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Decomposing and Valuing Callable Convertible Bonds:

A New Method Based on Exotic Options*

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Abstract: In the framework of Black-Scholes-Merton option pricing models, by employing exotic options instead of plain options or warrants, this paper presents an equivalent decomposition method for usual Callable Convertible Bonds (CCB). Furthermore, the analytic valuation formulae for CCB are worked out by using the analytic formulae for those simpler securities decomposed from CCB. Moreover, this method is validated by comparing with Monte Carlo simulation. Besides, the effects of call clauses, coupon clauses and soft call condition clauses are analyzed respectively. These give lots of new insights into the valuation and analysis of CCB and much help to hedge their risks.

Key words: Callable convertible bonds; Equivalent decomposition; Up-and-out calls; American binary calls; Derivative pricing

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1. Introduction

Convertible bonds have been playing a major role in the financing of companies because of their appealing hybrid feature that provides investors with both the downside protection of ordinary bonds and the upside return of equities. In practice, there are multifarious convertible bonds with diversified additional clauses, such as call clauses, put clauses, reset clauses, screw clauses and negative pledge clauses and so on. Although convertible bonds in the developed derivative markets such as American derivative market are generally very complex, those in the developing derivatives markets such as Chinese derivative market are relative simple. Anyway, callable convertible bonds are the most popular.

There are many literatures on the valuation of the callable convertible bonds. The Black-Scholes-Merton option pricing theory has become the definitive theoretic foundation for valuing the convertible bonds since the pioneer paper by Ingersoll (1977a). For the first time, he obtained the analytic formulae for the callable convertible bonds by employing the theoretically reasonable one-factor (i.e. firm value) no-arbitrage model. From then on, the theoretical equilibrium price of the callable convertible bond is defined as the one that offers no arbitrage opportunity to either the holders or the issuers, on the assumption that at each point in time the issuers execute the optimal call policy that maximizes the common shareholder's wealth (i.e. minimizes the value of this convertible bond) and that the holders execute the optimal conversion strategies that maximize the value of this convertible bond.

The vast majority of subsequent research has focused on either extending

Ingersoll's work to more complicated convertible bonds, or further relaxing his "ideal conditions". The two-factor (i.e. firm value and interest rate) no-arbitrage model was presented firstly by Brennan and Schwartz (1980) and then developed further by Buchan (1997), Carayannopoulos (1996) and Lvov et al. (2004). Although these models based on firm value are theoretically appealing, they are impractical because they involve some unobservable parameters (notably, the volatility of the firm value).

The more practical one-factor (i.e. stock price) no-arbitrage model was given for the first time by McConnell and Schwartz (1986). However, in order to capture default risk of convertible bonds, their model had to adopt the credit spread approach that would necessarily result into the theoretic inconsistence because a convertible bond as a kind of hybrid derivatives consists of a debt part that is subject to default risk and an equity part that is not. This theoretic inconsistence was reduced greatly by Goldman Sachs (1994) and Tsiveriotis and Fernandes (1998). Subsequently, the more reasonable two-factor (i.e. stock price and interest rate) no-arbitrage model was proposed firstly by Cheung and Nelken (1994) and developed further by introducing more reasonable interest rate models (Ho and Pfeffer, 1996; Yigitbasioglu, 2001).

Recently, the reduced-form approach has been adopted to consider default risk of the convertible bonds (Davis and Lischka, 1999; Takahashi et al. 2001; Ayache, Forsyth and Vetzal, 2003; Yigitbasioglu and Alexander, 2004, Liao and Huang, 2006). To sum up, with the development of these models, the pricing results have become more and more reasonable and accurate, and the mean of prediction errors can be less than 5% (Barone-Adesi, Bermudez and Hatgioannides, 2003).

However, these models above could not provide the investors with enough help to deeply understand the value components of the callable convertible bonds and the effect of every kind of typical clauses, and to conveniently replicate them so that their risk can be effectively hedged. Furthermore, solving these models generally has to adopt intricate numerical procedures that are very difficult for investors, especially in developing derivative markets. Obviously, those problems will be solved easily as long as we are able to completely decompose the callable convertible bonds into simple tradable securities in the actual market.

Since 1960s, researchers have attempted to reasonably decompose the convertible bonds into simple tradable securities. Baumol, Malkiel and Quandt (1966) proposed that a non-callable convertible bond could be regarded either as its corresponding ordinary bond (with the same principal and coupons and maturity) with a detachable call option struck at the value of this ordinary bond, or as stocks plus a put option struck at the value of this ordinary bond, which is greater. However, in light of later research, their conclusion is demonstrably incorrect. Ingersoll (1977a), under his “ideal conditions”, proved that a non-callable convertible bond had the same value as its corresponding ordinary bond plus an attached call warrant, and obtained its analytic valuation formula. Nyborg (1996) extended his decomposition by allowing the underlying stock to pay dividends and the capital structure to be more complex. However, both Ingersoll and Nyborg viewed the convertible bonds as contingent claims on the firm value. This makes parameter estimation very difficult since not all of firm assets are tradable. Connolly (1998, chapter 8) viewed them as derivatives on the underlying stock price, and completely decomposed a non-callable convertible bond into its corresponding ordinary bond and European call warrants. His decomposition is relative reasonable in principal.

However, in the existing literatures, until now is there no method to completely decompose the callable convertible bonds into simple securities trading in the actual market. To all appearances, one callable convertible bond can be directly decomposed into three simpler securities: one ordinary bond, one call option (i.e. the holders' convertible option) and another call option (i.e. the issuers' callable option). However, this direct decomposition is not valid because of the unnegligible interaction between the exercising of the embedded call option. As a result, the difference between the value of this callable convertible bond and that of the portfolio of these three securities can not be ignored (Ingersoll, 1977a; Ho and Pfeffer, 1996).

Ingersoll (1977a) proved that a callable convertible discount bond had the same value as its corresponding ordinary discount bond plus an attached stock call warrant minus an additional third term representing the cost of giving the callable option to the issuers. However, his model is impractical because he viewed the callable convertible discount bonds as contingent claims on the firm value. Ho and Pfeffer (1996) considered the callable convertible bonds as derivatives on the underlying stock price and presented that the value of one callable convertible bond was equal to its investment value (i.e. the value of its corresponding ordinary bond) plus its embedded warrant value minus its forced conversion value. However, they only demonstrated the importance of its forced conversion value and did not work out its analytic valuation formula.

In a word, none of these existing decompositions above is good enough to fully illustrate the value components of the callable convertible bonds and to conveniently replicate them so that their risk can be effectively hedged. As a matter of fact, due to the interactions between the embedded convertible option and the embedded callable option, one callable convertible bond is equivalent to its corresponding ordinary bond (with the same principal and coupons and maturity) plus an embedded peculiar path-dependent exotic option, whose exercise price and exercise time are indeterminate. Thus, inevitably, if a callable convertible bond is decomposed with only non-path-dependent plain options or warrants, there must be some unregular residual (e.g. the additional third term and the forced conversion value mentioned above) .

In this paper, in the framework of Black-Scholes-Merton option pricing models, according as the risk-neutral valuation principle, by employing simple exotic options instead of plain options or warrants, an equivalent decomposition method is presented for the Callable Convertible Bonds (CCB) defined in Subsection 3.1. Using this method, one callable convertible discount bond can be completely decomposed into its corresponding ordinary discount bond and three kinds of simple exotic options: regular American binary calls with an immediately-made fixed payment, regular up-and-out calls and regular American binary calls with a fixed payment deferred until maturity. Similarly, one coupon-bearing callable convertible bond can be completely decomposed into its corresponding ordinary bond and five kinds of simple exotic options. Intuitively and exactly, this method shows us the value components of CCB. Obviously it is very helpful to conveniently replicate CCB and effectively hedge their risks.

Furthermore, the analytic valuation formulae for CCB are worked out by making full use of the existing analytic valuation formulae for these simple securities decomposed from CCB. At the same time, these analytic formulae for CCB are validated by comparing with Monte Carlo simulation. Without doubt, these formulae can produce pricing results and corresponding Greeks more conveniently and quickly, because they need not to consume huge computational resources necessary for numerical procedures. Besides, they can be used to analyze the effects of call clauses, coupon clauses and soft call condition clauses respectively. These obviously give a lot of new insights into the valuation and analysis of CCB.

The remainder of this paper is organized as follows. In the next section, the assumptions and the rationale needed in this paper are explicated in detail. In Section 3, we present an equivalent decomposition method for CCB. In Section 4, the analytic valuation formulae are worked out. Subsequently, Section 5 validates these formulae by comparing with Monte Carlo simulation. In Section 6, we further analyze in detail the effect of every kind of typical clauses respectively. Section 7 concludes the paper.

2. Valuation framework

2.1. Assumptions

(a) The framework of Black-Scholes-Merton option pricing models is adopted.

It's well-known that this framework is very rigorous and has been relaxed gradually in order to value stock options more exactly. However, this framework has still often been adopted in order to obtain analytic valuation formulae for those complex derivative securities. As we know, in the Black-Scholes-Merton framework, capital market is both perfect and efficient; the term structure of the risk-free rate of interest is flat; there is no riskless arbitrage opportunity; and the underlying stock price follows the diffusion process below.

$$dS = \mu S d\tau + \sigma S dW \tag{1}$$

where the variable W follows a standard Wiener process under the probability measure P; μ and σ are the expected rate of return and volatility of the underlying stock price respectively. Let r denote the continuous risk-free interest rate and assume that r is constant[**]. This assumption is relatively reasonable since both Brennan and Schwarz (1980) and Carayannopoulos (1996) concluded that, for the reasonable range of parameters, the addition of an interest rate factor did not significantly improve the model's accuracy.

(b) All investors prefer more wealth to less. That is to say, the holders of the convertible bonds always seek to maximize the price of the convertible bonds; the issuers of the convertible bonds, as the deputies of the shareholders, act at all times to maximize the shareholders' wealth, i.e. the underlying stock price.

(c) Both the holders and the issuers behave with symmetric market rationality.

[**] Since Black and Scholes (1973) are only interested in the underlying asset price at maturity, they can allow r to be known functions of time. However, CCB and exotic options involved in this paper depend in complex ways on the time path of the variable r. Simply, we assume here that r is constant through time.

This implies that both the holders and the issuers are completely rational and one part can expect the optimal behaviors of the other. The same assumption was adopted in many literatures such as Ingersoll (1977a) and Barone-Adesi and Bermudez and Hatgioannides (2003).

(d) The potential dilution, which results from the possible conversion in the future, has already been reflected in the current underlying stock price. That is to say, the convertible bonds “can be valued without correction for dilution by using the volatility of the quoted share” (Connolly, 1998).

2.2. The rationale

According as the risk-neutral valuation principle, in the risk-neutral world, the expected return on all securities is the risk-free interest rate and the present value of any payoff can be obtained by discounting its expected value at the risk-free interest rate (Cox and Ross, 1976). Although the risk-neutral world is merely an artificial device for pricing derivative securities in the framework of the Black-Scholes-Merton option models, the valuation formulae obtained in the risk-neutral world are valid in all worlds. “When we move from a risk-neutral world to a risk-averse world, two things happen. The expected growth rate in the stock price changes and the discount rate that must be used for any payoff from the derivatives changes. It happens that these two changes always offset each other exactly (Hull, 2000, chapter 11).”

As seen in Harrison and Kreps (1979), in the risk-neutral world, the diffusion process that the underlying stock price follows becomes

$$dS = rSd\tau + \sigma Sd\tilde{W} \tag{2}$$

where the variable $\tilde{W}$ follows another standard Wiener process under the risk-neutral probability measure $\tilde{\mathrm{P}}$, which is equivalent to the probability measure P. Obviously, in the risk-neutral world, the expected return rate becomes the risk-free interest rate, but the expected volatility has no change.

3. Decomposing the callable convertible bond

3.1. Definition

In this paper, we focus on the usual Callable Convertible Bond (CCB) whose conversion feature and call feature are defined as follows. More specifically, (d1) they entitle the holders to convert them into common shares at the predetermined conversion price at any time in the future; (d2) they entitle the issuers to call them back at the predetermined call price at any time in the future; (d3) they have no call notice period (this limit is relative reasonable because the effect of the call notice period is relative little); (d4) both the conversion price and the call price are constant; (d5) they have the usual screw clauses, i.e. upon conversion the holders can not receive accrued interests any longer; (d6) they have no put clauses and reset clauses and other non-standard clauses. In Subsection 6.4, we will discuss further when they have the soft call condition clauses.

Although CCB with these clauses are relative simple, their value components are very similar with those of more complex convertible bonds with various flavor and forms. Therefore, if we completely decompose this kind of CCB into simple securities trading in the actual market, we will better understand the value components of CCB and better replicate them, even the more complex convertible bonds.

Consider one CCB defined above. For convenience, we denote its face value by B_F, conversion price by P_1, call price by B_c, remaining time to maturity by T. Then, its conversion ratio, i.e. the number of shares of the underlying common stocks into which it can be converted, is $\left(B_F/P_1\right)$.

Without loss of generality, assume that it still has N times payments of nominal coupons from now to maturity. Let $\tau_i\left(i=1,\cdots,N\right)$ denote correspondingly the time span from now to the ith ex-coupon date. Obviously, $\tau_N=T$. Let $C_i\left(i=1,\cdots,N\right)$ and $R_i\left(i=1,\cdots,N\right)$ denote respectively the coupon amount and the coupon rate at time τ_i. In this way, obviously $C_i=B_FR_i$. And let $Pv\left(T;C\right)$ denote the present value of all coming nominal coupons from now to maturity and $Fv\left(T;C\right)$ denote the future value of them at maturity. Let $Pv\left(\tau^*;C\right)$ denote the present value of all coming nominal coupons from now to the time τ^* at which the issuers will announce a call on their own initiative and $Fv\left(\tau^*;C,\right)$ denote the future value of them at time τ^*.

Besides, let S_0, S_τ and S_T denote the underlying stock price respectively at current time zero, at any future time τ and at maturity T, where $0<\tau\le T$. Let $CCB\left(S_0,T;C\right)$ denote its theoretical value at current time zero and $B\left(T;C\right)$ denote the theoretical value at current time zero of its corresponding ordinary bond (with the same principal and coupons and maturity), i.e. the so-called investment value.

3.2. Constraint Conditions

Based on the assumption (d) above, the conversion of CCB would not result in the immediate reduction of the underlying stock price since the underlying stock price has already reflected the potential dilution. Thus, its conversion value at any time τ will be exactly equal to $\left(B_F/P_1\right)S_\tau$. From McConnell and Schwartz (1986), its theoretical value must be at least as great as its conversion value and otherwise a riskless arbitrage opportunity exists. In addition, its so-called investment value can provide it with the downside protection at any time. Hence, the theoretical value of CCB at any time in the future before the call announcement and maturity must satisfy

$$CCB\left(S_\tau, T-\tau; C\right) \geq \max\left[B\left(S_\tau, T-\tau; C\right), \left(B_F/P_1\right)S_\tau\right] \tag{3}$$

Following McConnell and Schwartz (1986) and Barone-Adesi, Bermudez and Hatgioannides (2003), due to the callable option, its theoretical value will not be possible to exceed the predetermined call price.

$$CCB\left(S_\tau, T-\tau; C\right) \leq B_c \tag{4}$$

Putting (3) and (4) together, we can obtain

$$\max\left[B\left(S_\tau, T-\tau; C\right), \left(B_F/P_1\right)S_\tau\right] \leq CCB\left(S_\tau, T-\tau; C\right) \leq B_c \tag{5}$$

If a call were to be announced at time τ^* prior to maturity, since no call notice period (see Subsection 3.1), the holders would have to choose immediately the more attractive of the two options: accepting the call price B_c in cash or obtaining the conversion value $\left(B_F/P_1\right)S_\tau^*$, where S_τ^* denote the underlying stock price at τ^*.

$$CCB\left(S_\tau^*, T-\tau^*; C\right) = \max\left[\left(B_F/P_1\right)S_\tau^*, B_c\right] \quad \text{at call} \tag{6}$$

If no call were to be announced prior to maturity, according to the optimal conversion strategies given in the next subsection, CCB would be held until maturity. At maturity, the holders can accept the balloon payment or convert to obtain the conversion value, which is greater. Due to the usual screw clauses, the balloon payment is $B_F + C_N$. Therefore, the final condition is

$$CCB\left(S_T, 0; C\right) = \max\left[\left(B_F / P_1\right) S_T, B_F + C_N\right] \tag{7}$$

3.3. Optimal conversion strategies

The holders are entitled to convert one unit of CCB at any time in the future into $\left(B_F / P_1\right)$ units of shares of the underlying common stock. Based on the assumption (b) above, optimal conversion strategies of the holders are those strategies that maximize the theoretical value of CCB.

Theorem 1: *Given the assumptions in the subsection 2.1, it is optimal for the holders never to voluntarily convert the callable convertible bond defined in the subsection 3.1 except at maturity or the call announcement.*

The proof of this theorem sees Appendix A. In fact, this theorem is similar with Ingersoll's Theorem II (Ingersoll, 1977a) that "a callable convertible security will never be exercised except at maturity or call". The only difference is that he viewed CCB as the contingent claims on the firm value, but we view CCB as derivatives on the underlying stock price.

Prior to maturity, if a call were to be announced, from (6) the holders must choose immediately between accepting the call price in cash and converting. Based on the assumption (c) above, the holders can expect the optimal call policy of the issuers.

From Theorem 2 in the next subsection, it is optimal for the issuers to announce a call as soon as the underlying stock price reaches $S_\tau^* = \left(B_c / B_F\right) P_1$, i.e. the conversion value reaches the call price, $\left(B_F / P_1\right) S_\tau^* = B_c$. Therefore, upon the call announcement, the holders would be indifferent between accepting the call price in cash and converting.

If no call were to be announced prior to maturity, CCB would be held until maturity. At maturity, from the final condition (7), it is self-evident that the holders should voluntarily convert if the conversion value $\left(B_F / P_1\right) S_T$ is greater than the balloon payment $B_F + C_N$, i.e. the underlying stock price at maturity S_T is greater than the adjusted conversion price $\left(1 + \frac{C_N}{B_F}\right) P_1$, and otherwise claim the balloon payment.

3.4. Optimal call policies

The issuers are entitled to call CCB back at the predetermined call price at any time in the future. Based on the assumption (b), optimal call policies of the issuers are those policies that maximize the underlying stock price or, what is the same thing, minimize the theoretical value of CCB.

Theorem 2: *Given the assumptions in the subsection 2.1, it is optimal for the issuers to announce to call back the callable convertible bond defined in the subsection 3.1 as soon as the underlying stock price reaches* $S_\tau^* = \left(B_c / B_F\right) P_1$.

The proof of this theorem sees Appendix B. In fact, this theorem is similar with Ingersoll's Theorem IV (Ingersoll, 1977a). Upon the call announcement, the holders will be in the same way indifferent between accepting the call price in cash and converting, though he viewed CCB as the contingent claims on the firm value and we view CCB as derivatives on the underlying stock price,.

In practice, however, the call policies executed by the issuers are not consistent with these theoretical works. The issuers generally delay announcing a call until the conversion value is substantially higher than the call price (Ingersoll, 1977b; Constantinides and Grundy, 1987). Some reasons are demonstrated by Jalan and Barone-Adesi (1995) and Ederington, Caton and Campbell (1997) and so on. In order to consider this inconsistency, following Barone-Adesi, Bermudez and Hatgioannides (2003), the restriction condition (4) can be modified as:

$$CCB\left(S_{\tau}, T-\tau; C\right) \leq kB_c \tag{8}$$

where k is a conveniently-chosen factor bigger than one. In the same way, we can obtain that it is optimal for the issuers to announce a call as soon as the underlying stock price reaches $\hat{S}_{\tau}^{*} = k\left(B_c / B_F\right)P_1$.

3.5. The equivalent decomposition

Concerned with the ending of CCB, based on the assumptions in the subsection 2.1 and the optimal conversion strategies in the subsection 3.3 and the optimal call policy in the subsection 3.4, there exist only three possible cases. For convenience, let $P_2 = S_{\tau}^{*} = \left(B_c / B_F\right)P_1$.

In the first case, the underlying stock price will reach P_2 prior to maturity, and then the issuers will announce a call at once on their own initiative. At that time, the holders will be indifferent between accepting the call price in cash and converting. In the second case, the underlying stock price will not reach P_2 prior to maturity but at maturity will exceed the adjusted conversion price $\left(1+\frac{C_N}{B_F}\right)P_1$, and then CCB will be voluntarily converted at maturity by the holders on their own initiative. In the third case, the underlying stock price will neither reach P_2 prior to maturity nor at maturity exceed the adjusted conversion price, and then CCB will be redeemed at maturity by the issuers.

As a matter of fact, since the critical stock price P_2 can be regarded as the barrier of a regular American binary call with an immediately-made fixed payment, the payoff feature of CCB in the first case is similar with that to this regular American binary call. Furthermore, since the critical stock price P_2 and the adjusted conversion price $\left(1+\frac{C_N}{B_F}\right)P_1$ can be regarded respectively as the barrier and the exercise price of a regular up-and-out call, the payoff feature of CCB in the second case is similar with that to this regular up-and-out call. Therefore, firstly we can try to separate this American binary call and regular up-and-out call from CCB respectively. Finally, CCB can be completely decomposed into its corresponding ordinary bond and five kinds of simple exotic options through four steps as follows.

At the first step, off one unit of CCB, we strip $\left(B_F / P_1\right)$ units of long regular American binary calls, denoted as $ABC^i\left(S_0, T; P_2 - P_1, P_2\right)$, whose fixed payment $\left(P_2 - P_1\right)$ is made immediately when the underlying stock price reaches the barrier P_2 for the first time.

At the second step, from the rest, we separate $\left(B_F / P_1\right)$ units of long regular up-and-out calls, denoted as $UOC\left(S_0,T;\left(1+\frac{C_N}{B_F}\right)P_1,P_2\right)$, whose barrier is also P_2 and whose exercise price is the adjusted conversion price $\left(1+\frac{C_N}{B_F}\right)P_1$.

After two steps above, the residual can be completely decomposed into three simpler securities. One is a short non-regular American binary call, denoted as $ABC^d\left(S_0,T;B_F+Fv(T;C),P_2\right)$, with a time-varying payment $B_F+Fv(T;C)$ deferred until maturity when the underlying stock price reaches the barrier P_2 for the first time. Another is a long non-regular American binary call, denoted as $ABC^i\left(S_0,T;B_F+Fv\left(\tau^*;C\right),P_2\right)$, with an immediately-made indeterminate payment $B_F+Fv\left(\tau^*;C\right)$ when the underlying stock price reaches the barrier P_2 for the first time. And the third one is its corresponding ordinary bond $B\left(S_0,T;C\right)$.

In order to better demonstrate the value components of CCB, we continue the fourth step. In brief, $ABC^d\left(S_0,T;B_F+Fv(T;C),P_2\right)$ can be further completely decomposed into one regular American binary call with a fixed payment B_F deferred until maturity, denoted as $ABC^d\left(S_0,T;B_F,P_2\right)$, and one non-regular American binary call with a time-varying payment $Fv(T;C)$ deferred until maturity, denoted as $ABC^d\left(S_0,T;Fv(T;C),P_2\right)$. $ABC^i\left(S_0,T;B_F+Fv\left(\tau^*;C\right),P_2\right)$ can be further completely decomposed into one regular American binary call with an immediately-made fixed payment B_F, denoted as $ABC^i\left(S_0,T;B_F,P_2\right)$, and one non-regular American binary call with an immediately-made indeterminate payment $Fv\left(\tau^*;C\right)$, denoted as $ABC^i\left(S_0,T;Fv\left(\tau^*;C\right),P_2\right)$.

Theorem 3: *Given the assumptions in the subsection 2.1, one unit of the* *callable convertible bond defined in the subsection 3.1 has the same value at any time as the portfolio consisting of* (B_F/P_1) *units of long regular American binary calls* $ABC^i(S_0,T;P_2-P_1,P_2)$ *,* (B_F/P_1) *units of long regular up-and-out calls* $UOC\left(S_0,T;\left(1+\frac{C_N}{B_F}\right)P_1,P_2\right)$ *, one unit of short regular American binary call* $ABC^d(S_0,T;B_F,P_2)$ *, one unit of short non-regular American binary call* $ABC^d(S_0,T;Fv(T;C),P_2)$ *, one unit of long regular American binary call* $ABC^i(S_0,T;B_F,P_2)$ *, one unit of long non-regular American binary call* $ABC^i\left(S_0,T;Fv\left(\tau^*;C\right),P_2\right)$*, and its corresponding ordinary bond* $B(S_0,T;C)$*. This can be shown as the following equation.*

$$
\begin{aligned}
&CCB(S_0,T;C)\\
&\quad=(B_F/P_1)ABC^i(S_0,T;P_2-P_1,P_2)+(B_F/P_1)UOC\left(S_0,T;\left(1+\tfrac{C_N}{B_F}\right)P_1,P_2\right)\\
&\qquad+ABC^i(S_0,T;B_F,P_2)-ABC^d(S_0,T;B_F,P_2)\\
&\qquad+ABC^i\left(S_0,T;Fv\left(\tau^*;C\right),P_2\right)-ABC^d\left(S_0,T;Fv(T;C),P_2\right)\\
&\qquad+B(S_0,T;C)
\end{aligned}
\tag{9}
$$

The proof of this theorem is proved in Appendix C. Obviously the equation (9) demonstrates fully the value components of CCB. It is worth noting that $ABC^d(S_0,T;Fv(T;C),P_2)$ and $ABC^i\left(S_0,T;Fv\left(\tau^*;C\right)\right)$ are non-regular American binary calls. Fortunately, both of them result only from coupon payments and the holders take the short position in the former and the long position in the latter. It turns out that their total contribution to the value of CCB is relatively small, especially at near maturity and low current stock price.

In fact, $ABC^i(S_0,T;B_F,P_2)$ and (B_F/P_1) units of $ABC^i(S_0,T;P_2-P_1,P_2)$ may be merged into $ABC^i(S_0,T;(B_FP_2/P_1),P_2)$, whose fixed payment (B_FP_2/P_1)

is made immediately. Then, the equation (9) becomes

$$
\begin{aligned}
&CCB\left(S_0,T;C\right) \\
&\quad = ABC^i\left(S_0,T;\left(B_F P_2 / P_1\right),P_2\right)+\left(B_F / P_1\right)UOC\left(S_0,T;\left(1+\tfrac{C_N}{B_F}\right)P_1,P_2\right) \\
&\quad\quad - ABC^d\left(S_0,T;B_F,P_2\right)+ABC^i\left(S_0,T;Fv\left(\tau^*;C\right),P_2\right) \\
&\quad\quad - ABC^d\left(S_0,T;Fv\left(T;C\right),P_2\right)+B\left(S_0,T;C\right)
\end{aligned}
\tag{10}
$$

This equation implies that CCB can be completely replicated with only five kinds of exotic options and its corresponding ordinary bond.

Let $C_i = 0\left(i=1,\cdots,N\right)$, then CCB retrogresses to the callable convertible discount bond. Accordingly, the equation (10) becomes

$$
\begin{aligned}
CCDB\left(S_0,T;0\right) = {} & ABC^i\left(S_0,T;\left(B_F P_2 / P_1\right),P_2\right)+\left(B_F / P_1\right)UOC\left(S_0,T;P_1,P_2\right) \\
& - ABC^d\left(S_0,T;B_F,P_2\right)+DB\left(S_0,T;0\right)
\end{aligned}
\tag{11}
$$

This equation implies that the callable convertible discount bond can be completely replicated with only three regular exotic options and its corresponding ordinary discount bond, $DB\left(S_0,T;0\right)$.

Let $B_c \to +\infty$, then $P_2 \to +\infty$, the callable option will never be exercised. Then, CCB retrogresses to the non-callable convertible bond. Accordingly, the equation (10) becomes

$$
CB\left(S_0,T;C\right)=\left(B_F / P_1\right)W\left(S_0,T;\left(1+\tfrac{C_N}{B_F}\right)P_1\right)+B\left(S_0,T;C\right) \tag{12}
$$

where $W\left(S_0,T;\left(1+\tfrac{C_N}{B_F}\right)P_1\right)$ denotes a European call warrant with the exercise price $\left(1+\tfrac{C_N}{B_F}\right)P_1$ and the remaining time to maturity T. This equation implies that the non-callable convertible bonds can be completely replicated with European call warrants and its corresponding ordinary bond. In essence, this equation is the same as the one derived from the binomial tree method by Connolly (1998, Chapter 8).

4. Aanalytic valuation formulae

For regular American binary calls and up-and-out calls mentioned above, their analytic formulae have already been obtained in the Black-Scholes-Merton framework by Rubinstein and Reiner (1991a and 1991b). For the non-regular American binary call $ABC^i\left(S_0,T;Fv\left(\tau^*;C\right),P_2\right)$, its analytic formula has been derived in Appendix D. In short, the analytic formulae for these securities decomposed from CCB can be directly expressed below.

$$ABC^i\left(S_0,T;P_2-P_1,P_2\right)=\left(P_2-P_1\right)\left[\left(P_2/S_0\right)^{(\bar{\mu}+\tilde{\mu})/\sigma^2}N\left(-a_1\right)+\left(P_2/S_0\right)^{(\bar{\mu}-\tilde{\mu})/\sigma^2}N\left(-a_2\right)\right] \quad (13)$$

$$\begin{aligned}
&UOC\left(S_0,T;\left(1+\tfrac{C_N}{B_F}\right)P_1,P_2\right)\\
&=\left[S_0N\left(d_1\right)-\left(1+\tfrac{C_N}{B_F}\right)P_1e^{-rT}N\left(d_1-\sigma\sqrt{T}\right)\right]\\
&\quad-\left[S_0N\left(d_2\right)-\left(1+\tfrac{C_N}{B_F}\right)P_1e^{-rT}N\left(d_2-\sigma\sqrt{T}\right)\right]\\
&\quad+\left[S_0\left(P_2/S_0\right)^{2\hat{\mu}/\sigma^2}N\left(-d_3\right)-\left(1+\tfrac{C_N}{B_F}\right)P_1e^{-rT}\left(P_2/S_0\right)^{2\bar{\mu}/\sigma^2}N\left(-d_3+\sigma\sqrt{T}\right)\right]\\
&\quad-\left[S_0\left(P_2/S_0\right)^{2\hat{\mu}/\sigma^2}N\left(-d_4\right)-\left(1+\tfrac{C_N}{B_F}\right)P_1e^{-rT}\left(P_2/S_0\right)^{2\bar{\mu}/\sigma^2}N\left(-d_4+\sigma\sqrt{T}\right)\right]
\end{aligned} \quad (14)$$

$$ABC^d\left(S_0,T;B_F,P_2\right)=B_Fe^{-rT}\left[\left(P_2/S_0\right)^{2\bar{\mu}/\sigma^2}N\left(-a_3\right)+N\left(-a_4\right)\right] \quad (15)$$

$$ABC^i\left(S_0,T;B_F,P_2\right)=B_F\left[\left(P_2/S_0\right)^{(\bar{\mu}+\tilde{\mu})/\sigma^2}N\left(-a_1\right)+\left(P_2/S_0\right)^{(\bar{\mu}-\tilde{\mu})/\sigma^2}N\left(-a_2\right)\right] \quad (16)$$

$$ABC^d\left(S_0,T;Fv\left(T;C\right),P_2\right)=Pv\left(T;C\right)\left[\left(P_2/S_0\right)^{2\bar{\mu}/\sigma^2}N\left(-a_3\right)+N\left(-a_4\right)\right] \quad (17)$$

$$\begin{aligned}
&ABC^i\left(S_0,T;Fv\left(\tau^*;C\right),P_2\right)\\
&\quad=\sum_{i=1}^{N-1}\left\{B_FR_ie^{-r\tau_i}\left\{\begin{aligned}&\left[\left(P_2/S_0\right)^{2\bar{\mu}/\sigma^2}N\left(-a_3\right)+N\left(-a_4\right)\right]\\&\quad-\left[\left(P_2/S_0\right)^{2\bar{\mu}/\sigma^2}N\left(-a_5\right)+N\left(-a_6\right)\right]\end{aligned}\right\}\right\}
\end{aligned} \quad (18)$$

$$B\left(S_0,T;C\right)=B_Fe^{-rT}+\sum_{i=1}^{N}C_ie^{-r\tau_i} \quad (19)$$

where, $\bar{\mu}=r-\frac{1}{2}\sigma^2$, $\hat{\mu}=r+\frac{1}{2}\sigma^2$, $\tilde{\mu}=\left(\bar{\mu}^2+2r\sigma^2\right)^{1/2}$,

$a_1 = \left[\ln\left(P_2 / S_0\right) + \tilde{\mu}T\right] / \left(\sigma\sqrt{T}\right)$, $a_2 = \left[\ln\left(P_2 / S_0\right) - \tilde{\mu}T\right] / \left(\sigma\sqrt{T}\right)$, $d_1 = \left[\ln\left(S_0 / \left[\left(1 + C_N / B_F\right) P_1\right]\right) + \hat{\mu}T\right] / \left(\sigma\sqrt{T}\right)$, $d_2 = \left[\ln\left(S_0 / P_2\right) + \hat{\mu}T\right] / \left(\sigma\sqrt{T}\right)$, $d_3 = \left[\ln\left(P_2^2 / \left[S_0\left(1 + C_N / B_F\right) P_1\right]\right) + \hat{\mu}T\right] / \left(\sigma\sqrt{T}\right)$, $d_4 = \left[\ln\left(P_2 / S_0\right) + \hat{\mu}T\right] / \left(\sigma\sqrt{T}\right)$, $a_3 = \left[\ln\left(P_2 / S_0\right) + \bar{\mu}T\right] / \left(\sigma\sqrt{T}\right)$, $a_4 = \left[\ln\left(P_2 / S_0\right) - \bar{\mu}T\right] / \left(\sigma\sqrt{T}\right)$, $a_5 = \left[\ln\left(P_2 / S_0\right) + \bar{\mu}\tau_i\right] / \left(\sigma\sqrt{\tau_i}\right)$, $a_6 = \left[\ln\left(P_2 / S_0\right) - \bar{\mu}\tau_i\right] / \left(\sigma\sqrt{\tau_i}\right)$ and $N\left(x\right)$ is the cumulative probability distribution function for a variable x that is normally distributed with a mean of zero and a standard deviation of 1.0.

By substituting the equations (13) through (19) into the equation (10), the analytic formula for CCB can be obtained easily. Despite the seemingly complex form, this formula is theoretically rigorous. Moreover, its derivation requires only the same preconditions about capital markets as the Black-Scholes option pricing formulae. Besides, it needs to estimate only σ.

In practice, widespread use of this formula can be expected owing to its several obvious advantages below. First, it can be used to quickly estimate the value of CCB without consuming huge computation resource always required by numerical procedures. Second, base on it, the important Greeks (such as *delta* and *gamma*) for risk management can be directly calculated. Third, it may be used for sensitivity analysis that can give much help to design CCB. Four, it may also help investors seize possible riskless arbitrage opportunities between CCB and its duplicate portfolio mentioned in Theorem 3.

5. Comparison

To assess the validity of the equivalent decomposition above, we have compared the pricing results from our analytic formula with those from Monte Carlo simulation (Boyle, Broadie and Glasserman 1997), which has been widely considered as an essential method in the pricing of daily monitored derivative securities. In this paper, Monte Carlo prices are computed by using 10,000 simulation paths on assumption that there are 252 closing prices per year, i.e. $\Delta t = 1/252$. Moreover, the antithetic variable technique for variance reduction is adopted.

Since our analytic formula is obtained in the continuous context, its pricing results for the daily monitored CCB consequentially includes continuity errors. In order to remove the continuity errors, we have adopted the continuity correction by Broadie, Glasserman and Kou (1997). Specifically, the original barrier P_2 should be adjusted to be $P_2 \exp\left(\beta\sigma\sqrt{\Delta t}\right)$, where $\beta \approx 0.5826$.

Without loss of generality, consider a numerical example of the daily monitored CCB: $B_F = \$1000$, $R_i = 0.04\left(i = 1, \cdots, N\right)$, $P_1 = \$100$, $B_c = \$1200$, $r = 0.03$, $\sigma = 0.3$. Since both the current underlying stock price and the remaining time to maturity are state variables, comparisons are made in the following two different cases. In the first case, we set the remaining time to maturity to be five years and the current stock price to be variable within the reasonable range from \$30 to \$120, which is equally divided into 50 intervals, i.e. $\Delta S = (120-30)/50 = \$1.8$. In the second case, we set the current stock price to be \$100 (at the money) and the remaining time to maturity to be variable within the range from zero to five years, which is equally divided into 50 intervals too, i.e. $\Delta\tau = 5/50 = 0.1$.

As illustrated in Fig. 1 and 2, the pricing results from our analytic formula with the continuity correction (denoted as “Solution with correction”) are extremely close to those from Monte Carlo simulation (denoted as “Simulation”). The mean of percentage errors relative to the results from simulation is only 0.03% and the largest does not exceed 0.08% in magnitude. Moreover, with the number of simulation paths increasing, the percentage errors become smaller. Hence, our analytic formula is indeed valid.

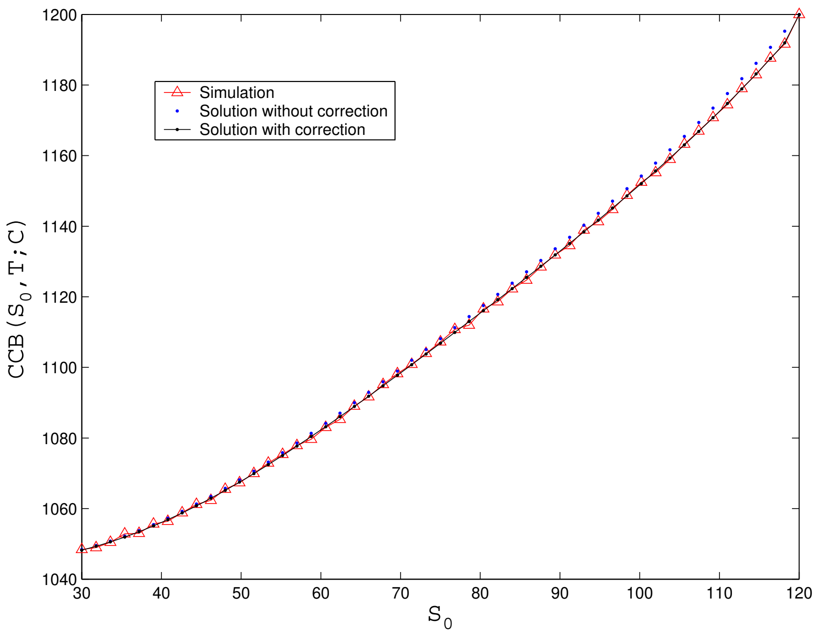

Fig. 1 Comparison when the current stock price is variable

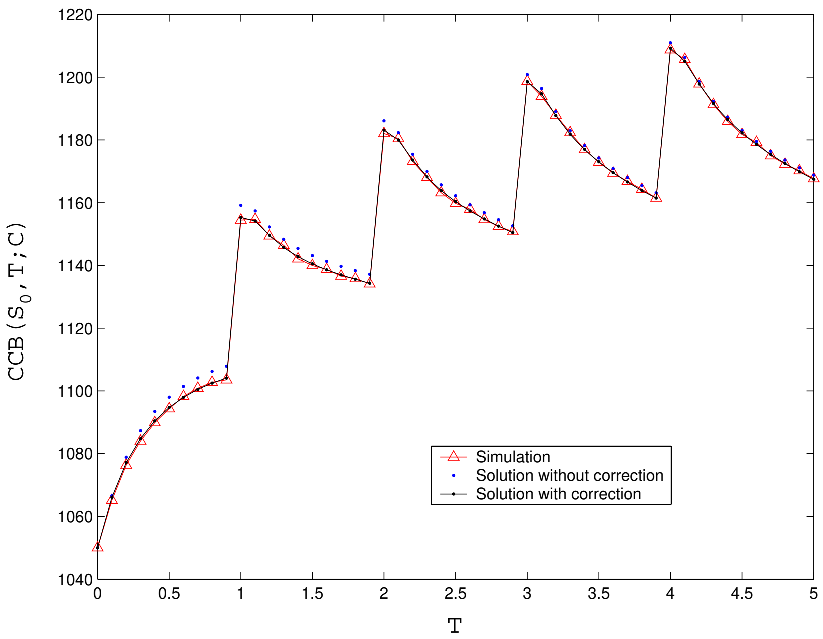

Fig. 2 Comparison when the remaining time to maturity is variable

To illuminate the effect of continuity errors, the pricing results from our analytic formula without the continuity correction (denoted as "Solution without correction") are also illustrated in Fig. 1 and 2. By comparison, it can be concluded that the uncorrected results are always greater than the corrected ones. Moreover, the closer the current stock price is to the barrier P_2, the larger their differences are. The mean of the percentage errors is 0.16% and the largest reaches 0.38%. Hence, it is better to adopt the continuity correction when our analytic formula is applied to the discretely monitored CCB.

6. Analyzing the callable convertible bond

6.1. Theoretical value and state variables

On the assumptions stated in the subsection 2.1, the theoretical value of CCB depends on two state variables: its remaining time to maturity and the current underlying stock price. By employing the same numerical example in the section 5, the three-dimensional graph (see Fig. 3) has been plotted to demonstrate the relationships between its theoretical value and two state variables. Fig. 3 shows clearly that its value increases with the current underlying stock price. Fig. 3 also shows that its value rupture downside shortly after the ex-coupon dates and increases gradually with the remaining time to maturity decreasing during the periods between two conjoint coupon dates except the last.

In the same way, based on the formulae from (13) to (19), the three-dimensional graphs can be plotted easily to demonstrate the relationships between the value of each component of CCB and two state variables.

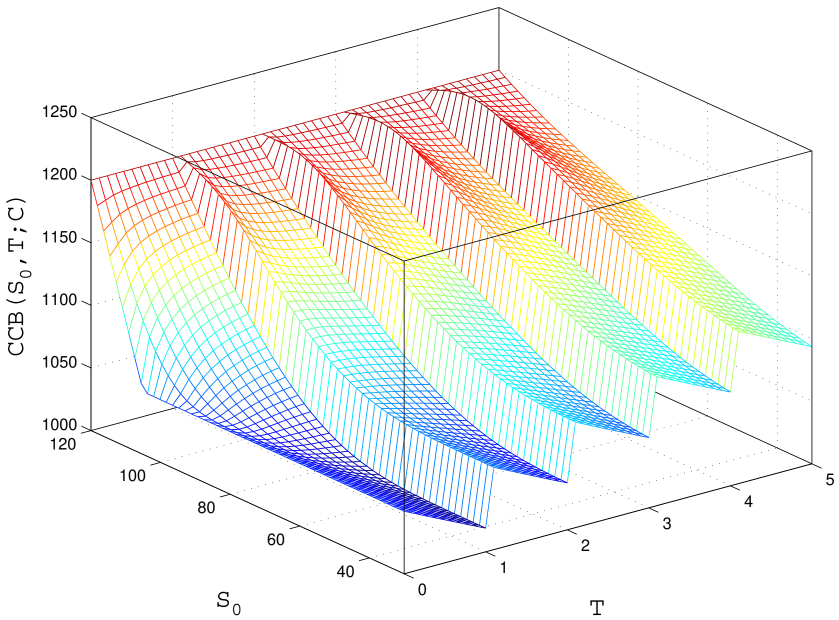

Fig. 3 Relationships between $CCB(S_0,T;C)$ and two state variables

6.2. The effect of coupon clauses

Without doubt, coupon payments must add the theoretical value of CCB. However, the added value by coupon payments is always less than the present value of all coming nominal coupons because of two reasons below. First, if CCB were to be called back prior to maturity, the nominal coupons hereafter would not be paid any more. Second, if it were to be voluntarily converted at maturity, due to the screw clauses the last nominal coupon would not be paid. In principal, the added value by coupon payments obviously should be the difference between $CCB(S_0,T;C)$ and $CCB(S_0,T;0)$. In terms of the equations (9) and (11), it can be expressed as

$$\begin{aligned} CCBCoupon(S_0,T) &= CCB(S_0,T;C) - CCB(S_0,T;0) \\ &= Pv(T;C) - \left\{ ABC^d\left(S_0,T;Fv(T;C),P_2\right) - ABC^i\left(S_0,T;Fv\left(\tau^*;C\right),P_2\right)\right\} \\ &\quad -\left(B_F/P_1\right)\left\{UOC\left(S_0,T;P_1,P_2\right) - UOC\left(S_0,T;\left(1+\tfrac{C_N}{B_F}\right)P_1,P_2\right)\right\} \end{aligned} \tag{20}$$

By employing the same example above, its three-dimensional graph (see Fig. 4) has been plotted too. Fig. 4 shows clearly that it decreases with the current stock price increasing. Moreover, the curves of the relationship between it and the remaining time to maturity look saw-toothed.

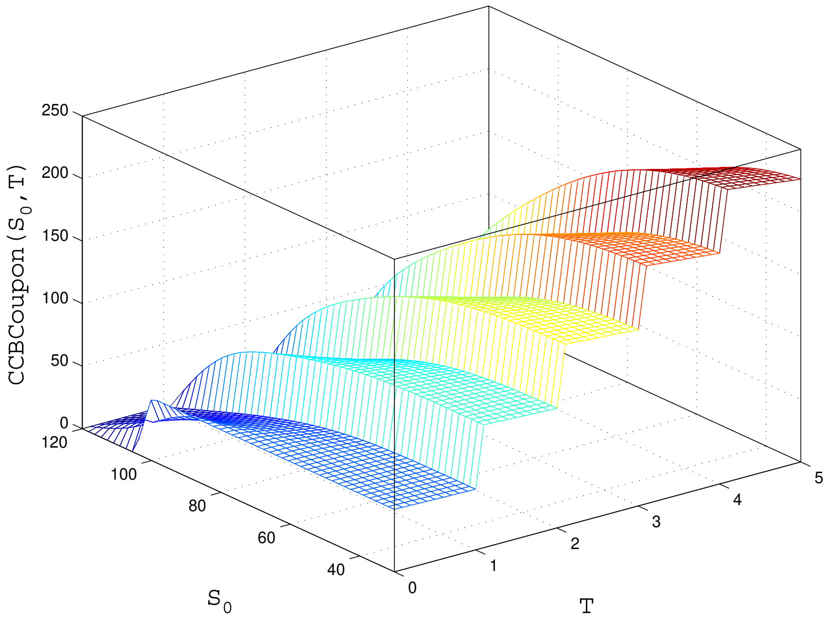

Fig. 4 Relationships between $CCBCoupon(S_0,T)$ and state variables

To further demonstrate the effect of coupon clauses, we have designed another indicator that is the ratio of the added value by coupon payments to the present value of all coming nominal coupons. It can be expressed as

$$Ratio\left(S_0,T;C\right)=CCBCoupon\left(S_0,T\right)/\,Pv\left(T;C\right) \tag{21}$$

Similarly, we plot its three-dimensional graph (see Fig. 5). Fig. 5 clearly shows that it decreases from 1 to 0 with the current stock price increasing. This is because the higher the current stock price is, the more possible it is for the issuers to call CCB back prior to maturity. In addition, it increases gradually with the remaining time to maturity decreasing during the periods between two conjoint coupon dates except the last, but ruptures downside shortly after the coupon dates, especially near at-the-money.

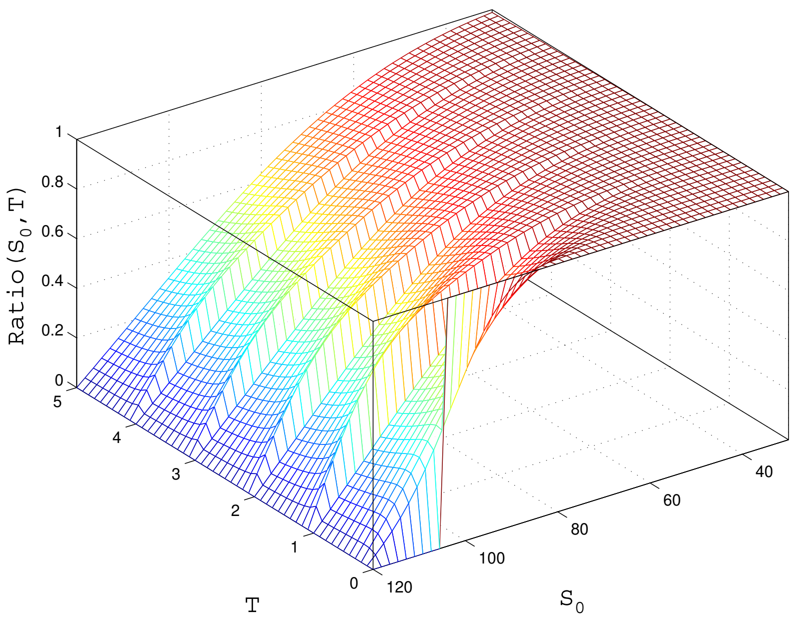

Fig. 5 Relationships between $Ratio\left(S_0,T;C\right)$ and state variables

6.3. The effect of call clauses

Since the only difference between CCB and its corresponding non-callable convertible bond rests with call clauses, the effect of call clauses can be obtained by subtracting the value of the former from that of the latter. In terms of the equations (10) and (12), its analytic formula can be derived below.

$$
\begin{aligned}
Call(S_0,T;C) &= CB(S_0,T;C) - CCB(S_0,T;C) \\
&= (B_F/P_1)W\left(S_0,T;\left(1+\tfrac{C_N}{B_F}\right)P_1\right) - ABC^i\left(S_0,T;(B_F P_2/P_1),P_2\right) \\
&\quad -(B_F/P_1)UOC\left(S_0,T;\left(1+\tfrac{C_N}{B_F}\right)P_1,P_2\right) + ABC^d\left(S_0,T;B_F,P_2\right) \\
&\quad - ABC^i\left(S_0,T;Fv\left(C,\tau^*\right),P_2\right) + ABC^d\left(S_0,T;Fv\left(C,T\right),P_2\right)
\end{aligned}
\tag{22}
$$

Its three-dimensional graph has also been plotted (see Fig. 6) by employing the same example above. Fig. 6 clearly shows that it increases with the current stock price and/or the remaining time to maturity.

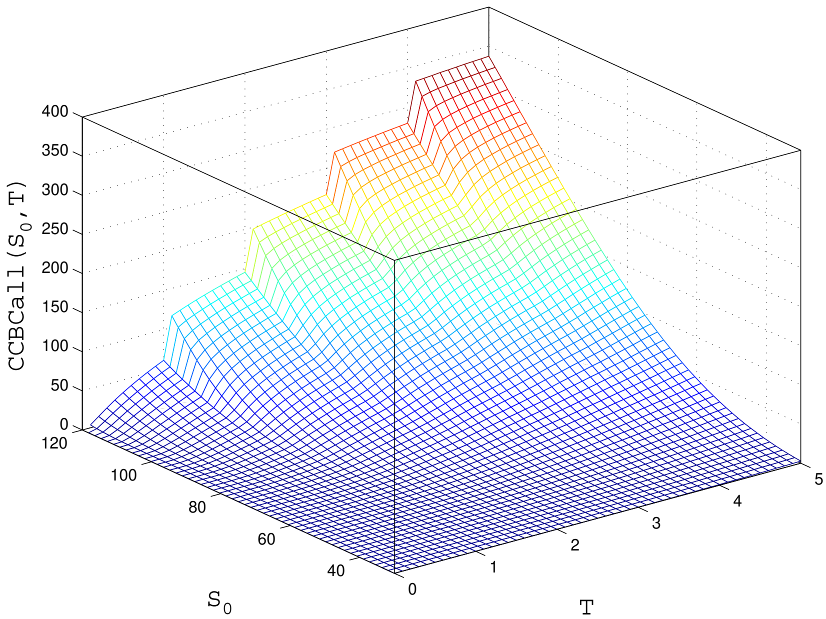

Fig. 6 Relationships between $CCBCall(S_0,T)$ and state variables

6.4. The effect of soft call condition clauses

Commonly, CCB are issued with soft call condition clauses that restrict the issuers to exercise the callable option. In this section, we analyze the effect of the soft call condition clauses where the issuers may call CCB back only if the underlying stock trades for no less than a predetermined trigger price (denoted as $\overline{P}_2$). Based on Theorem 2, $\overline{P}_2$ must be greater than the critical stock price $P_2 = S_\tau^* = \left(B_c / B_F\right) P_1$, i.e. $\left(B_F / P_1\right)\overline{P}_2 > \left(B_F / P_1\right) P_2 = B_c$, or else the issuers will not be restricted at all by the soft call condition clauses to exercise the callable option. Obviously, the soft call condition clauses benefit the holders.

Based on the analysis in the subsection 3.3 and 3.4, since $\overline{P}_2 > P_2 = S_\tau^*$, it is optimal for the issuers to announce a call immediately as soon as the underlying stock price reaches the trigger price $\overline{P}_2$; and then the holders must choose converting at once since at that time $\left(B_F / P_1\right)\overline{P}_2 > \left(B_F / P_1\right) P_2 = B_c$. Except at the call announcement, the soft call condition clauses have no effect on the conversion optimal strategies in the subsection 3.3. Therefore, with the same proof as the equation (10), the analytic valuation formula for CCB with the soft call condition clauses can be expressed as.

$$\begin{aligned} &CCB\left(S_0,T;\overline{P}_2\right) \\ &\quad = ABC^i\left(S_0,T;\left(B_F\overline{P}_2 / P_1\right),\overline{P}_2\right) + \left(B_F / P_1\right) UOC\left(S_0,T;\left(1+\tfrac{C_N}{B_F}\right)P_1,\overline{P}_2\right) \\ &\qquad - ABC^d\left(S_0,T;B_F,\overline{P}_2\right) + ABC^i\left(S_0,T;Fv\left(\tau^*;C\right),\overline{P}_2\right) \\ &\qquad - ABC^d\left(S_0,T;Fv\left(T;C\right),\overline{P}_2\right) + B\left(S_0,T;C\right) \end{aligned} \tag{23}$$

In this way, the effect of the soft call condition clauses can be expressed as

$$CCBSoft\left(S_0,T;P_2,\overline{P}_2\right) = CCB\left(S_0,T;\overline{P}_2\right) - CCB\left(S_0,T;P_2\right) \tag{24}$$

Its three-dimensional graph has also been plotted (see Fig. 7) by using the same example above and setting $\overline{P}_2 = \$130$. Fig. 7 clearly shows that it increases with the current stock price and/or the remaining time to maturity.

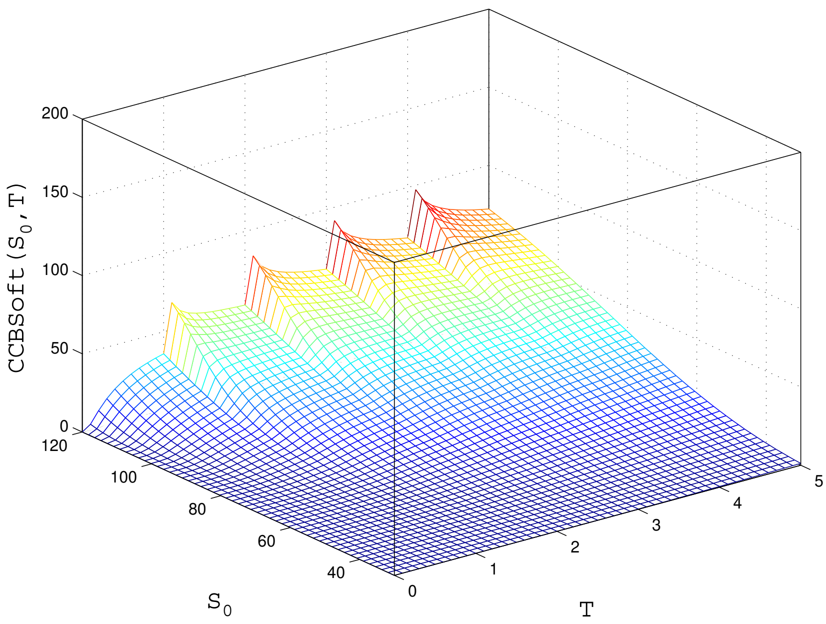

Fig. 7 Relationships between $CCBSoft\left(S_0, T; P_2, \overline{P}_2\right)$ and state variables

7. Conclusion

This paper presents an equivalent decomposition method for the callable convertible bonds (CCB) defined in Subsection 3.1, on the assumption that they are derivatives on their underlying stock prices according to Brennan and Schwarz (1980) and Carayannopoulos (1996). Using this method, the callable convertible discount bond can be completely replicated with its corresponding ordinary discount bond and three kinds of regular exotic options; the coupon-bearing callable convertible bond can be completely replicated with its corresponding ordinary bond and five kinds of exotic options. These are very helpful to understand the value components of various callable convertible bonds and to replicate them and hedge their risks.

Furthermore, the analytic valuation formulae for CCB have been obtained and validated by comparing with Monte Carlo simulation. These formulae can save huge computational resources required by numerical procedures. Moreover, although these formulae seem complicated, both the required assumptions about capital market and parameter estimations are the same as the Black-Scholes option pricing formulae. Therefore, widespread use of these formulae in practice would be expected, especially in the developing derivatives markets such as Chinese market.

In addition, we analyze in detail respectively the effects of coupon clauses, call clauses and soft call condition clauses on the theoretic value of CCB. These give a lot of new insights into the analysis of various callable convertible bonds.

A useful direction for further research is to analyze the impacts of other clauses such as put clauses or other factors such as default risk and dividends, which have not been considered in this paper.

Appendix A

Proof: Consider two investment portfolios: Portfolio I consists of only one unit of CCB; Portfolio II consists of $\left(B_F/P_1\right)$ units of shares of the underlying stocks. Since no dividend has been assumed in Subsection 2.1, Portfolio II always consists of $\left(B_F/P_1\right)$ units of shares of the underlying stock.

If no call were to be announced prior to maturity, from the inequality (3), prior to maturity Portfolio I would be worth at least as great as Portfolio II, even if there is no coupon. At maturity, in terms of the equality (7), the payoffs to these two portfolios are compared in Table 1. Table 1 shows clearly that Portfolio I is generally worth more than Portfolio II unless not only the holders voluntarily convert at maturity but also there is no coupon, in which case they have the same value.

Table 1. Demonstration that at maturity the payoff to Portfolio I will be at least as great as that to Portfolio II.

Portfolio	Current value	Stock price at maturity	
		$S_T < \left(1+\frac{C_N}{B_F}\right)P_1$	$S_T \geq \left(1+\frac{C_N}{B_F}\right)P_1$
I	$CCB(S_0,T;C)$	$B_F + Fv(T;C)$	$(B_F/P_1)S_T + Fv(T;C) - C_N$
II	$(B_F/P_1)S_0$	$(B_F/P_1)S_T$	$(B_F/P_1)S_T$
Relationship between terminal values of Portfolio I and II		$V_{\mathrm{I}} > V_{\mathrm{II}}$	$V_{\mathrm{I}} \geq V_{\mathrm{II}}$

If a call were to be announced prior to maturity, assuming at that time the underlying stock price is S_τ^*, from the equality (6) Portfolio I would be worth $\max\left[(B_F/P_1)S_\tau^*, B_c\right]$. The payoffs to these two portfolios at the call announcement are compared in Table 2. Table 2 shows that Portfolio I will never be worth less than Portfolio II and in some cases will be worth more, even if there is no coupon.

Table 2. Demonstration that at the call announcement the payoff to Portfolio I will never be less than that to Portfolio II.

Portfolio	Current value	stock price at the call announcement	
		$(B_F/P_1)S_\tau^* \geq B_c$	$(B_F/P_1)S_\tau^* < B_c$
I	$CCB(S_0,T;C)$	$(B_F/P_1)S_\tau^* + Fv(\tau^*;C)$	$B_c + Fv(\tau^*;C)$
II	$(B_F/P_1)S_0$	$(B_F/P_1)S_\tau^*$	$(B_F/P_1)S_\tau^*$
Relationship between the values of Portfolio I and II		$V_{\mathrm{I}} \geq V_{\mathrm{II}}$	$V_{\mathrm{I}} > V_{\mathrm{II}}$

To sum up, both conditions for dominance defined by Merton (1973) exist. Hence, unless the current value of Portfolio I exceeds the current value of Portfolio II, i.e. $CCB(S_0,T;C) > (B_F/P_1)S_0$, the former will dominate the latter. Obviously, CCB should never be voluntarily converted except at maturity or the call announcement.

Appendix B

Proof: Suppose that this theorem is not the case.

From the inequality (5), both the decline of interest rates and the rise of the underlying stock price can increase the lower limit of CCB. However, since the flat term structure has been assumed in Subsection 2.1, only the latter is relevant here. In terms of the inequality (5), it is very clear that the lower limit will approach the upper limit with the underlying stock price increasing. Therefore, the optimal call policy must yield a critical stock price S_τ^* so that it is optimal for the issuers to announce a call as soon as the underlying stock price reaches S_τ^*.

From the inequality (5) again, we can be sure $S_\tau^* \leq (B_c / B_F) P_1$.

Assume that it is optimal for the issuers to announce a call as soon as the underlying stock price reaches $\overline{S}_\tau < (B_c / B_F) P_1$. Let $\overline{\tau}$ denote the time at which the underlying stock price reaches $\overline{S}_\tau$ for the first time. According to this assumed optimal call policy, if $\overline{\tau} < T$, the issuers will immediately announce a call at time $\overline{\tau}$. From the equality (6) together with $\overline{S}_\tau < (B_c / B_F) P_1$, the holders must choose to accept the call price in cash when the issuers announce a call.

$$CCB\left(\overline{S}_{\tau}, T-\overline{\tau}; C\right) = \max\left[\left(B_F / P_1\right)\overline{S}_{\tau}, B_c\right] = B_c \quad \text{(B1)}$$

On the other hand, assume that the issuers do not follow the assumed optimal call policy and will announce a call as soon as the underlying stock price reaches $\left(B_c / B_F\right)P_1$. Let $\hat{\tau}$ denote the time at which the underlying stock price reaches $\left(B_c / B_F\right)P_1$ for the first time. Due to $\left(B_c / B_F\right)P_1 > \overline{S}_{\tau}$, we must get $\overline{\tau} < \hat{\tau}$. From the inequity (5), we can obtain at time $\overline{\tau}$

$$\max\left[B\left(T-\overline{\tau}; C\right), \left(B_F / P_1\right)\overline{S}_{\tau}\right] \leq CCB\left(\overline{S}_{\tau}, T-\overline{\tau}; C\right) \leq B_c \quad \text{(B2)}$$

From Barone-Adesi, Bermudez and Hatgioannides (2003), the equality $\left(B_F / P_1\right)\overline{S}_{\tau} = CCB\left(\overline{S}_{\tau}, T-\overline{\tau}; C\right) = B_c$ is valid only when $\overline{S}_{\tau} = \left(B_c / B_F\right)P_1$. However, $\overline{S}_{\tau} < \left(B_c / B_F\right)P_1$. Hence, if the issuers announce a call as soon as the underlying stock price reaches $S_{\tau}^* = \left(B_c / B_F\right)P_1$, we can obtain

$$CCB\left(\overline{S}_{\tau}, T-\overline{\tau}; C\right) < B_c \quad \text{(B3)}$$

From (B1) and (B3), we can know that the assumed optimal call policy can not result in the minimum price for CCB, so it is not optimal. Hence, it must be optimal for the issuers to call CCB back as soon as the underlying stock price reaches $S_{\tau}^* = \left(B_c / B_F\right)P_1$.

Appendix C

Let U denote the set of the paths where the underlying stock price will reach the critical value P_2 from below prior to maturity. Let V denote the set of the paths where the underlying stock price at maturity will exceed $\left(1+\frac{C_N}{B_F}\right)P_1$. In this way, the set U can be expressed as $U=\left\{\tau^{*}\le T\right\}$ where τ^{*} denotes the first time at which the underlying stock price reaches the critical value P_2 from below prior to maturity, the set V as $V=\left\{S_T>\left(1+\frac{C_N}{B_F}\right)P_1\right\}$, the intersection $\bar{U}V$ as $\bar{U}V=\left\{\tau^{*}>T, S_T>\left(1+\frac{C_N}{B_F}\right)P_1\right\}$ and the intersection $\bar{U}\bar{V}$ as $\bar{U}\bar{V}=\left\{\tau^{*}>T, S_T\le\left(1+\frac{C_N}{B_F}\right)P_1\right\}$. In terms of the description described in Subsection 3.5, the first, second and third case of the ending of CCB respectively corresponds to the set U, $\bar{U}V$ and $\bar{U}\bar{V}$. Let $1(A)$ denote the indicator function of the set A. Then, it's easy to get

$$E\left[1(U)+1\left(\bar{U}V\right)+1\left(\bar{U}\bar{V}\right)\right]=1 \tag{C1}$$

Based on these, in the risk-neutral world, the payoffs to the corresponding ordinary bond and exotic options decomposed from CCB can be expressed respectively as follows.

$$B\left(S_0,T;C\right)=B_F e^{-rT}+Pv\left(T;C\right) \tag{C2}$$

$$ABC^{i}\left(S_0,T;P_2-P_1,P_2\right)=\begin{Bmatrix} e^{-r\tau^{*}}\left(P_2-P_1\right), & \tau^{*}<T \\ 0, & otherwise \end{Bmatrix}=\left(P_2-P_1\right)E^{\tilde{P}}\left[e^{-r\tau^{*}}1(U)\right] \tag{C3}$$

$$\begin{aligned} UOC\left(S_0,T;\left(1+\tfrac{C_N}{B_F}\right)P_1,P_2\right)&=\begin{Bmatrix} 0, & \tau^{*}\le T \\ e^{-rT}\left[S_T-\left(1+\frac{C_N}{B_F}\right)P_1\right], & \tau^{*}>T, S_T>\left(1+\frac{C_N}{B_F}\right)P_1 \\ 0, & \tau^{*}>T, S_T\le\left(1+\frac{C_N}{B_F}\right)P_1 \end{Bmatrix} \\ &=e^{-rT}E^{\tilde{P}}\left[\left(S_T-\left(1+\tfrac{C_N}{B_F}\right)P_1\right)1\left(\bar{U}V\right)\right] \end{aligned} \tag{C4}$$

$$ABC^{i}\left(S_0,T;B_F,P_2\right)=\begin{Bmatrix} e^{-r\tau^{*}}B_F, & \tau^{*}\le T \\ 0, & otherwise \end{Bmatrix}=B_F E^{\tilde{P}}\left[e^{-r\tau^{*}}1(U)\right] \tag{C5}$$

$$ABC^{d}\left(S_0,T;B_F,P_2\right)=\begin{Bmatrix} e^{-rT}B_F, & \tau^{*}\le T \\ 0, & otherwise \end{Bmatrix}=B_F e^{-rT}E^{\tilde{P}}\left[1(U)\right] \tag{C6}$$

$$ABC^i\left(S_0,T;Fv\left(\tau^*;C\right),P_2\right)=\begin{Bmatrix} e^{-r\tau^*}Fv\left(\tau^*;C\right), & \tau^*\le T \\ 0, & otherwise \end{Bmatrix}=E^{\tilde{P}}\left[Pv\left(\tau^*;C\right)1\left(U\right)\right] \quad \text{(C7)}$$

$$ABC^d\left(S_0,T;Fv\left(T;C\right),P_2\right)=\begin{Bmatrix} e^{-rT}Fv\left(T;C\right), & \tau^*\le T \\ 0, & otherwise \end{Bmatrix}=Pv\left(T;C\right)E^{\tilde{P}}\left[1\left(U\right)\right] \quad \text{(C8)}$$

where

$$Pv\left(T;C\right)=\sum_{i=1}^{N}C_i e^{-r\tau_i} \quad \text{(C9)}$$

$$Fv\left(T;C\right)=\sum_{i=1}^{N}C_i e^{r\left(T-\tau_i\right)}=e^{rT}Pv\left(T;C\right) \quad \text{(C10)}$$

$$Pv\left(\tau^*;C\right)=\sum_{i=1}^{k}C_i e^{-r\tau_i} \qquad \tau_{\mathrm{k}}\le\tau^*<\tau_{\mathrm{k+1}} \quad \text{(C11)}$$

$$Fv\left(\tau^*;C\right)=e^{r\tau^*}Pv\left(\tau^*;C\right)=\sum_{i=1}^{k}C_i e^{r\left(\tau^*-\tau_i\right)} \qquad \tau_{\mathrm{k}}\le\tau^*<\tau_{\mathrm{k+1}} \quad \text{(C12)}$$

Obviously, the payoffs to CCB in the risk-neutral world are a lot more complex than these exotic options above. If the first case of its ending happens, its present value can be expressed as $e^{-r\tau^*}\left[\left(B_F/P_1\right)P_2+Fv\left(\tau^*;C\right)\right]$. If the second case happens, its present value is $e^{-rT}\left[\left(B_F/P_1\right)S_T+Fv\left(T;C\right)-C_N\right]$. If the third case happens, its present value is $e^{-rT}\left[B_F+Fv\left(T;C\right)\right]$. So the total payoffs to CCB can be expressed as follows.

$$
\begin{aligned}
&CCB\left(S_0,T;C\right)\\
&=\begin{cases}
e^{-r\tau^*}\left[\left(B_F/P_1\right)P_2+Fv\left(\tau^*;C\right)\right] & \text{if } \tau^*\le T\\
e^{-rT}\left[\left(B_F/P_1\right)S_T+Fv\left(T;C\right)-C_N\right] & \text{if } \tau^*>T,\ S_T>\left(1+\frac{C_N}{B_F}\right)P_1\\
e^{-rT}\left[B_F+Fv\left(T;C\right)\right] & \text{if } \tau^*>T,\ S_T\le\left(1+\frac{C_N}{B_F}\right)P_1
\end{cases}\\
&=\left(B_F/P_1\right)P_2E^{\tilde{P}}\left[e^{-r\tau^*}1\left(U\right)\right]+E^{\tilde{P}}\left[Fv\left(\tau^*;C\right)e^{-r\tau^*}1\left(U\right)\right]\\
&\qquad+e^{-rT}E^{\tilde{P}}\left\{\left[\left(B_F/P_1\right)S_T+Fv\left(T;C\right)-C_N\right]1\left(\bar{U}V\right)\right\}\\
&\qquad+e^{-rT}E^{\tilde{P}}\left\{\left[B_F+Fv\left(T;C\right)\right]1\left(\bar{U}\bar{V}\right)\right\}
\end{aligned}
\tag{C13}
$$

Substituting the equations (C1) through (C8) into the equation (C13) yields

$$
\begin{aligned}
&CCB\left(S_0,T;C\right)\\
&=\left(B_F/P_1\right)\left(P_2-P_1\right)E^{\tilde{P}}\left[e^{-r\tau^*}1\left(U\right)\right]+B_FE^{\tilde{P}}\left[e^{-r\tau^*}1\left(U\right)\right]+E^{\tilde{P}}\left[Pv(\tau^*;C)\ 1\left(U\right)\right]\\
&\quad+\left(B_F/P_1\right)e^{-rT}E^{\tilde{P}}\left[\left(S_T-\left(1+\frac{C_N}{B_F}\right)P_1\right)1\left(\bar{U}V\right)\right]+\left[B_F+Fv\left(T;C\right)\right]e^{-rT}E^{\tilde{P}}\left[1\left(\bar{U}V\right)\right]\\
&\quad+\left[B_F+Fv\left(T;C\right)\right]e^{-rT}E^{\tilde{P}}\left[1\left(\bar{U}\bar{V}\right)\right]\\
&=\left(B_F/P_1\right)ABC^i\left(S_0,T;P_2-P_1,P_2\right)+ABC^i\left(S_0,T;B_F,P_2\right)\\
&\quad+ABC^i\left(S_0,T;Fv\left(\tau^*;C\right),P_2\right)+\left(B_F/P_1\right)UOC\left(S_0,T;\left(1+\frac{C_N}{B_F}\right)P_1,P_2\right)\\
&\quad-\left[B_Fe^{-rT}+Pv\left(T;C\right)\right]E^{\tilde{P}}\left[1\left(U\right)\right]\\
&\quad+\left[B_Fe^{-rT}+Pv\left(T;C\right)\right]E^{\tilde{P}}\left[1\left(U\right)+1\left(\bar{U}V\right)+1\left(\bar{U}\bar{V}\right)\right]\\
&=\left(B_F/P_1\right)ABC^i\left(S_0,T;P_2-P_1,P_2\right)+\left(B_F/P_1\right)UOC\left(S_0,T;\left(1+\frac{C_N}{B_F}\right)P_1,P_2\right)\\
&\quad+ABC^i\left(S_0,T;B_F,P_2\right)-ABC^d\left(S_0,T;B_F,P_2\right)\\
&\quad+ABC^i\left(S_0,T;Fv\left(\tau^*;C\right),P_2\right)-ABC^d\left(S_0,T;Fv\left(T;C\right),P_2\right)\\
&\quad+B\left(S_0,T;C\right)
\end{aligned}
\tag{C14}
$$

Therefore, Theorem 3 holds in the risk-neutral world. According to the risk-neutral valuation principal, Theorem 3 still holds even if the assumption of the risk-neutral world is relaxed.

Appendix D

From Rubinstein and Reiner (1991b), we can get the expression of

$E^{\tilde{P}}\left[1\left(\tau_i \le \tau^* < \tau_j\right)\right]$. Based on this expression, the analytic formula for $ABC^i\left(S_0, T; Fv\left(\tau^*; C\right), P_2\right)$ can be derived below.

$$
\begin{aligned}
&ABC^i\left(S_0, T; Fv\left(\tau^*; C\right), P_2\right) \\
&= E^{\tilde{P}}\left[e^{-r\tau^*}\left(\sum_{i=1}^{k} C_i e^{r(\tau^*-\tau_i)}\right) 1\left(\tau^* \le T\right)\right] \quad \tau_{\mathrm{k}} \le \tau^* < \tau_{\mathrm{k+1}} \\
&= C_1 e^{-r\tau_1} E^{\tilde{P}}\left[1\left(\tau_1 \le \tau^* < \tau_2\right)\right] + \left(C_1 e^{-r\tau_1} + C_2 e^{-r\tau_2}\right) E^{\tilde{P}}\left[1\left(\tau_2 \le \tau^* < \tau_3\right)\right] + \cdots \\
&\qquad + \left(\sum_{i=1}^{N-1} C_i e^{-r\tau_i}\right) E^{\tilde{P}}\left[1\left(\tau_{N-1} \le \tau^* < T\right)\right] + \left(\sum_{i=1}^{N} C_i e^{-r\tau_i}\right) E^{\tilde{P}}\left[1\left(\tau^* = T\right)\right] \\
&= \sum_{i=1}^{N-1}\left\{C_i e^{-r\tau_i} E^{\tilde{P}}\left[1\left(\tau_i \le \tau^* < \tau_N\right)\right]\right\} \\
&= \sum_{i=1}^{N-1}\left\{C_i e^{-r\tau_i}\left[\Pr\left(0 < \tau^* < \tau_N\right) - \Pr\left(0 < \tau^* < \tau_i\right)\right]\right\} \\
&= \sum_{i=1}^{N-1}\left\{B_F R_i e^{-r\tau_i}\left\{\begin{array}{l}\left[\left(P_2 / S_0\right)^{2\bar{\mu}/\sigma^2} N\left(-a_3\right) + N\left(-a_4\right)\right] \\ \quad -\left[\left(P_2 / S_0\right)^{2\bar{\mu}/\sigma^2} N\left(-a_5\right) + N\left(-a_6\right)\right]\end{array}\right\}\right\}
\end{aligned}
\tag{D1}
$$

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