Dispersed Interactions of Urban Residents

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Abstract

Beckmann's interaction model has each resident touching base in face-to-face activity with every other resident at the other's residence per unit time. We re-work his resulting "interaction city" with each resident "operating with" a Cobb-Douglas utility function. Similar but somewhat "richer" outcomes occur. We also investigate a new case with intermediate dispersion of face-to-face activity, one with scale economies in trip-making.

Keywords Household spatial interactions · Dispersed residential activity

JEL Classification Numbers R14 · D11

1. Introduction

Beckmann (1976) set out an interesting model of an urban area, a model based on the interaction of each resident with each other, on a regular basis.[1] One might think of this as activity in the city based on face-to-face activity, here from home

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[1]The model is interpreted slightly differently (total cost of interaction rather than average cost) in Fujita and Thisse (2002, pp. 174-179) and re-presented clearly. We follow their "formulation".

to home rather than taking place in a central business district (CBD). Beckmann in fact felt he was capturing social or non-work interaction for the most part in his formulation but he did acknowledge that maybe his interactions combined work, shopping and purely social activities simultaneously. Each resident travels to the residence of every other resident, each period, in the model. An intuition for the Beckmann model is a tiny tropical village in which each person "wanders" about visiting each of her neighbors each period.[2] We place this travel activity in the familiar model of residential activity in space, the model in which transportation costs must be incurred in order that work activity can take place. Hence we gloss over the link between interactions and income earning.[3] Many modern cities exist "exporting" financial, legal and insurance services to a nearby hinterland and these services are produced with "face-to-face" activity. The contrast here is with older cities which grew up around a transshipment hub such as those with a port for ships or a railyard.

We make use of the familiar Cobb-Douglas form for a resident's utility of

[2] Things are similar for our "multiple interaction" model below but now at each place visited there are in general many persons to touch base with.

[3] It is reasonable to infer that the volume and quality of interactions for agent i will register as payoff in some sense. E. g., utility may be higher for an agent who interacts with more other agents or interacts in some more usefully intense way. Helsley and Strange (2005), e. g., have the utility of an agent higher with more interactions but at the cost of more travel. Beckmann assumed that each agent interacts with equal intensity with every other agent per unit time and thus the "interaction payoff" per agent is the same. This leads us to leave a variable capturing the payoff out of our analysis in the interest of conserving cleanliness. In addition one could envisage a macro community payoff to interacting in the sense that the larger the Beckmann community, the more interactions there are per agent in aggregate and one might infer, the better off each agent is. Then the utility of an agent would be a simple increasing function of city size. This is good to keep in mind but is somewhat peripheral to our agenda here.

home space and other consumption goods and thus depart from Beckmann's reliance on a special utility function, separable in the household's home space and other consumption goods. We are thus placing Beckmann's interesting interaction model more centrally in urban economics. We obtain a closed form solution for the special case of a household spending exactly half its after-transportation-cost income on "housing", $s(x)$ and half on "other consumption goods", $c(x)$. In this solution the equilibrium land rent function differs from the equilibrium density function, in contrast with Beckmann's solution. Departing from the 50-50 splitting of after-transportation-cost income leads us into an analysis without closed form solutions. We report very briefly on this.

The idea that the core of a city should be treated as a group of interacting (cross-visiting) firms followed directly from Beckmann, notably in Borukov and Hochman (1977), Imai (1982), Fujita and Ogawa (1982)[4], Tauchen and Witte (1983) and (1984), Kanemoto (1990), and more recently in Berliant, et al. (2000) and Helsely and Strange (2005). A central focus is on the inherent market failure associated with interactive cities. To a first approximation agent i locates to minimize her interaction costs without reckoning the costs she is imposing on the N-1 other firms by her choice of location. Each other firm faces a particular cost of interacting with her. One can envisage different degrees of interaction

[4]Lucas and Rossi-Hansberg (2002) rework the approach of Fujita and Ogawa (1982) in a very general model which ends up analyzable only by means of numerical simulations. Of interest is their discovery of "the extreme sensitivity of the nature of equilibria to small changes in assumed travel costs" (p. 1447).

corresponding to different departures of equilibria from first best outcomes, a topic we hope to pursue in the future.[5]

2. The Model

The city is located on a line with a resident at distance x from the center consuming $s(x)$ of land ("housing"). Land rent at x will be $R(x)$. Hence a household's budget constraint is

$$y - T(x) = cp + R(x)s(x)$$

with c other consumption goods (with price p set at unity), $T(x)$ interaction or total transportation costs per period per household, and y income per period. Consumption c will vary with distance x. The household has utility function $U = s(x)^{\alpha}c(x)^{1-\alpha}$. The utility level is fixed at $\overline{U}$ by free migration between cities (the open city assumption). Hence

$$s(x) = \frac{\overline{U}^{1/\alpha}}{c(x)^{(1-\alpha)/\alpha}}.$$

Since $c(x) = (1-\alpha)[y - T(x)]$, we have $s(x) = \xi \left[\frac{1}{[y-T(x)]^{(1-\alpha)/\alpha}}\right]$ for $\xi = \left[\frac{\overline{U}^{1/(1-\alpha)}}{(1-\alpha)}\right]^{(1-\alpha)/\alpha}$ and population density function $n(x) = 1/s(x)$ in

$$n(x) = \frac{1}{\xi}\left[y - T(x)\right]^{(1-\alpha)/\alpha}. \tag{2.1}$$

[5] Beckmann did not take up the issue of equilibria versus optima or a schedule of location taxes and subsidies that could implement a first best. Of interest would be the result that the first best Beckmann city was less technically complicated than the second best counterpart which we are reporting here.

And since $R(x)s(x) = \alpha[y - T(x)]$, we also have

$$R(x) = \frac{\alpha}{\xi}\,[y - T(x)]^{1/\alpha}\,.$$

Consider exogenous edge rent $\overline{R}$ at edge b, positive and unspecified.[6] Cobb-Douglasness of utility gives us

$$\begin{aligned} \alpha\,[y - T(b)] &= \overline{R}s(b) \\ \text{and } (1-\alpha)\,[y - T(b)] &= c\,(b) \\ \text{or } \alpha c(b) &= (1-\alpha)\overline{R}s(b). \end{aligned}$$

In addition we have $\overline{U} = s(b)^{\alpha}c(b)^{1-\alpha}$. Hence we can solve for edge values $s(b)$ ($=1/n(b)$) and $c(b)$. These values then allow us to solve for $T(b)$:

$$\begin{aligned} s(b) &= \left(\frac{\overline{R}}{\alpha}\right)^{\alpha-1}\cdot\xi^{\alpha}, \\ c(b) &= (1-\alpha)\cdot\left(\frac{\overline{R}\xi}{\alpha}\right)^{\alpha}, \end{aligned}$$

$$T(b) = y - \left(\frac{\overline{R}\xi}{\alpha}\right)^{\alpha}. \tag{2.2}$$

Observe that

$$\begin{aligned} R(x) &= \frac{\alpha[y - T(x)]}{s(x)} \\ &= \alpha[y - T(x)]\cdot n(x) \\ &= \frac{\alpha}{\xi}[y - T(x)]^{1/\alpha} \qquad (2.3) \\ &= \alpha\xi^{\alpha/(1-\alpha)}[n(x)]^{1/(1-\alpha)}. \end{aligned}$$

[6] With Cobb-Douglas utility one worries about a zero rent at the edge leading to an extremely large radius for a city. Hence $\overline{R}$ is treated as strictly positive.

Given $\overline{R}$ as rent at the edge, b, we can express $n(b)$ in terms of $\overline{R}$. That is

$$n(b) = \xi^{-\alpha} \left(\frac{\overline{R}}{\alpha} \right)^{1-\alpha}. \tag{2.4}$$

3. Monocentric City as Benchmark

We can fix ideas by appealing to the monocentric counterpart for comparison. Then $T(x) = t \cdot |x|$ and all interaction occurs at one point in the center. Given parameters α, y, ξ, t and edge rent $\overline{R}$, we can solve for edge, b in $\frac{\alpha}{\xi} [y - t \cdot b]^{1/\alpha} = \overline{R}$ and then city size N in

$$b = \frac{1}{t} [y - \left(\frac{\overline{R}\xi}{\alpha} \right)^{\alpha}]$$

$$2 \int_0^b \frac{1}{\xi} [y - t \cdot x]^{(1-\alpha)/\alpha} dx = \frac{2\alpha}{\xi t} \left\{ y^{1/\alpha} - (y - tb)^{1/\alpha} \right\} = N.$$

We interpret this as parameters α, ξ, y, t and $\overline{R}$ yielding geographic size b and then b and $n(x)$ yielding population, N.[7] This sequence of links is somewhat different for a Beckmann city.[8]

In Beckmann's city, interaction occurs by one-on-one visiting of each person to all others, one trip per person visited per period. Hence each household incurs $N - 1$ trips per period. Formally, then travel costs for interacting for a person at

[7] Somewhat parenthetically we note that $dR = Ndy - \overline{R}db$, for $R = 2 \int_0^b R(x)dx$. Roughly speaking, since the utility level is fixed, wage increments are fully capitalized in aggregate rent increments. This capitalization result turns on Leibnitz's Rule for differentiation of an integral.

[8] When we speak of a Beckmann city, we mean one generated with our Cobb-Douglas utility function, not one generated with Beckmann's utility function $u = \alpha \log s + c$. We have done no analysis with his utility function. In fact we started this analysis to see if we could re-work Beckmann's analysis with a Cobb-Douglas function.

x miles from the center, at zero, are

$$T(x) = \int_{-b}^{x} t(x-z)n(z)dz + \int_{x}^{b} t(z-x)n(z)dz. \tag{3.1}$$

The city ranges on the line from $-b$ to b. Observe that

$$\frac{d^2T(x)}{dx^2} = 2tn(x).$$

Hence we can substitute for $n(x)$ and obtain the fundamental equation for a Beckmann city,

$$\frac{d^2T(x)}{dx^2} = 2t\frac{1}{\xi}\left[y - T(x)\right]^{(1-\alpha)/\alpha}. \tag{3.2}$$

We turn to solving the model.

4. Solving the Model for $\alpha = 0.5$

Since the resident in the center at $x = 0$ will incur the least interaction costs, we have $T'(0) = 0$ and since $n(x)$ must be positive, we know that $T(x)$ is convex in x. For the case $\alpha = 1/2$, (3.2) is a linear nonhomogeneous equation of the second order

$$\frac{d^2T(x)}{dx^2} + \frac{2t}{\xi}T(x) = \frac{2t}{\xi}y$$

which solves[9] to the closed form

$$T(x) = y - c_1 \cos(x\sqrt{\frac{2t}{\xi}}). \tag{4.1}$$

[9] We find a general solution $T_h(x)$ of the corresponding homogeneous equation $T'' + \frac{2t}{\xi}T = 0$ (see Murphy, 1960, p. 84) and the particular integral $T_p(x)$ of the nonhomogeneous one [Murphy, 1960, p. 146], and then $T(x) = T_h(x)+ T_p(x)$, or by finding a solution of the equation with the missing $T'(x)$ [Murphy, 1960, p. 160]. We use $T'(x) = 0$ at 0.

c_1 is a constant of integration.

Using the definition (2.1) of $n(x)$ we have $n(x) = \frac{1}{\xi} c_1 \cos\left(x\sqrt{\frac{2t}{\xi}}\right)$. Substituting the last expression in definition (3.1) of $T(x)$ we obtain

$$\begin{aligned} T(x) &= \frac{c_1 t}{\xi}\left[\int_{-b}^{x} t(x-z)\cos\left(z\sqrt{\frac{2t}{\xi}}\right)dz + \int_{x}^{b} t(z-x)\cos\left(z\sqrt{\frac{2t}{\xi}}\right)dz\right] \\ &= c_1\left[\sqrt{\frac{2t}{\xi}}\cdot b\cdot\sin\left(b\sqrt{\frac{2t}{\xi}}\right) + \cos\left(b\sqrt{\frac{2t}{\xi}}\right) - \cos\left(x\sqrt{\frac{2t}{\xi}}\right)\right]. \end{aligned} \tag{4.2}$$

We equate (4.1) and (4.2) at $x = 0$ to get c_1 as a function of the new "parameter", b, temporarily unspecified:

$$c_1 = \frac{y}{\left[\sqrt{\frac{2t}{\xi}}\cdot b\cdot\sin\left(b\sqrt{\frac{2t}{\xi}}\right) + \cos\left(b\sqrt{\frac{2t}{\xi}}\right)\right]}. \tag{4.3}$$

Given boundary condition, $T(b) = y - \sqrt{2\overline{R}\xi}$, we have another nonlinear equation in c_1 and b.[10]

$$c_1 = \left[y - \sqrt{2\overline{R}\xi}\right] / \left[\sqrt{\frac{2t}{\xi}}\cdot b\cdot\sin\left(b\sqrt{\frac{2t}{\xi}}\right)\right], \quad b \in \left(0, \frac{\pi}{2}\sqrt{\frac{\xi}{2t}}\right). \tag{4.4}$$

Thus we have

Definition 4.1. *(Beckmann interactive city equilibrium): A positive pair (c_1^*, b^*) satisfying equations (4.3) and (4.4), given y, ξ, t and $\overline{R}$.*

[10] Here we must note that two different expressions for c_1 show us a connection between $y, \overline{R}, \xi, t$, and b. See Appendix 1.

To obtain b^*, we substitute for c_1 from (4.4) in (4.3) to get a nonlinear equation for b.

$$y - \sqrt{2\overline{R}\xi} \cdot \left[1 + \sqrt{\frac{2t}{\xi}} \cdot b \cdot \tan\left(b\sqrt{\frac{2t}{\xi}}\right)\right] = 0 \tag{4.5}$$

or

$$y - \sqrt{2\overline{R}\xi} = \sqrt{2\overline{R}\xi} \cdot \sqrt{\frac{2t}{\xi}} \cdot b \cdot \tan\left(b\sqrt{\frac{2t}{\xi}}\right)$$

which can be rewritten as

$$\frac{a}{\beta} = \tan(\beta)$$

where $\beta = b\sqrt{\frac{2t}{\xi}}$ and $a = \left[y - \sqrt{2\overline{R}\xi}\right] / \sqrt{2\overline{R}\xi}$. Since tan has intersections with the hyperbola only if $a > 0$, then we have a natural affordability condition[11] for the existence of positive root on b of equation (4.5), namely:

$$y - \sqrt{2\overline{R}\xi} \equiv T(b) > 0.$$

We denote the solution of (4.5) as b^*. Then $c_1^* = \left[y - \sqrt{2\overline{R}\xi}\right] / \left[\sqrt{\frac{2t}{\xi}} \cdot b^* \cdot \sin\left(b^*\sqrt{\frac{2t}{\xi}}\right)\right]$ from (4.4). We use this expression for c_1 in $T(x)$[12] to get $T(x; b^*)$ and then get $n(x; b^*) = \mu \cos(x\sqrt{2t/\xi})$ for $\mu = \frac{1}{\xi}\left[y - \sqrt{2\overline{R}\xi}\right] / \left[\sqrt{\frac{2t}{\xi}} \cdot b^* \cdot \sin\left(b^*\sqrt{\frac{2t}{\xi}}\right)\right]$. The integral for total population, N is then $2\mu \int_0^{b^*} \cos(x\sqrt{2t/\xi})dx$ which works out to be

$$\frac{1}{tb^*}(y - \sqrt{2\overline{R}\xi}) \equiv \frac{T(b^*)}{tb^*} = N. \tag{4.6}$$

[11] $y - T(b)$ is income available for housing and $c(b)$ at b and $\sqrt{2\overline{R}\xi}$ is the cost of achieving utility level $\overline{U}$ at b.

[12] $T(x; b^*) = y - \left\{\left[y - \sqrt{2\overline{R}\xi}\right] / \left[\sqrt{\frac{2t}{\xi}} \cdot b^* \cdot \sin\left(b^*\sqrt{\frac{2t}{\xi}}\right)\right]\right\} \cos\left(x\sqrt{\frac{2t}{\xi}}\right)$

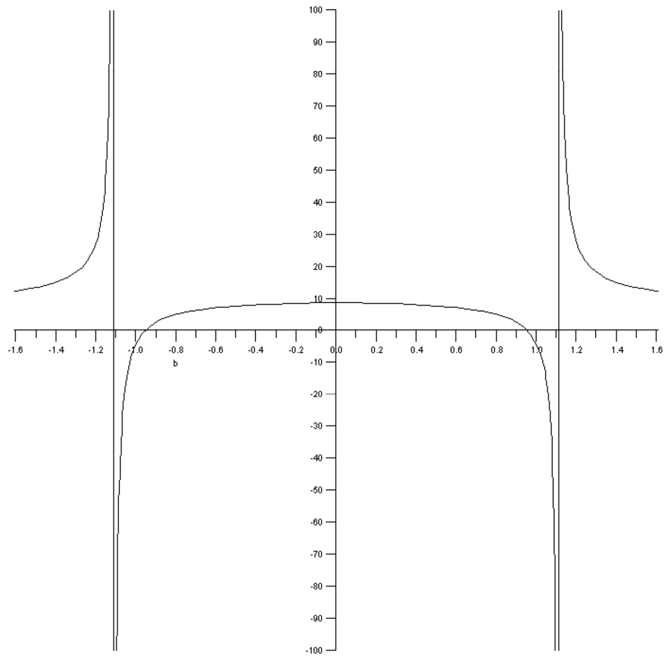

Figure 4.1: Beckmann b equation.

We plot the function on the left of (4.5) in Figure 4.1 for parameter values $y = 10$, $\xi = 1 = t = \overline{R}$.

It follows directly that $\frac{db}{dy} > 0$, $\frac{db}{d\overline{R}} < 0, \frac{db}{dt} < 0$, and $\frac{db}{d\xi} > 0$. The edge increases with income, decreases with edge rent, transportation cost and increases with the open city utility level.

Given $T(x; b^*)$ immediately above we have $R(x; b^*)$ defined in terms of x and b^*. Thus, we have

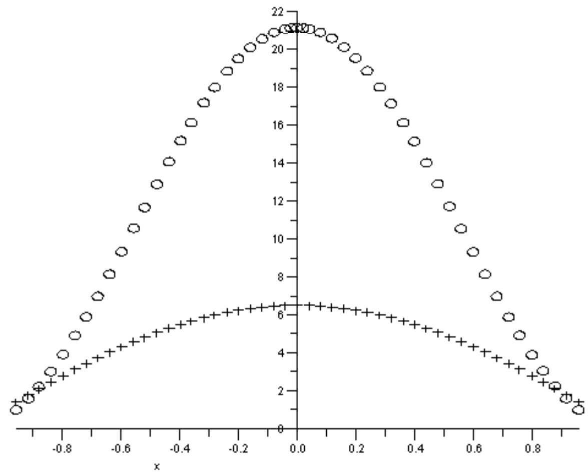

Figure 4.2: Beckmann density (+) and rent (o) functions.

Definition 4.2. *(Beckmann Interactive City): Functions* $T(x; b^*)$, $n(x; b^*)$, $R(x; b^*)$ *with positive values over* $(-b^*, b^*)$*, and population* N *in (4.6), given* b^* *an equilibrium value for* b*.*

Here is an example with $y = 10$, $\xi = 1 = t = \overline{R}$, which yields $b = .9558$ (see Fig. 4.1), and then $N = 8.98$. The functions $T(x), n(x)$, and $R(x)$ come out as:

$$\begin{aligned} T(x) &= 10 - 6.5 \cdot cos(x\sqrt{2}), \\ n(x) &= 6.5 \cdot cos(x\sqrt{2}), \\ R(x) &= 21.2 \cdot cos(x\sqrt{2})^2. \end{aligned}$$

$T(x)$ plots as a strictly convex function i.e. U shaped over $(-b^*, b^*)$, $n(x)$ as strictly concave (inverted U shaped) and $R(x)$ is generally bell-shaped with two points of inflection (see Fig. 4.2). The total transportation cost for this example is 43.002.

5. $\alpha \neq 0.5$

For α not 1/2 we can solve for $T(x)$ for an approximate solution by series methods.[13] In this case, we obtain

$$T(x) = T(0) + \frac{t(y-1)^{\frac{1-\alpha}{\alpha}}}{\xi} x^2 + \frac{(\alpha - 1)\left(t(y-1)^{\frac{1-\alpha}{\alpha}}\right)^2}{6\xi^2 \alpha (y-1)} x^4 + O(x^6) \tag{5.1}$$

For the case of $\alpha = \frac{1}{2}$ this solution tracks a plot of our exact solution above well in the neighborhood of $x = 0$ (see Fig. 5.1, with $T(x)$ analytical (circles) and $T(x)$ in form (5.1)(crosses)).

Comparison of $T(x)$ for different α is on Fig. 5.2 ($\alpha = .5$ - solid line, $\alpha = .45$ - crosses, and $\alpha = .55$ - circles). We do not pursue further analysis with $\alpha \neq \frac{1}{2}$ since we cannot obtain closed form solutions.

6. Scale Economies in Visiting

One could imagine scale economies in interaction costs. Once person i was at a site for a visit, she could visit all households there on just one trip from her home. This leads to a different "interaction city". We suppose that a resident travels each period to every other location in the city and once at a site, interacts

[13] This solution is a Taylor Series expansion in the neighborhood of $x = 0$.

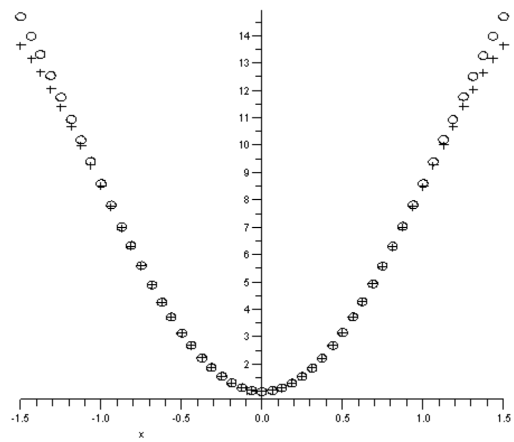

Figure 5.1: Taylor series (+) and analytical (o) solutions.

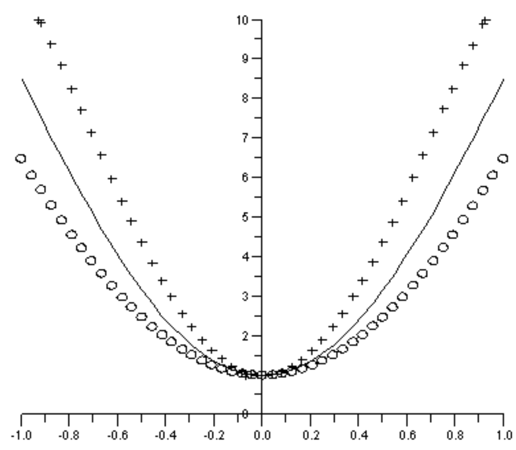

Figure 5.2: T(x), Taylor series: 0.45 (+), 0.5 (-), 0.55 (o).

or touches base with every resident at the site. Much transportation cost is thus economized on. We will call this type of city a "Spatial Multiple Interaction city" (MI), since any person from area x is able to interact with *all* persons at any place $z \in [-b, b]$ per single visit no matter what is the density $n(z)$. We depart from Beckmann, for a moment, by considering explicit costs of doing the interaction, once at the location.

Suppose that the interaction expenses of person from x, visiting z can be divided between a travel cost $t_1 |x - z|$ and a cost of actual interaction $t_2 \cdot n(z)$ with all residents at site z, where t_1 - travel cost per unit distance roundtrip and t_2 - cost of interaction at site per person. Since we assume that each resident has the same interaction field, i.e. $I_N(x) \equiv I_N$, then the total cost of interaction is also the same for each customer and depends only on N. We can think that it is already included into I_N and don't consider it further. Then we will set $t_2 = 0$ and denote $t \equiv t_1$. Hence we now have for the resident at x

$$
\begin{aligned}
T(x) &= \int_{-b}^{x} t(x-z)dz + \int_{x}^{b} t(z-x)dz \\
&= t(x^2 + b^2).
\end{aligned}
$$

As above, $T'(0) = 0$ and $T''(x) = 2t > 0$.

We can draw on our derivations above for the case of our utility function Cobb-Douglas. Hence now

$$
n(x) = \frac{1}{\xi}[\eta - t \cdot x^2]^{\frac{1-\alpha}{\alpha}}
$$

$$\text{and } R(x) \;=\; \frac{\alpha}{\xi}[\eta - t\cdot x^2]^{\frac{1}{\alpha}}$$

for $\eta = y - tb^2$, a constant. As we observed for the case of the monocentric city, we can obtain edge value b from $R(b) = \overline{R}$. The basic equation is

$$b = \sqrt{\frac{y - \left(\frac{\overline{R}\xi}{\alpha}\right)^{\alpha}}{2t}}$$

which has a plot between $-b$ and b similar to that for the Beckmann case in Figure 4.1. Existence of a positive b follows from condition $y - \left(\frac{\overline{R}\xi}{\alpha}\right)^{\alpha} \equiv T(b) > 0$. For $\alpha = 0.5$, we use this b to integrate $2\int_0^b n(x)dx$ to get population for the city $N = \frac{2b}{\xi}\left(y - \frac{4}{3}tb^2\right)$ or substituting b we have $N = \frac{2}{3\xi}\left(\frac{y-\sqrt{2\overline{R}\xi}}{2t}\right)^{\frac{1}{2}} \cdot \left(y + 2\sqrt{2\overline{R}\xi}\right)$.

Sometimes visits can be very time consuming and resident from site x is not able to interact with all the people at site z during one trip (one business day). For this case we can use a generalization of the model, where, as usual, the rule of transportation cost calculation reflects organization of the city:

$$T(x) = \int_{-b}^{b} [t_1|x - z|n(z)^{\gamma} + t_2 n(z)]\, dz,$$

where $\gamma \in [0, 1]$[14] - parameter of interaction activity. Note that for $\gamma = 0$ we have Multiple Interaction city and for $\gamma = 1$ - Beckmann (or Single Interaction) city. When we have no total time restrictions for interactions and the interaction field $I_N(x)$ is the same for all residents, we can neglect the term $t_2 n(z)$, otherwise it must depend on $I_N(x)$.

[14] γ can be greater then 1 if visiting a resident from z is so time consuming that takes more than one trip.

Example for MI city with $\xi = 1 = t = \overline{R}$, and $y = 6.035$ yields $b = 1.52$ and the same population as in Beckmann city $N = 8.98$. The functions $T(x), n(x)$, and $R(x)$ are:

$$\begin{aligned} T(x) &= x^2 + 2.31, \\ n(x) &= 3.725 - x^2, \\ R(x) &= 0.5 \cdot n(x)^2. \end{aligned}$$

The total transportation cost is 26.23. At the first glance it looks obvious that population of the same size N must always enjoy in MI city more vast area and totally spend less on transportation than in the Beckmann city (note that to have the same fixed utility $\overline{U} = s^{\alpha} c^{1-\alpha}$ with increased per capita area s we must adjust (decrease) level of income y to decrease c).

The scale economies (Multiple Interaction) approach to interaction really strikes us as a more plausible, quasi-empirical formulation than the alternate "many trip" version set out by Beckmann and the former is somewhat easier to analyze since one can work out the interaction-transportation cost function more easily. But of interest is the following comparison of total interaction costs for two cities of different types but the same population (and basic parameters, y, ξ, t, and $\overline{R}$). The parameters, $\xi = 0.05$, $\overline{R} = 0.01$, $t = 0.01$, and $y = 0.0916$, yield $b_{Beck} = 1.6777$ and $b_{MI} = 1.7324$ and the same population, $N = 3.58$. When we compare total transportation costs for these two distinct cities we observe that $T_{Beck} = 1.1304$

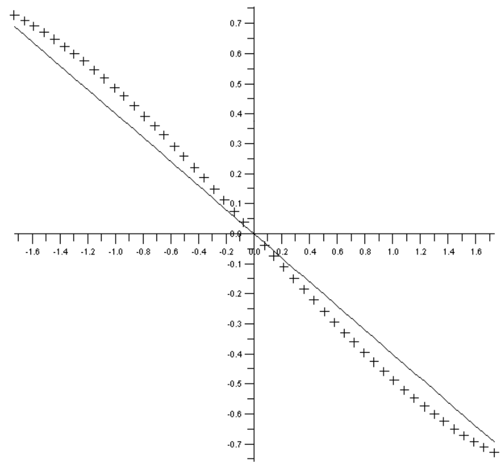

Figure 6.1: Density gradients: Beckmann (+) and MI (-).

and $T_{MI} = 1.1376$. Population density at the edge for each city is of course the same for the both cases at $n(b) = 0.6325$. (The density gradient for the Beckmann city dominates the one for MI in the absolute value, see Fig. 6.1). Hence a paradox of sorts: the total transportation costs are higher for the MI city, the city allowing for multiple visits per trip. The "paradox" goes away when one realizes that with $t_2 = 0$, the MI model reduces to a pure site-visiting model in contrast with the Beckmann model which is a combination site-visiting and person-visiting model. In brief, a visit in the Beckmann model is to a site weighted by a number of residents and with say 3 residents, this becomes 3 identical visits. In our numerical runs, virtually all sites have less than one person resident. Average density of residents is $N/2b$ or $3.58/3.3554$ for our Beckmann run and $3.58/3.4648$ for our

MI run. Hence while for the MI case, every site is visited by every resident, for the Beckmann case most sites receive a fraction, less than one, of a visit, given the low density of residents on land. We certainly do not expect our "paradox" to be observed for cases in which Beckmann's households were all located in densities at or above unity.[15] However, we have not been able to generate examples with identical parameters ξ, $\overline{R}$, t, y, N, for the both cities and with the required "high" densities (see Appendix 2).

7. Concluding Remark

We have brought the Beckmann model into mainstream urban economics by making use of a Cobb-Douglas utility function. We observed interesting density and rent functions for the case of a Cobb-Douglas utility function. The Beckmann model strikes one as highly inefficient since each visit by resident i to j requires a separate costly trip. Hence our introduction of the Multiple Interaction model, with scale economies in visiting. However when we compared numerically a Beckmann city and an MI city, we in fact observed the Beckmann city to have lower aggregate visiting (transportation) costs. This curious result turned on densities of residents being less than unity over much of space in the two models. Residential density less than unity is associated with a visiting scale economy in the Beckmann model.

[15] The paradox is rooted in the very low density phenomenon, not directly in the choice of form for the utility function.

References

[1] Beckmann, M., 1976, Spatial Equilibrium in the Dispersed City, in: G.J. Papageorgiou, ed., Mathematical Land Use Theory (Lexington Books, Lexington, MA) Chapter 8.

[2] Berliant, M., J. Reed, and P. Wang, 2000, Knowledge Exchange, Matching and Agglomeration. Vanderbilt Working Papers 33.

[3] Borukov, E. and O. Hochman, 1977, Optimum and Market in a Model of a City without a Predetermined Center, Environment and Planning A 9, 849-856.

[4] Fujita, M. and H. Ogawa, 1982, Multiple Equilibria and Structural Transition of Non-monocentric Urban Configurations, Regional Science and Urban Economics 12, 161-169.

[5] Fujita, M. and J.F. Thisse, 2002, Economics of Agglomeration (Cambridge University Press, New York).

[6] Helsley, R. W. and W. C. Strange, 2005, Social Interactions and Urban Spatial Structure. Centre for Urban Economics and Real Estate Working Paper $2005-03$.

[7] Imai, H., 1982, CBD Hypothesis and Economies of Agglomeration, Journal of Economic Theory 28, 275-299.

[8] Kanemoto, Y., 1990, Optimal Cities with Indivisibility in Production and Interaction between Firms, Journal of Urban Economics 27, 46-59.

[9] Lucas, R. and E. Rossi-Hansberg, 2002, On the Internal Structure of Cities, Econometrica 70, 1445-1476.

[10] Murphy, G., 1960, Ordinary Differential Equations and Their Solutions (D.Van Nostrad Company, Princeton, NJ).

[11] Tauchen, H. and A. D. Witte, 1983, An Equilibrium Model of Office Location and Contact Patterns, Environment and Planning A 15, 1311-1326.

[12] Tauchen, H. and A. D. Witte, 1984, Socially Optimal and Equilibrium Distributions of Office Activity: Models with Exogenous and Endogenous Contacts, Journal of Urban Economics 15, 66-68.

Appendix 1: Parameter Choice

Note, that besides natural limitations on the choice of parameters like $b \in \left(0, \frac{\pi}{2}\sqrt{\frac{\xi}{2t}}\right)$, (since we can have $\sin\left(b\sqrt{\frac{2t}{\xi}}\right)$ and $\cos\left(b\sqrt{\frac{2t}{\xi}}\right)$ in denominators) which can be important for creating an example with fixed b, ξ, and t in Beckmann's case, there are some implicit links between parameters which are somewhat different for different models.

Note also, that to obtain c_1 from (2.2): $T(b) = y - \sqrt{2\overline{R}\xi}$ we can also use the expression (4.1) and then

$$c_1 = \sqrt{2\overline{R}\xi} / \cos\left(b\sqrt{\frac{2t}{\xi}}\right). \tag{7.1}$$

With the use of (7.1) definition (3.1) becomes

$$T(x) = \sqrt{2\overline{R}\xi}\left[\sqrt{\frac{2t}{\xi}} \cdot b \cdot \tan\left(b\sqrt{\frac{2t}{\xi}}\right) + 1 - \frac{\cos\left(x\sqrt{\frac{2t}{\xi}}\right)}{\cos\left(b\sqrt{\frac{2t}{\xi}}\right)}\right], \tag{7.2}$$

which in a similar way to monocentric city does not contain y explicitly and then

$$R(x) = \frac{0.5}{\xi}\left\{y - \sqrt{2\overline{R}\xi}\left[\sqrt{\frac{2t}{\xi}} \cdot b \cdot \tan\left(b\sqrt{\frac{2t}{\xi}}\right) + 1 - \frac{\cos\left(x\sqrt{\frac{2t}{\xi}}\right)}{\cos\left(b\sqrt{\frac{2t}{\xi}}\right)}\right]\right\}^2.$$

Thus, from different expressions for c_1 in Beckmann city we can see interconnection between $y, \overline{R}, \xi, b$, and t :

$$y = \sqrt{2\overline{R}\xi}\left[1 + \sqrt{\frac{2t}{\xi}} \cdot b \cdot \tan\left(b\sqrt{\frac{2t}{\xi}}\right)\right].$$

For monocentric city dependence between these parameters is

$$y = \sqrt{2\overline{R}\xi} + tb,$$

and for Multiple Interaction city

$$y = \sqrt{2\overline{R}\xi} + 2tb^2.$$

Appendix 2: Comparison of Beckmann and MI Cities for $n(b) \geq 1$

Since for each model of the city we have some variables such as y, N, and b which are connected with each other via different expressions for $T(x)$, we must take it into account comparing cities with different organization (different $T(x)$). Thus the problem of comparison of Beckmann and MI cities with the same ξ, $\overline{R}$, t, N, and y (with different b) can be formulated as a problem of searching such a set of parameters ξ, $\overline{R}$, t, for which the system of equations

$$\begin{aligned} y_{MI} &= y_{Beck}, \\ N_{MI} &= N_{Beck} \end{aligned}$$

has at least one positive root $(y > 0, N > 0)$. Substituting corresponding expressions we have

$$\begin{aligned} 2tb_{MI}^2 &= 2\sqrt{\overline{R}t} \cdot b_{Beck} \cdot \tan(b_{Beck} \cdot \sqrt{\frac{2t}{\xi}}), \\ \frac{2b_{MI}}{\xi}\left[y_{MI} - \frac{4}{3}tb_{MI}^2\right] &= 2\sqrt{\frac{\overline{R}}{t}} \cdot \tan(b_{Beck} \cdot \sqrt{\frac{2t}{\xi}}). \end{aligned}$$

We can reduce it to one equation for b_{Beck} using expression for y_{MI}:

$$\frac{b_{Beck}}{\xi}\left[\sqrt{2\overline{R}\xi} + \frac{2}{3}\sqrt{\overline{R}t} \cdot b_{Beck} \cdot \tan(b_{Beck} \cdot \sqrt{\frac{2t}{\xi}})\right] = \sqrt{\sqrt{\frac{\overline{R}}{t}} \cdot b_{Beck} \cdot \tan(b_{Beck} \cdot \sqrt{\frac{2t}{\xi}})}.$$

Denote $\Delta(b_{Beck})$ a difference between the left and the right hand sides of this equation. Examination of the equation $\Delta(b_{Beck}) = 0$ shows that when ξ, $\overline{R}$ are such that $n(b) = \sqrt{\frac{2\overline{R}}{\xi}} < 1$, it always has two real roots: $b_{Beck1} = 0$ and one positive root $b_{Beck2} \in \left(0, \frac{\pi}{2}\sqrt{\frac{\xi}{2t}}\right)$. Choosing sequence of ξ, $\overline{R}$ such that $n(b) \to 1 - 0$ (from the left) we obtain sequence of $b_{Beck2} \to 0$, and correspondingly $y_{MI} = y_{Beck} \to 0$, $N_{MI} = N_{Beck} \to 0$. For parameters ξ, $\overline{R}$ such that $n(b) \geq 1$ we have only one trivial root $b_{Beck} = 0$.

Therefore in order to compare Beckmann and MI cities for larger amounts of population we can interpret the value of $n(x)$ as a number of tens, hundreds, or other groups of people. Then we must correct transportation cost function for MI city by dividing on correspondent factor (number of people in a group).

Note that we can not compare transportation costs of spatial continuous models with the monocentric one, which is discrete, because of difference in measure units (see Appendix 3).

Appendix 3: Correctness of Comparison with Monocentric City

Following Beckmann we compared shapes of density functions for different types of city given population N and edge of the city b. But we must note that generally speaking comparisons of the continuous models for the Beckmann's and the Multiple Interaction (MI) cities with the Monocentric one are inappropriate since in both spatial interaction cities (Beckmann's and MI) t is a cost of a travel not in some distance units but rather in areal units what can be seen from MI's transportation cost function $T(x) = t(x^2 + b^2)$.

Let's consider it on an example of a discrete model of the MI city. Assume that we have m internal points (buildings to visit) in a range from 0 to b, the cost of actual interaction t_2 is zero ($t \equiv t_1$), and $t = 1$. Since the city is symmetric, we can calculate travel expenses only for the right part of the city from 0 to b. For person in the center ($x = 0$) and for different m we have $T(0) =$

$$\begin{array}{ll}
m & T(0) \\
0 & b \\
1 & b + \frac{b}{2} = 1.5b \\
2 & b + \frac{2b}{3} + \frac{b}{3} = 2b \\
3 & b + \frac{3b}{4} + \frac{2b}{4} + \frac{b}{4} = 2.5b \\
\vdots & \qquad \ldots
\end{array}$$

In common case $T(0) = b + \frac{m}{2}b = b(1 + \frac{m}{2})$. Letting m to infinity we have $T(0) = \infty$. Hence even for countable infinite number of buildings m MI city becomes incomparable with the monocentric one.

As to calculating $T(0)$ for continuous case for the whole range $[-b, b]$, we can consider the process in the next way. At first we take a distance from the resident place ($x = 0$) to the right edge (b), then our trip will be to the point next to the edge, which we denote $b - \varepsilon$, and let the next trip will be to the left side of the city with "address" $-\varepsilon$. Sum of the last two trip distances is again b. Therefore, to calculate $T(0)$ means to summarize uncountable amount of ranges with the length b, which are the result of shifting of the original range $[0, b]$ to the left exactly on one point each time. The turn to use of areal units of measure after integrating for calculation of $T(x)$ in this case becomes more illustrative if we will shift these ranges not to the left but upward. Then we can imagine that in order to calculate all travel expenses for the person with address $x = 0$ we must take into consideration all points of the square with area $b \times b$.

Therefore, since the subset of points in a range $[0, b]$ which is the way for any person from monocentric city is a set of a measure null comparatively with measures of areas in spatial continuous models, we can compare these cities with very different organization only conditionally, taking into account their incomparable travel expenses. The comparison can be completely correct if to construct a discrete analogies of correspondent continuous models which will depend on one more parameter m - number of places for interaction (buildings).