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Optimal hedging strategies for multi-period guarantees in the presence of transaction costs: A stochastic programming approach ☆

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Abstract

Multi-period guarantees are often embedded in life insurance contracts. In this paper we consider the problem of hedging these multi-period guarantees in the presence of transaction costs. We derive the hedging strategies for the *cheapest* hedge portfolio for a multi-period guarantee that with certainty makes the insurance company able to meet the obligations from the insurance policies it has issued. We find that by imposing transaction costs, the insurance company reduces the rebalancing of the hedge portfolio. The cost of establishing the hedge portfolio also increases as the transaction cost increases. For the multi-period guarantee there is a rather large rebalancing of the hedge portfolio as we go from one period to the next. By introducing transaction costs we find the size of this rebalancing to be reduced. Transaction costs may therefore be one possible explanation for why we do not see the insurance companies performing a large rebalancing of their investment portfolio at the end of each year.

Keywords: Multi-period guarantee; Optimal hedging strategies; Transaction costs; Stochastic programming

1. Introduction

A life insurance company is mainly exposed to two types of risks; mortality risk and financial risk. Mortality risk is the risk that a policyholder lives longer or dies earlier than what is expected. This is a risk that can, by the law of large numbers, be diversified away by issuing many similar policies. The premiums collected by a life insurance company are typically invested in the financial market. A typical investment portfolio for an insurance company consists of bonds, stocks, and in some cases also real estate. The return on a life insurance policy is often a function of the return on the insurer's investment portfolio. Since many policies have a minimum guaranteed rate of return included, the uncertain

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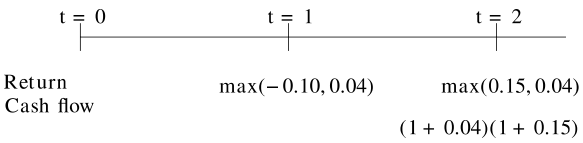

Fig. 1. The figure shows the return on and the cash flow from the insurance policy when one unit of account is invested in the insurer's investment portfolio and a two-period guarantee is embedded.

return on the investment portfolio exposes the insurers to financial risk. In contrast to mortality risk, financial risk *increases* in the number of policies issued. Thus, the financial risk is undiversifiable.

To reduce the probability of bankruptcy, the financial risk should be integrated in the company's overall risk management systems. In particular, the company should have a solid knowledge about how to hedge the financial risk inherent in the minimum rate of return guarantees.

Abstracting from mortality risk, a life insurance policy is basically a financial derivative that is written (on the return) on the life insurance company's investment portfolio. In a complete market, all financial derivatives can be replicated by self-financing trading strategies, the market value of the life insurance contract included. In the absence of arbitrage, it is clear that the market value of the insurance policy has to equal the initial cost of the replicating portfolio.

In this paper we focus on the hedging or replication of a policy with a so-called *multi-period* guarantee embedded.[2] In some countries the return on a life insurance contract is subject to an annual minimum guaranteed rate of return. For instance, if the annual minimum guaranteed rate of return is 4%, and the return on the insurance company's investment portfolio in year one and two are −10% and 15%, respectively, the return on and the cash flow from the insurance policy develop as in Fig. 1 (assuming one unit of account is invested with the guarantee embedded).

The multi-period guarantee has a sort of "ratcheting-effect", i.e., any good return in earlier periods will not be lost in a period with a "low" return on the investment portfolio. It turns out that this type of guarantee is rather expensive and exposes the life insurance companies to a considerable amount of financial risk. The "delta" of the guarantee can be rather large and can be much larger than one. It is therefore important that the issuers of such guarantees are aware of how to hedge them.

The hedging strategy for the multi-period guarantee distinguishes it self quite a bit from the hedging strategy for, say, a standard call option. From the Black and Scholes (1973) analysis we know that the hedging strategy for a call option is a portfolio of the underlying stock and the risk free asset. The exact holdings of the two assets are given by Ito processes with *continuous trajectories* over the life time of the option. This is not the case for the strategy for the multi-period guarantee. This strategy experiences a discontinuity, or jump, as one goes from one period to the next. That is, there is either a significant sell-off of the investment portfolio or the risk free asset, this depending on whether the guarantee is binding or not. Of course, since the replicating portfolio is self-financing, a sell-off of the risk free asset leads to an increased holding of the investment portfolio, and vice versa.

In practice, one does not observe the insurance companies performing such a major rebalancing of their balance sheet at the end of each year. There can be several reasons for this. One is that the policies they have issued are more complex than the one described above; another is that they do not create a perfect hedge for the guarantees they have issued. A third explanation, which is the one we investigate in this paper, is the presence of transaction costs. It is reasonable to expect that the volume of trading will diminish as the cost of trading increases. In particular, the focus in this paper is on the impact transaction costs have on the large rebalancing that is undertaken as we go from one period to the next.

The paper is organized as follows: in Section 2 we present assumptions underlying our economic model and the insurance policy. In Section 3 we review results regarding the hedging of the multi-period guarantee in the absence of transaction costs. In Section 4 we analyze the hedging strategies in the presence of transaction costs. Section 5 concludes.

2. The economic model and the insurance policy

As argued in Section 1, mortality risk can be diversified by issuing many similar policies. We therefore abstract from this type of risk and instead concentrate on financial risk.

[2] Sometimes we refer to the insurance policy with a guarantee embedded as simply *the guarantee.*

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We assume a very simple model for the financial market. Only two "assets" are considered; the insurance company's investment portfolio and a risk free asset. We assume that a binomial process gives the portfolio value. The market value of the investment portfolio is S_i in node i. In node $i+1$, the portfolio value has either increased to

$$S_{i+1} = S_i u \tag{1}$$

or decreased to

$$S_{i+1} = S_i d, \tag{2}$$

where

$$d = \frac{1}{u}, \tag{3}$$

see e.g., Cox et al. (1979). This is also illustrated in Fig. 2 (here S_0 is abbreviated to S). Let σ be the instantaneous standard deviation of the return on the investment portfolio in a continuous time model and let Δt be the time interval between node i and $i+1$. We then define the factor u as

$$u = \mathrm{e}^{\sigma\sqrt{\Delta t}}.$$

The risk free asset is a bank account accruing the constant short term interest rate $r = \ln R$, from one node to the next, thus

$$B_{i+1} = B_i R, \tag{4}$$

where $B_0 = 1$.

For simplicity we assume the company only has issued one contract. The contract lasts for two periods (each will typically be of one year). Let I be an even number. We divide each period into $I/2$ time steps, i.e., node $I/2$ is the end of the first period (also the beginning of the second period), while node I is the last node in the second period. Let i be the number of up-moves in the first period and j the number of up-moves in the second period. The return on the investment portfolio in the first period is then defined as

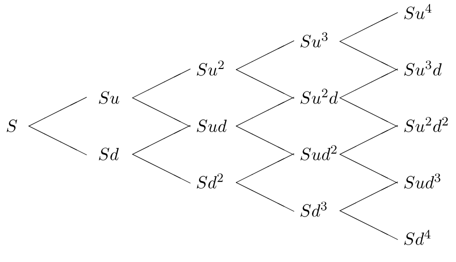

Fig. 2. Illustration of the development in the value of the investment portfolio for four time steps.

$$\delta_1 = u^i d^{I/2-i} - 1 \tag{5}$$

and for the second period

$$\delta_2 = u^j d^{I/2-j} - 1. \tag{6}$$

In each period the policyholder is guaranteed a minimum rate of return equal to g. This is known as a multi-period guarantee, or in this case a two-period guarantee. There are several interpretations of this type of guarantee, but we choose the same interpretation as in e.g., Miltersen and Persson (1999), i.e., the contract with the two-period guarantee embedded has a terminal cash-flow at the end of the second period of (i.e., in node I)

$$\pi_2 = \max(\Delta_1, G) \cdot \max(\Delta_2, G), \tag{7}$$

where $\Delta_i = 1 + \delta_i$, $i \in \{1,2\}$, and $G = 1 + g$. To emphasize the fact that the policy has a guarantee embedded, we will throughout denote the policy *the guarantee*.

As is evident from Eq. (7), the multi-period guarantee is a path dependent derivative asset. The terminal payoff is dependent on the return on the investment portfolio in the different periods. This is illustrated in Fig. 3 for the two-period guarantee when the investment portfolio evolves as in Fig. 2 and each period is of two time steps.

In a real world financial market, the trading of financial assets comes at a cost. Stockbrokers and other financial intermediaries charge their customers a fee when selling and buying financial assets. For simplicity we assume that there is only a cost associated with trading the investment portfolio, not the risk free asset. As will become clear later, to hedge the guarantee, the company has to construct a hedge portfolio consisting of the investment portfolio and the risk free asset. The proceedings from a sale of part of the investment portfolio, net of transaction costs, will be invested in the risk free asset and cash needed to increase the holding in the investment portfolio will have to be raised by a reduction in the holding of the risk free asset. No infusion or withdrawal of cash from the hedge portfolio is allowed for.

Let a_{i-1} be the number of units of the investment portfolio in the hedge portfolio that we "arrive" in node i with, and a_i the number we "leave" with. The corresponding quantities for the risk free asset are given by b_{i-1} and b_i, respectively. Total transaction cost in node i is given by

$$C_i = |a_i - a_{i-1}| S_i c,$$

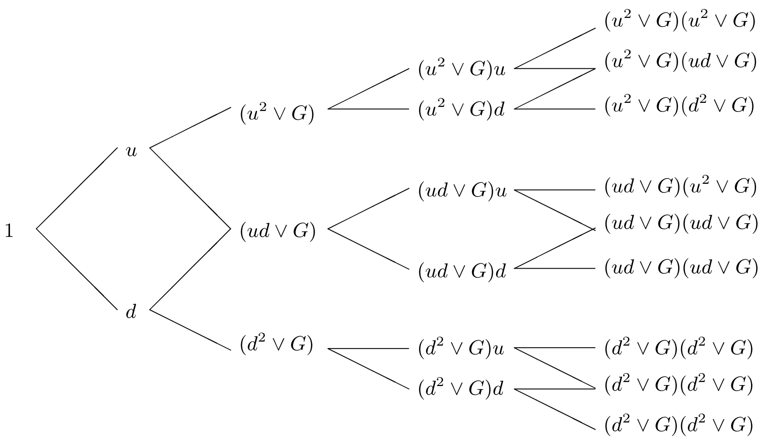

Fig. 3. Illustration of the gross return on the two-period guarantee given the development in the value of the investment portfolio in Fig. 2. $(a \vee b) = \max(a, b)$.

for some parameter $0 \leqslant c < 1$. Here c represents a proportional transaction cost. Thus, there are no fixed costs associated with rebalancing the hedge portfolio.

3. Hedging with zero transaction costs

The motivation for this paper is partially an observation in Lindset (2003). It is there observed, although in a continuous time setting, that there is a large rebalancing of the hedge portfolio as we go from one period to the next. This discontinuity is illustrated in Fig. 4.

In the case of zero transaction costs, it is straightforward to create a perfect hedge of the guarantee. Let $f(u)$ be the value of the guarantee in case the investment portfolio has had an up-move from node i to $i+1$ and $f(d)$ for a down-move. We then need a hedging strategy satisfying Eqs. (8) and (9)

$$a_i S_i u + b_i B_i R = f(u) \tag{8}$$

and

$$a_i S_i d + b_i B_i R = f(d). \tag{9}$$

The solution to these two equations is

$$a_i = \frac{f(u) - f(d)}{S_i(u-d)} \tag{10}$$

and

$$b_i = \frac{f(d) - \dfrac{(f(u) - f(d))d}{u-d}}{B_i R}. \tag{11}$$

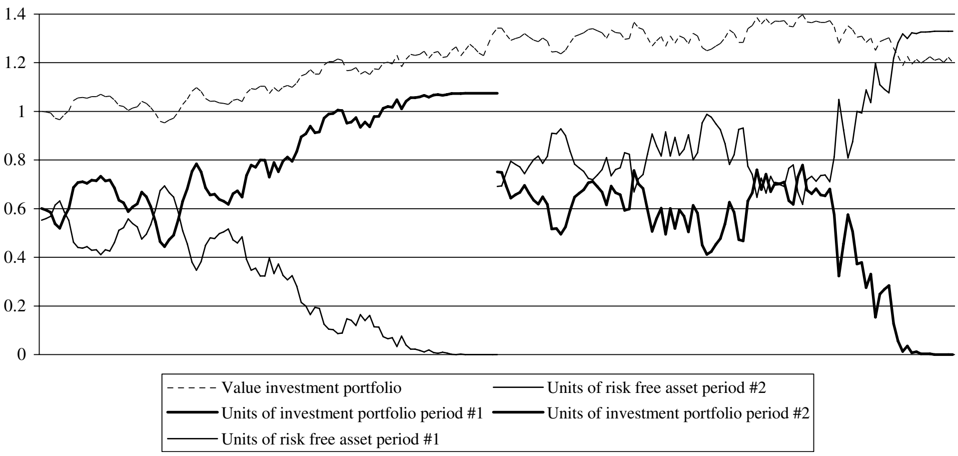

Fig. 4. Illustration of the discontinuity in the replicating portfolio (for a continuous time model) as we go from the first to the second period.

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Consider the situation where the investment portfolio has been performing badly in most of the first period, and any up-movements cannot prohibit the guarantee from becoming binding in the first period. This implies that[3] $f(u) - f(d) = 0$, thus,

$$a_i = 0 \quad \text{and} \quad b_i = \frac{f(d)}{B_i R}.$$

Similarly, if the investment portfolio has been performing well and any down-movements cannot make the guarantee become binding in the first period, we have that $f(u) - f(d) = S_i(u - d)\theta$, thus,

$$a_i = \theta \quad \text{and} \quad b_i = 0.$$

Here θ is the time 1 market value of a guarantee with time 2 payoff $\pi_2^m = \max(\Delta_2, G)$. This basically follows since Δ_1 and Δ_2 are statistically independent.

However, in the beginning of the second period, both a_i and b_i will be strictly positive if we exclude contract specifications where $G > u^{I/2}$ (the guarantee is always binding) or $G < d^{I/2}$ (the guarantee is never binding). We therefore conclude that there will also be a "large" change in the hedge portfolio as we go from the last node in the first period to the beginning of the second period.

4. Hedging with transaction costs

In this section we want to analyze the situation where we have a proportional transaction cost. It is reasonable to assume that it is more costly to rebalance the investment portfolio than it is to buy or sell the risk free asset. The costs may be both direct and indirect. The direct costs are such as commission fees to brokers and so on. For an insurance company with a large investment portfolio, the rebalancing of the investment portfolio may also have an influence on the liquidity in the market and represent an indirect cost. The total transaction cost may therefore be rather large.

That no transaction costs are imposed on selling or buying the risk free asset can also be justified by the fact that, for instance, any proceedings from selling parts of the investment portfolio, net of transaction costs, is used to buy the risk free asset. Therefore, c can be interpreted as also incorporating the costs associated with buying or selling the risk free asset. Also, as pointed out by Boyle and Vorst (1992), including transaction costs on the trading of the risk free asset, "...the model becomes much more complicated without providing new insight."

By imposing this transaction cost on the rebalancing of the hedge portfolio, we expect that less trading will take place than with zero transaction costs. In particular, what we would like to investigate in this paper is if the heavy rebalancing of the hedge portfolio at the end of the first period is still present when such a rebalancing is costly. And if so, does an increase in the cost of trading affect the hedge portfolio, and in particular, the size of the rebalancing at the end of the first period. When the management of the insurance company is trying to maximize the profit of the company, the cost of this rebalancing should be taken into account in their risk management routines.

From standard corporate finance literature we know that the only goal that should be pursued by the management is to maximize the value of the shareholders' stocks. Insurance companies are also subject to rather strict regulations. Here we assume that the insurer has to be able to meet the obligations imposed by the guarantee in every state of the world. The insurer does therefore have to create a hedge portfolio that will prevent it from defaulting on its obligations with certainty.[4] To maximize the market value of the share holders stocks, the management has to minimize the costs of establishing the hedge portfolio.[5]

Let $\mathscr{H}$ be the set of all possible trading strategies. We take **a** and **b** to be the trading strategy for the investment portfolio and the risk free asset, respectively.

The insurer's hedging strategy is the solution to the following optimization problem (for an introduction to stochastic programming, see e.g., Kall and Wallace, 1994).

$$\min_{\mathbf{a},\mathbf{b}\in\mathscr{H}} (a_0 S + b_0)$$

subject to

$$a_{I-1}S_I + b_{I-1}B_I \geqslant f_I$$
$$a_i S_i + b_i B_i = a_{i-1}S_i + b_{i-1}B_i - C_i, \quad 0 < i < I,$$

where f_I is the final payoff from the guarantee in node I.

Note that it is the portfolio we "arrive" in node I with, i.e., the portfolio constructed in node $I-1$,

[3] This follows since the terminal payoff at time t_2 (i.e., in node I) is the same both if the investment portfolio moves up or down.

[4] A buffer capital in form of equity would reduce the need for hedging. We do not take this into account.

[5] We assume that no funds are added or subtracted from the hedge portfolio between time zero and the end time (except for the transaction costs), i.e., we assume a "quasi" self-financing hedge portfolio.

that has to have a value greater or equal to the payoff from the guarantee (i.e., the insurance policy). There is no point in rebalancing this portfolio in node I, since this only would impose unnecessary transaction costs. The value of the portfolio we "leave" with in node i must be the same as the value of the portfolio we "arrive" with, subtracted the cost of rebalancing the hedge portfolio. Also, we have not included any transaction costs on the creation of the hedge portfolio at time 0 since we assume that the insurance company already is in possession of the investment portfolio.

4.1. A strictly dominating strategy

If there is a strategy dominating the dynamic hedge portfolio in the sense that the end value of the strategy is always greater or equal to the payoff from the guarantee, we say that this is a *dominating strategy*. If, in addition, the initial cost of the strategy is less than the cost of the dynamic hedge portfolio, we say that this strategy is a *strictly* dominating strategy. For instance, for a call option with terminal payoff $\max(S_T - X, 0)$, buying the stock at the time the option is written is a dominating strategy since it always gives a greater payoff than the call option (cf. Soner et al., 1995). For a maturity guarantee with payoff $\max(S_T, X)$, a portfolio consisting of the stock and the present value of X is a dominating portfolio.

For the two-period guarantee we have that the time 2 payoff is given by

$$\pi_2 = \max\left(\frac{S_{I/2}}{S_0}, G\right) \cdot \max\left(\frac{S_I}{S_{I/2}}, G\right). \tag{12}$$

This yields the possibility of four different payoffs (these are also illustrated in Fig. 5).

Payoff 1 $\frac{S_I}{S_0}$: This payoff can be replicated by buying $\frac{1}{S_0}$ units of the underlying asset at time 0 and holding on to the position until time 2. This situation corresponds to the case where the guarantee is not binding in any of the two periods.

Payoff 2 G^2: This payoff can be replicated by depositing $\frac{G^2}{R^2}$ in the bank (i.e., buying the risk free asset) at time 0 and holding on to the position until

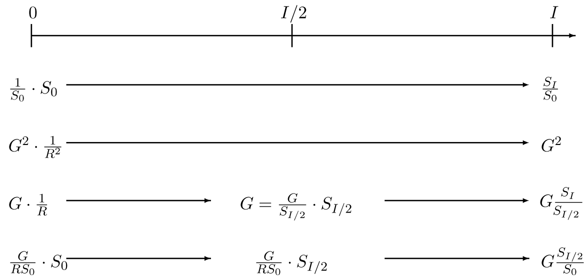

Fig. 5. The figure illustrates the four different parts of the dominating hedging strategy for a two-period guarantee. The first two parts are buy-and-hold strategies, while the last two require a reinvestment after the first period.

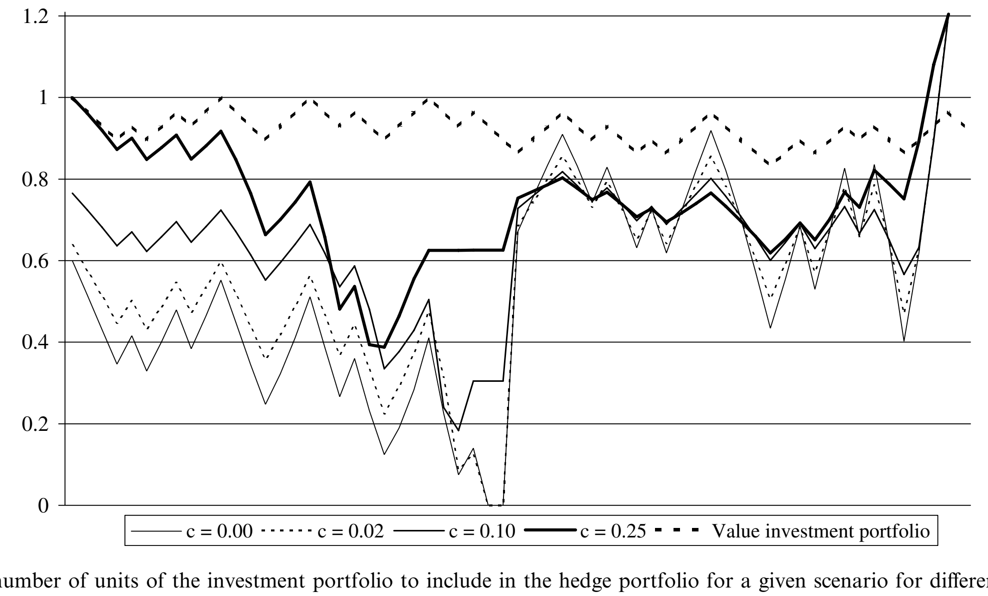

Fig. 6. The number of units of the investment portfolio to include in the hedge portfolio for a given scenario for different levels of the proportional transaction cost c.

time 2. This corresponds to the case where the guarantees are binding in both periods.

Payoff 3 $G\frac{S_I}{S_{I/2}}$: This payoff can be replicated by depositing $\frac{G}{R}$ in the bank at time 0 and holding on to the position until time 1. The amount has grown to G at time 1. We use this money to buy $G\frac{1}{S_{I/2}}$ units of the underlying asset and hold on to the position until time 2. This corresponds to the case where the guarantee is binding in the first period.

Payoff 4 $\frac{S_{I/2}}{S_0}G$: This payoff can be replicated by buying $\frac{1}{S_0}\frac{G}{R}$ units of the underlying asset at time 0 and holding on to the position until time 1. The position has grown to $\frac{S_{I/2}}{S_0}\frac{G}{R}$ at time 1. We sell the underlying asset and deposit the money in the bank and hold on to this position until time 2. This corresponds to the case where the guarantee is binding in the second period.

The cost of this strategy for $S_0 = 1$ is (without including transaction costs for the rebalancing at time 1 for payoff 3 and 4)

$$\pi_0^D = 1 + \frac{G^2}{R^2} + \frac{G}{R} + \frac{G}{R} = \left(1 + \frac{G}{R}\right)^2 \approx 4. \tag{13}$$

As we can see, this is a rather expensive strategy. The ratio $\frac{G}{R}$ will typically not be very far from 1, thus $\pi_0^D \approx 4$. By including transaction costs for the rebalancing at time 1 for payoff 3 and 4, the strategy becomes even more expensive.

If we find this strategy to be strictly dominating, we would prefer this strategy over the dynamic hedge since it is cheaper to establish and secures that the insurer is able to cover the liabilities imposed by the guarantee with certainty.

4.2. Example of two replicating strategies

In Figs. 6–9 we have illustrated the optimal hedging strategies for two given realizations of the development in the investment portfolio. The following parameters are used: $r = 0.05$, $g = e^{0.04} - 1$, $\sigma = 0.20$, and $\Delta t = 1/30$, i.e., each year is divided into 30 time steps. The initial investment to be subject to the guaranteed return is normalized to one. Figs. 6 and 8 show how the holdings of the investment portfolio changes as time passes by and the value of the investment portfolio changes. Figs. 7 and 9 show the corresponding holdings in the risk free asset. Although not easy to see from the figures, it appears that less trading takes place when the transaction cost increases. We can also see that there is a significant rebalancing in the middle of the figure (i.e., when we go from the first to the second period).

4.3. Is the "turn-over" in the hedge portfolio reduced when the cost increases?

To what extent is the total rebalancing of the hedge portfolio influenced by the transaction costs, and do the costs really have an influence on the major rebalancing at the end of the first year?

To answer this, we should take into account that some of the paths for the realization of the value of the investment portfolio may lead to more rebalancing than other paths. As a proxy for the probability of a given path or the probability for ending up in a particular node in the tree, we propose to use the

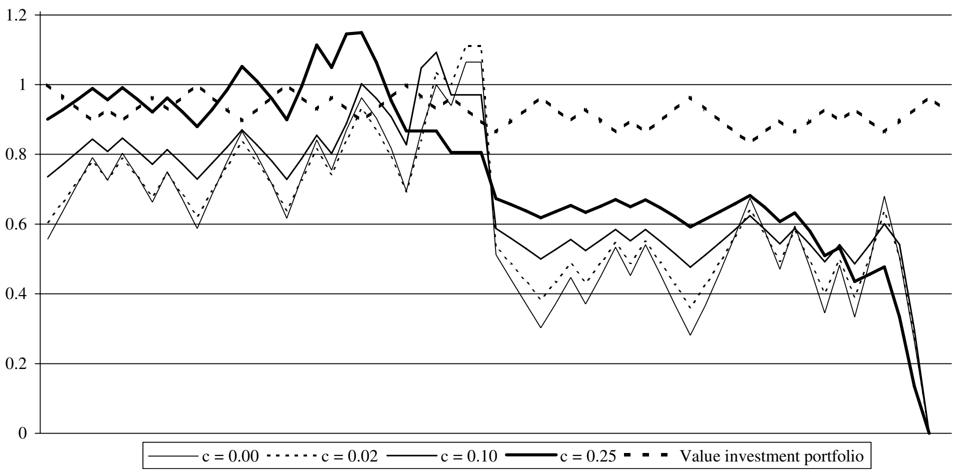

Fig. 7. The number of units of the risk free asset to include in the hedge portfolio for a given scenario for different levels of the proportional transaction cost c.

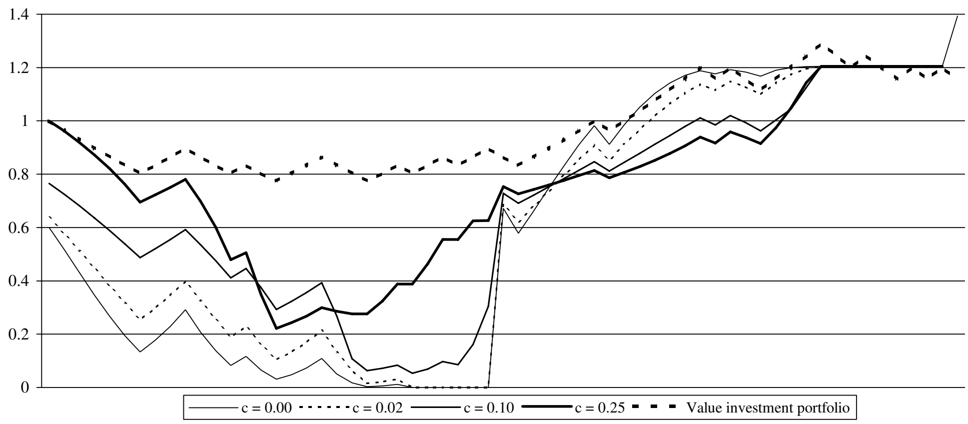

Fig. 8. The number of units of the investment portfolio to include in the hedge portfolio for a given scenario for different levels of the proportional transaction cost c.

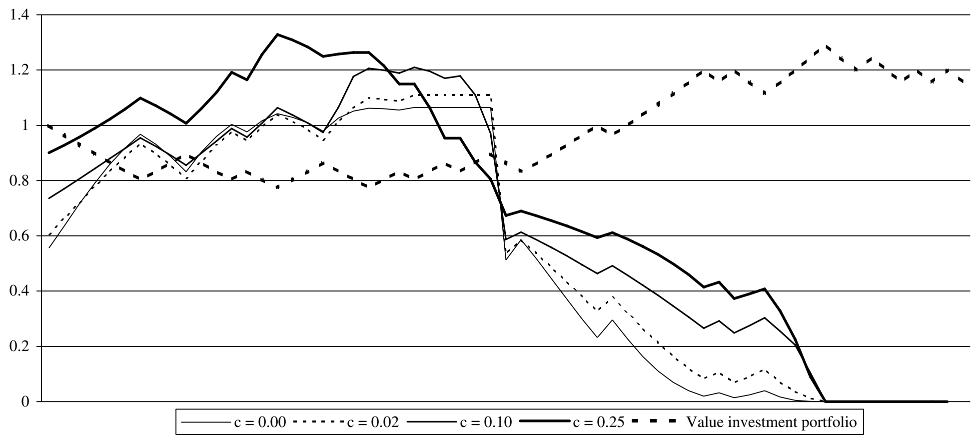

Fig. 9. The number of units of the risk free asset to include in the hedge portfolio for a given scenario for different levels of the proportional transaction cost c.

risk neutral up and down probabilities derived in Cox et al. (1979). They show that the up probability is given by

$$q = \frac{u - R}{u - d},$$

and the down probability by $1 - q$. We use these probabilities to calculate the expected rebalancing for a given level of the proportional transaction cost, c. Although this does not represent the real world probabilities, we think they suffice as a reasonable good approximation when calculating the effects on the hedge portfolio when introducing transaction costs.

From Table 1 we see that the expected rebalancing of the investment portfolio decreases when increasing the proportional transaction cost (line 7). The major rebalancing from period one to two is also expected to decrease. However, it is not decreasing monotonically; for small transaction costs, there is actually an increase in the expected rebalancing (line 2). It is interesting to notice that for large transaction costs, the expected jump between period one and two is significantly smaller, as a fraction of the total expected rebalancing (line 8), than for lower transaction costs. This indicates that the hedge portfolio is constructed so as to reduce the expensive jump. For reasonable transaction costs for a life insurance company, say, less than 6%, the jump represents a greater fraction of the total expected rebalancing than for the case with zero transaction costs. For both $c = 0.01$, $c = 0.02$,

Table 1
(1) is the proportional transaction cost; (2) is the (risk neutral) expected rebalancing of the investment portfolio between period 1 and 2; (3) is (2) normalized with $c = 0$ as the basis; (4) is the (risk neutral) expected rebalancing of the risk free asset between period 1 and 2; (5) is (4) normalized with $c = 0.6$ is the expected cost of rebalancing between period 1 and 2; (7) is the expected total rebalancing of the investment portfolio; (8) is the expected rebalancing between period 1 and 2 as a fraction of the total rebalancing; (9) gives the (smallest) initial cost for the hedge portfolio

(1)	Transaction cost c	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10	0.11	0.12
(2)	$E\left[\lvert a_{\frac{L}{2}} - a_{\frac{L}{2}-1}\rvert\right]$	0.523	0.534	0.544	0.553	0.481	0.486	0.488	0.418	0.326	0.326	0.312	0.298	0.295
(3)	(2) as a fraction of $c = 0$	1.000	1.020	1.039	1.056	0.920	0.929	0.933	0.798	0.623	0.623	0.597	0.569	0.565
(4)	$E\left[\lvert b_{\frac{L}{2}} - b_{\frac{L}{2}-1}\rvert\right]$	0.507	0.519	0.529	0.539	0.467	0.472	0.474	0.405	0.323	0.322	0.307	0.295	0.292
(5)	(4) as a fraction of $c = 0$	1.000	1.023	1.043	1.061	0.921	0.930	0.935	0.798	0.636	0.635	0.604	0.581	0.575
(6)	$E\left[\lvert a_{\frac{L}{2}} - a_{\frac{L}{2}-1}\rvert S_{\frac{L}{2}-1} c\right]$	0.000	0.005	0.011	0.017	0.020	0.025	0.030	0.029	0.027	0.030	0.031	0.033	0.036
(7)	$E[\lvert\Delta a\rvert]$	5.139	4.737	4.464	4.262	4.025	3.902	3.798	3.628	3.459	3.400	3.329	3.262	3.218
(8)	(2)/(7)	0.102	0.113	0.122	0.130	0.120	0.124	0.129	0.115	0.094	0.096	0.094	0.091	0.092
(9)	Price hedge portfolio	1.156	1.202	1.243	1.281	1.316	1.350	1.382	1.413	1.443	1.472	1.501	1.530	1.558
(1)	Transaction cost c	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.20	0.21	0.22	0.23	0.24	0.25
(2)	$E\left[\lvert a_{\frac{L}{2}} - a_{\frac{L}{2}-1}\rvert\right]$	0.254	0.251	0.194	0.194	0.191	0.156	0.156	0.155	0.153	0.143	0.117	0.117	0.115
(3)	(2) as a fraction of $c = 0$	0.485	0.480	0.370	0.371	0.366	0.298	0.299	0.296	0.292	0.274	0.223	0.223	0.220
(4)	$E\left[\lvert b_{\frac{L}{2}} - b_{\frac{L}{2}-1}\rvert\right]$	0.252	0.252	0.200	0.201	0.199	0.165	0.167	0.166	0.164	0.157	0.135	0.135	0.135
(5)	(4) as a fraction of $c = 0$	0.497	0.496	0.394	0.396	0.392	0.326	0.329	0.328	0.324	0.310	0.266	0.267	0.265
(6)	$E\left[\lvert a_{\frac{L}{2}} - a_{\frac{L}{2}-1}\rvert S_{\frac{L}{2}-1} c\right]$	0.032	0.034	0.029	0.031	0.033	0.027	0.028	0.029	0.030	0.030	0.025	0.026	0.027
(7)	$E[\lvert\Delta a\rvert]$	3.117	3.083	2.973	2.950	2.915	2.805	2.793	2.772	2.740	2.700	2.576	2.568	2.554
(8)	(2)/(7)	0.081	0.081	0.065	0.066	0.066	0.056	0.056	0.056	0.056	0.053	0.045	0.045	0.045
(9)	Price hedge portfolio	1.585	1.613	1.640	1.667	1.694	1.720	1.746	1.772	1.798	1.824	1.850	1.875	1.900

and $c = 0.03$, the expected jump is in fact greater than for $c = 0.00$. This is somewhat surprising and may be an indication that transaction costs are not the most likely explanation for why insurance companies are not performing a major rebalancing of their investment portfolio at the end/beginning of each year.

4.4. More advanced price processes

Returns on financial assets in real world financial markets often have heavier tails than normally distributed returns and volatilities are typically not constant over time. Kozubowski and Rachev (1994) find that the geometric stable distribution fits empirical returns well. This distribution typically has heavier tails than the normal distribution. There are also empirical evidence that volatilites follow stochastic processes (see e.g., Eraker et al., 2003) and that there are periods with high and low volatilities, often referred to as volatility clustering. Our analysis of the hedging strategy for the guarantee relies heavily on the assumed stochastic process for the investment portfolio. With more realistic price processes, e.g. exhibiting heavy tails and volatility clustering, there are no generally accepted ways of discretizing the processes into scenario trees. One possible approach is to use Monte Carlo simulation, and subsequently construct a scenario tree as in Heitsch and Römisch (2005). We leave the analysis of the hedging strategies using more advanced price processes for future research.

5. Conclusions

We have in this paper derived optimal hedging strategies for multi-period guarantees in the presence of transaction costs. We found the cost of establishing the hedge portfolio to increase as transaction costs increased. We also found the large rebalancing at the end of a period to decrease as a function of the transaction cost. Total rebalancing performed over the lifetime of the guarantee was also found to decrease when the transaction cost increases. However, for very small transaction cost, we actually found the total rebalancing to increase. This may be an indication that the presence of transaction costs does not explain, or is not the only explanation, for why life insurance companies are not performing a major rebalancing of their investment portfolio at the end of each year.

References

Black, F., Scholes, M., 1973. The Pricing of Options and Corporate Liabilities. Journal of Political Economy 81 (3), 637–654.

Boyle, P.P., Vorst, T., 1992. Option replication in discrete time with transaction costs. Journal of Finance 47 (1), 271–293.

Cox, J.C., Ross, S.A., Rubinstein, M., 1979. Option pricing: a simplified approach. Journal of Financial Economics 7, 229–263.

Eraker, B., Johannes, M., Polson, N., 2003. The impact of jumps in volatility and returns. Journal of Finance 58 (3), 1269–1300.

Heitsch, H., Römisch, W., 2005. Scenario tree modelling for multistage stochastic programs, Preprint 296, DFG Research Center Matheon.

Kall, P., Wallace, S.W., 1994. Stochastic Programming. Wiley, Chichester.

Kozubowski, T.J., Rachev, S.T., 1994. The theory of geometric stable distributions and its use in modeling financial data. European Journal of Operations Research 74 (2), 310–324.

Lindset, S., 2003. Discontinuous hedging strategies for multiperiod guarantees in life insurance. Journal of Risk Finance 5 (1), 51–63.

Miltersen, K., Persson, S.-A., 1999. Pricing rate of return guarantees in a Heath–Jarrow–Morton framework. Insurance: Mathematics and Economics 25 (3), 307–326.

Soner, H.M., Shreve, S.E., Cvitanic, 1995. There is no nontrivial hedging portfolio for option pricing with transaction costs. Annals of Applied Probability 5 (2), 327–355.

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