexity is a necessary and sufficient condition under which an increase in between-group dispersion will lead to an increase in the dispersion of Y in the entire population. This type of result is potentially useful for measuring inequality and analysing redistributive policies. We leave these possibilities for future research.

Appendix

Proof of Theorem 1

Sufficiency of Assumption A1

The proof of sufficiency is divided into two main steps. First, we construct a sequence of functions $\{\Omega_n(m)\}$ which converges pointwise to $(\Gamma\psi)(\theta) \equiv \int_S \psi(\omega)\, dF(\omega|\theta)$ for any given function $\psi \in \mathcal{C}'(S)$. Second, we show that under Assumption A1 each $\Omega_n(m)$ is a convex function defined on Θ. Hence, the limiting function $(\Gamma\psi)(\theta)$ is also convex.

Let ψ be an arbitrary function in $\mathcal{C}'(S)$. Let $\partial\psi(y)$ be the subdifferential of ψ at $y \in S$. Since ψ is continuous and convex, there exists a non-negative decreasing function $\eta : S \to \mathbb{R}_+$, $\eta(y) \in \partial\psi(y)$ for all $y \in$int S, such that

$$\psi(y) = \psi\left(\underline{y}\right) - \int_{\underline{y}}^{y} \eta(\omega)\, d\omega, \qquad \text{for all } y \in S. \tag{15}$$

For a proof of this statement, see for instance Niculescu and Persson (2006) Sections 1.5 and 1.6. For any positive integer $n \geq 1$, form the interval $S_n = \left[\underline{y}, \underline{y} + n\right)$ and partition it into subintervals of equal length 2^{-n}. The end-points of these subintervals are denoted by $\{\varepsilon_i^n\}$, where

$$\varepsilon_i^n = \underline{y} + \frac{i-1}{2^n}, \qquad \text{for } i = 1, 2, ..., n2^n + 1.$$

Define a sequence of functions $\{\eta_n(\cdot)\}$ according to

$$\eta_n(\omega) = \begin{cases} \eta\left(\varepsilon_{i+1}^n\right) & \text{if } \omega \in \left[\varepsilon_i^n, \varepsilon_{i+1}^n\right), \\ 0 & \text{if } \omega \geq \underline{y} + n. \end{cases}$$

for each $n \geq 1$. This function can be rewritten as a linear combination of simple functions, i.e.,

$$\eta_n(\omega) = \sum_{i=1}^{n2^n} \lambda_{i,n} \mathcal{I}_{\left[\omega \leq \varepsilon_{i+1}^n\right]}, \tag{16}$$

where $\mathcal{I}_{\left[\omega \leq \varepsilon_{i+1}^n\right]} = 1$ if $\omega \leq \varepsilon_{i+1}^n$ and zero otherwise and the coefficients are given by

$$\lambda_{i,n} = \begin{cases} \eta\left(\varepsilon_{i+1}^n\right) - \eta\left(\varepsilon_{i+2}^n\right) & \text{for } i = 1, 2, ..., n2^n - 1, \\ \eta\left(\varepsilon_{i+1}^n\right) & \text{for } i = n2^n. \end{cases}$$

Since $\eta(\cdot)$ is non-negative and decreasing, we have $\lambda_{i,n} \geq 0$ for all i. Hence, $\eta_n(\omega) \geq 0$ for all $\omega \in S$. In addition, the sequence $\{\eta_n(\cdot)\}$ will converge pointwise to $\eta(\cdot)$.

In order to apply the monotone convergence theorem, we need to show that $\{\eta_n(\cdot)\}$ is a monotonically increasing sequence of functions, i.e., $\eta_n(\omega) \leq \eta_{n+1}(\omega)$ for any $n \geq 1$ and for any $\omega \in S$. There are three possible cases to consider: (i) $\omega \geq \underline{y} + n + 1$, (ii) $\underline{y} + n + 1 > \omega \geq \underline{y} + n$, and (iii) $\omega \in S_n$. In the first scenario, we have $\eta_n(\omega) = \eta_{n+1}(\omega) = 0$. In the second scenario, we have $\eta_n(\omega) = 0 \leq \eta_{n+1}(\omega)$. In the third scenario, if $\omega \leq \left(\varepsilon_i^n + \varepsilon_{i+1}^n\right)/2$, then

$$\eta_{n+1}(\omega) = \eta\left(\frac{\varepsilon_i^n + \varepsilon_{i+1}^n}{2}\right) \geq \eta\left(\varepsilon_{i+1}^n\right) = \eta_n(\omega),$$

If $\omega > \left(\varepsilon_i^n + \varepsilon_{i+1}^n\right)/2$, then $\eta_{n+1}(\omega) = \eta_n(\omega)$. Hence, $\{\eta_n(\cdot)\}$ is a monotonically increasing sequence of non-negative functions.

Define a sequence of functions $\{\varphi_n(\cdot)\}$ according to

$$\varphi_n(y) \equiv \int_{\underline{y}}^{y} \eta_n(\omega)\, d\omega. \tag{17}$$

By the monotone convergence theorem,

$$\lim_{n\to\infty} \varphi_n(y) = \int_{\underline{y}}^{y} \left[\lim_{n\to\infty} \eta_n(\omega)\right] d\omega = \int_{\underline{y}}^{y} \eta(\omega)\, d\omega, \qquad \text{for all } y \in S.$$

Note that $\{\varphi_n(\cdot)\}$ is itself a monotonically increasing sequences of non-negative functions.

Finally, for each $n \geq 1$, define a function $\Omega_n(\cdot) : \mathbb{R}_+ \to \mathbb{R}$ according to

$$\Omega_n(\theta) \equiv \psi\left(\underline{y}\right) - \int_S \varphi_n(y)\, dF(y \mid \theta). \tag{18}$$

Applying the monotone convergence theorem on $\{\varphi_n(\cdot)\}$ gives

$$\begin{aligned}
\lim_{n\to\infty} \Omega_n(\theta) &= \psi\left(\underline{y}\right) - \lim_{n\to\infty} \int_S \varphi_n(y)\, dF(y \mid \theta) \\
&= \psi\left(\underline{y}\right) - \int_S \left[\lim_{n\to\infty} \varphi_n(y)\right] dF(y \mid \theta) \\
&= \int_S \left[\psi\left(\underline{y}\right) - \int_{\underline{y}}^{y} \eta(\omega)\, d\omega\right] dF(y \mid \theta) \\
&= \int_S \psi(y)\, dF(y \mid \theta),
\end{aligned}$$

for any $\theta \in \Theta$. The last equality is obtained by using (15). This completes the first step of the proof, which is to construct a sequence of functions $\{\Omega_n(\cdot)\}$ that converges pointwise to $\Gamma\psi$.

Fix $n \geq 1$. We now establish the convexity of $\Omega_n(\cdot)$. First, combining (16) and (17) gives

$$\varphi_n(y) = \sum_{i=1}^{n2^n} \lambda_{i,n} \int_{\underline{y}}^{y} \mathcal{I}_{\left[\omega \leq \varepsilon_{i+1}^n\right]} d\omega,$$

where

$$\int_{\underline{y}}^{y} \mathcal{I}_{\left[\omega \leq \varepsilon_{i+1}^n\right]} d\omega = \min\left\{y, \varepsilon_{i+1}^n\right\} - \underline{y}.$$

Substituting these into (18) gives

$$\Omega_n(\theta) = \psi\left(\underline{y}\right) + \underline{y}\sum_{i=1}^{n2^n} \lambda_{i,n} - \sum_{i=1}^{n2^n} \lambda_{i,n} \int_S \min\left\{\omega, \varepsilon_{i+1}^n\right\} dF(\omega \mid \theta), \tag{19}$$

where

$$\begin{aligned}
\int_S \min\left\{\omega, \varepsilon_{i+1}^n\right\} dF(\omega \mid \theta) &= \int_{\underline{y}}^{\varepsilon_{i+1}^n} \omega dF(\omega \mid \theta) + \varepsilon_{i+1}^n \int_{\varepsilon_{i+1}^n}^{\infty} dF(\omega \mid \theta) \\
&= \varepsilon_{i+1}^n F\left(\varepsilon_{i+1}^n \mid \theta\right) - \int_{\underline{y}}^{\varepsilon_{i+1}^n} F(\omega|\theta)\, d\omega + \varepsilon_{i+1}^n \left[1 - F\left(\varepsilon_{i+1}^n \mid \theta\right)\right] \\
&= \varepsilon_{i+1}^n - \Phi\left(\theta; \varepsilon_{i+1}^n\right).
\end{aligned} \tag{20}$$

The second line is obtained after using integration by parts on the first integral. Finally, combining (19) and (20) gives

$$\begin{aligned}
\Omega_n(\theta) &= \psi\left(\underline{y}\right) + \underline{y}\sum_{i=1}^{n2^n} \lambda_{i,n} - \sum_{i=1}^{n2^n} \lambda_{i,n} \left[\varepsilon_{i+1}^n - \Phi\left(\theta; \varepsilon_{i+1}^n\right)\right] \\
&= \psi\left(\underline{y}\right) - \sum_{i=1}^{n2^n} \lambda_{i,n} \left(\varepsilon_{i+1}^n - \underline{y}\right) + \sum_{i=1}^{n2^n} \lambda_{i,n} \Phi\left(\theta; \varepsilon_{i+1}^n\right).
\end{aligned}$$

By Assumption A5, $\Phi(\theta; y)$ is convex in θ for every $y \in S$. Since $\lambda_{i,n} \geq 0$, it follows that $\Omega_n(\cdot)$ is a convex function. Hence, we have constructed a sequence of convex functions $\{\Omega_n(\cdot)\}$ that converges pointwise to $(\Gamma\psi)(\theta) \equiv \int_S \psi(\omega)\, dF(\omega \mid \theta)$. By Theorem 10.8 in Rockafellar (1970), the limiting function $\Gamma\psi$ is also a convex function. This establishes the sufficiency of Assumption A1.

Necessity of Assumption A5

Suppose $\Gamma\psi$ is a convex function in θ for all $\psi \in \mathcal{C}'(S)$. Fix $y \in S$ and define $\psi_y(\omega) = \max\{y-\omega, 0\}$ for all $\omega \in S$, which is decreasing and convex. Applying Γ on $\psi_y(\cdot)$ gives

$$\left(\Gamma\psi_y\right)(\theta) = \int_S \max\{y-\omega, 0\}\, dF(\omega \mid \theta) = \int_{\underline{y}}^{y} F(\omega \mid \theta)\, d\omega.$$

Since $\left(\Gamma\psi_y\right)(\theta)$ is convex in θ for any $y \in S$, the condition in Assumption A1 is satisfied. This establishes the necessity part and also completes the proof of Theorem 1.

Proof of Theorem 3

Let $u(\cdot)$ be an increasing concave function in $\mathcal{C}(S)$. Since both $G'(\cdot)$ and $G''(\cdot)$ are taken from $\mathcal{L}(\Theta)$, the expectations in (10) are finite. Consider the "if" part. Suppose Assumption A1 is satisfied. Then by the corollary of Theorem 1, the expected value $\int_S u(y)\, dF(y \mid \theta)$ is a concave function in θ. Equation (10) follows immediately from the assumption that $G' \leq_{cx} G''$.

Next, consider the "only if" part. Suppose (10) holds for all the distributions in $\mathcal{L}(\Theta)$ and for all increasing concave $u(\cdot)$. Pick any two points θ_1 and θ_2 in Θ. For any $\alpha \in (0,1)$, define $\theta_\alpha = \alpha\theta_1 + (1-\alpha)\theta_2$. Since Θ is convex, it also contains θ_α. Take $G'(\cdot)$ to be the Dirac distribution at θ_α and $G''(\cdot)$ be the distribution that assigns probability α to θ_1 and probability $(1-\alpha)$ to θ_2. Then for any real-valued convex function ζ on Θ, we have

$$\int_\Theta \zeta(\theta)\, dG'(\theta) = \zeta(\theta_\alpha) \leq \alpha\zeta(\theta_1) + (1-\alpha)\zeta(\theta_2) = \int_\Theta \zeta(\theta)\, dG''(\theta),$$

which means $G' \leq_{cx} G''$ holds. The condition in (10) can now be rewritten as

$$\int_S u(y')\, dF(y' \mid \theta_\alpha) \geq \alpha \int_S u(y'')\, dF(y'' \mid \theta_1) + (1-\alpha) \int_S u(y'')\, dF(y'' \mid \theta_2),$$

or equivalently

$$(\Gamma u)(\theta_\alpha) \geq \alpha(\Gamma u)(\theta_1) + (1-\alpha)(\Gamma u)(\theta_2).$$

Since $-u(\cdot)$ is an arbitrary member of $\mathcal{C}'(S)$, the above condition implies that the operator Γ maps every function in $\mathcal{C}'(S)$ to a convex function in θ. By Theorem 1, this is true if and only if Assumption A1 is satisfied. This completes the proof of Theorem 3.

Proof of Theorem 5

This proof uses the same line of reasoning as in the proof of Theorem 3. Let $u'(\cdot)$ be an arbitrary function in $\mathcal{C}'(S)$ and let s'' be *any* solution of (P1) under y''. Suppose Assumption A1 is satisfied. Then by Theorem 1, the expected marginal utility

$$\int_S u'\left[y + (1+r)\, s''\right] dF(y \mid \theta),$$

is a convex function in θ. The inequality in (13) follows immediately from the hypothesis that $G' \leq_{cx} G''$.

Next, suppose (13) holds for all the distributions in $\mathcal{L}(\Theta)$ and for any arbitrary function in $\mathcal{C}'(S)$. Pick any two points θ_1 and θ_2 in Θ and define the mixture $\theta_\alpha = \alpha\theta_1 + (1-\alpha)\,\theta_2$ for $\alpha \in (0,1)$. Take $G'(\cdot)$ to be the Dirac distribution at θ_α and $G''(\cdot)$ be the distribution that assigns probability α to θ_1 and probability $(1-\alpha)$ to θ_2. The inequality in (13) can now be rewritten as

$$\begin{aligned} & \alpha \int_\Theta \int_S u'\left[y'' + (1+r)\, s''\right] dF\left(y'' \mid \theta_1\right) + (1-\alpha) \int_\Theta \int_S u'\left[y'' + (1+r)\, s''\right] dF\left(y'' \mid \theta_1\right) \\ \geq \quad & \int_\Theta \int_S u'\left[y' + (1+r)\, s''\right] dF\left(y' \mid \theta_\alpha\right). \end{aligned}$$

In words, this means the operator Γ maps every function in $\mathcal{C}'(S)$ to a convex function in θ, which is true if and only if Assumption A1 is satisfied. This completes the proof of Theorem 5.

Proof of Lemma 6

Since the objective function is continuous and the constraint set is compact, (P2) has at least one solution. Since $v(\cdot)$, $\psi(\cdot)$ and $\phi(\cdot)$ are all increasing concave functions, the objective function is also concave in s. Hence, the Kuhn-Tucker first-order conditions are both necessary and sufficient to identify the solutions of (P2). What remains is to show that we can rule out the corner solution $s = z$ with the help of $\beta(1+r) < 1$. First, since $\phi(\cdot)$ is increasing, we have

$$\int_\Theta \int_S \phi\left[y + (1+r)\, z\right] dF(y \mid \theta)\, dG(\theta) \geq \phi(0)$$

$$\Rightarrow M\left(y + (1+r)\, z\right) \geq 0. \tag{21}$$

Second, note that the derivative of $\psi(\cdot)$ is given by

$$\psi'(x) = \frac{v'\left[\phi^{-1}(x)\right]}{\phi'\left[\phi^{-1}(x)\right]}.$$

Since $\psi'(\cdot)$ is decreasing and $\phi^{-1}(\cdot)$ is strictly increasing, it follows that $v'(q)/\phi'(q)$ is decreasing in q. This, together with (21) implies

$$\psi'\left[\int_\Theta \int_S \phi\left[y + (1+r)z\right] dF(y \mid \theta)\, dG(\theta)\right] = \frac{v'\left[M(y + (1+r)z)\right]}{\phi'\left[M(y + (1+r)z)\right]} \leq \frac{v'(0)}{\phi'(0)}. \tag{22}$$

Finally, using (21)-(22) and $\beta(1+r) < 1$ gives

$$\beta(1+r)\psi'\left[\int_\Theta \int_S \phi\left[y + (1+r)z\right] dF(y \mid \theta)\, dG(\theta)\right] \int_\Theta \int_S \phi'\left[y + (1+r)s\right] dF(y \mid \theta)\, dG(\theta) < v'(0).$$

This condition states that the marginal benefit of increasing c_1 from zero outweighs the marginal cost of doing so. Hence, it is not optimal to choose $c_1 = 0$. Thus, any solution of (P2) must be strictly less than z and is characterised by the Euler equation in (14) which is implied by the Kuhn-Tucker conditions. This completes the proof of Lemma 6.

Proof of Theorem 7

As in Theorem 5, let $G'(\cdot)$ and $G''(\cdot)$ be two distributions in $\mathcal{L}(\Theta)$ such that $G' \leq_{cx} G''$. Let y' and y'' be the random incomes generated by $\{\mathcal{F}, G'\}$ and $\{\mathcal{F}, G''\}$, respectively. Let s'' be *any* solution of (P2) under y''. Suppose Assumption A1 is satisfied. We want to show that

$$\begin{aligned} & \psi'\left[\int_\Theta \int_S \phi\left[y'' + (1+r)s''\right] dF\left(y'' \mid \theta\right) dG''(\theta)\right] \int_\Theta \int_S \phi'\left[y'' + (1+r)s''\right] dF\left(y'' \mid \theta\right) dG''(\theta) \\ \geq\ & \psi'\left[\int_\Theta \int_S \phi\left[y' + (1+r)s''\right] dF\left(y' \mid \theta\right) dG'(\theta)\right] \int_\Theta \int_S \phi'\left[y' + (1+r)s''\right] dF\left(y' \mid \theta\right) dG'(\theta) \end{aligned} \tag{23}$$

By Theorem 4 (and the preceding discussion), we have

$$\int_\Theta \int_S \phi\left[y'' + (1+r)s''\right] dF\left(y'' \mid \theta\right) dG''(\theta) \leq \int_\Theta \int_S \phi\left[y' + (1+r)s''\right] dF\left(y' \mid \theta\right) dG'(\theta).$$

The concavity of $\psi(\cdot)$ then implies

$$\begin{aligned}
& \psi'\left[\int_\Theta \int_S \phi\left[y'' + (1+r)\, s''\right] dF\left(y'' \mid \theta\right) dG''(\theta)\right] \\
\geq\ & \psi'\left[\int_\Theta \int_S \phi\left[y' + (1+r)\, s''\right] dF\left(y' \mid \theta\right) dG'(\theta)\right]. \qquad (24)
\end{aligned}$$

On the other hand, since $\phi'(\cdot)$ is a decreasing convex function in $\mathcal{C}(S)$, so by Theorem 1 we can get

$$\begin{aligned}
& \int_\Theta \int_S \phi'\left[y'' + (1+r)\, s''\right] dF\left(y'' \mid \theta\right) dG''(\theta) \\
\geq\ & \int_\Theta \int_S \phi'\left[y' + (1+r)\, s''\right] dF\left(y' \mid \theta\right) dG'(\theta). \qquad (25)
\end{aligned}$$

Since all the quantities involved in (24) and (25) are nonnegative, these two inequalities together imply (23).

Next, suppose (23) holds for all convex $\phi'(\cdot)$ and all concave $\psi(\cdot) \equiv v\left[\phi^{-1}(\cdot)\right]$. Take ψ to be any linear function with strictly positive slope. Then (23) can be simplified to become

$$\int_\Theta \int_S \phi'\left[y'' + (1+r)\, s''\right] dF\left(y'' \mid \theta\right) dG''(\theta) \geq \int_\Theta \int_S \phi\left[y' + (1+r)\, s''\right] dF\left(y' \mid \theta\right) dG'(\theta).$$

The rest of the proof is essentially the same as the proof for the “only if” part of Theorem 5. This completes the proof of Theorem 7.

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