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Continuous Representation of Preferences

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1. INTRODUCTION

The problem of constructing a representation of preferences appears when it is needed to assign utility functions to preferences in a well-defined way, since in general many utilities will represent the same preference. In models where the preferences of the agents vary, and the behaviour of the model depends on limiting properties of preferences, it is important that utilities be assigned to preferences so as to vary continuously as preferences change.

In this paper we study a utility representation for preferences, and we prove its continuity, using a topology for preferences introduced by Chichilnisky (1977). Such utility representations were constructed under restrictive conditions first by Y. Kannai (1970), W. Hildenbrand (1970), W. Neuefeind (1972) and more recently by K. Mount and S. Reiter (1974, 1975). Our results are related more closely to those of Mount and Reiter (1974 and 1975). Starting from restrictive conditions, these works have attempted to enlarge the class of preferences that could be continuously represented.[1] The class of preferences studied here is much larger than those considered earlier, and it therefore applies to a wider class of problems. Our preferences include ones which are not necessarily convex or monotone, and which may be locally satiated; furthermore, no completeness of the preferences is required. The assumption made in Mount and Reiter (1975) of the existence of an ε-threshold is also not required here. These results are possible due to the properties of the order topology introduced in Chichilnisky (1977). As we shall now discuss, this topology has quite desirable features for the study of preferences, and these make it a natural choice for the problem at hand.

The order topology is finer than the Hausdorff metric, and, as opposed to the Hausdorff metric, it is sensitive to the measure of the graphs of the preferences. This sensitivity is the main property that allows one to prove continuity of the representation, which is based on measure.

2. CONTINUITY OF THE REPRESENTATION

The Mount and Reiter indicator (constructed in Mount and Reiter 1974 and 1975) has certain desirable properties. It applies to a large class of preferences such as those that appear in models with public goods: non-convex, with possible satiation, not necessarily monotone. In addition, it has the property that if two different agents α and β should have the same lower contour and indifferent set for two different commodity points, the function gives those points the same utility.[1] However, Mount and Reiter showed that their indicator is *not* continuous on the full class of continuous preferences endowed with the closed convergence topology. They find that the problem to obtain continuity for a larger class of preferences is the fact that the Hausdorff metric and the closed convergence topology generally used for preferences are too " coarse "; in particular, they count as neighbouring agents some of which are quite different from the point of view of the utility indicator which is based in measure. They also indicate that a finer topology for preferences seems appropriate, Mount and Reiter (1975). These types of utility indicators can be described intuitively as real valued functions which assign to a point x in the commodity space and a preference θ the measure of the set of commodities that are less preferred than x according to the preference θ. Thus, continuity of such utilities depends in part on the sensitivity of the topologies on the space of preferences to the measures of the lower contour sets for the preferences' graphs. It is known that the Hausdorff metric (Debreu (1969)) or the closed convergence topology (Hildenbrand (1974)) are not sensitive enough in this sense (see Chichilnisky (1977), pp. 167–168).

The order topology was introduced for the study of a space of continuous preferences. This topology has certain desirable features; in particular, it is finer than both the Hausdorff metric and the closed convergence topology, when restricted to their common spaces of definition, and the measure of the graphs of the preferences with this topology is actually a continuous function on the space of continuous preferences (Chichilnisky (1977), Theorem 1). In the following we consider both an order topology and a refinement of the Hausdorff or closed convergence topologies by an order topology and we show that the utility indicator of Mount and Reiter (1975) can be extended to a class of all continuous preferences satisfying certain minimal topological restrictions on their graphs. No convexity, monotony or local non-satiation is required on the preferences. Also, neither completeness nor the existence of an ε-threshold as in Mount and Reiter (1975) is required of the preferences so that our class of preferences is strictly wider than that of Mount and Reiter (1975).

3. NOTATION AND DEFINITIONS

Let μ_1 be a bounded measure on R^l which is absolutely continuous with respect to the Lebesgue measure and in which open sets have positive measure, and let $\mu = \mu_1 \times \mu_1$ be the product measure on $R^l \times R^l$. Let v be the measure induced by μ on the *diagonal* $D = \{(x, y) \in R^l \times R^l : x = y\}$.

A *preference* or *relation* on R^l is given by its graph, a subset θ of $R^l \times R^l$. For any θ in $R^l \times R^l$, $\bar{\theta}$ indicates the closure of the set θ; θ^0 indicates its interior and $\partial\theta$ the boundary of θ; $\theta_1 - \theta_2$ the set of points in θ_1 not in θ_2; and $\theta_1 \Delta \theta_2$ the set $(\theta_1 - \theta_2) \cup (\theta_2 - \theta_1)$. For a given θ, $(u, v) \in \theta$ when v is preferred or indifferent to u according to θ. Define the indifference relation $I(\theta)$ as the set $\{(u, v) \in \theta : (v, u) \in \theta\}$. We say that $u \in R^l$ is in the *consumption set* of θ denoted $C\theta$ if either $(u, v) \in \theta$ or $(v, u) \in \theta$ for some $v \in R^l$.

The next step is to define the space of preferences and its topology. The space of continuous preferences is defined by: $\theta \in \mathcal{O}$ if and only if θ is closed and not empty, $\theta \cap D = \overline{(\theta \cap D)^0}$ and $v(\partial(\theta \cap D)) = 0$. The regularity condition $\theta \cap D = \overline{(\theta \cap D)^0}$ is needed for the definition of the order topology σ; this is discussed in more detail in Chichilnisky (1977). The condition $v(\partial(\theta \cap D)) = 0$ is used in the proof of the theorem below to insure the desired sensitivity of the topology to the measure of the graphs of the indifference relations associated to the preferences. In order to define a topology τ_0 for the space of preferences $\mathcal{O}$ we use the *order topology* σ (Chichilnisky (1977)). This topology is defined on the set $\mathcal{B}$ of all closed subsets θ of R^l with $\overline{\theta^0} = \theta$, by giving a sub-base of neighbourhoods $\mathcal{S} = \{U_{\alpha,\beta}\}$, where $U_{\alpha,\beta} \in \mathcal{S}$ if $U_{\alpha,\beta} = \{\theta \in \mathcal{S} : \theta \subset \theta_\alpha^0$ and $\theta^0 \supset \theta_\beta$, θ_α and θ_β in $\mathcal{B}\}$. Another topology used in the literature is the *closed convergence* topology; it is defined on closed subsets θ of R^l (as a natural extension of the Hausdorff metric on compact commodity spaces) by giving a sub-base $\mathcal{S}c$ where $U \in \mathcal{S}c$ if

$$U = \{\theta : \theta \cap K = \varnothing \quad \text{and} \quad \theta \cap G \neq \varnothing\}$$

where K is a compact subset and G an open subset of R^l (see Hildenbrand (1974)). Let τ_c denote the closed convergence topology on $\mathcal{O}$. We now define τ_0 by the convergence rule:

$$\{\theta^j\}\xrightarrow{\tau_0}\theta\Leftrightarrow\{\theta^j\}\xrightarrow{\tau_c}\theta \quad \text{and} \quad \{\theta^j\cap D\}\xrightarrow{\sigma}\theta\cap D.$$

If $\theta\cap D = \varnothing$, then

$$\{\theta^j\}\xrightarrow{\tau_0}\theta \quad \text{if and only if } \{\theta^j\}\xrightarrow{\tau_c}\theta$$

since σ-convergence is not defined when $\theta\cap D = \varnothing$.) Alternatively, a topology τ based only on the order topology on both the graphs of the preferences and their intersection with the diagonal is defined on the subspace of preferences θ in $\mathcal{O}$ with $\bar{\theta}^0 = \theta$ as follows:

$$\{\theta^j\}\xrightarrow{\tau}\theta\Leftrightarrow\{\theta^j\}\xrightarrow{\sigma}\theta \quad \text{and} \quad \{\theta^j\cap D\}\xrightarrow{\sigma}\theta\cap D.$$

In Chichilnisky (1977) it was shown that the order topology σ is strictly finer than the closed convergence topology when restricted to their common domains. An example of a sequence of preferences $\{\theta^j\}$ that converges in the Hausdorff metric to another θ but such that $\{\theta^j\}\nrightarrow\theta$ in the order topology is given in Chichilnisky (1977), Figure 1, page 168. This example shows also the insensitivity of the Hausdorff metric to the measure of the graphs of the preferences (which makes the closed convergence and Hausdorff topologies less useful in our case, as explained above). τ_0 is therefore strictly weaker than τ, and thus, if an indicator on preferences is proven to be continuous under the topology τ_0, it follows that it will also be continuous under the order topology τ.[2]

We now define a representation, which was introduced in Mount and Reiter (1975). Let $L(\theta, x)$ be the lower contour set of θ, i.e. $L(\theta, x) = \{y\colon (y, x)\in\theta\}$, and let

$$S(\theta, x) = \{(v, u)\in\theta\colon u\in L(\theta, x)\}.$$

Let $I(\theta, y)$ denote the set $\{u\in R^l\colon (u, y)\in I(\theta)\}$ and

$$\mathscr{I}(\theta) = \bigcup_{y\in R^l} I(\theta, y)^0.^3$$

The map $U\colon \mathcal{O}\times R^l\to R^+$ is defined by

$$U(\theta, x) = \mu\{S(\theta, x)\}-\mu\{S(\theta, x)\cap(R^l\times\mathscr{I}(\theta))\} = \mu\{S(\theta, x)\cap(R^l\times\mathscr{I}(\theta)^c\}.^4$$

The following conditions on preferences in $\mathcal{O}$ will now be assumed:

(C1) θ is reflexive and transitive, the consumption set $C\theta$ is convex and the graph of θ is connected as a set.

Note that here we assume from the preference θ that the graph of the set θ is connected, i.e. that the graph cannot be the union of sets which are both open and closed. In general, this corresponds to the intuitive fact that if $(x, y)\in\theta$ and $(z, u)\in\theta$ then there is a path starting in (x, y) and ending in (z, u) which is contained in the set θ. Note that this differs from the other uses of the word " connected " for preferences: for example, the preferences considered here *are not necessarily complete* even though they are reflexive and transitive and their graph is a connected set.

(C2) (Cantor condition) For each open set V in $C\theta$ with $V\times V\not\subset I(\theta)$,

$$\mu\{\theta^c\cap(C\theta\times\mathscr{I}(\theta)^c)\cap(C\theta\times V)\}\neq 0.$$

This is condition (ii) of Mount and Reiter (1975), translated to our context. Note that condition (C2) is satisfied in particular when $C\theta$ has a non-empty interior and $\mathscr{I}(\theta)^c$ is open, since by the definition of $\mathcal{O}$, θ^c is open, and therefore the set whose measure is considered in (C2) will then be open.

4. THE RESULT

Let Γ be the subset of $\mathcal{O}\times R^l$ defined by: $(\theta, x)\in\Gamma$ if θ is in $\mathcal{O}$ and satisfies (C1) and x is in the consumption set $C\theta$, and let $\bar{\mathcal{O}}$ be defined as the set of preferences in $\mathcal{O}$ which satisfy

(C1) and (C2).[5] The next result extends Theorem 1 of Mount and Reiter (1975) on continuous representation of preferences. No convexity or local non-satiation are required here on the preferences. In addition here we do not require the conditions of completeness on preferences or the existence of an ε-threshold required in Mount and Reiter (1975), so that our results prove continuity of the representation on a much larger space of preferences.

Theorem. *For each preference θ in $\bar{\mathcal{O}}$ the utility representation given by the function $U(\theta, \cdot)$ is order preserving on the consumption set $C\theta$ according to θ; i.e.*

$$U(x, \theta) < U(y, \theta) \quad \text{if } (x, y) \in \theta \text{ and } (y, x) \notin \theta$$

and

$$U(x, \theta) = U(y, \theta) \quad \text{if } (x, y) \text{ and } (y, x) \text{ are both in } \theta.$$

If (θ^j, x^j) is a sequence of preferences and commodity vectors in Γ which converges to (θ, x) in Γ, then

$$\lim_j U(\theta^j, x^j) = U(\theta, x).$$

Proof.[6] In order to prove this result, we shall rely in part on arguments of Mount and Reiter (1975) and of Chichilnisky (1977). As shown in Theorem 1 of Mount and Reiter (1975), to prove the desired continuity of U it is necessary to prove that

$$\{\theta^j\} \xrightarrow{\tau_0} \theta \text{ in } \bar{\mathcal{O}} \text{ implies } v[(I(\theta^j)^0 \cap D) \triangle (I(\theta)^0 \cap D)] \to 0. \qquad \ldots(1)$$

In Lemma 5 of Mount and Reiter (1975) it was proven that within the space of closed graph preferences θ with $\overline{\theta^0} = \theta$ and $\mu(\partial\theta) = 0$, if $\{\theta^j\} \to \theta$ in the order topology, then

$$\lim_j \mu(\theta^j \triangle \theta) = 0. \qquad \ldots(2)$$

Since by assumption (C1) θ is reflexive it follows that $I(\theta) \cap D = \theta \cap D$.

$$\{\theta^j\} \xrightarrow{\tau_0} \theta \text{ implies } \{\theta^j \cap D\} \xrightarrow{\sigma} \theta \cap D$$

by definition of τ_0. Now, using (2) above applied to the sequence of relations $\{\theta^j \cap D\}$ and $\theta \cap D$, this implies $v\{(\theta^j \cap D) \triangle (\theta \cap D)\} \to 0$ which, by reflexivity, is equivalent to

$$v\{[I(\theta^j) \cap D] \triangle [I(\theta) \cap D]\} \to 0. \qquad \ldots(3)$$

Now, for any two sets A and B, $\partial(A \cap B) \supset \partial A \cap B^0$, therefore

$$\partial(I(\theta) \cap D) \supset \partial[I(\theta)] \cap D^0. \qquad \ldots(4)$$

Since θ is reflexive, the condition $v(\partial(\theta \cap D)) = 0$ is equivalent to $v(\partial(I(\theta) \cap D))) = 0$. By (4), $v(\partial(I(\theta) \cap D))) = 0$ implies $v(\partial[I(\theta)] \cap D^0) = 0$, so that

$$v(\partial[I(\theta)] \cap D) = 0. \qquad \ldots(5)$$

From (3) and (5) we immediately obtain (1), and thus, the admissibility of Definition 2 required in Mount and Reiter (1975) has been proven to be satisfied here. Therefore by the definition of $\bar{\mathcal{O}}$ and by Lemma 4 of Mount and Reiter (1975), $U(\theta, \cdot)$ is monotone with respect to θ in the consumption set $C\theta$. To obtain joint continuity of U, recall that τ_0 and τ are both finer than the closed convergence topology of Debreu (1969), which implies that $\theta^j \to \theta$ in the closed convergence topology also. Therefore, by the above results, we can apply Theorem 1 of Mount and Reiter (1975) to our case. This completes the proof. ||

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NOTES

1. It appears that a non-constructive proof of existence of a jointly continuous utility indicator for the class of continuous preferences can be obtained from an application of a result of Michael (1956), see A. Mass-Colell (1976). However, as discussed in Mount and Reiter (1975) (Introduction and Section 4) such a utility indicator based on Michael's selection theorem may not have certain desirable properties such as, for instance, that its value at a preference commodity couple (θ_1, x) be equal to its value at another (θ_2, y) if the lower contour set and indifference set θ_1 at x is equal to the lower contour set and indifference set of θ_2 at y (a form of the axiom of independence of irrelevant alternatives).

2. In a previous version of this paper we considered the topology τ only. A referee suggested that the result could be extended when the (weaker) topology τ_0 was used.

3. Here $I(\theta, y)^0$ denotes the interior of $\mathscr{I}(\theta, y)$ (not the relative interior). Note that $\mathscr{I}(\theta) = \Pi_1(I(\theta)^0)$, when θ satisfies the condition $(C1)$ below, where Π_1 denotes the projection which carries a couple (y, z) to its first component y, see Lemma 1, page 7, of Mount and Reiter (1975).

4. This is definition 1 of Mount and Reiter (1975), in our context. Note that the indifference classes of the preferences satisfying the assumptions of Neuefeind (1972) have measure zero and hence, it follows that for the preferences of Neuefeind (1972), $U(\theta, x) = \mu(S(\theta, x))$, see Mount and Reiter (1975).

5. The class of preferences satisfying the assumptions of Neuefeind (1972) also satisfy (C2), see Mount and Reiter (1975), Section 3.

6. This present shorter line of proof has been suggested by a referee.

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