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By

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Abstract

The received theory of aggregation has been erected on certain fundamental hypotheses. One of them is that producer durables deteriorate exponentially, which implies that their replacement is proportional to the corresponding capital stocks. However the proportionality hypothesis conflicts with most of the available theoretical and empirical evidence. So an effort to relax it is long overdue. To this end the present paper investigates the conditions for consistent aggregation in a two-sector vintage capital model with exogenous technological change and endogenous useful lives. In the model aggregation is achieved by adaptation of the procedure first suggested by Haavelmo (1960). From the simulations of the solution with data from the United States in the post-war period it is found that the conventional approach to aggregation may be responsible for significant biases in the measurement of the economy-wide capital stock.

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1. Introduction

Producer durables last for many years. But as more and more services are extracted from them through utilization or pure deterioration and technological obsolescence, their earning capability declines until eventually they are scrapped or replaced. Hence efforts to derive an index of "capital-in-general" from the wide variety of durables that are usually employed in the economy should take into consideration the way in which they deteriorate. Yet even a cursory perusal would suffice to reveal that the relevant literature either ignores the deterioration of producer durables altogether or it presumes that it proceeds at a constant rate per unit of time, which implies that their replacement is modeled as a constant proportion of the outstanding stocks. To ascertain that this is the case, consider the contributions by Solow (1955-1956) and Fisher (1965). The necessary and sufficient conditions that they derive for constructing an index of aggregate capital emanate from models in which the capital goods are presumed to be infinitely durable, thus abstracting completely from the difficulties associated with their decay. On the contrary, when Samuelson (1962) introduced the model of "surrogate capital" and Green (1966) revisited the problem of aggregation both postulated that the physical depreciation is always proportional to the outstanding physical stocks.

Certainly the proportionality hypothesis had a lot to recommend it. Jorgenson (1965) supported it with appeals to renewal theory. The results by such eminent economists as Preinreich (1940), Terborgh (1952) and Smith (1961) pointed favorably in its direction; and not less attractive was that it facilitated the construction of models that permitted simple and elegant solutions.[1] However, despite its overwhelming acceptance by economic theorists and practitioners, the proportionality hypothesis has at least one fundamental shortcoming. This is its implication that the decay of capital goods is invariant with respect to their ages. As a result, soon after it was introduced in the 1950s it became clear from the writings of Haavelmo (1960) and others that this conceptualization rendered aggregation feasible by ignoring the durability of capital goods. So while a group of academic economists headed by Robinson (1953-54, 1959) rejected the possibility of deriving an aggregate index of "capital-in-general", another group continued to toil over the issues involved on the conviction that some satisfactory middle ground was possible.

As could be expected the researchers in the latter group built on the solid finding by the great Austrian and Swedish economists that the time structure of aggregate capital is indispensable for explaining the contribution to productivity, income shares, business cycles, etc., of producer durables. To liberate themselves from the narrow point-input point-output analyses of yesteryears, initially they abandoned the concept of the *period of production* and instead focused on the *longevity* or *useful life* of capital goods. A characteristic example of the progress that was accomplished on account of this change is found in the contributions by Blitz (1958) and Westfield (1958), who showed how optimal longevity of capital could be computed in the presence of technological obsolescence. Two years later Haavelmo (1960) made the first ever attempt to achieve consistent aggregation of two capital goods that differ in quantity and longevity.[2] But even though his approach was promising, his analysis was restricted by two assumptions. Namely, that no technological change took place and that the useful lives of the aggregated capital goods were exogenously given. Thus further progress required that these two assumptions be relaxed. My objective here is to demonstrate that considerable headway can be made in this direction by combining a two-sector generalization of the one-sector capital vintage model presented by Brems (1968) with the approach to aggregation suggested by Haavelmo (1960).

To this end, I consider an economy with two sectors. The representative firms *X* and *Y* that operate in them are characterized by three fundamental differences. The first of them is that, whereas firm *X* supplies electricity, which is a necessity with relatively inelastic demand that lasts forever, firm *Y* supplies tennis rackets, which is a luxury with highly elastic demand that may vanish at any time due to shifts in tastes. The second difference springs from the implication that, because of the inherent difference in the nature of their products, the two firms are bound to view their re-investment opportunities differently. Firm *X* would plan for the indefinite future by adopting a capital policy of perpetual replacements, whereas firm *Y* would adopt a scrapping policy, which would give it an option to decide at the end of the useful life of its current investment whether to exit or reinvest, depending on the demand for tennis rackets at that time.[3] Finally, the third difference is that technical progress increases the productivity of more recent vintages of the durables in each sector at different constant and exogenous rates. Otherwise firms *X* and *Y* are similar. In particular, they face downward sloping demand curves, implying that they behave as monopolists. They deter other firms from entering into their markets to take advantage of the higher productivity of newer durables by applying a pricing rule that transfers all benefits from technological change to final consumers; and last, but not list, while the durables they build internally are fixed in the sense that they cannot be moved from the one sector to the other, workers move freely in the economy.

Due to the structural and behavioral differences of firms *X* and *Y*, the model that emerges leads to different useful lives for their durables. To be sure, drawing on Bitros and Flytzanis (2005), this finding would be expected even if the firms differed only with respect to their capital policies, i.e. replacement vs. scrapping. But in the richer modeling environment of this paper the differences in the useful lives arise also because the two firms operate in markets with different elasticities of demand and different rates of embodied technical change. Thus, as soon as the attention turns from microeconomics to macroeconomics, the analysis confronts the question of how to aggregate the two durables, since: a) they are not substitutes and hence their physical quantities cannot be translated into an index of homogeneous units; b) older vintages differ from newer vintages because the latter embody the more recent advances in technology, and c) depending on the elasticities of demand for electricity and tennis rackets, the rates of embodied technological change, and other market parameters, the durables of firm X may last longer than those of firm Y. To tackle it, the investigation starts from the realization that at the sectoral level the quantities of the two durables are expressed in uniform monetary values of constant prices. This implies that, if they did not differ in any other respect, adding their purchase values would give an index of the quantity of "capital-in general". But the two durables differ also in quality as well as durability and this approximation would be open to serious objections from both the theoretical and the empirical standpoints. Therefore, taking into consideration that the useful lives of the two durables account endogenously for the differences in their quality, the model is endowed with a Haavelmo (1960, 95-102) type mechanism, which, by expressing the two durables in units of standard durability, permits their aggregation in a consistent manner. Moreover, drawing on the results from a comparative evaluation of the traditional and the proposed approach to aggregation, it is established that insisting on the former may be responsible for significant biases in the estimates of the economy-wide capital stock.

The rest of the paper is organized as follows. Section 2 initially highlights the key role of the proportionality hypothesis in the dominant theory of aggregation; then it conducts a brief survey of the available literature, and lastly it makes a case for replacing the proportionality hypothesis with models linking the decay of producer durables to their ages. Section 3 suggests an approach for doing so by proposing a two-sector vintage capital model with exogenous technological change and endogenous useful lives. Section 4 characterizes the properties of its solution for the aggregate capital stock, and Section 5 contains a summary of the main findings and conclusions.

2. The theory of aggregation and the proportionality hypothesis

Before the mid 1960s, the evidence in support of the proportionality hypothesis was mostly indirect in the sense that no results had been reported linking it to the necessary and sufficient conditions for achieving consistent aggregation. At the time economic theorists adopted it because all indications from neighboring areas of research were in its favor. Direct evidence started to emerge with the contributions by Whitaker (1966) and Hall (1968) that highlighted the conditions for aggregation in a one-sector vintage model of capital. Just to indicate why the proportionality hypothesis proved paramount in this respect, consider the following analysis.

In the absence of technological change of any kind, an aggregate capital stock J is said to exist at time *t*, if it is computed according to the formula:

$$J(t) = \int_{t-T}^{t} \psi(t,\upsilon)I(\upsilon)d\upsilon = \int_{t-T}^{t} \psi(t-\upsilon)I(\upsilon)d\upsilon\,, \tag{1}$$

where $I(\upsilon)$ represents gross investment at time $\upsilon < t$, the function $\psi(t-\upsilon)$ is an approximation to the function $\psi(t,\upsilon)$ and stands for the deterioration or loss of output efficiency by an investment aged $t-\upsilon$ periods, and T is a constant useful life uniform across all vintages. If $S_T(t)$ denotes the durables of age T that are scrapped at t and $\psi'(t-\upsilon)$ represents the derivative of $\psi(t-\upsilon)$ with respect to time we have:

$$\frac{dJ}{dt} = I(t)\psi(0) - S_T(t) + \int_{t-T}^{t} \psi'(t-\upsilon)I(\upsilon)d\upsilon\,. \tag{2}$$

Next, introducing the normalization $\psi(0)=1$ and expressing the integral in (2) as:

$$\int_{t-T}^{t} \psi(t-\upsilon)\frac{\psi'(t-\upsilon)}{\psi(t-\upsilon)}I(\upsilon)d\upsilon = \kappa(t,T)J(t)\,, \tag{3}$$

where the function $\kappa(t,T)$ describes the percentage deterioration of the durables of all ages that remain in the capital stock, expression (2) can be written as:

$$\frac{dJ(t)}{dt} = I(t) - S_T(t) + \kappa(t,T)J(t)\,. \tag{4}$$

From this it follows that the existence of $J(t)$ depends on $\kappa(t,T)$, and hence eventually on the time, *t*, the useful life, T, and the form of the function κ. Therefore, assuming that we know or can approximate the form of this function, any time the useful life T changes, the differential equation in (4) will give a different stock of capital, thus prohibiting aggregation without taking into consideration the useful life of aggregated durables. But now consider the very special case where the deterioration function takes the form $\psi(t-\upsilon) = \alpha e^{-\delta(t-\upsilon)}$, in which α and δ are constants independent of υ. Then the producer durables deteriorate by radioactive decay, thus implying that $S_T(t)=0$ and expression (3) yields:

$$\int_{t-T}^{t}\psi(t-\upsilon)\frac{\psi'(t-\upsilon)}{\psi(t-\upsilon)}I(\upsilon)d\upsilon=-\delta J(t)\,. \tag{3'}$$

Consequently, substituting (3′) into (4) we obtain the well-known differential equation:

$$\frac{dJ(t)}{dt}=I(t)-\delta J(t)\ . \tag{5}$$

This proves that the aggregate capital stock $J(t)$ can be defined uniquely if and only if deterioration for all vintages of investment proceeds exponentially at the constant rate δ.[4]

The link of the proportionality hypothesis to the conditions for consistent aggregation in the case of differentiated producer durables was highlighted by Zarembka (1975), who investigated the question of collapsing a static multisectoral economy into a two-sector model of heterogeneous capital. In particular, and because of the importance of the issue involved, here is how he summarized his findings:

> "… In a steady-state model it is reasonable to assume that depreciation[5] is some constant fraction of the stock of a particular capital good [my note: in a footnote he refers to Jorgenson (1974)] and that the rate does not vary substantially according to the goods produced (with some exceptions)… But if the depreciation rate varies substantially among capital goods, then the reduction of equation (10) to (11) in the capital goods sectors does not obtain (and similarly for the consumer goods sectors). Therefore, in comparing steady-state equilibria, it is not possible to aggregate capital goods with different depreciation rates (and thus one reason why capital in structures and equipment needs to be disaggregated)" (p. 113).

On this account in order to collapse a multisectoral economy into two sectors, one producing capital goods and another producing consumer goods, the capital goods within each sector must depreciate at constant uniform rates. However, as full aggregation in the sense of reducing the multisectoral economy into one sector fell outside the scope of his research, this author stopped short of considering the interesting question whether the depreciation rates in the two sectors had to be equal or not for their aggregation.

This question was raised and answered a year later by Brown and Chang (1976) in the confines of a static general equilibrium model of production. More specifically, these authors investigated the conditions for intrasectoral, intersectoral and full aggregation and found that, as long as the depreciation of capital goods is proportional to the respective stocks, such aggregation can be achieved even if the depreciation rates are unequal. Apparently this finding contradicted the results obtained by Zarembka (1975). But it ascertained beyond any doubt that that the proportionality hypothesis is one of the cornerstones of the received theory of aggregation.

Having arrived at this conclusion it was natural to assess its standing in economic theory and empirical research. The results of this endeavor are presented in Bitros (2008). From this survey it follows that the proportionality hypothesis is in conflict with both theoretical and empirical evidence. In particular, the dominant view of replacement theorists is that the conditions for a constant replacement/capital stock ratio are highly restrictive and unlikely to hold in reality. In the area of economic growth and business cycles the hypothesis is being abandoned in favor of an economic theory of replacement. All theory of industrial organization is based on the view that how sturdy producer durables are built is decided at the time of their production on the basis of economic criteria and that their useful lives are determined eventually by such deliberate economic processes as the intensity of utilization and maintenance; and last, but not least, the implication that firms cannot affect the manner in which their durables decay is completely alien with the modes of thinking in neighboring fields like capital budgeting, operations management and accounting. Moreover, the empirical evidence shows that the replacement investment/capital stock ratio varies over the business cycle under the influence of key economic variables; the age-price profiles of durables do not support the view that depreciation rates are geometric, and the scrappage rate is determined to a significant extent by market forces. Therefore, the abandonment of the proportionality hypothesis is long overdue. How this may be done and what would be the implications for aggregation theory are the issues of focus below.

3. Aggregation with exogenous technological change and endogenous useful lives

Once the proportionality hypothesis is abandoned, in order for the aggregate capital stock $J(t)$ to exist and be well defined we need an analytical framework to explain the determination of the useful live of capital stock, T. For then, assuming that the form of the function $\kappa(t,T)$ is known or can be approximated, the terms $\kappa(t,T)K(t)$ and $S_T(t)$ in expression (3) are fully determined. The objective in this section is to present a model in which the useful live of producer durables is decided along with other key variables in the presence of exogenous but embodied technological change.

3.1 The model

Before embarking on the presentation of the model, it is convenient to clarify the meaning of the symbols used to denote its variables and parameters. This is done in Table 1 below.

(Please insert here Table 1)

Now consider an economy with two firms and any number of workers. Each firm consists of two lines of production, one constructing an intermediate durable called capital solely by means of labor[6] and another producing a final good by combining each unit of capital with one unit of labor. Let firm X produce electricity and firm Y produce tennis rackets. In year υ, firm X uses electric generators capable of producing $K_X(\upsilon)$ units of electricity, whereas firm Y employs a lathe capable of producing $K_Y(\upsilon)$ thousand rackets per year. Usage does not wear capital because its effects are exactly offset by maintenance. But from the one period to the next $K_X(\upsilon)$ and $K_Y(\upsilon)$ become more productive because newer vintages of capital embody the most recent advances in science and technology. So to capture the impact of embodied technological change, let the productivity of newer vintages of capital increase respectively at the constant and exogenous rates μ_X and μ_Y. Then newer vintages of capital would present a competitive advantage to firms that might wish to enter into the two sectors. For this reason, assume finally that to deter potential entrants firms X and Y reduce the prices of their products at the rate of technological change. The issue I want to investigate here is how to derive a consistent measure of economy-wide capital. To this end I proceed as follows.

3.1.1 Microeconomics

The representative firms that operate in the two sectors of the economy are characterized by the fundamental differences that were described in the introduction. Hence, I will analyze their economics separately.

Representative Firm X

Assume that firm X faces a demand curve for electricity of the constant elasticity type:

$$X(\upsilon) = N_X[P_X(\upsilon)]^{\eta_X} , \tag{6}$$

where $\eta_X < -1,\ X(\upsilon) > 0,\ N_X > 0,\ P_X(\upsilon) > 0$.

During year υ the firm uses $K_X(\upsilon)$ units of electricity generating capacity, all of which are equally productive because they embody the same technology. Hence, let its production function take the form:[7]

$$X(\upsilon) = \frac{1}{b_X(\upsilon)} K_X(\upsilon) \Rightarrow b_X(\upsilon) = \frac{K_X(\upsilon)}{X(\upsilon)} . \tag{7}$$

As it will be useful below, observe that the inverse of the capital-output coefficient, $b_X(\upsilon)$, gives the marginal productivity of capital of vintage υ.

Electricity capacity built after year υ is expected to be more productive because of technological progress. So to allow for this consideration, and following the demonstration in Appendix A, let the capital-output coefficient of firm X decline through time as follows:[8]

$$b_X(t) = b_X(\upsilon)e^{\mu_X(t-\upsilon)}, \tag{8}$$

where $\upsilon < t$ and μ_X<0.

Next, regarding the minimum amount of labor required to build a unit of electricity generating capacity, I assume that:

$$\beta = M_X[b_X(0)]^{\gamma}, \tag{9}$$

where $M_X > 0$ and $\gamma < -1$. This implies that, if the firm tried to cut its capital-output coefficient by half, the minimum amount of labor required to build a unit of electricity generating capacity would increase by more than half, thus prohibiting the firm from growing to such an extent that it might become a monopoly in the whole economy.

Finally, recalling that the useful life of electric generators is T_X, $K_X(\upsilon)$ is kept in operation for the time interval $\upsilon < t < \upsilon + T_X$. During these years another firm may enter the market by purchasing newer, and hence more productive, electricity generators. So to discourage potential competition firm X reduces the price of electricity at the rate of technological progress by setting:

$$P_X(t) = P_X(\upsilon)e^{\mu_X(t-\upsilon)}. \tag{10}$$

At this point one may ask: how do we know that this pricing rule does deter new entrants? To ascertain that it does, divide (10) by (8) and set $\upsilon = 0$ to obtain:

$$\frac{P_X(t)}{b_X(t)} = \frac{P_X(0)}{b_X(0)}. \tag{11}$$

What this equation signifies is that by following (11) the representative firm prices the electricity produced by the various vintages of electricity generators so as to equate the value of the marginal Kilowatt-Hour produced by the most recent vintage of electricity generators to that produced from the initial vintage. But according to the proof in Appendix B the price of electricity from the initial vintage is calculated to reduce the unit net worth of electricity generators to zero.

Consequently, the same must hold for every vintage up to t, and hence no potential competitor should have an incentive to enter, because no potential competitor can expect to make any profits by taking advantage of more productive electricity generators. In essence, under the pressure to protect its market from potential competitors, firm X is forced to pass all benefits of technological change to the consumers of electricity. This is the miracle of potential competition.

Drawing on the above and the step-by-step explanations found in Appendix C, if the salvage value of equipment on retirement is zero, the unit net worth of new electricity generators at $\upsilon = 0$ is given by:

$$n_X(0) = \int_0^{T_X} [\frac{P_X(0)}{b_X(0)} e^{\mu_X t} - w] e^{-\sigma t} dt - \beta w = \frac{P_X(0)}{b_X(0)} \frac{1-e^{(\mu_X-\sigma)T_X}}{\sigma-\mu_X} - w\frac{1-e^{-\sigma T_X}+\beta\sigma}{\sigma}. \qquad (12)$$

Observe that the term $P_X(0)/b_X(0)$ in the parenthesis is multiplied by $e^{\mu t}$. This implies that the unit value of the marginal product of the electricity generators declines at the rate μ per unit of time, because $\mu_X < 0$. The obvious reason is that as time goes by the earning capability of these electricity generators becomes inferior relative to the new ones that are more productive, since they embody the most recent advances in technology. Subtracting from the unit value of the marginal product the fixed unit labor cost, w, we obtain a declining stream of net unit income. Finally, discounting the latter over the useful life of the electricity generators with the help of the positive discount factor σ, we arrive at the unit net worth, $n_X(0)$.

Firm X is justifiably presumed to behave as if its monopoly will last forever on two grounds. The first is that by pricing electricity according to (10) it deters all competition from new potential entrants, whereas the second springs from the realization that, since electricity is a necessity, it will be always in demand. By implication, at any period the firm must have no more and no less than the necessary electricity generating capacity to meet this demand. For if it has less it will be losing sales and if it has more it will be wasting resources.[9] As a result, since reinvestment opportunities will repeat indefinitely, the firm is led to maximize the present value of profits from an infinite series of equidistant replacements.[10] Using (6) and (12) in conjunction with the expression **(D.2)** derived in the Appendix D, the objective function becomes:

$$\Pi(T_X, P_X(0)) = \frac{b_X(0) n_X(0) X(0)}{1-e^{-(\mu_X-\sigma)T_X}} = b_X(0) N_X(0) [P_X(0)]^{\eta_X} \left[\frac{P_X(0)}{b(0)} \frac{1}{\sigma-\mu_X} - \frac{w}{\sigma} \frac{1-e^{-\sigma T_X}+\beta\sigma}{1-e^{-(\mu_X-\sigma)T_X}} \right] \qquad (13)$$

From the first order conditions for $P_X(0)$ and T_X we obtain:[11]

$$P_X(0) = \frac{\eta_X}{1+\eta_X}\frac{\sigma-\mu_X}{\sigma}\frac{1-e^{-\mu_X T_X}+\beta\sigma}{1-e^{(\mu_X-\sigma)T_X}}b_X(0)w, \tag{14}$$

$$\sigma e^{-\mu_X T_X} - \mu_X e^{-\sigma T_X} = (1+\beta\sigma)(\sigma-\mu_X). \tag{15}$$

Equation (15) does not permit an explicit solution for T_X. However, it can be established that one and only one positive solution for T_X exists.

To sketch the proof, consider Figure 1. Let the left-hand side of (15) be denoted by the function $g(T_X)$. If we set $T_X = 0$, we see that this function takes the value $\sigma - \mu_X$. Next, as the useful

(Please insert here Figure 1)

life T_X rises above zero, the curve $g(T_X)$ rises because it holds that $\partial(\sigma e^{-\mu_X T_X} - \mu_X e^{-\sigma T_X})/\partial T_X = -\mu_X\sigma e^{-\mu_X T_X} + \sigma\mu_X e^{-\sigma T_X} > 0$. To ascertain that this is the case, observe that the first term of this derivative has e raised to the positive power $-\mu_X T_X$ and multiplied by the positive coefficient $-\mu_X\sigma$; thus when T_X rises without bound the first term also does so. The second term has e raised to the negative power $-\sigma T_X$; therefore as T_X rises without bound the second term vanishes. These findings are depicted by the bold upward sloping curve $g(T_X)$.[12] Finally, looking at the right-hand side of (15), notice that it does not contain T_X. This implies that the right-hand side defines a horizontal line, labeled as FF', which cuts the vertical axis above the value $\sigma - \mu_X$ because $(1+\beta\sigma)(\sigma - \mu_X) > (\sigma - \mu_X)$. Therefore, the curve $g(T_X)$ is bound to cut the horizontal line just once, giving the optimal service life T_X^*.

At this point it is interesting to pose the following question: how does the optimal useful life T_X^* change when the parameters in equation (15) change? To answer it, I computed T_X^* for various combinations of empirically reasonable values of σ, μ_X and β from the United States in the postwar period. These values are exhibited in Table 2 below. Looking across each row, we observe that for every pair of values assumed by the parameters μ_X and β the optimal useful live of

(Please insert here Table 2)

electricity generators would be longer the higher the interest rate. Next, focusing in a single column, we notice two results. The first has to do with the cases where the parameters σ and β remain constant and only μ_X changes. In all cases it turns out that the useful life is uniformly longer, the slower is technological progress. As to the second result, this corresponds to the cases where the parameters σ and μ_X are held constant and only β changes. From them it emerges that, the costlier the acquisition of electricity generators in terms of the minimum labor required for their construction, the longer their useful live. As we would expect, these results make a lot of sense because the costlier the producers' goods and the higher the interest rate, the more urgent it becomes to save capital cost by lengthening their useful lives, whereas the slower the technological progress, the less difference between the efficiencies of producers' goods of consecutive vintages, and hence the lower the pressure of retirement.

Introducing T_X^* and (9) into (15) and using the resulting expression in conjunction with (6), (7) and (12) we find:

$$n_X^*(0) = \left[\frac{\eta_X}{1+\eta_X} - 1\right]\frac{1 - e^{-\sigma T_X^*} + \beta\sigma}{\sigma} w \tag{16}$$

$$K_X^*(0) = N_X\left[\frac{\eta_X}{1+\eta_X}\frac{\sigma-\mu_X}{\sigma}\frac{1-e^{-\sigma T_X^*}+\beta\sigma}{1-e^{(\mu_X-\sigma)T_X^*}}w\right]^{\eta_X}\left[\frac{\beta}{M_X}\right]^{\frac{(1+\eta_X)}{\gamma}}. \tag{17}$$

From these we observe that both the unit net worth $n_X^*(0)$ and the quantity $K_X^*(0)$ of electricity generating capacity depend also on T_X^*. But from (15) we know that T_X^* depends on the capital policy adopted by the firm. Consequently, under a policy of perpetual equidistant replacements the construction cost and the market value of the surviving physical capital employed by firm X would be given respectively by $\beta w K_X^*(0)$ and $n_X^*(0)K_X^*(0)$.

Representative Firm Y

Now let me turn to firm Y. Above it was indicated that this firm plans either to exit at the end of the useful life of the lathe or re-invest, if market conditions warrant it. Moreover, since by assumption firm Y applies pricing rule (5), it defends its market from potential competitors as securely as firm X. So the natural question is why should firm Y behave differently regarding its re-investment opportunities? The answer is that tennis rackets is a luxury good whose demand is very sensitive to income and fashion trends, and hence demand conditions may have changed drastically at the end of the useful life of its current stock of capital. By implication, acting rationally firm Y retains an option to decide whether to exit or re-invest in the light of the demand conditions that will prevail at T_Y.[13] Following then the same analysis as for firm X, but without an infinite series of re-investments, firm Y maximizes:[14]

$$\Pi(T_Y,P_Y(0))=b_Y(0)n_Y(0)Y(0)=b_Y(0)N_Y(0)[P_Y(0)]^{\eta_Y}\left[\frac{P_Y(0)}{b(0)}\frac{1\text{-}e^{-(\mu_Y-\sigma)T_Y}}{\sigma-\mu_Y}-w\frac{1-e^{-\sigma T_Y}+\beta\sigma}{\sigma}\right], \qquad (18)$$

with respect to T_Y and $P_Y(0)$. In **(13)** it should be observed that β is the same as that in equation **(4)**. This implies that:

$$\beta = M_Y[b_Y(0)]^{\gamma}. \qquad (9')$$

The rationale for this assumption is that the minimum required labor to build a unit of productive capacity should be the same across the two representative firms, because differences in productivity in their capital building departments would tend to vanish through a competitive reallocation of workers among them.

From the first order conditions for maximization of (18) we obtain:

$$P_Y(0)=\frac{\eta_Y}{1+\eta_Y}\frac{(\sigma-\mu_Y)}{\sigma}\frac{1-e^{-\sigma T_Y}+\beta\sigma}{1-e^{(\mu_Y-\sigma)T_Y}}b_Y(0)w \qquad (19)$$

$$\frac{1+\eta_Y}{\eta_Y}\sigma e^{-\mu_Y T_Y}-\frac{\eta_Y\mu_Y+\sigma}{\eta_Y}e^{-\sigma T_Y}=(1+\beta\sigma)(\sigma-\mu_Y). \qquad (20)$$

Denoting the left-hand side of (20) by the function $h(T_Y)$, we observe that as T_Y goes to zero this function takes the value $\sigma-\mu_Y$. Hence, if we set $\mu_X=\mu_Y$ in order to highlight the differences that emerge in the useful lives of the stocks of capital in the two sectors due to the different capital policies applied by the two representative firms, both functions $g(T_X)$ and $h(T_Y)$ start from the same point on the vertical axis in Figure 1. Next let T_Y rise above zero and take the derivative of $h(T_Y)$. As T_Y rises without bound, this derivative remains positive, which means that $h(T_Y)$ always rises. Then the important question is whether the function $h(T_Y)$ rises to the left or to the right of function $g(T_X)$. To tackle it, let us compare the derivatives of $g(T_X)$ and $h(T_Y)$ with respect to the corresponding useful lives. These derivatives are:

$$\frac{\partial(\sigma e^{-\mu_X T_X} - \mu_X e^{-\sigma T_X})}{\partial T_X} = -\mu_X \sigma e^{-\sigma T_X}(e^{(\sigma-\mu_X)T_X} - 1), \tag{21}$$

$$\frac{\partial(\frac{1+\eta_Y}{\eta_Y}\sigma e^{-\mu T_Y} - \frac{\eta_Y\mu_Y+\sigma}{\eta_Y}e^{-\sigma T_Y})}{\partial T_Y} = -\mu_Y\sigma e^{-\sigma T_Y}(e^{(\sigma-\mu_Y)T_Y}-1)+\frac{\sigma}{\eta_Y}(-\mu_Y e^{-\mu_Y T_Y}+\sigma e^{-\sigma T_Y}). \tag{22}$$

Now, given that $\mu_X = \mu_Y$, if we let $T_X = T_Y$ expression (22) differs from (21) only in that it includes the second term, which is negative because $\mu_Y < 0$ and $\eta_Y < -1$. This ascertains that (22) is smaller than (21) at any optimal useful life, and hence that $h(T_Y)$ rises always to the right of $g(T_X)$. Consequently, if the rates of embodied technological change are the same in both sectors, in Figure 1 the function $h(T_Y)$ will cut the horizontal line FF' to the right of T_X^*, say at T_Y^*, and the optimal useful life of the lathe will be longer solely because firm *Y* applies scrapping. Thus, having computed T_Y^* from (20), the values for $n_Y^*(0)$ and $K_Y^*(0)$ can be derived with the help of:

$$n_Y(0) = \int_0^{T_Y}[\frac{P_Y(0)}{b_Y(0)}e^{\mu_Y t} - w]e^{-\sigma t}dt - \beta w = \frac{P_Y(0)}{b_Y(0)}\frac{1-e^{(\mu_Y-\sigma)T_Y}}{1-\mu_Y} - w\frac{1-e^{-\sigma T_Y}+\beta\sigma}{\sigma}, \tag{12'}$$

in conjunction with (6), (7) and (19). In particular, we obtain:

$$n_Y^*(0) = \left[\frac{\eta_Y}{1+\eta_Y} - 1\right]\frac{1-e^{-\sigma T_Y^*}+\beta\sigma}{\sigma}w, \tag{23}$$

$$K_Y^*(0) = N_Y(0)[\frac{\eta_Y}{1+\eta_Y}\frac{\sigma-\mu_Y}{\sigma}\frac{1-e^{-\sigma T_Y^*}+\beta\sigma}{1-e^{(\mu_Y-\sigma)T_Y^*}}w]^{\eta_Y}[\frac{\beta}{M_Y}]^{\frac{(1+\eta_Y)}{\gamma}}. \tag{24}$$

From them it turns out that, once we find T_Y^*, the other two key variables, i.e. the unit net worth $n_Y^*(0)$ and the stock of physical capital $K_Y^*(0)$, are fully determined.

3.2 Macroeconomics

Let us turn now from microeconomics to macroeconomics. Since the two representative firms produce their goods by means of different capital, the question that arises is how to define and measure the capital employed in the economy.[15] If electricity generators and lathes were perishable goods like lemons and oranges, the answer would be very easy. Simply, we would multiply their prices by their quantity and we would sum the results to compute their aggregate value in the economy. But this approach is untenable under the present circumstances because the durables employed in the economy are determined by three variables, i.e. acquisition cost, quantity and useful live. Hence, we must devise a different approach.

Following Haavelmo (1960, pp. 100-101), a reasonably consistent index of the economy-wide stock of capital in the absence of technological change may be derived by: a) deflating appropriately the money values of the various components of the capital stock to obtain "constant-dollar" denominated series; b) converting the deflated value figures to an "equal-durability" basis, and c) adding the resulting series. In the confines of the present model, the unit cost $w\beta$ of building each type of capital is constant. Hence there is no need for deflation to obtain "constant dollar" figures. But electricity generators and lathes have different useful lives and must be converted to an "equal-durability" basis. To express the longevity of $K_Y^*(0)$ in terms of the longevity of $K_X^*(0)$, I start from the realization that the construction cost of the capital goods is equal to the discounted present value of the stream of net revenues over their useful lives. On this ground I write:

$$w\beta K_Y^*(0)=\int_0^{T_Y^*}[\frac{P_Y(0)}{b_Y(0)}e^{\mu_Y t}-w]e^{-\sigma t}dt\,. \tag{25}$$

If K_Y^* had been of the same longevity as K_X^*, it would earn annually the same income but over the useful life T_X^*. Therefore, for the hypothetical capital stock $K^{**}(0)$, (25) would transform into:

$$w\beta K_Y^{**}(0)=\int_0^{T_X^*}[\frac{P_Y(0)}{b_Y(0)}e^{\mu_Y t}-w]e^{-\sigma t}dt\,. \tag{26}$$

Next, dividing (26) by (25) and rearranging we obtain:

$$K_Y^{**}(0)=K_Y^*(0)\frac{\int_0^{T_X^*}[\frac{P_Y(0)}{b_Y(0)}e^{\mu_Y t}-w]e^{-\sigma t}dt}{\int_0^{T_Y^*}[\frac{P_Y(0)}{b_Y(0)}e^{\mu_Y t}-w]e^{-\sigma t}dt}=zK_Y^*(0)$$

$$\text{with } z=\frac{\frac{P_Y(0)}{b_Y(0)}\frac{1-e^{(\mu_Y-\sigma)T_X^*}}{1-\mu_Y}-w\frac{1-e^{-\sigma T_X^*}}{\sigma}}{\frac{P_Y(0)}{b_Y(0)}\frac{1-e^{(\mu_Y-\sigma)T_Y^*}}{1-\mu_Y}-w\frac{1-e^{-\sigma T_Y^*}}{\sigma}}. \tag{27}$$

Finally, by adding the stocks of capital in the two sectors, I obtain:

$$K^*(0) = w\beta[K_X^*(0) + K_Y^{**}(0)] = w\beta[K_X^*(0) + zK_Y^*(0)]. \tag{28}$$

From this expression it is clear that the conventional approach of obtaining $K^*(0)$ as $w\beta[K_X^*(0)+K_Y^*(0)]$ corresponds to $z=1$. But is this justified? From the theoretical point of view it would be if and only if it transpired that $T_X^* = T_Y^*$ and $\mu_X = \mu_Y$. Yet as I will argue shortly none of these conditions is likely to be met individually and certainly not both at the same time. Therefore, the only justification would be if z were reasonably close to 1 at least empirically. Under what circumstances would such an approximation be warranted, if at all, is the subject of the analysis in the following section.

4. The aggregate stock of capital

Turning first to the main properties of the solution, recall from Section 2 that researchers in the area of aggregation assume traditionally that firms replace their durables as per the proportionality hypothesis and that the rate of embodied technological change is uniform throughout the economy. So an issue that arises concerns the implications for aggregation if the rates of embodied technological change in the two sectors of the economy differ. To highlight it, assume that the two representative firms apply replacement but $\mu_X \neq \mu_Y$. Then from equation (15) it would follow that $T_X^* \neq T_Y^*$ and it would be impossible to achieve consistent aggregation without expressing the two types of durables in terms of equal durability. Not unexpectedly the same conclusion would hold if $\mu_X = \mu_Y$ and firm *X* applied replacement, whereas firm *Y* applied scrapping. For then the equations (15) and (20) would give $T_X^* \neq T_Y^*$. Finally, observe that the solution for T_Y^* from equation (20) involves the elasticity η_Y^*, which varies over the business cycle. As a result the useful life of capital in the *Y* sector of the economy varies with it, thus shifting the time structure of aggregate capital. Hence, it is no wonder that scrapping is a prime mechanism for the propagation of business cycles, irrespective of the causes that start them.

From the above it follows that on the basis of the proposed model the value of z would be expected to differ from 1, thus invalidating the conventional approach to aggregation. But even so it may be sufficiently close to 1 to save us from the complexities of having to deal with the non-stationarities that would be involved in the computation of deterioration-age profiles. To investigate this possibility, some of the variables and parameters in (27) were given arbitrary values, some others were approximated with empirically plausible values from the United States in the postwar period, and still some others were allowed to vary over reasonable ranges. In particular, what values the variables and parameters in (27) received and why is explained below:

- The marginal product in the Y sector, $P_Y(0)/b_Y(0)$, was normalized to 1 and the wage rate w was set equal to 0.30.
- The value of the elasticity of output demand in the Y sector, η_Y, which enters in the computation of T_Y from (20), was set equal to –15. In the computations this parameter could be allowed to vary. But here it was fixed in order to focus exclusively on the effects on z of the interest rate and the relative rates of technological change.
- The approximation to the minimum building labor β was derived as follows. Drawing on the assumption that each unit of capital in the model is combined with one unit of labor and the fact that $\beta w = p$, where p is the construction cost of producer durables, I was able to write:

$$\beta = \frac{w\beta K^*(0)}{wK^*(0)} = \frac{pK^*(0)}{wL^*(0)}.$$

  For the United States Evans (2000) estimated that in the postwar period the mean value of the undepreciated capital stock was 2.06 times the real gross domestic product, whereas the mean share of real gross domestic product paid to labor during the same period was roughly 0.8. Hence the value of β was in the vicinity of 2.5.
- The interest rate σ was approximated by the return on capital. According to the estimates by Evans (2000), the mean return on capital over the period 1947-1998 was 9%.
- Finally, the rates of embodied technological change, μ_X and μ_Y, may be approximated by the growth rate of the average labor productivity in the economy as a whole. For the United States in the postwar period this rate has been in the neighborhood of 2%. By implication, the reciprocal of the capital coefficient $1/b(\upsilon) = X(\upsilon)/L(\upsilon)$ declined at the same rate, so we can set that on the average $\bar{\mu}_X = \bar{\mu}_X = -0.02$. In short, in the simulations the variables and the parameters involved in equations (15), (20) και (27) take the following values:

$$\frac{P_Y(0)}{b_Y(0)} = 1 \quad \text{w=0.3} \quad \eta_Y = -15, \ \beta\text{=2.5}$$

$$\sigma = \{0.07 \ \ 0.08 \ \ 0.9 \ \ 0.10\}$$

$$\mu_X, \mu_Y = \{-1.5 \ \ -2 \ \ -2.5 \ \ -3\}.$$

The solution procedure involved two steps. In the first one, equation (20) was solved iteratively to obtain the values for T_Y^*. As in the case of equation (15), which was solved for T_X^* and gave the results shown in the row $\beta = 2.5$ of Table 2, this produced the 16 solutions that are exhibited in Tables 3. In the second step, the solutions for T_X^* and T_Y^*, together with

(Please insert here Table 3)

the assumed values for the other variables and parameters, were introduced into (27) to calculate z. From the calculations emerged the 64 values that appear in Table 4. So the task now is to reflect on these findings.

(Please insert here Table 4)

Searching for rows and columns where z is approximately equal to 1, observe that this occurs only in the two instances where $\{\sigma = 0.08, \mu_X = -0.015\}$ and $\{\sigma = 0.09, \mu_X = -0.02\}$. Moreover, notice that in these instances the values of z remain nearly invariant with respect to μ_Y, thus suggesting that the rate of embodied technological change in the *Y* sector does not matter. From this it follows that, since in the United States in the postwar period the average rates of return on capital and the growth rate in the productivity of labor, as a proxy for technological progress, were close to these ranges, the conventional scheme of aggregation might not be inappropriate and we could forget about the differences in useful lives of producer durables in the various sectors of the economy. But according to the calculations by Evans (2000) and others, the rate of return on capital and the economy-wide growth rate in the productivity of labor varied significantly from one five-year period to the next, let alone from one year to the next. In turn this implies that using the averages of the whole period instead of the annual observations may bias the aggregates of the capital stock and replacement investment.

To get a feeling of the possible order of magnitude of these biases, consider the case of the capital stock. Its biases would depend on two factors, i.e. the value of the parameters involved in the calculation of useful lives and the relative quantities of capital in the two sectors of the economy. For an example let: a) $T_X^* \neq T_Y^*$, either because $\mu_X \neq \mu_Y$ or due to differences in the applied capital policies (i.e. replacement vs. scrapping); b) the quantity of capital in the *Y* sector is large relative to *X* sector, and c) the values of the parameters entering into the calculation of useful lives are close to $\{\sigma = 0.07,\ \mu_X = -0.030,\ \mu_Y = -0.015\}$. Then the conventional approach to aggregation will overestimate the aggregate capital stock a great deal because the value of z will be in the vicinity of its lower bound 0.836. On the contrary, if the values of the same parameters are $\{\sigma = 0.10, \mu_X = -0.015, \mu_Y = -0.015\}$, the conventional approach to aggregation will underestimate the aggregate capital stock because the value of z will be at its upper bound of 1.066. Moreover, observe from the latter case that, even though $\mu_X = \mu_Y$, the aggregation bias would still exist because $T_X^* \neq T_Y^*$, since the one representative firm applies replacement and the other scrapping. Therefore, from this evidence it follows that, if we do not allow for the age structure of capital, the conventional approach to aggregation may result in a correct measure of "capital-in general" only by numerical accident; and even then it will be conceptually faulty, because it does not allow properly for the effects of embodied technological change and the other determinants of useful lives.

5. Summary of findings and conclusions

With regard to the aggregation of producer durables, the dominant theory has been erected on two fundamental hypotheses and a result of theoretical deduction. Referring to the hypotheses, first and foremost among them is that the deterioration of producer durables is independent of their age. The second hypothesis is that the process by which producer durables deteriorate follows an exponential distribution, which implies that the amount of replacement investment is proportional to the outstanding capital stocks; and lastly the result of theoretical deduction is that in order to achieve consistent aggregation under the proportionality hypothesis all producer durables must deteriorate at the same proportional rate. Certainly these conceptualizations make life easy because, by allowing us to approximate the decay of producer durables by a single-parameter function, we are able to bypass the analytical and computational complexities that would be involved if the decay of producer durables were conceived to vary with their age. But the proportionality hypothesis has been found to be in conflict with most of the available theoretical and empirical evidence and the need for alternative approaches to aggregation that would allow explicitly for the ability of firms to determine endogenously the useful lives of their durables in the presence of embodied technological change is long overdue.

To this end in the present I constructed a two-sector vintage capital model in which the main thrust was to investigate the conditions for consistent aggregation in the presence of embodied technological change and endogenous useful lives. On the theoretical plain the results showed that because of the differences in the rates of embodied technological change and/or in the capital policies applied, the conventional approach to aggregation is conceptually untenable. In view of this finding, I then adapted the aggregation mechanism first proposed by Haavelmo (1960). In particular, to achieve consistent aggregation the stock of capital in the one sector was expressed in units of equal longevity with the stock of capital in the other sector. This procedure gave a measure of "capital-in-general" in the form of $w\beta[K_X^*(0)+zK_Y^*(0)]$, where z is a complex function of the variables and parameters that enter into the determination of the useful lives of capital stocks in the two sectors of the economy. Finally, I simulated this function by assigning empirically plausible values to its determinants from the United States in the postwar period.

From these results there emerged two main conclusions. The first of them was that, if the variables and parameters in the model are approximated by their mean values over the postwar period, then the conventional aggregation scheme gives approximately correct results, and hence we do not need to bother with the suggested adjustments for the longevity of the capital stocks in the economy. This corroborates the evidence presented by Hulten and Wykoff (1989). However, as these variables and parameters varied significantly from one year to the other, in all probability the useful lives varied significantly as well. Consequently, the second conclusion was that, depending on the distribution of capital stocks among the various sectors as well as their respective useful lives, the conventional approach to aggregation might be liable for significant biases in the economy-wide measurement of capital.

Endnotes

[1] On the side of econometric applications here is how Hulten et al. (1989) expressed the usefulness of the proportionality hypothesis: " We have found that a major event like energy crises, … did not in fact result in a systematic change in age-price profiles. This lends confidence to procedures that assume stationarity in order to achieve a major degree of simplification (and because of nonstationarity is so difficult to deal with empirically). Or, put simply, the use of a single number to characterize the process of depreciation (of a given type of capital asset) seems justified in light of the results…" (p. 255).

[2] Three year later Griliches (1963) did take notice of the problems that were posed for aggregation from the durability of capital, but he failed to advance the art beyond the state Haavelmo (1960) had left it.

[3] At this point the choice of capital policy may appear to be imposed on the two firms exogenously. But it is not, because in an earlier stage the two firms solve the problem posed by Bitros and Flytzanis (2005), where the horizon of re-investment is determined endogenously. In other words, here it is assumed that, if the firms solved this problem in the light of the differences in the nature of and the demand for their products, firm X would apply replacement and firm Y scrapping.

[4] In addition, the proportionality hypothesis has several powerful advantages. According to Jorgenson (1974), one of them is that the theory of replacement that emerges is characterized by price-quantity duality. In particular: "… The level of acquisition of capital goods is dual to the rental price of capital services. Capital stock is dual to the acquisition price of capital goods. Replacement requirements, a component of investment expenditures are dual to depreciation, a component of the rental price of capital services…" (p. 219). Another, demonstrated by Hall (1968, p. 41), is that it renders the price of durables in the capital stock independent of the future values of the interest rate and the acquisition prices of new investment; and still another, pointed out by Harper (1982), is that under this hypothesis the age-efficiency and the age-price profiles coincide and there is no need to distinguish between the capital stock as a factor of production and the capital stock as a store of value or wealth. Moreover, and perhaps most importantly for empirical research and policy applications, the proportionality hypothesis has the advantage that it simplifies significantly the computation of capital stock series through the perpetual inventory method.

[5] Just for the clarity of this term it should be noted that depreciation is dual to replacement investment, which constitutes a component of gross investment equal to retirements plus the loss of efficiency in the surviving part of the stock of capital.

[6] Even though telephone companies, electric utilities, and other network industries construct a good deal of their capital internally, this conceptualization is quite removed from the actual economy. However, it is adopted in order to avoid the opening of a third sector, which would complicate the analysis significantly without adding much to the explanatory power of the model.

[7] The production of electricity is presumed to take place by combining one unit of capital with one unit of labor, whereas electricity-generating capacity is built within the firm solely by means of labor. Hence the addition of an

employment equation would turn the model into one of general equilibrium in labor and capital. However, as the ensuing analysis would not be affected, the employment of labor sector of the model is ignored.

8 Regarding the impact of technological change, there are two possibilities. Technological change may affect the capital-output coefficient either positively or negatively. Normally new production techniques take the form of innovations that reduce the capital-output coefficient. But one cannot preclude isolated episodes of technological regression or of some heavily capital-intensive innovation, which for some period may increase the capital-output coefficient above its downward trend. In this paper I focus only on the innovations that as a rule and on the average reduce the capital-output coefficient.

9 To be sure, given that the demand for electricity is actually stochastic, in reality Firm *X* will need to keep adequate slack capacity to avoid blackouts. Assuming that electric utilities have developed a rule by which they determine the percentage of stand-by capacity they must have to meet peak loads, the analysis is not affected because the optimal capacity may be defined to include this percentage of spare capacity.

10 This policy is one of many that can possibly result when solving for the optimal re-investment horizon. Tightening a bit further the assumption introduced in footnote 6, I assume for reasons of simplicity that, if firm *X* solved for the horizon of re-investments, the optimal capital policy would turn out to be one of infinite replacements at equal time distances.

11 The detailed derivations of the first order conditions from the objective functions (8) and (13) are available on request from the author.

12 Please note that the vertical axis in Figure 1 depicts the left-hand side of equations (15) and (20). Thus the axis measures the values taken by the functions $g(T_X)$ and $h(T_Y)$. The latter function, shown in Figure 1 by the corresponding bold upward sloping curve, will be explained below.

13 The option to decide whether to exit or re-invest at the end of the initial investment cycle is not free. To ascertain the existence of the costs involved, assume first that firm *Y* decides to re-invest. In this event, given that $T_X^* < T_Y^*$, firm *Y* renews its capital slower than firm *X*. But by doing so it foregoes the benefit of taking advantage of technological change at a quicker pace. Moreover, firm *Y* may have to absorb the costs that accompany the shutting down of business operations.

14 Clearly, since in this case there is no infinite series of re-investments, the objective function of firm *Y* consists solely of a single acquisition of capital stock. This explains why the denominator in equation (13) is missing from equation (18).

15 In an economy with a small number of distinct kinds of capital goods it may be possible to sidestep the issues discussed immediately below by resorting to a fully disaggregated analysis. However, as the model under consideration is intended to be generalizable to an economy with any number of heterogeneous capital goods, the adopted approach to aggregation is of particular methodological importance.

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Appendix A

Suppose that embodied technological progress reduces the capital-output coefficient $b_X(t)$ at the constant and exogenous rate μ. This rate scales with time because it depends on the interval over which it is considered. So suppose that we are at time t and we observe a vintage of capital that was built υ periods ago. On average, from υ to t the capital coefficient must have declined as follows:

$$\frac{b_X(t)-b_X(\upsilon)}{b_X(\upsilon)} = \mu(t-\upsilon), \tag{A.1}$$

with $\mu < 0$.

From this we get:

$$b_X(t) = b_X(\upsilon)[1+\mu(t-\upsilon)]. \tag{A.2}$$

Now assume that the process of technological progress begins at $t=\upsilon$. At that moment the capital coefficient will be equal to $b(\upsilon)$. Next divide the interval $t-\upsilon$ into m equal timesteps. At the end of the first timestep the capital coefficient will be:

$$b_X(\upsilon+\frac{t-\upsilon}{m}) = b_X(\upsilon)[1+\mu\frac{t-\upsilon}{m}]. \tag{A.3}$$

At the end of the second timestep the capital coefficient will be:

$$b_X(\upsilon+2\frac{t-\upsilon}{m}) = b_X(\upsilon+\frac{t-\upsilon}{m})[1+\mu\frac{t-\upsilon}{m}] = b_X(\upsilon)[1+\mu\frac{t-\upsilon}{m}]^2. \tag{A.4}$$

And at the end of the *m* timestep the capital coefficient will have declined to:

$$b_X(t) = b_X(\upsilon+(m-1)\frac{t-\upsilon}{m})[1+\mu\frac{t-\upsilon}{m}]^m. \tag{A.5}$$

Now let $m \to \infty$. Then, the timestep $(t-\upsilon)/m$ will go to zero and we will have:

$$b_X(t) = b_X(\upsilon)[1+\mu\frac{t-\upsilon}{m}]^m = b_X(\upsilon)e^{\mu(t-\upsilon)}, \tag{A.6}$$

which is equation (8) in the text.

Appendix B

Inserting **(10)** into **(9)** and rearranging we obtain:

$$P_X(0) = \frac{\eta_X}{1+\eta_X}\frac{\sigma-\mu}{\sigma}\frac{1-e^{-\mu T_X}+\beta\sigma}{1-e^{-(\mu-\sigma)T_X}}b_X(0)w\,. \tag{B.1}$$

In turn, if this is introduced into **(7)** yields:

$$n_X(0) = -\frac{1}{1+\eta_X}\frac{1-e^{\mu T_X}+\beta\sigma}{\sigma}w\,. \tag{B.2}$$

But the representative firm is presumed to defend its monopoly by restricting itself through pricing rule **(5)** to making only normal profits, i.e. profits that would result in an industry in perfectly competitive long-run equilibrium. So it will let the price elasticity of demand η_X approach to minus infinity and use the approximation:

$$\frac{\eta_X}{1+\eta_X} = 1\,. \tag{B.3}$$

Now, if we deduct 1 from both sides of this equation and insert the result into (B.2) we obtain:

$$n_X(0) = 0\,. \tag{B.4}$$

This proves that under the adopted pricing rule the unit net worth of equipment in the initial vintage is set equal to zero and corroborates the claim made in the text.

Appendix C

At υ the worth of revenue minus the labor cost of operating a unit of equipment per small fraction dt of a year located at time t is given by:

$$[\frac{P(\upsilon)}{b(\upsilon)}e^{\mu(t-\upsilon)} - w]e^{-\sigma(t-\upsilon)}dt \tag{C.1}$$

where w and σ denote respectively the economy wide rates of wages and interest.

Consequently at υ the worth of the sum total of revenue minus operating labor cost of a unit of such equipment over its entire useful life T is:

$$n_X(0)=\int_{\upsilon}^{\upsilon+T}[\frac{P(\upsilon)}{b(\upsilon)}e^{\mu(t-\upsilon)}-w]e^{-\sigma(t-\upsilon)}dt-\beta w=\frac{P_X(\upsilon)}{b_X(\upsilon)}\frac{1-e^{(\mu-\sigma)T}}{\sigma-\mu}-w\frac{1-e^{-\sigma T}}{\sigma}. \tag{C.2}$$

Now let p be the purchase price of a new unit of electricity generators. Assuming its salvage value upon retirement is zero, its net worth is:

$$n(\upsilon)=\frac{P(\upsilon)}{b(\upsilon)}\frac{1-e^{(\mu-\sigma)T}}{\sigma-\mu}-w\frac{1-e^{-\sigma T}}{\sigma}-p\,. \tag{C.3}$$

Finally, since by **(4)** the minimum labor required to build a new unit of electricity generating capacity is β, under perfectly competitive transfer prices from the capital building to the capital using department of the representative firm, plus the assumption that electricity generating capacity is built solely by means of labor, it will hold that:

$$p=\beta w\,. \tag{C.4}$$

Thus substituting (C.4) into (C.3) gives (12).

Appendix D

Let a capital stock be replaced forever every υ years. At time *t*=0 the firm acquires the vintage zero capital stock $K_X(0)=b_X(0)X(0)$. At time $t=T_X$ the vintage zero capital stock is retired and replaced by vintage υ capital stock. But vintage T_X capital stock is more efficient due to technological progress. How more efficient it is relative to vintage zero capital stock is given by $b_X(T_X)=b_X(0)e^{\mu T_X}$. Hence, to keep the capacity constant from vintage to vintage, at $t=\upsilon$ the capital stock that is needed for replacement is less. In particular, its quantity is given by $K_X(T_X)=b_X(T_X)X(0)=b_X(0)X(0)e^{\mu T_X}$. At $t=2T_X$ the capital stock that will be needed for replacement will be $K_X(2T_X)=b_X(2T_X)X(0)=b_X(0)X(0)e^{2\mu T_X}$; at time $t=jT_X$, the capital stock that will be needed is $K_X(jT_X)=b_X(jT_X)X(0)=b_X(0)X(0)e^{j\mu T_X}$, and so on.

Now, consider the net worth of the acquisitions. At time $t=0$ the net worth of the capital stock is $n_X(0)K_X(0)=b_X(0)n_X(0)X(0)$. At time $t=\upsilon$ the net worth of the acquired capital stock is $n_X(T_X)K_X(T_X)=b_X(0)n_X(0)X(0)e^{\mu T_X}$. Notice in this expression that $n(\upsilon)$ has been substituted for $n(0)$. This has been done on account of the expressions **(6)** and **(7)** in the text. So, if the net worth of the capital stock acquired at $t=\upsilon$ is discounted to $t=0$ using the discounting term $e^{-\sigma T_X}$ we obtain $b_X(0)n_X(0)X(0)e^{(\mu-\sigma)T_X}$. Finally, repeating the preceding steps for the net worth of capital stock acquired at $t=jT_X$, gives $b_X(0)n_X(0)X(0)e^{(\mu-\sigma)jT_X}$. Consequently, since in addition to the initial acquisition there take place j replacements, the net present value of the $1+j$ acquisitions of capital stock is given by:

$$b_X(0)n_X(0)X(0)[1+e^{(\mu-\sigma)T_X}+...+e^{(\mu-\sigma)jT_X}]. \tag{D.1}$$

The expression in the brackets is a geometric progression with $1+j$ terms, each of which is equal to the preceding one multiplied by $e^{(\mu-\sigma)jT_X}$, where $(\mu-\sigma)T_X<0$. This implies that the bracketed expression constitutes a geometric progression with declining terms. Thus, if we set $j\to\infty$, at $t=0$ the sum of the infinite series of replacements $K_X(0),\ K_X(T_X), K_X(2T_X),...$ is given by:

$$A_X(0)=\frac{b_X(0)n_X(0)X(0)}{1-e^{(\mu-\sigma)T_X}}. \tag{D.2}$$

This is equation (13) in the text.

Table 1: Notation of the variables and parameters of the model

Variables	**Parameters**
X, Y = Representative firms in the two sectors of the model.	N_X, N_Y = Multiplicative constants in the demand equations for the products produced by firms *X* and *Y*.
$X(\upsilon)$, $Y(\upsilon)$ = Quantities of consumer goods produced by firms *X* and *Y* in year υ.	M_X, M_Y = Multiplicative constants in the equations that determine the minimum labor requirements in the construction of durables by firms *X* and *Y*
$K_X(\upsilon), K_Y(\upsilon)$ = Stocks of new physical capital built in year υ by firms *X* and *Y*.	η_X, η_Y = Price elasticities of demand for the outputs produced by firms *X* and *Y*.
$K(\upsilon)$ = Economy wide stock of physical capital in the economy in year υ	b_X, b_Y = Physical capital coefficients pertaining to firms *X* and *Y*.
$P_X(\upsilon), P_Y(\upsilon)$ = Prices of consumer goods produced by firms *X* and *Y*.	μ_X, μ_Y = Rates of technological progress per annum pertaining to firms *X* and *Y*.
$p_X(\upsilon), p_Y(\upsilon)$ = Unit prices of new capital goods constructed in year υ by firms *X* and *Y*.	β = Minimum labor required to build one unit of physical capital in the economy.
$n_X(\upsilon), n_Y(\upsilon)$ = Unit present net worth of new producers' durables built by firms *X* and *Y* in year υ.	γ = Elasticity in the equation that determines the minimum labor requirements in the construction of physical capital.
T_X, T_Y = Useful lives of durables	w = Money wage rate-*numeraire*
	σ = Rate of interest per annum.

Table 2: Useful life of $K^*_X(0)$ from equation (15)

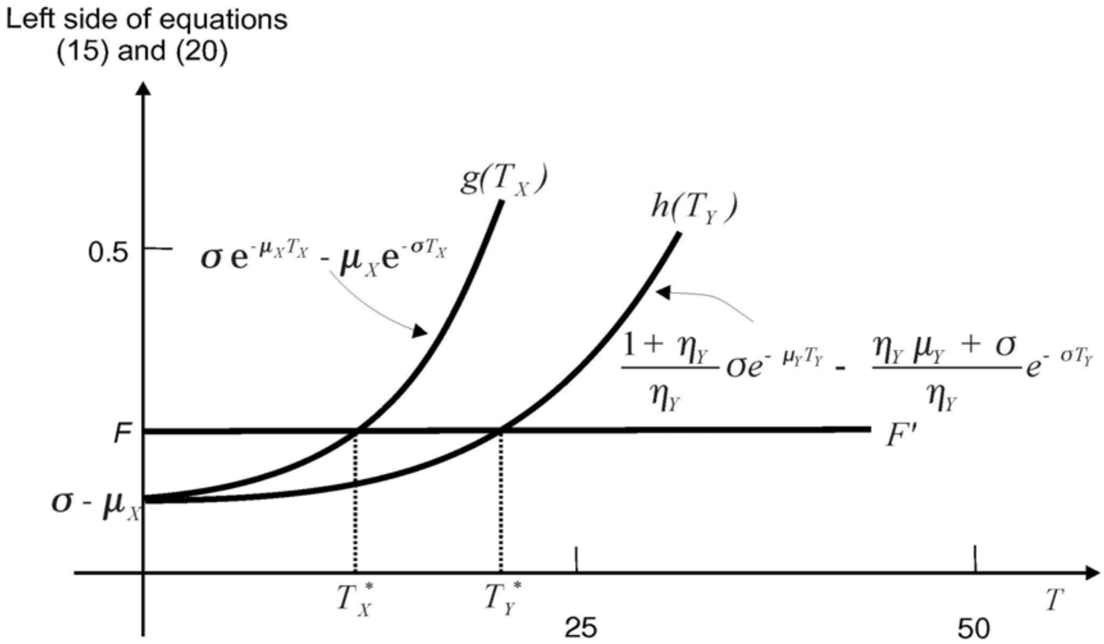
		σ			
	μ_X	0.07	0.08	0.09	0.10
$\beta=2$	-0.015	18.8	19.4	19.9	20.5
	-0.020	13. 8	14.1	14.3	14.6
	-0.025	12.3	12.6	12.8	13.0
	-0.030	11.2	11.4	11.6	11.8
$\beta=2.5$	-0.015	21.4	22.1	22.8	23.5
	-0.020	15.5	15.9	16.2	16.6
	-0.025	13.9	14.2	14.4	14.7
	-0.030	12.7	12.9	13.1	13.3
$\beta=3.0$	-0.015	23.8	24.6	25.4	26.3
	-0.020	17.1	17.5	18.0	18.4
	-0.025	15.3	15.6	16.0	16.3
	-0.030	13.9	14.2	14.4	14.7
$\beta=3.5$	-0.015	26.0	27.0	27.9	29.0
	-0.020	18.6	19.1	19.6	20.1
	-0.025	16.6	17.0	17.4	17.8
	-0.030	15.0	15.4	15.7	16.0

Table 3: Useful life of $K^*_Y(0)$ from equation (20)

$\beta=2.5$		σ			
$\eta_Y=-15$		0.07	0.08	0.09	0.10
μ_Y	-0.015	30.7	31.6	32.6	33.6
	-0.020	25.0	25.7	26.3	27.0
	-0.025	21.4	21.9	22.4	22.9
	-0.030	18.9	19.3	19.7	20.1

Table 4: Values of z from equation (27)

$\frac{P_Y(0)}{b_Y(0)}=1 \quad w=0.3 \quad \eta_Y=-15 \quad \beta=2.5$					
		μ_Y			
	μ_X	-0.015	-0.020	-0.025	-0.030
$\sigma=0.07$	-0.015	0.968	0.976	0.982	0.987
	-0.020	0.919	0.934	0.947	0.958
	-0.025	0.875	0.894	0.912	0.928
	-0.030	0.836	0.858	0.879	0.899
$\sigma=0.08$	-0.015	1.008	1.008	1.008	1.007
	-0.020	0.964	0.971	0.977	0.982
	-0.025	0.922	0.933	0.944	0.954
	-0.030	0.884	0.899	0.913	0.926
$\sigma=0.09$	-0.015	1.040	1.036	1.031	1.026
	-0.020	1.001	1.002	1.003	1.004
	-0.025	0.962	0.968	0.973	0.978
	-0.030	0.926	0.935	0.944	0.952
$\sigma=0.10$	-0.015	1.066	1.058	1.050	1.042
	-0.020	1.031	1.029	1.026	1.023
	-0.025	0.995	0.998	0.999	1.000
	-0.030	0.961	0.966	0.971	0.976



Useful lives of durables in the sectors *X* and *Y*
assuming $\mu_X = \mu_Y$

Figure 1