Productive government expenditure and fiscal sustainability

Arai, Real

Graduate school of Economics, Kyoto University

2 May 2008

Productive Government Expenditure and Fiscal Sustainability

Real Arai

First Draft: April 29th, 2008.

Abstract

We consider an overlapping generations model in which public spending directly contributes to grow up productivity as Barro (1990) and a government comforts the constant spending-GDP and debt-spending ratio rules. We analyse policy effects on fiscal sustainability, growth rate and welfare. This paper gives some remarks as follows: First, we demonstrate that when spending-GDP ratio rises it may be more sustainable fiscal policy. Second, we show analytically that if higher spending-GDP ratio is more sustainable fiscal policy, it brings higher growth rate in both short-term and long-term. Third, such policy change is Pareto improving. These remarks are not obtained in previous researches on fiscal sustainability.

Key Words: fiscal sustainability, productive government spending, public debt, endogenous growth

JEL Classification: E62, H54, H63

1 Introduction

We consider an economy where public spending directly contributes to grow up productivity and investigate what type of fiscal policy is sustainable and how change of fiscal policy affects the economy. We analyse policy effects on fiscal sustainability, growth rate and welfare and obtain some different remarks in comparison with previous works. We demonstrate it with analytical and numerical methods in what follows.

The relation between productive public spending and economic growth is analysed in many papers. Barro (1990) considers the case where public spending is used to raise productivity, directly, and shows the inverse U shaped relationship between public spending-GDP ratio and growth rate. Futagami et al. (1993) considers the case where public capital investment raises it and analyses the same as Barro (1990). However, these papers investigate only the situation in which the government follows balanced-budget rule and do not take account into public debt.

In recent years, fiscal sustainability has been analysed by some papers, which assume that the government can issue public debt under some fiscal policy rules, and analyses whether stock of public debt diverges or not in an overlapping generations model. Chalk (2000) and Rankin and Roffia (2003) suppose that public spending is fixed and check the condition acheiving fiscal sustainability in exogenous growth OLG models. Braüninger (2005) also analyses the issue in OLG model including endogenous growth in which the government conforms constant public spending-GDP ratio and public debt-GDP ratio rules. In these papers, increasing in spending or issurance of public debt leads to less sustainability of fiscal policy and, in addition, Braüninger (2005) says that it makes growth rate lower.

Futhermore, Yakita (2008) tackles the analysis of fiscal sustainability when the government expenditure is productive through investiment of public capital and follows the fiscal policy rules as Braüninger (2005). The effect of public spending has not been considered in preceding researches of fiscal sustainability as above. He concludes that i) increasing in public spending-GDP ratio or public debt-GDP ratio leads to less fiscal sustainability, and ii) increasing in debt-GDP ratio declines growth rate but increasing in spending-GDP ratio may raise.

However, he assumes that the government invests no-depreciate public capital, which corresponds to the situation of Futagami et al. (1993). We focus on the situation corresponding to Barro (1990). In other words, we suppose that public spending directly grows productibity and the government follows the fiscal policy rules used in Braüninger (2005) and Yakita (2008), and investigate policy effects on fiscal sustainability and economic growth.

We obtain following main remarks in this paper: i) if spending-GDP ratio is small, increasing in the ratio brings more fiscal sustainability, ii) if higher spending-GDP ratio improves fiscal sustainability, it makes growth rate higher, iii) when raising spending-GDP ratio gives more fiscal sustainability, such policy change is Pareto improving.

First remark is different one in contrast to previous studies on fiscal sustainability as above. Good policy effect for sustainability tends to occur when public spending-GDP ratio is small, as we will see. In other words, there is non-monotonic, inverse-U shape relationship between public spending and fiscal sustainability in our model, which was not obtained in the previous researches. The reason is that productivity depends on concave function of public spending-GDP ratio. Hence, when the ratio increases, if the ratio is sufficiently small then outputs sufficiently enlarges and its effect is dominated the effect of costs of increasing in public spending. This implies that we have to check not only the amount of public debt but also public spending-GDP ratio in order to attain fiscal sustainability by some policy change.

Second remark corresponds to the conclusion of Barro (1990), which shows the inverse U-shape relationship between spending-GDP ratio and growth rate. In previous studies of fiscal sustainability, this relationship was not obtained and the relationship considerably affects our remarks.

Third remark is also divergent from the remark of preceding works. There are a few researches which analyses the welfare effect by change of fiscal policy rules. Futagami and Shibata (2003) shows that any change of public spending cannot Pareto improve in an endogenous growth OLG model [1)]. However, our paper shows that change of public spending policy can improve welfare of all generations, when the policy change leads to more fiscal sustainability.

The rest of this paper consists as follows: Section 2 gives framework of model, shows the existence of balanced growth path steady state and illustrates dynamics of public debt. Section 3 derives the policy effects on fiscal sustainability and growth rate. Finally, section 4 is conclusion.

2 The Model

We consider an overlapping generations model which consists of individuals, firms and the government.

Individuals live for two periods. The size of each generations is same, which is normalized to unity. Individuals which are born at $t = 0, 1, 2, \cdots$ has identical utility function as below,

$$\ln c_t^y + \beta \ln c_{t+1}^o \tag{1}$$

where c_t^y is consumption when they are young, c_{t+1}^o is when old and $\beta \in (0, 1)$ is subjective discount rate. When indiviuals are young, they supply

[1)] Note that they suppose that the government comforms different fiscal policy rules from the previous works as above which are constant public spending-capital and *tax*-capital ratio.

their own labor inelastically, receive wage and consume or save it. On the one hand, when old, they unpile their saving. There are no bequests. Each household maximizes its own utility subject to intertemporal budget constraint.

Firms product goods in perfect competitive market. They have identical production technology,

$$y_t = Ak_t^{\alpha}(g_t l_t)^{1-\alpha} \tag{2}$$

where A is technological parameter (constant), k_t is stock of physical capital, g_t is the size of the government expenditure and l_t is unit of labor. Firms aim to maximize their profit in perfect competitive market and then we can write down the firm's profit maximization problem as

$$\max_{k_t, l_t} \pi_t = y_t - r_t k_t - w_t l_t. \tag{3}$$

The government spends on public expenditure to raise productivity of labor. It is financed by flat-rate income tax or issuance of public debt.

$$\begin{aligned} b_{t+1} &= (1+R_t)b_t + g_t - T_t \\ T_t &= \tau_t[r_t s_{t-1} + w_t] \end{aligned} \tag{4}$$

where we denote b_t as the stock of public debt at initial of period t, R_t interest rate of public debt, and T_t total tax revenue. We assume that government conforms the fiscal policy rules as follows,

$$g_t = \xi y_t \tag{5}$$

$$b_{t+1} - b_t = \chi g_t. \tag{6}$$

where $\xi, \chi \in [0,1]$ are policy parameters. Equation (5) means that the government fixes the ratio of public spending to GDP and equation (6) says that public debt is issued with constant proportion to government expenditure. These fiscal rules are used in many previous works, Braüninger (2005), Yakita (2008) and so on.

Then we define competitive equilibrium and consider the path only which attains the equilibrium in every period.

Definition 1 (competitive equilibrium)**.** *The sequences of predetermined variables $\{k_t, b_t\}_{t=0}^{\infty}$ and price system $\{r_t, R_t, w_t\}_{t=0}^{\infty}$ are* ***competitive equlibrium*** *if, for any t, they satisfy the following conditions:*

1. *Household's utility maximization conditions,*

$$s_t = \frac{\beta}{1+\beta}(1-\tau_t)w_t. \tag{7}$$

2. *Firm's profit maximization conditions,*

$$r_t = \alpha A k_t^{\alpha-1} g_t^{1-\alpha} \tag{8}$$

$$w_t = (1-\alpha) A k_t^{\alpha} g_t^{1-\alpha}. \tag{9}$$

3. *The government's budget constraint,* (4)*, and the fiscal policy rules,* (5) *and* (6).

4. *No-arbitrage condition,*

$$(1-\tau_t)R_t = (1-\tau_t)r_t. \Leftrightarrow R_t = r_t \tag{10}$$

5. *Capital market clearing condition,*

$$s_t = k_{t+1} + b_{t+1}. \tag{11}$$

6. *Labor market clearing condition,*

$$l_t = 1. \tag{12}$$

Furthermore, we denote the balanced growth path steady state as below.

Definition 2 (Balanced growth path steady state)**.** *The sequences of predetermined variables $\{k_t, b_t\}_{t=0}^{\infty}$ and price system $\{r_t, R_t, w_t\}_{t=0}^{\infty}$ are **balanced growth path steady state (BGP)** if they are competitive equilibrium and if there exists some γ such that for any t,*

$$\frac{k_{t+1}}{k_t} = \frac{b_{t+1}}{b_t} = \gamma. \tag{13}$$

Defining $\tilde{b}_t := b_t/k_t$, we can write down the dynamic system [2] as

$$\frac{\tilde{b}_{t+1}}{\tilde{b}_t} := \frac{b_{t+1}/b_t}{k_{t+1}/k_t} = \frac{1 + \chi \tilde{b}_t^{-1} A^{1/\alpha} \xi^{1/\alpha}}{\frac{\beta}{1+\beta} \frac{1+\xi\chi-\xi}{1+\alpha\tilde{b}_t} A^{1/\alpha} \xi^{1/\alpha-1}(1-\alpha) - \tilde{b}_t - \chi A^{1/\alpha} \xi^{1/\alpha}}. \tag{14}$$

By the definition, it is equivalent for the BGP that $\tilde{b}_t = 1$.

We investigate the condition for sustainable fiscal policy. The government's fiscal policy is represented as the set of policy parameters, (ξ, χ) and it is sustainable fiscal policy if there exists BGP in this system, $\tilde{b}_{t+1}/\tilde{b}_t = 1$ because, when the system has BGP, debt-capital ratio can converge for some appropriate initial stocks of debt and capital, which is shown in section 2.1. Lemma 1 is preparation and Proposition 1 tells what kind of fiscal policy is sustainable.

[2] We show the derivation of equation (14) in Appendix.

Lemma 1. *If A is sufficient large, there exists ξ^{MIN} and ξ^{MAX} such that $\xi \in [\xi^{MIN}, \xi^{MAX}]$ is equivalent to exist BGP for some χ.* [3].

Proof. See Appendix. □

Proposition 1. *Given sufficiently large A such that ξ^{MIN} and ξ^{MAX} exist, and fixed $\xi \in [\xi^{MIN}, \xi^{MAX}]$, there exists a critical value of χ to garantee the existence of BGP. Furthermore, i) if χ is smaller than critical one, the system has two BGPs. ii) if χ equals, has only one BGP. iii) if χ is larger, has no BGP.*

Proof. Define the functions, Φ and Ψ;

$$\Phi(\tilde{b};\chi) = \frac{\beta}{1+\beta}\frac{1+\xi\chi-\xi}{1+\alpha\tilde{b}_t}A^{1/\alpha}\xi^{1/\alpha-1}(1-\alpha) - \chi A^{1/\alpha}\xi^{1/\alpha}, \tag{15}$$

$$\Psi(\tilde{b};\chi) = 1 + \chi A^{1/\alpha}\xi^{1/\alpha}\tilde{b}_t^{-1} + \tilde{b}. \tag{16}$$

Φ and Ψ satisfy the properties,

$$\Phi(0) \in (0,\infty), \lim_{\tilde{b}\to 0}\Psi(\tilde{b}) = +\infty, \lim_{\tilde{b}\to\infty}\Phi(\tilde{b}) < 0, \lim_{\tilde{b}\to\infty}\Psi(\tilde{b}) = +\infty. \tag{17}$$

We obtain $\partial\Phi/\partial\chi < 0$ and $\partial\Psi/\partial\chi > 0$ and so that the larger χ we choose, the more BGPs become nonexistent [4]. This is all of the proof. □

Figure.1 illsurates whether there are BGPs or not. When χ is lower, graph of Φ lies upside and Ψ downside and then BGPs tend to exist. If higher χ is chosen, Φ moves downward and Ψ up over and then BGPs become nonexistent.

We aim to investigate what kind of fiscal policy is sustainable. Figure 2 illustrates the range of the sustainable fiscal policy parameters with $\alpha = 0.2, \beta = 0.55$ and $A = 12$. These parameters are the same as Braüninger (2005) [5]. In Figure 2, downside region of the dotted line represents sustainable policy in the economy considered by Braüninger (2005) in which public spending is nonproductive and wasted. On the other hand, downside of the solid line indicates sustainable one in our model, where public spending is productive.

[4] Strictly speaking, we can prove this with the saddle-note bifurcation theorem. See Devaney (2003), for example.

[5] Here, we want to compare the set of policy parameters which attains fiscal sustainability, and then, we adopt the same parameters as Braüninger (2005) and do not consider whether these parameters are plausible or not in our model. In following part, we set the plausible parameters again.

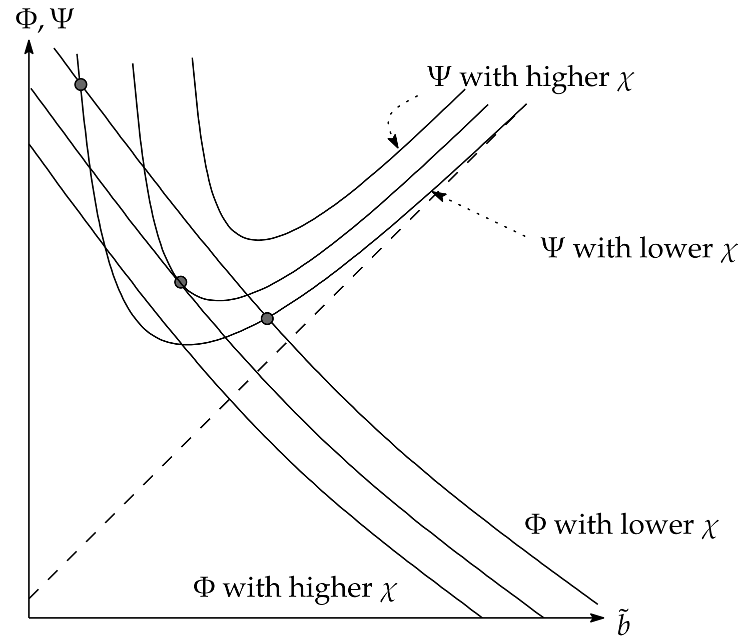

Figure 1: Exsitence of BGPs.

We can intuitively interpret the set of parameters of sustainable fiscal policy as follows. First, we demonstrate why spending-GDP ratio, ξ, must be moderate for fiscal sustainability. If ξ is too small, marginal productivity is insufficient and it negatively affects fiscal sustainability. On the other hand, if ξ is much higher, the 'cost', which contains of tax-rate and issurance of public debt, becomes too much higher and it worsens sustainability. These facts corresponds to Barro (1990), which shows the inverse U-shape relationship between spending-GDP ratio and growth rate. Second, we explain the relationship between debt-spending ratio χ and fiscal sustainability. When the government adopts higher χ, the government issues so much more public debt that the payment of interest of public debt grows and private capital is crowded out in long-term. This means that the higher χ the government chooses, the less fiscal sustainability the economy attains.

The figure implies that our model derives that increasing in ξ may lead good effects on sustainability and when ξ is relatively small, which is not appeared in previous researches about fiscal sustainability. The upward sloping graph in the left side area of Figure 2 means that fiscal policy with higher ξ is more sustainable, because raising χ worsens sustainability as shown. We will show the insight at the section of policy effect.

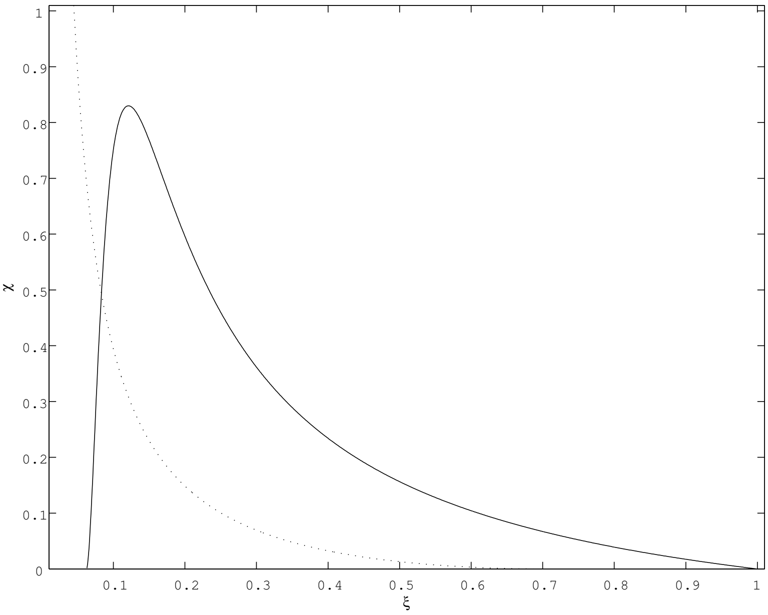

Figure 2: The range of the sustainable fiscal policy parameters. Downside of the line illustrates the area of sustainable fiscal policy, (ξ, χ). [Solid line: productive public spending, Dotted line: wasteful public spending.]

2.1 Stability of BGP

Next, we check the stability of BGP. We obtain the following proposition.

Proposition 2. *When there exists two BGPs, one BGP with smaller $\tilde{b}$ is stable and the other with larger $\tilde{b}$ unstable.*

Proof. By definition of Φ and Ψ, we get that

$$\frac{\tilde{b}_{t+1}}{\tilde{b}_t}\begin{cases}\geq 1 \\ < 1\end{cases} \iff \begin{cases}\Phi(\tilde{b}_t) \leq \Psi(\tilde{b}_t) \\ \Phi(\tilde{b}_t) > \Psi(\tilde{b}_t).\end{cases} \tag{18}$$

Therefore, Figure. 3 shows the statement. □

Proposition 2 tells that figure 3 illustrates that initial ratio of public debt to capital, $\tilde{b}_0$, influences the sustainability. Let the ratio at stable BGP be $\tilde{b}^S$ and at unstable one be $\tilde{b}^U$, respectively. From Figure 3, if $\tilde{b}_0 \leq \tilde{b}^U$ then $\tilde{b}_t$ converges to $\tilde{b}^S$ (or continues to stay $\tilde{b}^U$). On the other hand, if $\tilde{b}_0 > \tilde{b}^U$ then $\tilde{b}_t$ diverses and this means that such fiscal policy is unsustainable. This

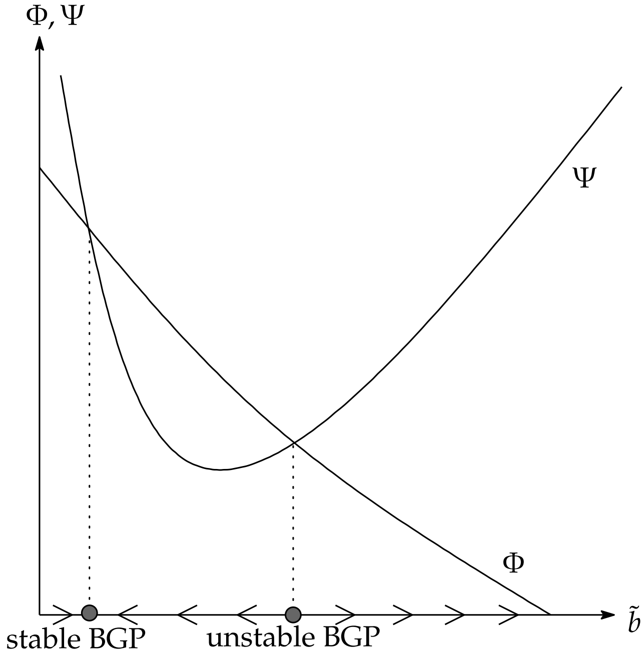

Figure 3: The stability of BGP when there exists two BGPs.

remark implies that it is important for fiscal sustainability that not only policy parameters but also initial stock of public debt. It is consistent for other previous researches.

3 Policy Effect

In this section, we analyse the effect of change of fiscal policy. Concretely, we will investigate the effects on fiscal sustainability and growth rate when the government changes policy parameters, ξ and χ.

Policy Effect on Sustainability

We have already analysed the policy effect on sustainability when the government changes χ. From Prosition 1, the larger χ the government chooses, the more BGPs become nonexistent. Intuitively, increasing χ brings that public debt growth rate rises and capital is crowded out, so that there rarely tends to exist BGPs.

When the government changes ξ under constant χ, policy effect on sustainability may not be the same as the case where changes χ. To analyse it, we differentiate $\Phi = \Psi$ and obtain

$$\frac{d\tilde{b}}{d\xi} = \frac{\Phi_\xi - \Psi_\xi}{-\Phi_{\tilde{b}} + \Psi_{\tilde{b}}}. \tag{19}$$

where Φ_x, and Ψ_x, is partial differentiation of Φ, and Ψ, with respect to x, respectively. Equation (19) tells how changing ξ (with fixed χ) affects $\tilde{b}$ at BGP. The sign of denominator of (19) is decided by whether BGP is stable or unstable; if $\tilde{b}$ is the level at unstable BGP then $-\Phi_{\tilde{b}} + \Psi_{\tilde{b}} > 0$. The sign of numerator of (19) is calculated as

$$\Phi_\xi - \Psi_\xi = A^{1/\alpha}\xi^{1/\alpha - 1}\left[\frac{\beta}{1+\beta}(1-\alpha)\frac{(1/\alpha - 1)\xi^{-1} - 1/\alpha(1-\chi)}{(1+\alpha\tilde{b}_t)} - \frac{\chi}{\alpha}(\tilde{b}_t^{-1} + 1)\right]. \tag{20}$$

and so that if (20) is positive, increasing in ξ gives more sustainable fiscal policy in the sense that that the range of initial debt-capital ratio which achieves fiscal sustainability widens [6]. (20) can be positive when ξ is small. However, it is hard for us to obtain analytical remark on $\partial\tilde{b}/\partial\xi$ and hence we take the numerical example of the sign of $\partial\tilde{b}/\partial\xi$. Figure 4 illustrates the relationship between debt-spending ratio and $\tilde{b}$ at stable BGP.
We calculate with the deep parameters which are $A = 4, \alpha = 0.2, \beta = 0.55$ and policy parameter, $\chi = 0.01$. We can observe the non-monotonic relationship between ξ and $\tilde{b}$ at stable BGP. This was not be got in previous works on fiscal sustainability.

We consider the reason why the non-monotonic relationship arises. Suppose that spending-GDP ratio is much small. The situation means that increasing in public spending-GDP ratio drastically raises interest rate and wage rate through ascenting . This effect dominates the other effects of public costs, raising tax rate and/or more issurance of public debt, and the effect yields net benefits. Therefore, when ξ is much small, raising ξ leads to more fiscal sustainability. Inversely, if spending-GDP ratio is relatively large, when the government enlarges ξ, the effects of costs dominates the effect of marginal productivity. Then enlarging ξ brings less sustainable fiscal policy.

Policy Effect on Growth Rate

Next, we analyse the policy effect on growth rate. Concretely, when we change ξ or χ, how does it make effect on growth rate.

[6] The reason is that debt-capital ratio at unstable BGP means the upper bound of sustainable initial debt-capital ratio.

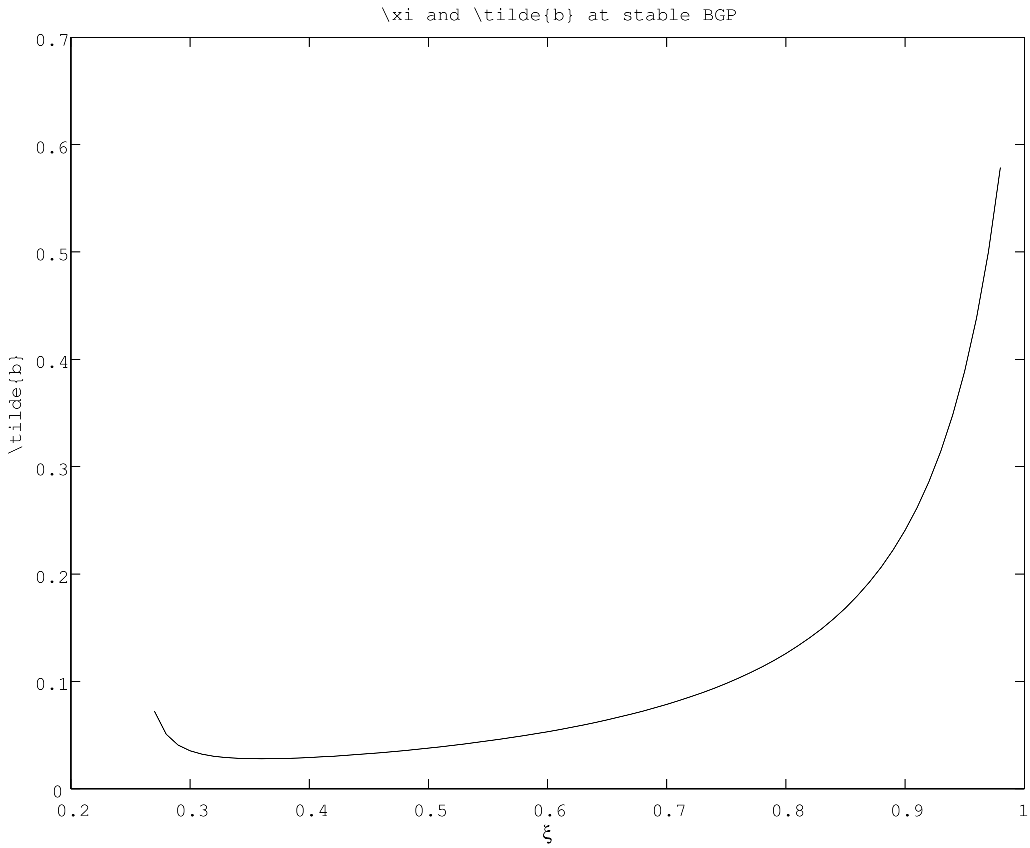

Figure 4: The relation between $\tilde{b}$ at stable BGP and ξ.

Gross growth rate of GDP is represented as those of capital [7],

$$\frac{y_{t+1}}{y_t} = \frac{k_{t+1}}{k_t} = \frac{\beta}{1+\beta}\frac{1+\xi\chi-\xi}{1+\alpha\tilde{b}_t}A^{1/\alpha}\xi^{1/\alpha-1}(1-\alpha) - \tilde{b}_t - \chi A^{1/\alpha}\xi^{1/\alpha}. \tag{21}$$

Hence, we can obtain the growth rate by equation (21).

At first, we investigate the policy effect when change ξ with fixed χ.

[7] This is lead because public spending per GDP is constant from the fiscal policy rule. If the policy rules is changed, this may not be held.

Change of ξ leads to alter growth rate at stable BGP as

$$\frac{\partial\gamma}{\partial\xi} = A^{1/\alpha}\xi^{1/\alpha-1}\left[\frac{\beta}{1+\beta}(1-\alpha)\frac{(1/\alpha-1)\xi^{-1}-1/\alpha(1-\chi)}{1+\alpha\tilde{b}_t} - \frac{\chi}{\alpha}\right] - \left[\frac{\beta}{1+\beta}\frac{1-\xi(1-\chi)}{(1+\alpha\tilde{b}_t)^2}A^{1/\alpha}\xi^{1/\alpha-1}\alpha(1-\alpha)+1\right]\frac{\partial\tilde{b}}{\partial\xi} \quad (22)$$

If $\partial\tilde{b}/\partial\xi$ is negative, the first term of equation (22) is positive [8] and so that $\partial\gamma/\partial\xi$ also is. However, if $\partial\tilde{b}/\partial\xi$ is positive, the first term of equation (22) may be positive or negative. Then the growth rate on the ratio of public spending to GDP cannot be determined analytically.

Proposition 3. *If debt-capital ratio at stable BGP decreases when spending-GDP ratio raises (and debt-spending ratio is fixed), growth rate increases in both short-term and long-term.*

The reason why positive growth effect is generated by raising ξ is that ξ much affects labor productivity. Suppose that the government alters ξ to higher and it makes debt-spending ratio low at stable BGP. This generates three effects: raising wage, income tax-rate and interest rate in short-term. When higher ξ leads to more fiscal sustainablility, the effect of income tax rate is dominated by the others and so that raising ξ brings household's saving in short-term. In addition, in long-term, growth rate of capital rises because $\tilde{b}_t$ gets smaller to $\tilde{b}$ at new stable BGP level and it declines the amount of roll over and the payment of interest of public debt. As the result, we hold proposition 3.

However, if larger ξ means less sustainable fiscal policy, we cannot guarantee holding the same remark, that is, larger ξ can make growth rate lower. We take a numerical examples to demonstrate it. Figure 5 illustrates the relation between growth rate and the ratio of public spending to GDP. We set the policy parameters as $\chi = 0.01$.

Next, we investigate the policy effect when the government changes χ with fixed ξ. The growth rate at BGP is represented by (21),

$$\frac{\partial\gamma}{\partial\chi} = A^{1/\alpha}\xi^{1/\alpha}\left[\frac{\beta}{1+\beta}\frac{1-\alpha}{1+\alpha\tilde{b}} - 1\right] - \left[\frac{\beta}{1+\beta}\frac{1-\xi(1-\chi)}{(1+\alpha\tilde{b}_t)^2}A^{1/\alpha}\xi^{1/\alpha-1}\alpha(1-\alpha)+1\right]\frac{\partial\tilde{b}}{\partial\chi} \quad (23)$$

where γ is the growth rate at BGP. Then $\partial\gamma/\partial\chi < 0$ at stable BGP because $\partial\tilde{b}/\partial\chi > 0$ at stable. This tells that growth rate is decreased in long-term

[8] See the subsection of policy effects on sustainability.

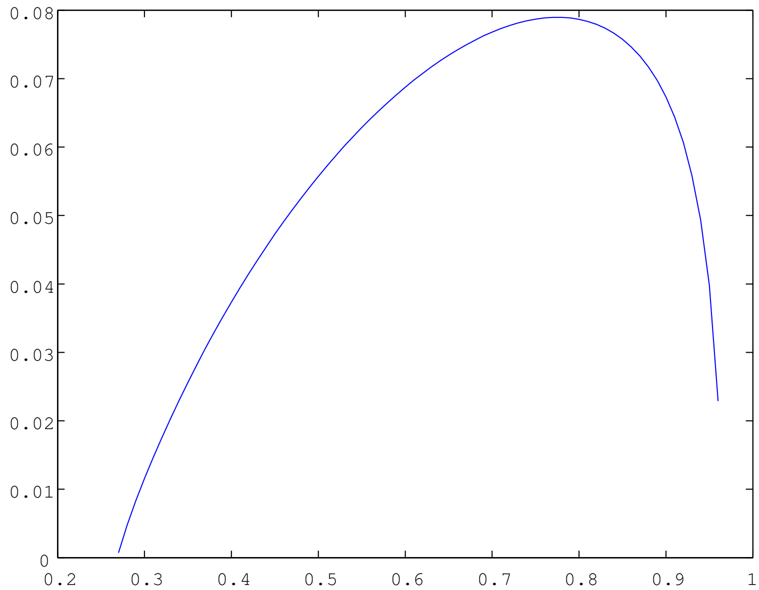

Figure 5: The relation between growth rate at stable BGP and ξ.

when the ratio of the government debt to public spending is increased with constant spending-GDP ratio. Growth rate is also reduced even in short-term.

The reason of change of growth rate is as follows: Suppose that the economy is at BGP at initial period and consider the case where the government chooses larger χ. Household's saving increases from tax-cut but issuance of public debt in next period increases more. Therefore private capitals are crowded out in next period and growth rate descends in short-term. Furthermore $\tilde{b}$ at new stable BGP grows up and growth rate goes down in long-term, because of less saving from tax-raise and crowding out of private capital [9].

Pareto Improving Policy Change

Though raising ξ brings more fiscal sustainability and higher growth rate if ξ is rather small as previous part, this does not mean directly that such policy change is Pareto improving. Increasing in ξ has four channels as we have seen in previous section: it changes interest rate, wage rate, income tax rate and debt-capital ratio. We analyse welfare effect through them when fiscal policy rule changes in follows.

Household's born at t has the indirect utility, V_t, which is derived as

$$\begin{aligned} V_t := (1-\beta)\ln\frac{1}{1+\beta} + \beta\ln\beta + (1+\beta)\ln\left[\frac{1-\xi(1-\chi)}{(1+\tilde{b}_t)}(1-\alpha)A^{1/\alpha}\xi^{1/\alpha-1}\right] \\ + (1+\beta)\ln k_t + \beta\ln\left[1+\frac{1-\xi(1-\chi)}{(1+\alpha\tilde{b}_{t+1})}\alpha A^{1/\alpha}\xi^{1/\alpha-1}\right] \quad (24) \end{aligned}$$

Note that for initial stocks of capital, k_0, we can write $k_t = \prod_{i=0}^{t-1}\gamma_i k_0$ where γ_t^k is capital growth rate, that is,

$$\gamma_t^k := k_{t+1}/k_t = \frac{\beta}{1+\beta}\frac{1-\xi(1-\chi)}{1+\alpha\tilde{b}_t}A^{1/\alpha}\xi^{1/\alpha-1}(1-\alpha) - \tilde{b}_t - \chi A^{1/\alpha}\xi^{1/\alpha}. \quad (25)$$

Suppose that the government announces and sets ξ to higher at period $t = T$ and such policy change brings more fiscal sustainability. We will consider the welfare effect of change of ξ in follows. At first, check the welfare of household born at period $T-1$. $\tilde{b}_T$ and k_T are already determined and then its welfare is affected through change of interest rate and income tax rate. We can show that the impact of interest is larger than of income tax

[9] From Firure 3, we can show that $\tilde{b}_t$ grows up to new stable BGP on a transition path.

because we hold $\partial[(1-\tau_T)r_T]/\partial\xi > 0$ [10)]. In other words, raising ξ makes current generation household happier. Second, the welfare of household born at T depends on not only the same channel as above but also change of wage and growth rate. Nontheless, we obtain the remark that household born at period T improves its welfare because raising ξ leads growth rate to higher, which is shown by proposition 3, and we hold $\tilde{b}_t \leq \tilde{b}_T$ for all $t > T$ by our suppostition. Hence, its welfare must rise by increasing in ξ. Furthermore, we can show that households born after period $T+1$ improve their welfare in similar way. As a result, we obtain the following proposition.

Proposition 4. *If raising spending-GDP ratio brings more sustainable fiscal policy, such policy change is Pareto improving.*

Proof. See Appendix C □

However, when raising ξ brings less fiscal sustainability, we cannot decide the sign of welfare effect analytically even in short-term. In short-term, there are opposite effects through between interest and/or wage rate and income tax rate and, hence, welfare effect may be positive or negative. Additionally, in long-term, $\tilde{b}_t$ increases and converges to $\tilde{b}$ which is debt-capital ratio at 'new' stable BGP. Hence, in general, the government cannot attain Pareto improving by fiscal policy change [11)].

4 Conclusion

We construct a overlapping generations model in which the government expenditures productive public spending and comforts the constant spending-GDP and debt-spending rules, and then analyse the policy effects on fiscal sustainability, growth rate and welfare of households which belongs each generations.

In this model, we obtain three analytical remarks as follows: i) when spending-GDP ratio is small, raising the ratio brings more sustainable fiscal

[10)] This partial differentiation is equivalent for first term of equation (20) and it must be positive if $\partial\tilde{b}/\partial\xi > 0$ at unstable BGP.

[11)] When the government alters χ, it is obvious that this policy change cannot attain Pareto improving. The reason is that raising χ improves welfare of current generation, which occurs through cutting income, tax but worsens welfare of future generations, which generates through lowering growth rate and increaseing in payment of interest of public debt by increase in $\tilde{b}_t$.

policy, ii) if raising spending-GDP ratio leads to more fiscal sustainability, increasing in the ratio makes growth rate higher, and iii) if increasing in spending-GDP ratio gives more sustaianble fiscal regime, such policy change is Pareto improving. In sum, Our contribution is that we reveal the new relationship between spending-GDP ratio and fiscal sustainability. These remarks depends on the assumption which is that government spending directly and instantenously grows marginal productivity, rather than through accumulation of public capital.

Appendix

A Proof of Lemma 1

We prove lemma 1 in Appendix A.

Proof. We prove the statement by following steps.

1. Fix a policy (ξ, χ). If there exists BGPs in the system, we have
$$\frac{\beta}{1+\beta}(1-\alpha)A^{1/\alpha}\xi^{1/\alpha-1}(1-\xi) > 1. \tag{26}$$
2. If ξ satisfies (26), there exists some BGP for some $\chi > 0$.
3. if A is sufficiently large, there exists ξ which satisfies (26).

We can show easily the statment of last step and hence we give the proof of the first and second statements.

Proof of first statement: Fix (ξ, χ) and, by assumption, we have some $\tilde{b}$ such that
$$\frac{\beta}{1+\beta}(1-\alpha)A^{1/\alpha}\xi^{1/\alpha-1}\frac{1-\xi(1-\chi)}{1+\alpha\tilde{b}} - \chi A^{1/\alpha}\xi^{1/\alpha} = 1 + \tilde{b} + \chi A^{1/\alpha}\xi^{1/\alpha}\tilde{b}^{-1}. \tag{27}$$
The left side of (27) is decreasing in χ and $\tilde{b}$. Hence,
$$\frac{\beta}{1+\beta}(1-\alpha)A^{1/\alpha}\xi^{1/\alpha-1}\frac{1-\xi(1-\chi)}{1+\alpha\tilde{b}} - \chi A^{1/\alpha}\xi^{1/\alpha} < \frac{\beta}{1+\beta}(1-\alpha)A^{1/\alpha}\xi^{1/\alpha-1}(1-\xi). \tag{28}$$
As the same way, we have on the right side of (27),
$$1 + \tilde{b} + \chi A^{1/\alpha}\xi^{1/\alpha}\tilde{b}^{-1} > 1. \tag{29}$$
Combining the two inequlities, we obtain equation (27).

Proof of second statement: Fix ξ which satisfies (**??**). Then we can denote $\epsilon > 0$ as

$$\epsilon := \frac{\beta}{1+\beta}(1-\alpha)A^{1/\alpha}\xi^{1/\alpha-1}(1-\xi) - 1 > 0. \tag{30}$$

We want to show that equation (27) has solution(s) for ξ if some small χ. Difference between the left hand and right hand of (27) is calculated with ϵ as

$$\frac{1+\epsilon}{1+\alpha\tilde{b}_t} - (1+\tilde{b}_t) - \chi A^{1/\alpha}\xi^{1/\alpha}\left[\frac{\beta}{1+\beta}(1-\alpha)\frac{1}{1+\alpha\tilde{b}_t} - (1+\tilde{b}_t^{-1})\right]. \tag{31}$$

It is obvious that if χ is adequately small, the difference can be zero for some $\tilde{b}_t$, which means that there exists BGPs for some ξ which satisfies (27) and χ.

□

B Derivation of Equations (14)

In this section, we give an explanation how to obtain equation (14).

First, we compute $(1-\tau_t)$. Total tax revenue, T_t, is

$$T_t = \tau_t\left(r_t s_{t-1} + w_t\right) = \tau_t(y_t + r_t b_t) \tag{32}$$

from equations (8), (9) and (11). Then the government's budget constraint, (4), and policy rules, (5) and (6), leads to

$$\xi\chi y_t = r_t b_t + \xi y_t - \tau_t(y_t + r_t b_t). \tag{33}$$

We solve the equation (33) for τ_t and obtain

$$1 - \tau_t = \frac{1+\xi\chi-\xi}{1+\alpha\tilde{b}_t}. \tag{34}$$

Second, we calculate k_{t+1}/k_t and b_{t+1}/b_t. k_{t+1}/k_t is derived by substitution equations (7), (34) and (6) to (11) as

$$\frac{k_{t+1}}{k_t} = \frac{\beta}{1+\beta}\frac{1+\xi\chi-\xi}{1+\alpha b_t/k_t}A^{1/\alpha}\xi^{1/\alpha-1}(1-\alpha) - \frac{b_t}{k_t} - \chi A^{1/\alpha}\xi^{1/\alpha}. \tag{35}$$

And we obtain b_{t+1}/b_t from equation (6),

$$\frac{b_{t+1}}{b_t} = 1 + \chi\left(\frac{b_t}{k_t}\right)^{-1}A^{1/\alpha}\xi^{1/\alpha} \tag{36}$$

Finally, combining (35) and (36), we get equation (14),

$$\frac{b_{t+1}/b_t}{k_{t+1}/k_t} = \frac{\tilde{b}_{t+1}}{\tilde{b}_t} = \frac{1 + \chi \tilde{b}_t^{-1} A^{1/\alpha} \xi^{1/\alpha}}{\frac{\beta}{1+\beta} \frac{1+\xi\chi-\xi}{1+\alpha\tilde{b}_t} A^{1/\alpha} \xi^{1/\alpha-1}(1-\alpha) - \tilde{b}_t - \chi A^{1/\alpha} \xi^{1/\alpha}}. \tag{37}$$

C Proof of Proposition 4

In this section, we give the proof of Proposition 4 as below.

Proof. Suppose that government fixes χ, raises ξ suddenly at period $t = T$, and this policy change gives more sustainable fiscal policy. We show Proposition 4 by following three steps.

Step1 Welfare of household born at $t = T - 1$ can improve by this policy change.

Step2 Welfare of household born at $t = T$ can improve .

Step3 Welfare of household born after $t = T + 1$ can improve.

Proof of Step1.

The indirect utility of household born at $T - 1$ is given as,

$$\begin{aligned} V_{T-1} := (1-\beta)\ln\frac{1}{1+\beta} + \beta\ln\beta + (1+\beta)\ln\left[\frac{1-\xi(1-\chi)}{(1+\alpha\tilde{b}_{T-1})}(1-\alpha)A^{1/\alpha}\xi^{1/\alpha-1}\right] \\ + (1+\beta)\ln k_{T-1} + \beta\ln\left[1 + \frac{1-\xi(1-\chi)}{(1+\alpha\tilde{b}_T)}\alpha A^{1/\alpha}\xi^{1/\alpha-1}\right] \end{aligned} \tag{38}$$

The household takes $\tilde{b}_{T-1}$, k_{T-1} and $\tilde{b}_T$ as given because they are predetermined. Therefore, when the policy change arises at period T, we have

$$\frac{dV_{T-1}}{d\xi} = \frac{d}{d\xi}\ln\left[1 + \frac{1-\xi(1-\chi)}{(1+\alpha\tilde{b}_T)}(1-\alpha)A^{1/\alpha}\xi^{1/\alpha-1}\right] > 0. \tag{39}$$

This inequlity is held because of the assumption that the policy change brings more fiscal sustainability [12)].

[12)] To check this, see equation (19) and (**??**). It is obvious that if the above inequlity is not held, raising ξ must make fiscal sustainability worse.

Proof of Step2.

The indirect utility of household born at T is got as,

$$V_T := (1-\beta)\ln\frac{1}{1+\beta} + \beta\ln\beta + (1+\beta)\ln\left[\frac{1-\xi(1-\chi)}{(1+\alpha\tilde{b}_T)}(1-\alpha)A^{1/\alpha}\xi^{1/\alpha-1}\right] + (1+\beta)\ln k_T + \beta\ln\left[1+\frac{1-\xi(1-\chi)}{(1+\alpha\tilde{b}_{T+1})}\alpha A^{1/\alpha}\xi^{1/\alpha-1}\right] \quad (40)$$

The household takes $\tilde{b}_T$ and k_T as given. However, $\tilde{b}_{T+1}$ can alter by the policy change. Then, marginal change of the indirect utility is led as

$$\frac{dV_T}{d\xi} = \frac{\partial}{\partial\xi}\ln\left[1+\frac{1-\xi(1-\chi)}{(1+\alpha\tilde{b}_{T+1})}(1-\alpha)A^{1/\alpha}\xi^{1/\alpha-1}\right]. \quad (41)$$

We hold $\partial\tilde{b}_{T+1}/\partial\xi > 0$ and $\partial/\partial\xi[(1-\xi(1-\chi))\xi^{1/\alpha-1}] > 0$ because we suppose that the policy change brings more sustainable fiscal policy. Hence, we get $dV_T/d\xi > 0$.

Proof of Step3.

To prove Proposition 4, we have only to show that the welfare of household born at $T+1$ can improve [13]. The indirect utility of household born at $T+1$ is as follow.

$$V_{T+1} := (1-\beta)\ln\frac{1}{1+\beta} + \beta\ln\beta + (1+\beta)\ln\left[\frac{1-\xi(1-\chi)}{(1+\alpha\tilde{b}_T)}(1-\alpha)A^{1/\alpha}\xi^{1/\alpha-1}\right] + (1+\beta)\ln k_{T+1} + \beta\ln\left[1+\frac{1-\xi(1-\chi)}{(1+\alpha\tilde{b}_{T+1})}\alpha A^{1/\alpha}\xi^{1/\alpha-1}\right] \quad (42)$$

By the discussion in the proof of step1 and 2, the third and last term of (42) is increasing in ξ. Then we aim to show $dk_{T+1}/d\xi > 0$. We rewrite k_{T+1} as below,

$$k_{T+1} = \gamma_T^k k_T = \left[\frac{\beta}{1+\beta}\frac{1-\xi(1-\chi)}{1+\alpha\tilde{b}_T}A^{1/\alpha}\xi^{1/\alpha-1}(1-\alpha) - \tilde{b}_T - \chi A^{1/\alpha}\xi^{1/\alpha}\right]k_T. \quad (43)$$

[13] As the same way, we can show that the welfare of household born after $T+2$ can improve by the policy change.

where $\tilde{b}_T$ and k_T are given. Therefore,

$$\frac{dk_{T+1}}{d\xi} = [\Phi_\xi - \Psi_\xi]k_T > 0. \tag{44}$$

The inequlity is held because the policy change brings more fiscal sustainability. □

References

Barro, R. J. (1990) "Government Spending in a Simple Model of Endogeneous Growth," *Journal of Political Economy*, Vol. 98, pp. S103–S125.

Braüninger, Michael. (2005) "The Budget Deficit, Public Debt, and Endogenous Growth," *Journal of Public Economic Theory*, Vol. 7, No. 5, pp. 827–840.

Chalk, N.A. (2000) "The sustainability of bond-financed deficits: An overlapping generations approach," *Journal of Monetary Economics*, Vol. 45, No. 2, pp. 293–328.

Devaney, R.L. (2003) *An Introduction to Chaotic Dynamical Systems*: Westview Press.

Futagami, K. and A. Shibata (2003) "Budget deficits and economic growth," *Public Finance*, Vol. 53, No. 2-3, pp. 331–354.

Futagami, K., Y. Morita, and A. Shibata (1993) "Dynamic Analysis of an Endogenous Growth Model with Public Capital," *The Scandinavian Journal of Economics*, Vol. 95, No. 4, pp. 607–625.

Rankin, N. and B. Roffia (2003) "Maximum Sustainable Government Debt in the Overlapping Generations Model," *The Manchester School*, Vol. 71, No. 3, pp. 217–241.

Yakita, Akira. (2008) "Sustainability of public debt, public capital formation, and endogenous growth in an overlapping generations setting," *Journal of Public Economics*, Vol. 92, No. 3-4, pp. 897–914.