Characterizing Pure-strategy Equilibria in Large Games

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Abstract

In this paper, we consider a generalized large game model where the agent space is divided into countable subgroups and each player's payoff depends on her own action and the action distribution in each of the subgroups. Given the countability assumption on its action or payoff space or the Loeb assumption on its agent space, we show that that a given distribution is an equilibrium distribution if and only if for any (Borel) subset of actions the proportion of players in each group playing this subset of actions is no larger than the proportion of players in that group having a best response in this subset. Furthermore, we also present a counterexample showing that this characterization result does not hold for a more general setting.

Keywords: Large games; Pure strategy Nash equilibrium; equilibrium distribution; Characterization

1 Introduction

In this paper, we consider a generalized large game model where the agent space is divided into countable (finite or countably infinite) different subgroups and each player's payoff depends on her own action and the action distribution in each of the subgroups.[1] In such a large game, a pure-strategy action profile that assigns an

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[1]The large game discussed here is a generalization to the large non-anonymous games discussed in Khan and Sun (2002) (Section 3).

action to each player is called a *pure-strategy Nash equilibrium* if no player has the incentive to deviate. A distribution vector which records the action distributions of all subgroups is called a *pure-strategy equilibrium distribution* if it is induced by a pure-strategy Nash equilibrium of the game.

Previous studies on large games are productive on the existence or nonexistence of the Nash equilibria.[2] But very few studies pay attention to the characterization of the Nash equilibria, or equivalently, the equilibrium distributions. This paper aims to make some contributions in filling this gap, that is, we hope to find some good characterization results which may broaden our understanding of the equilibria and also provide practical and useful guidance in determining the equilibria. In particular, this paper presents three characterization results and a counterexample for the equilibrium distributions in large games.

Our first result characterizes the equilibrium distributions in large games with *countable actions*. We show that a distribution vector on the action space is an equilibrium distribution if and only if for any (finite) subset actions the proportion of players in each group playing this subset of actions is no larger than the proportion of players in that group having a best response in this subset.

Our second result studies large games with *countable homogeneous groups of players*, where the homogeneousness assumption means that the players in each subgroup share a common payoff function and a common action set. Our third result is for large games endowed with *atomless Loeb agent spaces*. These two results are in the same form which also parallels the first result. Both of the results show that a given distribution vector is an equilibrium distribution if and only if for any Borel [Open or closed] subset of actions the proportion of players in each group playing this subset of actions is no larger than the proportion of players in that group having a best response in this subset.

Next we show through a simple counterexample that if both actions and payoffs are uncountable and the agent space is a general probability space, say the Lebesgue unit interval, then a similar characterization result *does not* hold anymore for such a large game.

The proof of our first result uses Bollobas and Varopoulos (1974)'s extension of the famous marriage theorem (or the Hall's theorem) and the proof of the third result relies on Sun (1996)'s result on the distributional properties of correspondence on Loeb spaces.

[2] For a detailed survey, see also Khan and Sun (2002)

The paper is organized as follows. Section 2 introduces the game model and Section 3 presents three characterization results for three settings of large games. Section 4 gives a counterexample showing that such a characterization result fails in a more general setting. Section 5 contains some concluding remarks and all the proofs are given in Section 6.

2 The model

Let $(T, \mathscr{T}, \lambda)$ be an atomless probability space of *agents* and I a countable (finite or countably infinite) index set. Let $(T_i)_{i\in I}$ be a measurable partition of T with positive λ-measures $(\alpha_i)_{i\in I}$. For each $i \in I$, let λ_i be the probability measure on T_i such that for any measurable set $B \subseteq T_i$, $\lambda_i(B) = \lambda(B)/\alpha_i$.

Let A be a Polish space[3] of *actions*, $\mathscr{B}(A)$ the Borel σ-algebra of A, $\mathscr{M}(A)$ the set of all Borel probability measures on A, endowed with the topology of weak convergence of measures, and $\mathscr{M}(A)^I$ the product space of $|I|$ copies of $\mathscr{M}(A)$, endowed with the usual product topology. Suppose that each player $t \in T$ chooses her own action from an action set $K(t) \in A$, where $K : T \to A$ is a compact valued measurable correspondence. Since A is Polish, $\mathscr{M}(A)$ is Polish[4] and hence $A \times \mathscr{M}(A)^I$ is also Polish. For easy notation, we now let $\Omega := A \times \mathscr{M}(A)^I$.

Unless otherwise specified, any topological space discussed in this paper is tacitly understood to be equipped with its Borel σ-algebra, i.e., the σ-algebra generated by the family of open sets, and the measurability is defined in terms of it.

Definition 1. A *large game* is a Carathéodory function[5] $U : T \times \Omega \to R$ such that for each $\omega \in \Omega$, the function $U^\omega = U(\cdot, \omega) : T \to R$ is measurable and for each $t \in T$, the function $U_t = U(t, \cdot) : \Omega \to R$ is continuous. A measurable function $f : T \to A$ is called a *pure-strategy profile* if $f(t) \in K(t)$ for all $t \in T$. A pure-strategy profile f is called a *(pure-strategy) Nash equilibrium*[6] if

$$U[t, f(t), (\lambda_i f_i^{-1})_{i\in I}] \geq U[t, a, (\lambda_i f_i^{-1})_{i\in I}] \text{ for all } a \in K(t) \text{ and all } t \in T,$$

[3] A Polish space is a topological space homeomorphic to some complete separable metric space.

[4] See, eg, Theorem 14.15 in Aliprantis and Border (1999).

[5] A large game is also often defined to be a measurable function from T to the space of payoff functions, which is the space of all continuous real-valued functions on Ω here. Since such a measurable function can always be transformed to be a Carathéodory function, our definition here is more general.

[6] Throughout this paper, we deal only with pure-strategy Nash equilibrium and pure-strategy equilibrium distribution. Thus we suppress the adjective 'pure-strategy' hereafter.

where f_i is the restriction of f to T_i and $\lambda_i f_i^{-1}$ the induced distribution on A. A distribution vector μ in $\mathscr{M}(A)^I$ is called a *(pure-strategy) equilibrium distribution*[7] if $\mu = (\lambda_i f_i^{-1})_{i \in I}$ for some Nash equilibrium f.

Recall that a correspondence F from T to A is said to be *measurable* if for each closed subset C of A, the set $F^{-1}(C) = \{t \in T : F(t) \cap C \neq \emptyset\}$ is measurable in $\mathscr{T}$. A function f from T to A is said to be a *measurable selection* of F if f is measurable and $f(t) \in F(t)$ for all $t \in T$. When F is measurable and closed valued, the classical Kuratowski-Ryll-Nardzewski Theorem (see, eg, Aliprantis and Border (1999, p.567)) shows that F has a measurable selection.

Given $\mu \in \mathscr{M}(A)^I$, let $B^\mu(t) = \arg\max_{a \in K(t)} U(t, a, \mu)$ be the set of best responses for player t given action distribution μ. By the Measurable Maximum Theorem in Aliprantis and Border (1999, p.570), B^μ is a measurable correspondence from T to A, has nonempty compact values and admits a measurable selection. Let $B_i^\mu : T_i \twoheadrightarrow A$ be the restriction of B^μ to T_i. It is straightforward to check that μ is an equilibrium distribution if and only if for each $i \in I$ there exists a measurable selection f_i of B_i^μ such that $\mu = (\lambda_i f_i^{-1})_{i \in I}$.

3 The results

Our first result is for large games with countable actions and is formulated as follows.

Theorem 1. *In a large game U, if the action space A is a countable and complete metric space, then the following statements are equivalent:*

(i) $\mu = (\mu_i)_{i \in I} \in \mathscr{M}(A)^I$ *is an equilibrium distribution;*

(ii) *for each* $i \in I$, $\mu_i(C) \leq \lambda_i[(B_i^\mu)^{-1}(C)]$ *for every subset C in A;*

(iii) *for each* $i \in I$, $\mu_i(D) \leq \lambda_i[(B_i^\mu)^{-1}(D)]$ *for every finite set D in A.*

Literally, the above theorem says that a distribution vector on the action space is an equilibrium distribution if and only if for any subset or any finite subset C of the actions, there are less players in each group playing the actions in C than

[7]More precisely, μ should be called an *equilibrium distribution vector*.

having a best response in C.[8] The special case that $|I| = 1$ and A is finite in Theorem 1 is the main result in Blonski (2005).

Our next result considers a situation where all the players in each group are *homogeneous*, that is, all the players in each subgroup share a common payoff function and a common action set. Before we state our result, let's define the concept of homogeneousness.

Definition 2. A large game U is said to have *countable homogeneous groups of players* if for each group $i \in I$, U_t and $K(t)$ do not change for all $t \in T_i$.

Theorem 2. *If a large game U has countable homogeneous groups of players, then the following statements are equivalent:*

(i) $\mu = (\mu_i)_{i \in I} \in \mathscr{M}(A)^I$ *is an equilibrium distribution;*

(ii) *for each $i \in I$, $\mu_i(C) \leq \lambda_i[(B_i^{\mu})^{-1}(C)]$ for every Borel set C in A;*

(iii) *for each $i \in I$, $\mu_i(F) \leq \lambda_i[(B_i^{\mu})^{-1}(F)]$ for every closed set F in A;*

(iv) *for each $i \in I$, $\mu_i(O) \leq \lambda_i[(B_i^{\mu})^{-1}(O)]$ for every open set O in A.*

Clearly, the homogeneousness assumption also implies that there are totally countably many payoffs in the game. Thus both Theorem 1 and 2 adopt a countability restriction which is either on the action space or on the payoff space. Our third result shows that if we replace the usual agent space by an atomless Loeb probability space,[9] then we can remove all the countability restrictions.

Theorem 3. *If the agent space $(T, \mathscr{T}, \lambda)$ of a large game U is an atomless Loeb probability space, then the result in Theorem 2 is still valid.*

This result is shown by applying a proposition on the distributional properties of correspondences on Loeb spaces from Sun (1996).

Both Theorems 2 and 3 implies that a given distribution vector is an equilibrium distribution if and only if for any Borel [open or closed] subset C of actions more players in each subgroup have a best response in C than play actions in C.

[8] If μ is an equilibrium distribution, then $\mu_i(C) = \lambda_i(f_i^{-1}(C)) = \lambda_i\{t \in T_i : f_i(t) \in C\}$, where $f_i \in B_i^{\mu}$, is the proportion of players playing the actions in C.

[9] The usage of hyperfinite Loeb spaces in modeling large games is systematically studied in Khan and Sun (1996, 1999). For more information about Loeb spaces, see also Loeb and Wolff (2000).

4 A counterexample

The previous section presents characterization results for large games restricted by the countability assumption on the action or payoff space or the Loeb assumption on the agent space. It would be good if we can obtain a similar characterization result for a general game without the above restrictions. However, as we can see from the example below, a similar characterization result does not hold for a general large game endowed with uncountable actions, uncountable payoffs and a Lebesgue measure space of agents. For simplicity, we only need to consider a case where there is no partition on the agent space.

Example 1. Consider a large game U in which the space of players is the Lebesgue unit interval $T = [0, 1]$ with the Lebesgue measure denoted by λ, the action set A is the interval $[-1, 1]$ and the payoffs are given by $U(t, a, \mu) = -|t - |a||$[10] where $t \in T$, $a \in A$ and $\mu \in \mathscr{M}(A)$, which, obviously, is a Carathéodory function.

Let the uniform distribution on $[-1, 1]$ be denoted by η. Thus, given η, the best response set for player t is:

$$B^{\eta}(t) = \arg\max U(t, a, \eta) = \{t, -t\}.$$

Let C be any Borel set in A and define $C_1 = C \cap (0, 1]$ and $C_2 = C \cap [-1, 0]$. Then

$$\begin{aligned}
\lambda[(B^{\eta})^{-1}(C)] &= \lambda(\{t \in T : B^{\eta}(t) \cap C \neq \emptyset\}) \\
&= \lambda\{t \in T : t \in C_1 \text{ or } -t \in C_2\} \\
&\geq \max\{\lambda(C_1), \lambda(C_2)\} \\
&\geq \frac{\lambda(C_1) + \lambda(C_2)}{2}.
\end{aligned}$$

Since η is the uniform distribution on $[-1, 1]$, $\eta(C) = \frac{\lambda(C)}{2} = \frac{\lambda(C_1 \bigcup C_2)}{2} = \frac{\lambda(C_1)+\lambda(C_2)}{2}$. Therefore, we have

$$\lambda[(B^{\eta})^{-1}(C)] \geq \eta(C).$$

Now we shall prove by contradiction that η can not be an equilibrium distribution.

Suppose η is an equilibrium distribution. Then, by definition, there exists a measurable selection f of B^{η} such that $\lambda f^{-1} = \eta$ and $f(t) \in B^{\eta}(t)$ for all $t \in T$.

[10] This payoff function is similar to a payoff function used in Khan et al. (1997).

Let $D = f^{-1}((0,1])$. Then

$$f(t) = \begin{cases} t, & t \in D \\ -t, & t \notin D. \end{cases}$$

Note that $f^{-1}(D) = \{t : f(t) \in D\} = \{t : t \in D\} = D$. Hence, $\lambda(D) = \lambda f^{-1}(D) = \eta(D) = \frac{\lambda(D)}{2}$, which is a contradiction. Therefore, η cannot be an equilibrium distribution. ■

5 Concluding remarks

The three characterization results presented in this paper are all in the same form and the characterizing counterparts are easy to understand. Therefore these results could be served as a practical tool to determine the pure-strategy Nash equilibria, and they also provide an alternative way of showing the existence of Nash equilibria by showing the existence of their characterizing counterparts. The counterexample shows that our characterization results are actually quite sharp.

6 Proofs

6.1 Proof of Theorem 1

To prove this theorem, we need the following lemma from Bollobas and Varopoulos (1974), which is an extension of the famous marriage theorem.

Lemma 1 (Bollobas and Varopoulos (1974))**.** *Let $(T, \mathcal{T}, \lambda)$ be an atomless probability space, I a countable index set, $(T_i)_{i\in I}$ a family of sets in $\mathcal{T}$, and $(\alpha_i)_{i\in I}$ a family of non-negative numbers. Then the following two statements are equivalent*

(i) $\lambda(\bigcup_{i\in D} T_i) \geq \sum_{i\in D} \alpha_i$ *for all finite subsets D of I;*

(ii) *there is a family $(S_i)_{i\in I}$ of sets in $\mathcal{T}$ such that for all $i, j \in I, i \neq j$, one has $S_i \subseteq T_i$, $\lambda(S_i) = \alpha_i$ and $S_i \cap S_j = \emptyset$.*

Proof of Theorem 1 For (i)⇒(ii), let μ be an equilibrium distribution. Then by definition, there exists a Nash equilibrium $f : T \to A$ such that $\mu =$

$((\lambda_i f_i^{-1})_{i\in I})$. Notice that for each $i \in I$, $f_i(t) \in B_i^\mu(t)$ for all $t \in T_i$. Thus, for any $i \in I$ and for every $C \subseteq A$,

$$\begin{aligned}\mu_i(C) &= \lambda_i(f_i^{-1}(C)) = \lambda_i(\{t \in T_i : f_i(t) \in C\}) \\ &\leq \lambda_i(\{t \in T_i : B_i^\mu(t) \cap C \neq \emptyset\}) = \lambda_i[(B_i^\mu)^{-1}(C)].\end{aligned}$$

It is clear that (ii) $\Rightarrow$ (iii).

It remains to prove (iii)$\Rightarrow$(i). To see this, fix any $i \in I$. Let $A := \{a_j\}_{j\in\mathbb{N}}$. For each $j \in \mathbb{N}$, let $\beta_j = \mu_i(\{a_j\})$ and $T_i^j := (B_i^\mu)^{-1}(\{a_j\}) = \{t \in T_i : a_j \in B_i^\mu(t)\}$. Let D be an arbitrary finite subset of $\mathbb{N}$. Observe that $(B_i^\mu)^{-1}(\bigcup_{j\in D}\{a_j\}) = \bigcup_{j\in D} T_i^j$. By assumption, we have $\sum_{j\in D}\beta_j = \mu_i(\bigcup_{j\in D}\{a_j\}) \leq \lambda_i(\bigcup_{j\in D} T_i^j)$. Thus we can apply Lemma 1 to assert that there exist, for all $j \in \mathbb{N}$, $S_j \subseteq T_i^j$ such that $\lambda_i(S_j) = \beta_j$ and $S_j \cap S_k = \emptyset$ for all $k \neq j$.

Now we define a measurable function $h_i : T_i \to A$ such that for all $j \in \mathbb{N}$ and for all $t \in S_j, h_i(t) = a_j$. Since, for any $j \in \mathbb{N}$, $t \in S_j$ implies that $a_j \in (B_i^\mu)(t)$, we have $h_i(t) \in B_i^\mu(t)$ for all $t \in T$. Furthermore, $\lambda_i(h_i^{-1}(\{a_j\})) = \lambda_i(S_j) = \beta_j = \mu_i(\{a_j\})$ for all $j \in \mathbb{N}$, which implies $\lambda_i h_i^{-1} = \mu_i$. Repeat the above arguments for all $i \in I$ and define a measurable function $h : T \to A$ by letting $h(t) = h_i(t)$ if $t \in T_i$. Thus it is clear that h is a pure strategy Nash equilibrium and $\mu = (\mu_i)_{i\in\mathbb{N}} = (\lambda_i h_i^{-1})_{i\in\mathbb{N}}$ is the equilibrium distribution induced by h. ■

6.2 Proof of Theorem 2

To prove this theorem, we need to use the following lemma which is well know in this field and can be obtained by appropriately adjusting the proof of Theorem 3.11 in Skorokhod (1956).

Lemma 2. *(Skorokhod, 1956, Theorem 3.11) Let $(T, \mathcal{T}, \lambda)$ be an atomless probability space and A a Polish space. Then for any $\nu \in \mathscr{M}(A)$ there exists a measurable function $f : T \to A$ such that $\lambda f^{-1} = \nu$.*

Proof of Theorem 2. Let $\mu = (\mu_i)_{i\in I}$ be an element of $\mathscr{M}(A)^I$. Firstly, we need to show that for each $i \in I$ and every $C \in \mathscr{B}(A)$, $(B_i^\mu)^{-1}(C)$ is measurable. To see this, fix any $i \in I$. The fact that U_t and $K(t)$ do not change for all $t \in T_i$ implies that $B_i^\mu(t)$ also does not change for all $t \in T_i$. Thus we can let $C_i := B_i^\mu(t)$

for all $t \in T_i$. Then, for any $C \in \mathscr{B}(A)$, we have

$$(B_i^\mu)^{-1}(C) = \{t \in T_i : B_i^\mu(t) \cap C \neq \emptyset\} = \begin{cases} T_i & \text{if } C_i \cap C \neq \emptyset; \\ \emptyset & \text{otherwise,} \end{cases}$$

which is measurable.

To see (i)⇒(ii), let $\mu = (\mu_i)_{i\in I}$ be an equilibrium distribution. By assumption, there exists a Nash equilibrium $f : T \to A$ such that $\mu = (\lambda_i f_i^{-1})_{i\in I} \in \mathscr{M}(A)^I$ and $f(t) \in B^\mu(t)$ for all $t \in T$. Therefore, for any $C \in \mathscr{B}(A)$,

$$\begin{aligned} \mu_i(C) &= \lambda_i f_i^{-1}(C) = \lambda_i(\{t \in T_i : f_i(t) \in C\}) \\ &\leq \lambda_i(\{t \in T_i : B_i^\mu(t) \cap C \neq \emptyset\}) \\ &= \lambda_i[(B_i^\mu)^{-1}(C)]. \end{aligned}$$

It is clear that (ii) ⇒ (iii).

To see (iii) ⇒ (iv), let O be an open set in A. Then there is an increasing sequence $\{F_n\}_{n=1}^\infty$ of closed sets in A such that $O = \bigcup_{n=1}^\infty F_n$. For each n, we have $(B_i^\mu)^{-1}(F_n) \subseteq (B_i^\mu)^{-1}(O)$, which implies that $\mu_i(F_n) \leq \lambda_i[(B_i^\mu)^{-1}(F_n)] \leq \lambda_i[(B_i^\mu)^{-1}(O)]$. Thus, $\mu_i(O) \leq \lambda_i[(B_i^\mu)^{-1}(O)]$.

It remains to show (iv) ⇒ (i).

Recall that for all $i \in I$, the set $C_i := B_i^\mu(t)$ for any $t \in T_i$ is compact and hence also complete and separable. Fix any $i \in \mathbb{N}$. By the fact that the set $(A - C_i)$ is open, we have that

$$1 - \mu_i(C_i) = \mu_i(A - C_i) \leq \lambda_i[(B_i^\mu)^{-1}(A - C_i)] = 0, \tag{1}$$

which gives $\mu_i(C_i) = 1$ for all i. Therefore, by Lemma 2, there exists a measurable function $f_i : T_i \to C_i$ such that $\mu_i = \lambda_i f_i{}^{-1}$. By definition, $f_i \in B_i^\mu$.

Define $f : T \to A$ by letting $f(t) = f_i(t)$ for all $t \in T_i$ and all $i \in I$. Thus f is a measurable selection of B^μ and $\mu = (\mu_i)_{i\in I} = (\lambda_i f_i^{-1})_{i\in I}$ is an equilibrium distribution. ∎

6.3 Proof of Theorem 3

To prove this theorem, we need to use the following lemma in Sun (1996).

Lemma 3. *(Sun, 1996, Proposition 3.5) Let Γ be a closed valued measurable correspondence from an atomless Loeb probability space $(\Omega, \mathscr{F}, P)$ to a Polish space X. Let ν be a Borel probability measure on X. Then the following are equivalent:*

(i) *there is a measurable selection f of Γ such that $Pf^{-1} = \nu$;*

(ii) *for every Borel set C in X, $\nu(C) \leq P(\Gamma^{-1}(C))$;*

(iii) *for every closed set F in X, $\nu(F) \leq P(\Gamma^{-1}(F))$;*

(iv) *for every open set O in X, $\nu(O) \leq P(\Gamma^{-1}(O))$.*

PROOF OF THEOREM 3. For any $i \in I$, notice that B_i^{μ} is a compact valued (and hence closed valued) measurable correspondence from an atomless Loeb probability space $(T_i, \mathscr{T}_i, \lambda_i)$ to the Polish space A. Thus, by applying Proposition 3.5 in Sun (1996) to B_i^{μ}, we see that $\mu_i = \lambda_i f_i^{-1}$ for some f_i being a measurable selection of B_i^{μ} if and only if for every Borel (closed, or open) set H in A, $\mu_i(H) \leq \lambda_i[(B_i^{\mu})^{-1}(H)]$.

Since the above result holds for all $i \in I$, thus $\mu = (\mu_i)_{i \in I}$ is an equilibrium distribution if and only if for each $i \in I$ and every Borel (closed, or open) set H in A, $\mu_i(H) \leq \lambda_i[(B_i^{\mu})^{-1}(H)]$. ■

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