# Markets with endogenous uncertainty: theory and policy

Chichilnisky, Graciela

UNESCO, Stanford Institute for International Studies, Columbia Business School

1996

GRACIELA CHICHILNISKY

# MARKETS WITH ENDOGENOUS UNCERTAINTY THEORY AND POLICY

*Dedicated to Tjalling C. Koopmans*

ABSTRACT. Classic formulations of markets regard uncertainty as originating from acts of nature. I extend this to a formulation of markets which face risks induced by the economy itself, such as the environmental risks of atmospheric and climate change induced by CFC and $CO_2$ emissions.

I formulate and prove the existence of a general competitive equilibrium where the state space and the probabilities of events are endogenously determined as part of the equilibrium. Traders take optimal positions with respect to the uncertainty which their own actions induce. The equilibrium allocations are efficient in a restricted sense. I show that scientific uncertainty can be fully hedged. However uncertainty induced by the unknown level of output at an equilibrium cannot be hedged fully. I discuss applications for CAT Futures, recently introduced on the Chicago Board of Trade, and to international environmental strategies.

KEY WORDS: endogenous uncertainty, markets, general equilibrium, financial innovation, CAT futures.

## 1. MARKETS WITH ENDOGENOUS UNCERTAINTY

For several centuries financial activity has evolved around the need for hedging weather risks, for example through mutual insurance within agricultural societies.[1] Corresponding to this, classical formulations of a market economy regard uncertainty as arising from acts of nature.

Today's markets face a new type of risk, such as the environmental risks induced by emissions of CFC's and greenhouse gases, which can deplete the atmosphere of its ozone layer and induce a climate change.[2] The tables have been turned on nature: for the first time in history economic activity has reached levels at which it risks changing the planet's atmosphere and its climate.

Risks which are partly induced by economic actions[3] are called "endogenous uncertainty". The aim of this paper is to extend the

*Theory and Decision* **41:** 99–131, 1996.

Arrow–Debreu theory to incorporate resource allocation under endogenous uncertainty. The concerns about endogenous uncertainty break new ground, going beyond the scope of the classical formulations of market economies.[4] The problem of uncertainty which is induced by economic activity has a distinguished ancestry: it was originally suggested by T.C. Koopmans to K.J. Arrow many years ago, while both were at the Cowles Foundation of the University of Chicago.[5]

How does economic activity induce uncertainty? It does so in many ways. For example, the use of chlorofluorocarbons (CFC's) for industrial production and the burning of fossil fuels to generate energy produce emissions can pierce the planet's protective ozone layer, and induce climate change. Economic activity can alter the probabilities of different states of the world's climate. Other examples arise in financial markets, where new instruments can induce new risks. The introduction of derivative securities helps heighten the interdependence of financial markets and market participants, and disturbances are rapidly transmitted throughout the world economy.[6] Financial innovation can therefore cause more interconnected transactions and by itself lead to new risks.[7]

The practical importance of this new type of uncertainty is best exemplified by two questions which are the subject of scientific and policy debate today: How much should we pay to decrease the probability of a catastrophic risk, such as the destruction of the ozone layer or global warming? How should the use of derivative securities be regulated in order to control the financial risks which they induce?

Implicit in these questions is the acknowledgement that the risks we face are induced by human action, and that they are to a certain extent under our control. Such questions cannot be posed, let alone answered, within an Arrow–Debreu formulation of markets, where risks represent solely nature's moves and are beyond our control. Within the framework of Arrow–Debreu markets traders cannot alter the actual risks which they face. We cannot use the Arrow–Debreu framework to decide how much to pay for policies which are impossible within that framework. Yet the questions posed by environmental policy are compelling and will not go away: they deserve answers. It is possible to formulate these questions precisely in the context of markets with endogenous uncertainty as formalized here. The results in this paper are oriented to the analysis of such questions. Global environmental risks are good examples. The United Nations, the World Bank and the OECD all regard the risk of global warming as centrally important, and consider the allocation of very sizable budgets to hedge this risk.[8] In the conclusions I offer a framework for environmental policy analysis.

### 1.1. *Equilibrium with Endogenous Uncertainty: Results*

The following sections formulate and prove the existence of a general equilibrium where the state space and the probabilities of the events are endogenously determined as part of the equilibrium. The aggregate activity of the economy induces the uncertainty which the traders face, namely the state space and the probabilities of each state. Traders take optimal positions with respect to risks which their own actions induce. A competitive equilibrium is a set of states, probabilities, prices, outputs and net trades, where the traders' actions are optimal with respect to the risks which these same actions induce. Theorem 1 shows that under very general conditions an equilibrium with endogenous uncertainty exists generically on technologies and endowments. The proof depends on topological arguments; these are useful because the standard conditions which would allow the use of fixed point theorems are not met in this context.[9]

### 1.2. *Scientific Uncertainty and Endogenous Uncertainty*

Markets with endogenous uncertainty include Arrow–Debreu markets as special cases: this is when all endogenous uncertainty is trivial. However, I show below that except for such trivial cases these markets can be fundamentally different from Arrow–Debreu markets.

Scientific uncertainty is part of the problem. By *scientific uncertainty* I mean the uncertainty derived from the fact that scientists do not know exactly how industrial activity alters the risks we face. Each vector of aggregate output could induce one of several probabilities over states of natures; we do not know which is the true probability. It is established below that this type of uncertainty can be hedged fully with the use of CAT (for "catastrophe") futures, Section 5. These financial instruments were introduced theoretically in [13] and [15] and have started recently trading in the Chicago Board of Trade. Financial innovation can deal effectively with scientific uncertainty. There is however a more pervasive and complex, type of endogenous uncertainty: ignorance about the output of the economy in equilibrium. This occurs, for example when there are several possible market equilibria, each with a different aggregate output, and one is aware that one of these equilibria may occur without knowing which. This latter type of uncertainty I call *strict endogenous uncertainty*, because it depends strictly on the functioning of the economy, rather than on scientific research. Theorem 3 shows that strict endogenous uncertainty cannot be hedged fully, no matter how many securities are introduced. Pareto efficient allocations of risk are therefore not possible.

### 1.3. *Explaining Incomplete Markets*

The difficulties involved in hedging endogenous uncertainty are revealing. By themselves they can provide an explanation of why markets are incomplete. As already mentioned, Theorem 3 proves that markets with strict endogenous uncertainty cannot hedge uncertainty fully: they are essentially incomplete. The incompleteness is a different phenomenon from that which appears in the standard theory of markets with exogenous uncertainty and with incomplete asset structures.[10] The latter assumes by fiat that some assets do not exist, and therefore that the transfer of wealth from one state of nature to another is not possible. The incompleteness of markets is not explained in such models: it is assumed. Furthermore, this standard incompleteness can always be removed by the introduction of a number of "Arrow securities", which pay a unit of account for each state of nature into which it was not possible to transfer wealth before.

By contrast, the formulation presented here explains incomplete markets as arising from endogenous uncertainty. When incomplete markets arise from strict endogenous uncertainty as defined here, it does not go away by adding more assets: in general it is not possible to complete a market with endogenous uncertainty.[11]

Markets with endogenous uncertainty lead to allocations which are efficient in a constrained sense (Proposition 2). However if an economy with endogenous uncertainty has $k \geqslant 2$ distinct equilibria, and the traders are aware of this fact, in general the corresponding equilibria allocations are inefficient (Proposition 3). This issue is analyzed in Section 5 which explores the possibility of completing a market with endogenous uncertainty by financial innovation, i.e. by the introduction of additional financial instruments. Theorem 2 shows that if the uncertainty is solely scientific, then the introduction of a number of instruments akin to CAT (catastrophe) futures will suffice to reach a Pareto efficient allocations. Theorem 4 shows that when the uncertainty is about the equilibrium aggregate output in the economy, i.e. strictly endogenous, Pareto efficient allocations cannot be reached. Section 6 discusses policy implications.

## 2. MARKETS WITH ENDOGENOUS UNCERTAINTY

A market economy $E$ has $H \geqslant 2$ traders, and $J \geqslant 2$ firms which produce $N \geqslant 2$ commodities over $T \geqslant 1$ periods of time. There are $S \geqslant 1$ states of exogenous uncertainty representing acts of nature. To simplify the exposition, and to isolate the essential features which distinguish endogenous uncertainty, the formulation of the market $E$ will be identical to the classic Arrow–Debreu formulation in every possible way except in the treatment of uncertainty. In particular, in the market $E$ all traders are competitive and there is symmetric information.[12]

$E$ has a complete set of financial markets for *exogenous* uncertainty, which is formalized by the realizations of random variables which represent acts of nature.[13] Each trader $h$ has an initial endowment of goods and assets, $\Omega_h \in R^{(N\times S\times T)+}$, and an initial endowment of shares in the $J$ firms, $\theta_h = (\theta_h^1 \dots \theta_h^J) = 1$. The economy's technology is described within every state $i = 1 \dots S$, by a convex, compact, closed production possibility set[14] $Y^i \subset R^{N\times T}$, and

$$(1) \qquad Y = \Pi_{i=1}^{S} Y^i \subset R^{N\times T\times S}.$$

For each price vector $p \in \Delta_{N\times T\times S}$,[15] let $y(p) \in Y$, denote the aggregate supply vector – or set of vectors – which obtain from profit maximizing behavior on the part of the producers. Corresponding to each price $p$ trader $h$ receives profits $\chi_h(p) \geqslant 0$. Because all uncertainty up to now is exogenous, the economy described so far is identical to a standard Arrow–Debreu economy with commodity space $R^{N\times T\times S}$ and with no uncertainty.[16]

### 2.1. *Endogenous Uncertainty*

This section introduces a new aspect of the economy $E$, its endogenous uncertainty, which takes it beyond the Arrow–Debreu structure and gives it drastically different properties.

In the Arrow–Debreu treatment of markets[17] probabilities are fixed. There is either a *fixed* probability distribution over the given set of natural events $S$, or, alternatively, there are different subjective distributions for each trader, each of which is also fixed. Here, instead the set of events and the probabilities over these states are all *variable*: both vary over the set of all possible states and all possible probabilities.[18]

In the following I formalize uncertainty arising from economic activity. To motivate this, consider the environmental problem discussed above: each vector of aggregate output of goods in the world economy induces a level of emissions of $CO_2$ or of CFCs. Corresponding to each level of emissions new states of nature may develop, such as for example a state where the ozone layer is 50% damaged, or where there is a disruption of the planet's weather pattern known as global climate change. Of course today's industrial production could induce changes in tomorrow's state of nature: nature could react with a lag.[19] Alternatively, the impact could be simultaneous: tomorrow's industrial production induces changes in tomorrow's climate.

All of these possibilities are formalized within one simple, general framework: the endogenous uncertainty in the economy $E$ is expressed as a relation between the vector of aggregate output of the economy and the associated states of the economy with their corresponding probability distributions, describing the risks which traders face for each vector of aggregate output. The set $\{1 \ldots D\}$ represent all possible states, current or latent, so that $D \geqslant S$.[20] Clearly, endogenous uncertainty can be represented simply by a relation between output vectors $y$ and those density or probability functions $\{\pi(y)\}$ associated to $y$; the corresponding set of states are simply those states in $\{1 \ldots D\}$ which have non-zero probabilities.[21] The following formulation includes lagged effects: some of the coordinates of the output vector $y$ can be indexed by time, as it is standard in Arrow–Debreu markets, and changes in the components of the vector $y$ in time periods prior to $n$ may affect the states of nature in periods $n + k$, where $k$ is the lag. A possible interpretation is that traders inflict externalities on each other: their consumption and production decisions influence total output levels and therefore other traders' utilities through changes in their probabilities.

ASSUMPTION 1. There exists a scientific relation between the aggregate output of the economy and the probabilities over possible events, given by an implicit function linking aggregate output vectors $y \in R^{N \times T \times D}$ and probabilities $\pi \in \Delta_D$:

$$\begin{aligned} &\Psi(y, \pi) = 0 \\ &\text{where } \Psi = (\Psi^1, \ldots, \Psi^D) : R^M \times \Delta_D \to R^D, \\ (2) \qquad &\qquad M = N \times T \times D \\ &\text{and } \Delta_D = \left\{ \pi = (\pi_1 \ldots \pi_D) : \pi_i \geqslant 0 \text{ and } \sum_{i=1}^{D} \pi_i = 1 \right\}; \end{aligned}$$

$\Psi$ is smooth on a neighborhood of its domain,[22] it has a solution for every $y \in R^M$ and if $\Psi(y, \pi) = \Psi(y', \pi') = 0$, $\|y'\| > \|y\|$ then $\min_i(\pi'_i) > \min_i(\pi_i)$.[23]

The economy with endogenous uncertainty has $M = N \times T \times D$ markets. Therefore its prices are vectors in the unit simplex $\Delta_M$.

The following assumption requires smooth supply behavior:

ASSUMPTION 2. The aggregate supply function of the economy, $y : \Delta_M \to R^M$, and, for each trader $h$, the trader's profit function $\chi_h : \Delta_M \to R$, are smooth maps on $\Delta_M$.

The following assumption focuses the analysis on endogenous uncertainty, as distinct from the exogenous uncertainty which arises with changes in resources or production; this assumption is unnecessary for the results in this paper and can be removed at no cost:

ASSUMPTION 3. The total endowments and the production technologies of the economy $E$ are the same across all states of endogenous uncertainty:

$$\forall i, j \in \{S + 1, \ldots, D\}$$

$$\sum_{h=1}^{H} \Omega_{hi} = \sum_{h=1}^{H} \Omega_{hj}, \text{ and } Y^i = Y^j,$$

where $\Omega_{hi}$ is trader $h's$ endowment in state $i$, and $Y^i$ is the production set in state $i$. Labor is a commodity in this economy so that $\Omega_{hi} > 0 \forall i \in D$.

The following assumption formalizes the intuition that in "small" societies uncertainty is about acts of nature: such societies' have relatively small levels of output, and their effects on nature can be considered negligible:

ASSUMPTION 4. Small levels of output are associated to probability distributions which represent natural risks.[24] There exists a probability $\pi^0 \in \Delta_D$, such that

$$\Psi(0, \pi^0) = 0,$$

where $\pi^0$ gives positive probability to all exogenous states, $\pi_i^0 > 0 \forall i \in \{1, \ldots S\}$, and $\Psi$ is locally constant on its second term at $(0, \pi^0)$.

The latter implies that at small output levels scientific instruments cannot distinguish between infinitesimal changes in probabilities.

Assumption 4 implies:

PROPOSITION 1. *When the economy's production structure is very "small", i.e. $Y = \{0\}$, then there exists a standard Arrow–Debreu pure exchange economy with exogenous uncertainty about $S$ acts of nature, and with commonly shared probabilities $\pi^0$, which equals $E$.*

This assumption can be easily extended to assume that the (fixed) probabilities $\pi^0$ are not commonly shared: the model and its results hold when each trader $h$ has possibly different subjective probabilities[25] $\pi_h^0$.

## 2.2. *Market Equilibrium with Endogenous Uncertainty*

How do rational traders behave in the face of endogenous uncertainty? This depends on the structure of information of the economy. I assume here the simplest possible structure in order to isolate the essential features of the problem.[26]

### 2.2.1. *The Structure of Information*

Traders know that endogenous uncertainty exists, and no more. This paper is about market equilibria, not about the process by which the economy arrives at one. The formulation presented below is consistent with several different interpretations. One interpretation is that traders know the aggregate output $y$ and the map $\Psi$ which is the scientific relations between aggregate output and risks faced by the economy, so that they also know the possible probability distributions, at any price vector. One can consider a different interpretation: a process similar to the price discovery process in the Arrow–Debreu theory. The endogenous states and their probabilities could be announced by an auctioneer the same way that prices are announced by an auctioneer in a Walrasian market. The role of the Walrasian auctioneer in the Arrow–Debreu theory is to announce prices; and ensure that no trading takes place until an equilibrium is reached. Here the (expanded) role of the auctioneer would be to announce prices, states and probabilities, and ensure that no trading takes place until an equilibrium with endogenous uncertainty has been reached. The auctioneer's announcements about states and their probabilities are neither correct nor false, in the same way that a Walrasian auctioneer announces any prices, and not just the equilibrium prices. In the same vein, here the announced states and probabilities may or not be the ones which will eventually emerge in a market equilibrium. In this paper there is no assumption about perfect foresight, nor is there any other assumption or formulation of expectations. But in any case one assumes:

ASSUMPTION 5. Each competitive trader considers the world's endogenous uncertainty states and their probabilities as independent of her/his individual actions.

This is a realistic assumption in economies where uncertainty has some of the characteristics of a "public good";[27] as was already indicated, the economy can be thought of as having "externalities" because each trader's actions can alter the probabilities of events which enter in other trader's utilities. The intuition for Assumption 5 is that traders are "small" so that while the aggregate output of the economy does affect endogenous uncertainty, each trader takes the world's endogenous uncertainty as a parameter. This mimics the situation with respect to prices in the standard competitive markets: each trader takes prices as given even though everyone's actions has some influence on the prices at an equilibrium.

2.2.2. *The trader's choice under endogenous uncertainty*
Having established the structure of information, the trader's problem of choice under uncertainty is straightforward. Under the standard von Neumann–Morgenstern axioms for choice under uncertainty:

ASSUMPTION 6. For any given price vector $p \in \Delta_M$ and any given set of states of uncertainty[28] $e \subset D = \{1 \dots D\}$ with probabilities[29] $\pi = \{\pi_i\}_{i \in D}$, each trader $h$ chooses a consumption vector $d_h(p, \pi) = (d_{h1} \dots d_{hD}) \in R^M$, $d_{hi}(p, \pi) \in R^{N \times T}$ which maximizes the expected utility of consumption

$$(3) \qquad \sum_{i=1}^{D} \pi_i u_h^i(d_{hi}(p, \pi)) = Max \left( \sum_{i=1}^{D} \pi_i u_h^i(z_i(p)) \right)$$

where the maximization is over the set of consumption vectors $(z_1(p), \dots, z_D(p))$ having a value equal to that of the trader's endowments plus the trader's share of profits:[30]

$$(4) \qquad \sum_{i=1}^{D} \langle p_i, z_i(p) \rangle = \sum_{i=1}^{D} [\langle p_i, \Omega_{hi} \rangle + \chi_h(p_i)],$$

where $p = (p_1, \dots, p_D)$, and $\forall i, p_i \in \Delta_{N \times T}$. Since each trader takes the probability vector $\pi$ as given[31] the maximization problem in (3) and (4) has a unique solution as a function of the price vector $p$ when $u_h^i$ is strictly concave. Observe that in a states $i$ with zero probability, there is no demand for goods with positive prices.

Furthermore, by suitable boundary approximations, the solutions can be made to vary smoothly with $u_h^i$ sufficiently smooth:

ASSUMPTION 7. For each vector of probabilities in $\Delta_M$, the solution to problem (3) defines a smooth demand function of prices $d_h : \Delta_M \to R^M$[32] for each trader $h$.

ASSUMPTION 8. Boundary behavior: For any probability vector $\pi \in \Delta_F^0 \subset \Delta_D$, with $F \leqslant D$, if $\lim_{n\to\infty} p^n \to \partial\Delta_{N\times T\times F} \subset \Delta_D$, then the corresponding aggregate demand vector increases[33] beyond the economy's bounded resources: $\lim_{n\to\infty} \| \sum_{h=1}^{H} d_h(p^n,\pi) - \Omega_h \| > \|Y\|$.

DEFINITION 2. A market economy $E$ with endogenous uncertainty is an economy with $H$ traders, $J$ profit maximizing firms which produce $N$ goods in $T$ periods of time, a production technology $Y$ as described in Assumption 2 above, and a structure of endogenous uncertainty described by a function $\Psi$ as in Assumption 11, where each trader maximizes preferences as described in (3) and 4, and satisfying Assumptions 1–8.

## 3. EXISTENCE OF A COMPETITIVE EQUILIBRIUM WITH ENDOGENOUS UNCERTAINTY

The following definition of a competitive equilibrium with endogenous uncertainty formalizes the notion that the set of states of uncertainty and their probabilities are determined as part of the equilibrium:

DEFINITION 3. A *competitive equilibrium with endogenous uncertainty* is: a price vector $p^* \in \Delta_M$, an aggregate production vector of the economy $y^* \in Y$, a set of states describing uncertainty $e^*$, $e^* \subset \{1 \ldots D\}$, each state with a corresponding probability $\pi_i^* > 0$, $i \in e^*, \sum_{i\in e^*} \pi_i^* = 1, \pi^* = \{\pi_i^*\}$, and for each trader $h$ a consumption vector $d_h(p^*,\pi^*) \in R^M$, $d_h(p^*,\pi^*) = \{d_{hi}(p^*,\pi^*)\}_{i\in e^*}$, $\forall i, d_{hi}(p^*,\pi^*) \in R^{N\times T}$, such that:

1. For each trader $h$ the consumption vector $d_h(p^*,\pi^*)$ is optimal for problem (3) with constraint (4), for the set of endogenous states $e^*$ with associated probabilities $\pi_i^*, i \in e^*$,

2. The aggregate production vector $y^*$ is profit maximizing within $Y$ at the equilibrium prices $p^*$:

$$y^* = y(p^*),$$

3. All markets clear at each state $i$:

$$\forall i \in e^*, \sum_{h=1}^{H} d_{hi}(p^*, \pi^*) - \Omega_h^i = y^i(p^*),$$

and the states of endogenous uncertainty $e^*$ with probabilities $\pi^* = \{\pi_i^*\}_{i \in e^*}$ are within those induced by the aggregate output of the economy at the equilibrium:

$$\Psi(p^*, \pi^*) = 0.$$

The existence of a market equilibrium with endogenous uncertainty can be established under general conditions generically on technologies, namely for an open dense set of production technologies $Y$. To make this precise one needs to define a concept of proximity of technologies: two technologies are close to each other if the feasible vectors under one are close to the feasible vectors under the other in the euclidean metric of $R^M$.[34]

Consider the set of all market economies $E_Y$ with endogenous uncertainty which differ solely on their production technologies $Y$ and the traders' initial endowments, $\Omega_h$, $h = 1 \ldots H$. An open dense set of such economies consists of economies whose technologies vary over an open dense set of technologies $Y$ according to the distance defined in the Appendix, and whose endowments vary over an open dense subset of $R^{M \times H}$. The following result establishes that for an open and dense set of such economies, there exist an equilibrium with endogenous uncertainty:

THEOREM 4. *Under Assumptions 1–8, there exists a competitive equilibrium for a market with endogenous uncertainty, generically on technologies and endowments.*

*Proof.* In the appendix. ∎

## 4. SCIENTIFIC UNCERTAINTY AND STRICT ENDOGENOUS UNCERTAINTY

It seems worth distinguishing between two types of endogenous uncertainty. The first is the simplest: *scientific uncertainty*. This is uncertainty induced by ignorance about the scientific relations between economic activity and environmental risks, rather than about how the economy itself will perform. I formalize this type of uncertainty by considering an economy which has only one possible equilibrium aggregate output $y \in Y \subset R^M$ which clears the markets, but corresponding to this unique vector of output there exists several possible probability densities over the set of states $\{1 \ldots D\}$, and science does not know which is the true distribution. I show below that in such an economy the uncertainty is endogenously induced, but minimally so: the endogeneity is rather simple. If the environment is relatively stable, with enough observations and through learning, the traders could in principle find out the set of possible densities over $D$. Economies of this type are relatively simple and well behaved. I show below that by introducing certain financial instruments to hedge scientific uncertainty, the economy becomes equivalent to an Arrow–Debreu market, and its equilibria are efficient allocations.

It can be argued, however, that such simple economics do not embody "true" endogenous uncertainty, in the sense that there is no uncertainty about which vector of aggregate economic output will emerge at an equilibrium: only one is possible. A more realistic situation is when the economy has more than one competitive equilibrium each with a different aggregate output, and correspondingly different densities over the states in $D$. In such economies it is not possible to "learn the true probabilities": there is always more than one equilibrium, and therefore the uncertainty persists. This motivates the following definitions:

DEFINITION 5. A market economy $E$ as in Definition 2 is said to have *scientific uncertainty* if it has only one equilibrium level of aggregate output as in Definition 3, corresponding to which there are several possible probability distributions over the states in $D$.

DEFINITION 6. A market economy $E$ as in Definition 2 is said to have *strict endogenous uncertainty* if it has more than one equilibrium as in Definition 3, and at least two of these equilibria have different probabilities over $D$.

The following sections will examine resource allocation in these two types of markets with endogenous uncertainty. Markets with *strict endogenous uncertainty* are more complex than the rest; they embody some of the conceptual issues which one identifies with the concept of endogenous uncertainty.

## 5. RISK ALLOCATION WITH ENDOGENOUS UNCERTAINTY

This section analyzes risk allocation in economies with endogenous uncertainty. The concept of Pareto efficiency can be ambiguous in such economies. This is because the traders' von Neumann–Morgenstern preferences are defined with reference to the probabilities of the events. Within the Arrow–Debreu framework these are either subjective or objective, but in any case *fixed*. Here matters are different. The probabilities are now endogenously defined as part of an equilibrium. Therefore the *traders' preferences could themselves vary with the equilibrium*. Pareto efficiency of an allocation could become a self-referential concept, in the sense the allocation itself induces the probabilities in the preferences by which the optimality of the allocation is evaluated. It is however possible to define a restricted concept of efficiency in markets with endogenous uncertainty:

DEFINITION 7. An allocation of the economy $E$ is a vector $x \in R^{M \times H}$; where $x = (x_h)_{h=1\ldots H}$, $x_h \in R^M$; $x$ is called *feasible* if $\sum_{h=1}^{H} (x_h - \Omega_h) \in Y$, i.e. when the sum of what is allocated in excess of the economy's endowments is produced under profit maximization.

DEFINITION 8. A feasible allocation $x = (x_h)_{h=1\ldots H} \in R^{M \times H}$ in $E$ is called *restricted* or *r-Pareto efficient* when it is Pareto efficient relative to allocations according to the preferences prevailing at $x$, i.e., when there exists no other feasible allocation $y = (x_h)_{h=1\ldots H} \in R^{M \times H}$ in $E$ such that for all $h = 1 \ldots H$,

$$\sum_{i \in e(y)} \pi_i(x) u_h^i(y_h) \geqslant \sum_{i \in e(y)} \pi_i(x) u_h^i(x_h)$$

with strict inequality for some $h$.

PROPOSITION 9. *A competitive equilibrium of a market with endogenous uncertainty is restricted Pareto efficient.*

*Proof.* This follows immediately from the first welfare theorem. ∎

DEFINITION 10. A feasible allocation is *Pareto efficient* in the market $E$ if there exists no other feasible allocation in $E$ which everyone prefers, and which someone prefers strictly.

The following proposition addresses risk allocation in economies which have several equilibria, but each equilibrium has the same set of probability density over $D$. How can two equilibria of the economy $E$ have the same probabilities over states in $D$ and still be different? This happens when the aggregate output vector is the same in each of the equilibria. [35] In such an economy two equilibria can only differ in terms of a redistribution of the same total consumption across individuals; in addition there must be no scientific uncertainty as defined above. In any case, as the probabilities are the same in each equilibrium of the economy $E$, then the preferences prevailing at any of the equilibria are also the same, and one obtains:

PROPOSITION 11. *Let $E$ be an economy with endogenous uncertainty having the same probability distributions over states in $D$ at every equilibrium. Then every equilibrium of $E$ is Pareto efficient.*

*Proof.* This is an immediate corollary of Proposition 9 above. ■

## 6. INEFFICIENT EQUILIBRIA

The following results exhibit a main difference between the Arrow–Debreu market, and markets with endogenous uncertainty. In Arrow–Debreu economies a competitive equilibrium is always Pareto efficient. This property is very general: it is the first theorem of welfare economies and is true whether or not the economy has more than one Walrasian equilibrium, and whether or not the equilibria prices are different for different goods in different states: in all cases each competitive equilibrium is Pareto efficient.[36] This property can break down in markets with endogenous uncertainty. Intuitively this could be due to the "externalities" which each trader causes on others.

DEFINITION 12. Two equilibria are called *distinct* if they assign different prices to one state of endogenous uncertainty.

PROPOSITION 13. *If a market economy $E$ with endogenous uncertainty has $k \geqslant 2$ distinct market equilibria, then all its equilibria are Pareto inefficient.*

*Proof.* If the market economy $E$ has two distinct equilibria, say with price vectors $p_1^*$ and $p_2^* \in \Delta_M$ respectively, then each of the two equilibria assigns a different state-contingent price vector to some state $i$ of endogenous uncertainty, i.e. $\exists i > S : p_1^{i*} \neq p_2^{i*}$. But profit maximization and strictly convexity implies then that the total output produced in equilibrium 1 at state $i$ differs from that produced in equilibrium 2 in state $i$, $y_1^{i*} \neq y_2^{i*}$. Since in equilibrium all markets clear at each state, this implies that at least one trader, say trader $h$, consumes a different amount in state $i$ in equilibrium 1 than the amount he/she consumes in equilibrium $2 : x_{hi1}^* \neq x_{hi2}^*$. I shall now show that in that case the equilibria cannot be Pareto efficient. Since the production sets are concave, then the new allocation $z^* = (z_h^*)_{h=1...H}$ defined $\forall h$ by: $z_{hj}^* = x_{hj1}^*$ for all $j \neq i$ and $\lambda x_{hi1}^* + (1-\lambda)x_{hi2}^*$ for some $\lambda \in (0,1)$, is feasible, because the production sets are convex and $Y^i = Y^j$ in all states $i, j$ of endogenous uncertainty, Assumption 3. Furthermore, since preferences are strictly concave, $z^*$ is preferred at each equilibrium by all traders to the initial allocation and strictly preferred by trader $h$. This completes the proof. ∎

Consider an economy with scientific uncertainty, in which all traders know that one of several probabilities will emerge, and traders satisfy von Neumann–Morgenstern axioms for choice under uncertainty with respect to all possible probabilities.

PROPOSITION 14. *The economy $E$ with scientific uncertainty has Pareto inefficient equilibrium.*

*Proof.* This follows from the fact that there are no securities to transfer wealth from the states corresponding to different probability distributions; for a proof see Cass, Chichilnisky and Wu [4], and Chichilnisky and Heal [15]. ∎

## 7. FINANCIAL INNOVATION AND ENDOGENOUS UNCERTAINTY

In the previous sections I showed that markets with endogenous uncertainty have competitive equilibria very generally, and that their competitive equilibria give rise to Pareto efficient allocations in a restricted sense. However I also showed that the market allocations of an economy with endogenous uncertainty can be Pareto inefficient if the economy has several distinct equilibria, of if the economy has scientific uncertainty. This is true even though the conditions usually required for the first welfare theorem are satisfied: traders and firms are competitive, symmetric information prevails, and traders' preferences are strictly increasing in consumption.

One reason for the inefficiency of the markets considered until now is that they have no specific financial assets to hedge scientific uncertainty. Traders know that several possible probabilities can emerge, but have no means for hedging this risk.

PROPOSITION 15. *The market with scientific uncertainty E is incomplete, in the sense that it has no assets to hedge against the different possible densities which can emerge at an equilibrium.*

It seems therefore natural to introduce in our economy new assets which pay contingent on the realization of different probabilities. The introduction of new assets is called *financial innovation*.

## 7.1. *Financial Innovation*

In standard markets with incomplete asset structures [12] it is always possible to introduce new assets and therefore complete the market. Indeed, through the introduction of Arrow securities with allow the transfer of wealth across states, incomplete markets (with exogenous uncertainty) can always be completed, in the sense that they become identical to an Arrow–Debreu market. When all possible such assets have been introduced and the economy is identical to an Arrow–Debreu market, one says that the markets have been *completed*.

DEFINITION 16. Consider a standard market economy with *exogenous* uncertainty describing acts of nature, and with $S$ states of nature.[37] Assume that in this economy it is not possible to shift income across $S - T$ of its states (i.e. the span of its asset matrix is $T < S$): the economy is called *incomplete with exogenous uncertainty*.[38] It is possible to model such an economy has $S - T + 1$ budget constraints: one for each of the $S - T$ states and one overall. The act of introducing $S - T$ *Arrow securities* which pay a unit of a numeraire in each of the $S - T$ states and zero in all others can be called *completing the market*. A completed market is by definition one which is identical to a standard Arrow–Debreu model. In particular, a competitive equilibrium of a completed market is always Pareto efficient.

In parallel with the results just quoted on incomplete markets with exogenous uncertainty, in the following section I explore the possibility of completing the markets with scientific uncertainty. Each state of scientific uncertainty is represented by an (endogenously determined) probability function $\pi$ over events in the set $D$. To complete this market one therefore introduces financial instruments which pay contingent on such distributions. The assets I have in mind mimic Arrow-securities, but their payoffs are contingent on probabilities: they pay a unit of the numeraire if one probability arises at an equilibrium, and zero otherwise. Since I have assumed that there is no problem of information, everyone can observe ex-post the realized probability at an equilibrium, for example, by observing the actual incidence or the frequency of an event in large groups of independent events.[39] It is possible to introduce and trade such instruments. The matter may appear at first sight to be somewhat theoretical; for this reason it is desirable to discuss a practical example where such instruments have been introduced and are currently traded.

## 7.2. *CAT Futures*

Assets which pay contingent on the observed frequencies of occurrence of natural events were first introduced and analyzed in Cass, Chichilnisky and Wu [4] and Chichilnisky and Heal [15]. Following this, assets which pay contingent on the realization of frequencies of natural risks have been recently introduced in the Chicago Board of Trade, called *CAT*[40] *Futures*. These are instruments whose payoffs depend *inter alia* on the yearly incidence of tropical storms in the United States as measured, for example, by the *Insurance Service Organization* index. The catastrophes included in CAT futures are: earthquakes on the West coast, tornadoes on the East coast and floods in the Midwest. The frequencies of these events are unknown ex-ante and can therefore be treated as risks.

## 7.3. *Completing the market with scientific uncertainty*

Cass, Chichilnisky and Wu [4] and Chichilnisky and Heal [13], [15] showed that under certain conditions such instruments lead to Pareto efficient allocations of risks in markets with unknown risks, i.e. where the probabilities or the frequencies of certain events are unknown.

Floods, earthquakes and tornadoes are all exogenous physical events, they are not risks induced by economic actions. Our markets, instead, face endogenous risks. But one can still try to mimic the introduction of CAT futures by introducing Arrow securities which are contingent on the possible probabilities of event in the set $D$ at a given equilibrium. To complete the market one needs to define the informational structure in this market.

Consider an economy $E$ with scientific uncertainty, and assume that it has one equilibrium output vector, associated to which there are several probabilities over states in $D$. An announcement is made about the probabilities which could arise in equilibrium. Traders can now trade contingent on this set of probabilities, denoted $\Lambda$; assume for simplicity that all elements in $\Lambda$ are equally likely. In the case of CAT Futures the set of probabilities represents, for example, the yearly incidence of tornadoes in a given area. The results of Chichilnisky and Heal [13], [15] then establish that after trading the new contingent securities a Pareto efficient allocation of risks would be realized, one which is identical to that in an Arrow–Debreu market economy. The market is thus completed by the introduction of these new assets, CAT futures:

THEOREM 17. *An economy $E$ with scientific uncertainty can be completed by the introduction of CAT futures. The corresponding allocations are Pareto efficient.*

*Proof*. For a proof see Chichilnisky and Heal and Cass [15], Chichilnisky and Wu [4]. ■

## 7.4. *Strict endogenous uncertainty*

I turn now to *strict endogenous uncertainty*. This is not induced by scientific uncertainty about the connection between economic activity and the environment, but is rather uncertainty about the performance of the economy itself. To represent this type of uncertainty, consider the case where the initial economy has several possible equilibria with a different aggregate output vector each. The following result is obtained under the most favorable conditions on our ability to complete this market: when corresponding to each level of output there is a single probability over events in $D$, and is known by all. This assumption makes our result stronger: the same result follows immediately, and more readily so, when each equilibrium vector of output is associated instead to several densities.

THEOREM 18. *It is not possible to complete the market $E$ with strict endogenous uncertainty, in the sense that no matter how many securities are introduced the economy is not identical to an Arrow–Debreu market and in particular its equilibria allocations are not Pareto inefficient.*[41]

*Proof.* Consider an economy $E$ with strict endogenous uncertainty. Since the economy has strict endogenous uncertainty, the set $\Lambda$ of probabilities $\{\pi_i\}_{i\in D}$ over the events $\{1\ldots D\}$ is not a singleton. Assume that all possible contracts contingent on the elements in $\Lambda$, have now been introduced. Equivalently, assume that at an equilibrium traders trade contingent on the realization of each possible probability distribution $\{\pi_i\}_{i\in D} \in \Lambda$, which they know, for example, to be all equally likely. Any other commonly shared distribution can be considered as well.

I will now show that the assumption that the market has been completed leads to a contradiction. Since $\Lambda$ is not a singleton, it has $K \geqslant 2$ elements; there exist by assumption at least two different equilibria of the economy $E$, with prices $p_1^* \neq p_2^*$ respectively, and with corresponding aggregate output levels $y(p_1^*) \neq y^*(p_2^*)$. The equilibria allocations are $x_1^* \neq x_2^*$. Let $\pi^k = \pi(y(p_k^*))$ be the probability prevailing over $\{1\ldots D\}$ in the $k-th$, equilibrium, $k = 1\ldots K$. If the market were now a complete Arrow–Debreu market, then by definition it could reach an allocation corresponding to that of an Arrow–Debreu economy with complete asset markets. Let $x_{h,k}^*$ be the allocation of the $h$ trader if equilibrium $k$ is realized; let $x_{hki}^*$ denote the equilibrium allocation of the $h'th$ trader when equilibrium $k$ is realized and the state of the economy is $i \in D$. For each trader $h$ an equilibrium in the completed economy consists of the vector $x^{h*} = \{x_{h,k}^*\}_{k\in\Lambda}$. I will now show that the $h'th$ trader must be fully insured across states in $\Lambda$ if the market has now been completed. Suppose not. If $h$ is not fully insured, define a new allocation $y^* = (y_1^* \dots y_H^*)$ by $y_h^* = E(x_{h,k}^*) = \sum_{k \in \Lambda} \mu_k(x_{h,k}^*)$ where $\mu_k = 1/k$. This new allocation can be achieved by the $h-th$ trader, who by assumption can now transfer income across all states in $\Lambda$, and it is different from $x_h^*$. Furthermore expected utility satisfies:

$$Eu_h(y_h^*) = Eu_h\left(\sum_{k \in \Lambda} \mu_k x_{h,k}^*\right) > \sum_{k \in \Lambda} \mu_k Eu_h(x_{h,k}^*) .$$

In addition,

$$\sum_{h=1}^{H} y_h^* = \sum_{h=1}^{H}\left(\sum_{k \in \Lambda} \mu_k x_{h,k}^*\right) = \sum_{k \in \Lambda} \mu_k \left(\sum_{h=1}^{H} x_{h,k}^*\right),$$

so that $y_h^*$ is feasible in $E$ because for each $k \in \Lambda$, $(x_{1,k}^* \dots x_{H,k}^*)$ is a feasible allocation. Since the equilibria of Arrow–Debreu economies are Pareto efficient, this implies a contradiction; therefore the $h$'th trader must be fully insured. But this in turn implies that

$$\forall h = 1 \dots H, \forall k, l, x_{h,k}^* = x, \text{ and}$$

$$x_{hik}^* = x_{hi1}^* \text{ for all } i = 1 \dots D ,$$

so that at each $i$ the prices are the same across elements of $\Lambda$:

$$\forall l, l' \in \Lambda, p_l^* = p_{l'}^* ,$$

and therefore $\Lambda$ is a singleton, a contradiction. The contradiction arises from the assumption that the allocations are Pareto efficient, namely that the markets can be completed. Therefore the market cannot be completed if the economy has strict endogenous uncertainty. ■

## 8. CONCLUSIONS AND A FRAMEWORK FOR POLICY

I formulated and proved existence of a competitive equilibrium in markets with endogenous uncertainty, where the traders' actions induce changes in the state space which represents uncertainty, and in the probabilities of each state. The equilibria of such markets exist very generally, generically on technologies and endowments. These markets are natural extensions of Arrow–Debreu markets; indeed when the economy's activity is relatively small and it does not induce changes in nature, the two concepts coincide. Arrow–Debreu markets are therefore a strict subset of the set of markets with endogenous uncertainty.

It is possible to distinguish two types of endogenous uncertainty. The first is called *scientific uncertainty*: markets with this type of uncertainty have a unique equilibrium output vector, but associated to this there are several possible sets of states and probabilities over these states, any of which may arise. I showed that in markets with scientific uncertainty all uncertainty can be hedged by introducing certain financial instruments. With this financial innovation, the markets behave as Arrow–Debreu markets and reach Pareto efficient allocations. The type of instruments which can hedge scientific uncertainty pay contingent on the possible probability distributions over states. Such assets now exist: they have been introduced recently in the Chicago Board of Trade, denoted CAT futures. Their creation was anticipated in Chichilnisky and Heal [13] and [15] and Cass, Chichilnisky and Wu [4]. Such instruments can improve the economy's allocation and lead to Pareto efficiency when the economy faces scientific uncertainty. A practical use of CAT futures is to hedge a large insurer's correlated risks, namely the risk faced by an insurer who may have to pay on a large number of contracts of individuals affected by the same natural disaster.

A second type of endogenous uncertainty arises in economics which have more than one competitive equilibrium, and a different level of output in each. These I call markets with *strict endogenous uncertainty*. Their behavior is quite different from that of Arrow–Debreu markets, and they provide a well founded reason for the current preoccupation about incomplete markets. These markets are fundamentally incomplete: I showed that no hatter how many state contingent securities are added, these markets will not reach Pareto efficient allocations. It is possible, however, to reach fully insured allocations, but this requires a completely different market structure, involving derivative securities, as established in Chichilnisky, Dutta and Heal [17], and Chichilnisky [6].

Finally consider a policy issue which arises in the case of environmental risks which are the by-product of industrial activity. If one seeks constrained Pareto efficiency, it is possible to compute precisely the cost which is worth incurring to decrease the probability of a catastrophic environmental risk. Assuming a known (or even an approximately known) scientific relation between industrial by-products and the probability distribution on environmental states (i.e. the map $\Psi$) one computes the manifold of equilibria of the economy with endogenous uncertainty, and finds within it a new equilibrium with the desired value $\Psi(y, \pi) = 0$. The Appendix provides a proof of existence of a competitive equilibrium which can be used to compute an equilibrium. At this new equilibrium one computes for each aggregate output $y$ the utility levels achieved by the traders at their new consumption. The difference in the welfare of the traders at the first and at the second equilibrium provides an upper bound for the willingness to pay for a decreasing a global environmental risks such as that of global climate change.

## 9. APPENDIX

### 9.1. *The distance between two technologies*

DEFINITION 19. The distance between two technologies $Y, Z \subset R^M$ is the Lebesgue measure of the symmetric difference of the two production sets:

$$d(Y, Z) = \mu(Y \Delta Z)$$

where $Y \Delta Z = (Y - Z) \cup (Z - Y)$ and $Y - Z = Y \cap Z^c$.

This definition relies on a prior result: the existence of a complete metric topology on the space of all closed sets in $R^M$ which are the closure of their interior, called the "order topology", introduced and characterized in Chichilnisky [5]. The "order topology" yields this metric $d$ a special case, namely when restricted to the family of closed, bounded convex technologies in $R^M$. The order topology is strictly finer than the Hausdorff metric and is sensitive to measure, a property that the Hausdorff metric does not have, see [5].

## 9.2. *The degree of a continuous map*

Let $\Lambda$ denote an open set in $R^N$ and $f$ a smooth mapping of class $C^p, p \geqslant 2, f : \Lambda \to R^M$. If at a point $x_0$, the rank of the Jacobian $f'(x_0) = M$, then $f$ maps a small neighborhood of $x_0$ onto a small neighborhood of $f(x_0)$.

DEFINITION 20. If at a point $x_0$, the rank of the Jacobian $f'(x_0) = M$, such a point is called regular with respect to $f$.

DEFINITION 21. The complement in $\Lambda$, i.e. the set

$$C = \{x : x \in \Lambda, \textit{rank}\, f'(x) < M\}$$

is called the critical set of $f$ and a point $x \in C$ is called a critical point.

The set $C$ is closed in $\Lambda$, since if $x_n \to x \in \Lambda$, rank $f'(x_n) \geqslant$ *rank* $f(x)$.

SARD'S THEOREM. *If $f(x)$ is a $p-$ times continuously differentiable mapping of $\Lambda$ into $R^M$, then the critical values $f(C)$ have measure zero in $R^M$, provided $N - M + 1 \leqslant p$.*

See, e.g. M. Berger [3], p. 52–53.

### 9.2.1. *The degree of a continuous map*

Sard's theorem can be used to define the degree of a continuous mapping $f : \Lambda \to R^N$. This integer provides an "algebraic count" of the number of solutions of the equation $f(x) = p$ in $\Lambda$, provided $f(x) \neq p$ on $\partial\Lambda$, and $\Lambda$ is bounded. Intuitively, the degree defines the number of solutions to the equation $f(x) = p$ when $p$ is a regular value. The degree can also be defined for continuous maps. A formal definition is as follows.

DEFINITION 22. The definition of the degree of a continuous map $f$ is given in three steps:

(i) Suppose $f$ is a $C^1$ mapping of $\Lambda \to R^N$ and that whenever $f(x_0) = p$ the Jacobian determinant of $f$ at $x_0 |J_f(x_0)| \neq 0$. Then we define the degree of $f$ at $p$ relative to $\Lambda$ as

$$d(f, p, \Lambda) = \sum_{f(x)=p} sgn|J_f(x_0)|\,.$$

This sum is finite by virtue of the compactness of $\bar{\Lambda}$, the closure of $\Lambda$, and by the inverse function theorem.

(ii) Suppose now that $f$ a $C^1$ mapping of $\Lambda \to R^N$. Then by Sard's theorem we can find a sequence of regular points $\{p\}$ (with respect to $f, \Lambda$) such that $p_n \to p \in R^N$. We then define the degree of $f$ at $p$ as

$$d(f, p, \Lambda) = \lim_{n \to \infty} d(f, p_n, \Lambda).$$

(iii) Finally if $f$ is only known to be continuous in $\Lambda$, there is a sequence of $C^1$ mappings $f_n \to f$ uniformly on $\bar{\Lambda}$ and we set

$$d(f, p, \Lambda) = \lim_{n \to \infty} (f_n, p, \Lambda).$$

The function $d(f, p, \Lambda)$ can be shown to be well defined in (ii) and (iii) and the limits to exist and to be independent of the approximating sequences.

DEFINITION 23. A map $H : D \times [0, 1] \to R^N$ is called a *homotopy* between the maps $H_0 : D \to R^N$ and $H_1 : D \to R^N$, where $H_0(x) = H(x, 0)$ and $H_1(x) = H(x, 1)$.

For a bounded domain $D \subset R^N$, the following is a basic property:

LEMMA 24. (Homotopy invariance). *Suppose $H : D \times [0, 1] \to R^N$ is a continuous function and that $H(x, t)$ has no solution $x \in \partial D$ for any $t \in (0, 1]$. Then the degree of $H$, $d(H(x, t), p, D)$, is a constant independent of $t \in [0, 1]$.*

See, e.g. Berger [3], p. 53.

### 9.3. *Proof of Existence of an Equilibrium*

THEOREM 1. *Under Assumptions 1 to 8, there exists a competitive equilibrium with endogenous uncertainty generically in technologies and endowments.*

*Proof.* An intuitive explanation of the proof is as follows. One shows that the economy $E$ can be deformed continuously into a particularly simple economy called $E_0$ which has an equilibrium. A homotopy argument is then used to show that since $E$ is a continuous deformation of $E_0$, then $E$ has at least one equilibrium. [42]

The deformation of $E$ into $E_0$ consists of "shrinking" gradually the technology of the economy $E$ while leaving all its other features unchanged, until it reaches a small enough level of output when it is an Arrow–Debreu economy, i.e. an economy with exogenous uncertainty, see Proposition 1, for which an equilibrium is easily established. This is $E_0$. The homotopy invariance lemma stated above is then used to show that because the simple economy $E_0$ has a competitive equilibrium, the initial economy $E$ does too.

Let $Y$ be the production possibility set $Y$ the economy $E$. Now define a family of economies $\{E_t\}_{t\in[0,1]}$, all identical to $E$ except for their production possibility sets: for each $t \in [0,1]$, let the production set of the economy $E_t$ be $Y_t = tY + (1-t)\{0\}$ where $\{0\} \in Y$. The structure of endogenous uncertainty is given by the map $\Psi$ defined in Assumption 1. When $t = 1$, $E_t = E$, the original economy, and when $t = 0$, $E_0$ is a pure exchange Arrow–Debreu economy with exogenous uncertainty because of Assumption 4 and Proposition 1.

Denote by $y_t(p)$ the profit maximizing production vector associated with the price vector $p \in \Delta_M$ in economy $E_t$ which exists by Assumption 2; by renormalizing units of measurement and without loss of generality[43] we may assume that $\|y_t(p)\| > \|y_{t'}(p)\|$ when $t > t'$. Let $d_h^t(p,\pi)$ be the demand vector of trader $h$ corresponding to price $p \in \Delta_M$ and probability $\pi \in \Delta_D$, as in Assumption 6.

Now define the map $\Phi(p, \pi_1, \ldots \pi_D, t) : \Delta_M \times \Delta_D \times [0,1] \to R^M \times R^D$ by:

$$\begin{aligned} &\Phi(p, \pi_1, \ldots, \pi_D, t) = \\ &\quad = \left( \sum_{h=1}^{H} (d_h^t(p, \pi_1, \ldots, \pi_D) - \Omega_h) - \right. \\ &\quad\quad \left. - y_t(p), \Psi(y_t(p), \pi_1, \ldots, \pi_D) \right) \in R^M \times R^D \end{aligned} \tag{5}$$

where for each $t$ the demand function $d_h^t(p,\pi)$ is defined as in (3) and (4) for each probability vector $\pi = (\pi_1, \ldots, \pi_D) \in \Delta_D$ and where the supply function $y_t(p)$ is as in Assumption 2. Consider for each $t \in [0,1]$, the set of zeros of the map $\Phi(.,t)$, namely the set $\{\Phi(.,.,t)^{-1}(0)\}$, a set which may be in principle empty. This set of zeros consists of those prices and probability densities which, when taken as given by the traders, leads to aggregate demand and supply vectors at which all markets clear, and at which simultaneously the relation between of aggregate output levels and densities is consistent with scientific knowledge as defined by the map $\Psi$. Therefore, by construction, each zero of the map $\Phi(p, \pi, t)$ is an equilibrium with endogenous uncertainty of the economy $E_t$. To prove the existence of an equilibrium with endogenous uncertainty for the economy $E$ one must therefore show that the set of zeros of $\Phi(p, \pi, t)$ is not empty when $t = 1$, i.e. that

$$\{\Phi(.,.,1)^{-1}(1)\} \neq \phi .$$

The proof uses the homotopy invariance lemma stated above. To do so, one checks that the conditions of the lemma are satisfied. For $t = 0$ we know by Assumption 4 that there exists a probability vector $\pi^0 = (\pi_1^0, \ldots, \pi_D^0) \in \Delta_D$ such that $\Psi(0, \pi^0) = 0$ and $\pi_i^0 > 0, \forall i = 1, 2, \ldots, S$; $\pi^0$ could assign zero probability to states of endogenous uncertainty. Consider now the standard pure exchange Arrow–Debreu with exogenous uncertainty defined by the set of states $\{1, \ldots, D\}$, the commonly shared probability vector $\pi^0$, and with the same traders, preferences and initial endowments as those in $E$. Since $\Psi(0, \pi^0) = 0$, a zero of the map

$$\Phi(.,.,0) : \Delta_M \times \Delta_D \to R^M \times R^D$$

can be located by finding a competitive equilibrium price $p^*$ of this standard Arrow–Debreu economy, called $E_0$, where all traders have von Neumann–Morgenstern utilities with the same probabilities, $(\pi_1^0, \ldots \pi_D^0)$, since by the definition of $\Phi$ the equilibrium price $p^*$ will satisfy:

$$\Phi(p^*, \pi_1^0, \ldots, \pi_D^0, 0) = 0 .$$

An equilibrium price $p^*$ always exists, given the boundary Assumption 8, by standard arguments applied to the pure exchange Arrow–Debreu economy $E_0$. Therefore there exists a solution to the equation $\Phi(.,.,0) = 0$. By Assumption 4 there exists another probability, $\pi'$, solving the equation $\Psi(0, \pi') = 0$, $\pi' \in \Delta_D^0$. Since when $Y = \{0\}$ the economy is, by Assumption 4 and Proposition 1, an Arrow–Debreu economy with exogenous uncertainty, the same argument used above proves that there exists a solution to $\Phi(.,.,0)$ whose second coordinate is in the interior of $\Delta_D$, and whose first coordinate, by Assumption 8, must therefore be in the interior of $\Delta_M$. Therefore for $t = 0$ the map $\Phi(.,.,t)$ has a solution in $\Delta_M^0 \times \Delta_D^0$. Since as shown above, $\forall p, \|y_{t'}(p)\| > \|y_t(p)\|$ when $t' > t$, it follows by Assumption 1 that for $t > 0$, no solution of $\Phi(.,.,t)$ has a probability vector $\pi_1^*, \ldots, \pi_D^*$ in $\partial\Delta_D$. By Assumption 8 this implies that for $t > 0$ there can be no solutions to $\Phi$ where the corresponding prices are in the boundary of $\Delta_M$, $\partial\Delta_M$.

Therefore the conditions of the homotopy invariance lemma stated above are satisfied: the map $\Phi : \Delta_M \times \Delta_D \times [0,1] \to R^M \times R^D$ has no solution in the boundary of the set $\Delta_M \times \Delta_D$ for $t > 0$, and generically 0 is a regular value at $t = 1$. Therefore both maps $\Phi_0$ and $\Phi_1$ have the same degree. In particular the set of solutions is not empty for $t = 1$. Therefore the economy $E_1$, which coincides with the economy $E$, has an equilibrium. ■

## ACKNOWLEDGEMENTS

Research support from UNESCO, NSF grants No. 92-16028 and 94-08798, the Stanford Institute for International Studies and the Sloan Foundation Project on Productivity and Information Technology at Columbia Business School, is gratefully acknowledged. This paper was presented as Plenary Session III of FUR VII, The International Conference on the Foundation and Application of Utility, Risk and Decision Theory, Norwegian School of Management, Sandvika, Norway, July 1, 1994.

## NOTES

[1] See Chichilnisky and Heal [13].

[2] See Chichilnisky and Heal [13].

[3] See also Dasgupta and Heal [21], Hahn [24], [25], [26], and Kurz [29]. Optimal growth paths with endogenous uncertainty were first studied in Heal [27], [28]. The first theorems on existence of a general market equilibrium with endogenous uncertainty are in Chichilnisky and Wu [16], Chichilnisky, Dutta and Heal [17], Chichilnisky, Hahn and Heal [18], and Chichilnisky [6]. Hahn [24], and Chichilnisky, Hahn and Heal [18] established that in a two-period world, markets with price uncertainty are essentially incomplete. Endogenous uncertainty comes in many forms: it can be due to price uncertainty, [24], [18], [17], or risks of default [16], [20], or about the interconnectedness of trading patterns (Chichilnisky and Wu [16], Chichilnisky, Heal and Tsomocos [20]) or, as in this paper, about the set of states and their probabilities (see also Chichilnisky

and Heal [13]), or it can be also induced by expectations as in a temporary equilibrium framework: for example in Fudenberg, Holstrom and Milgrom [23], in the "asymptotic rational expectations" formulation introduced by Stein [34] and [35], or the related concept of "rational beliefs" developed by Kurz [29], [30]. Moral hazard is also an instance of endogenous uncertainty, see, e.g. Arrow [2]; it depends however on asymmetric information. Endogenous uncertainty is more general, occurring even when information is fully symmetric.

4 Volatility could also be a manifestation of this type of uncertainty; that the introduction of one asset can alter the volatility of another was established in Dasgupta and Heal [21], Chapter 13.

5 This information was provided by Kenneth Arrow during a Theory Seminar at the Department of Economics of Stanford University, February 10, 1994, where this paper was presented.

6 This quotes from "Fed chief warns of risks posed by derivatives" by John Gapper, Banking Editor in London, *The Financial Times*, No. 32,282, February 2 1994, page 1. See also "Regulators move to limit risks of OTC derivatives" by N. Cohen, T. Corrigan and L. Morse in *Financial Times*, March 16, 1994, p. 1.

7 This is Theorem 2 in Chichilnisky and Wu [16], see also Chichilnisky, Heal and Tsomocos [20].

8 For example, the OECD has discussed a global carbon tax that would raise $150 billion in revenues yearly, see Chichilnisky [7].

9 The relevant maps are non-convex correspondences, so that fixed points arguments cannot be used in general.

10 Also called general equilibrium with incomplete markets, or GEI, see, e.g. Chichilnisky and Heal [12].

11 At least not within a standard Arrow–Debreu market. Hahn [24] and Chichilnisky, Hahn and Heal [18] have obtained related results in another model of endogenous uncertainty, in an economy with two periods where the uncertainty is about prices, not about states and probabilities, and where the level of output does not affect the states or their probabilities. It is however possible to "complete" markets with endogenous (price) uncertainty, but this requires a different market structure, effectively one in which derivative securities are traded in a different way than commodities are traded in an Arrow–Debreu model, see Chichilnisky, Dutta and Heal [17].

12 Therefore there is no "moral hazard", see, e.g. Arrow [2] and Stiglitz [33]. As already mentioned, moral hazard is a special case of endogenous uncertainty, but it can only arise with asymmetric information, while endogenous uncertainty is a more general phenomenon which can arise with or without symmetric information.

13 For example: when $T = 2$, there are $S$ assets each of which pays in terms of a numeraire good $n$, and the span of the economy's asset matrix is $S$.

14 $Y^s$ is the closure of its interior.

15 $\Delta_{N\times T\times S} = \{(p_1 \dots p_m) \in R^{N\times T\times S} : p_i \geqslant 0 \text{ and } \sum p_i = 1\}$.

16 A possible difference is that short sales are allowed. Necessary and sufficient conditions for the existence of an equilibrium with or without short sales are in [8], [9], [10] and [11].

17 With exogenous uncertainty.

18 this constitutes an infinite domain.

[19] Heal [27] and [28] formalizes and analyzes the properties of a dynamic economy where the level of consumption at time $t$ induces changes in the climate and therefore affects the production possibility set of the economy, at a future random date which varies with today's consumption.

[20] $D$ is the set of all possible or "latent" states; its cardinality could be infinite ($D = \infty$) without changing the results in any way, but at the cost of more notation. In the case that the cardinality is infinite one needs to work in economies with infinite dimensional commodity spaces, ideally in Sobolev spaces; see, e.g. Chichilnisky and Heal [14] for a general theorem of existence and characterization of a competitive equilibrium in Sobolev spaces with or without short sales.

[21] In general $D \geqslant S$ but it would suffice to consider the case where $D = S$ by allowing density functions which assign zero probability to some states. In general, however, it seems preferable to differentiate between exogenous and endogenous states ($S \neq D$) because they could have different properties.

[22] Twice continuously differentiable or $C^2$.

[23] This assumption ensures that endogenous uncertainty in the market is not trivial, i.e. changes in aggregate output can change the states and their probabilities, and that a probability which has become positive remains so as output expands.

[24] One could also require that the initial endowments of the economy should be small for the probabilities to be constant: such an assumption would not change the results, but would require more notation.

[25] I owe this observation to Mark Machina.

[26] More realism can be introduced at the cost of more notation, but the main features are captured by this formulation. For example, moral hazard can lead to endogenous uncertainty but only when there is asymmetric information, which is not essential in our formulation of endogenous uncertainty.

[27] This assumption was suggested during discussions with Robert Wilson, and is crucial to the formalization of the problem.

[28] To simplify notation, and without loss of generality we set from nowon $e \equiv \{1 \dots D\}$, by allowing some probabilities to be zero, i.e. for some $i$, $\pi_i = 0$.

[29] Where $\pi_i = 0$ when $i \notin e$.

[30] This budget constraint means that there exists markets to trade contingent on all states in $\{1, \dots, D\}$.

[31] Obviously, we may assume that each trader has different subjective probabilities, and formulate this assumption in terms of trader-specific subjective probabilities. This would not change the results in any way but complicates the notation.

[32] As is standard, one considers an appropriate approximation of the solution to (3) which is continuously differentiable in a neighborhood of $\Delta_M$, and which does not add new equilibria to the model.

[33] This condition is sufficient but not necessary to prove the existence of a competitive equilibrium in an Arrow–Debreu economy. A necessary and sufficient condition for existence is *limited arbitrage* as defined and established in Chichilnisky [8], [9], [10], [11].

[34] See the Appendix.

[35] This happens when at some state $i$ the equilibrium allocation of trader $h$ in the first equilibrium, $x^{i*}_{1,h}$, is quite different than the allocation of same trader $h$ at

$i$ in the second equilibrium, $x^{i*}_{2,h} \neq x^{i*}_{1,h}$, although across equilibria the aggregate supply is the same in each state, i.e. $\forall i, y^{i*} = y^{i*}_1 \in Y$.

[36] The first welfare theorem is true whether the economy has concave preferences or not, and whether the economy has convex production functions or not.

[37] E.g. Chichilnisky and Heal [12].

[38] Also called GEI, general equilibrium with incomplete markets.

[39] The Insurance Service Organization (ISO) performs such tasks, see below.

[40] For "catastrophe".

[41] Fully insured allocations are possible if one changes drastically the market structure, allowing for regulated derivative markets with have a different trading structure from Arrow–Debreu markets, see Chichilnisky [6], and Chichilnisky, Dutta and Heal [17].

[42] See Eaves [22] on the use of homotopy methods for locating fixed points.

[43] The technology $Y$ is a convex body.

## REFERENCES


1. Arrow, K.J.: 1953, 'The role of securities in the optimal allocation of risk-bearing', in French: *Econometric: Proceedings of the Colloque sur les Fondments et Applications de la Theorie du Risque en Econometrie,* Centre National de la Recherche Scientifique, Paris. English Translation: *Review of Economic Studies* (1964) **31**: 91–96.
2. Arrow, K.J.: 1963, 'Uncertainty and the Welfare Economics of Medical Care', *American Economic Review* **LIII**(5), 941–973.
3. Berger, Melvyn S.: 1977, *Nonlinearity and Functional Analysis, Lectures on Nonlinear Problems in Mathematical Analysis.* Academic Press, New York, San Francisco, London.
4. Cass, D.G. Chichilnisky, and H-M Wu: 1996, 'Individual Risks and Mutual Insurance', *Econometrica* **64**(2), 333–342.
5. Chichilnisky, G.: 1977, 'Spaces of Economic Agents', *Journal of Economic Theory* **15**(1), pp. 106–73.
6. Chichilnisky, G.: 1992, 'Existence and Optimality of a Competitive Equilibrium with Endogenous Uncertainty', Working Paper, Columbia University, December 1992.
7. Chichilnisky, G.: 1993, 'Comment: The Taxation of Carbon Emissions in Industrial and Developing Countries', in *OECD: The Economies of Climate Change* (ed. T. Jones), Paris, 1994, pp. 159–170.
8. Chichilnisky, G.: 1993, 'Social Diversity, Arbitrage and Gains from Trade: A Unified Perspective on Resource Allocation', *American Economic Review,* May 1994, 427–433.
9. Chichilnisky, G.: 1991, 'Limited Arbitrage is Necessary and Sufficient for the Existence of a Competitive Equilibrium With or Without Short Sales', Working Paper, Columbia University, *Economic Theory,* January 1995, Vol 5, No 1, pp. 79–108.
10. Chichilnisky, G.: 1994, 'A Unified Perspective on Resource Allocation: Limited Arbitrage is Necessary and Sufficient for the Existence of a Competitive Equilibrium, the Core and Social Choice', Invited Presentation at the

International Economic Association (IEA) Round Table on Social Choice, Schloss Hernstein, Austria, May 25-28, 1994, forthcoming in *Social Choice Reexamined* (ed. K. Arrow, A. Sen and K. Suzumura), MacMillan.

11. Chichilnisky, G.: 1994, 'Limited Arbitrage is Necessary and Sufficient for the Existence of a Competitive Equilibrium and the Core, and Limits Voting Cycles', *Economic Letters*, **46**, pp. 321–331, to reappear as "Limited Arbitrage is Necessary and Sufficient for the Nonemptiness of the Core", *Economic Letters*, 1996.
12. Chichilnisky, G., and G.M. Heal: 1993, 'The Existence and Structure of the Pseudoequilibrium Manifold in Incomplete Asset Markets', forthcoming, *Journal of Mathematical Economics*, 1996.
13. Chichilnisky, G. and G.M. Heal: 1992, 'Global Environmental Risks', *Journal of Economic Perspectives*, 7(4), pp. 65–86.
14. Chichilnisky, G. and G.M. Heal: 1993, 'Existence of a Competitive Equilibrium in Sobolev Spaces without Bounds on Short Sales', *Journal of Economic Theory*, **59**(2) pp. 364–384.
15. Chichilnisky, G. and G.M. Heal: 1992, 'Financial Markets with Unknown Risks', Working Paper, Columbia University.
16. Chichilnisky, G. and H.-M. Wu.: 1991, 'Financial Innovation and Endogenous Default in Incomplete Asset Markets', Stanford Institute for Theoretical Economics Technical Report No. 50, 1992, forthcoming in *Information and Uncertainty: New Directions in Economic Theory*, Essays in Honor of Kenneth Arrow (ed. G. Chichilnisky), Cambridge University Press.
17. Chichilnisky, G., J. Dutta, G.M. Heal: 1991, 'Price Uncertainty and Derivative Securities in General Equilibrium', Working Paper, Columbia University, revised 1993.
18. Chichilnisky, G., G.F. Hahn and G.M. Heal: 1992, 'Price Uncertainty and Incomplete Markets', Working Paper, Columbia Business School.
19. Chichilnisky, G., G.M. Heal, P. Streufert, and J. Swinkels: 1992, 'Believing in Multiple Equilibrium', Stanford Institute for Theoretical Economics Technical Report No. 49, August 1992, and First Boston Working Paper FB-92-32, Columbia Business School, August 1992.
20. Chichilnisky, G., G.M. Heal, and D. Tsomocos: 1994, 'Option Values and Endogenous Uncertainty in ESOP's, MBO's and Asset Backed Securities', Working Paper, Columbia Business School and Yale University, *Economic Letters*, 1995, Vol 48. No 3–4, pp. 379–388.
21. Dasgupta, P. and G.M. Heal: 1979, *Economic Theory and Exhaustible Resources*, Cambridge University Press.
22. Eaves, C.: 1972, 'Homotopies for Computation of Fixed Points', *Math. Programming* 3.
23. Fudenberg, D.B. Holstrom and P. Milgrom: 1990, 'Short-term Contracts and Long-term Agency Relationships', *Journal of Economic Theory*, **51**(1), June 1990.
24. Hahn, F.: 1991, 'A Remark on Incomplete Market Equilibrium', Working Paper Cambridge University and University of Siena, to appear in *Information and Uncertainty: New Directions in Economic Theory*, Essays in Honor of Kenneth Arrow (ed. G. Chichilnisky), Cambridge University Press.

25. Hahn, F.: 1992, 'Making General Equilibrium Theory More Plausible', *The Second Annual Arrow Lecture Series*, Department of Economics, Stanford University, May 5 and 6.
26. Hahn, F.: 1973, *The Concept of Equilibrium in Economics*, Inaugural Lecture, Cambridge University, Cambridge University Press.
27. Heal, G.: 1984, 'Interaction of the Economy and Climate: A Framework for Policy Design under Uncertainty' in Smith, Kerry and Ann Dryden White eds. *Advances in Applied Macroeconomics*, Greenwich, J.A.I. Press, 151–158.
28. Heal, G.M.: 1990, 'Economy and Climate: A Preliminary Framework for Microeconomic Analysis', in Just, Richard and Nancy Bockstael, eds. *Commodity and Natural Resource Policies in Agricultural Systems*, New York and Oxford: North Holland 1993, 855–880.
29. Kurz, M.: 1974, 'The Kesten-Stigum Model and the Treatment of Uncertainty in Equilibrium Theory', in M.C. Balch, D.McFadden, and S. Wu, (eds.) *Essays in Economic Behavior Under Uncertainty*, North Holland.
30. Kurz, M.: 1991, 'On the Structure and Diversity of Rational Beliefs', Technical Report No. 31, Stanford Institute of Theoretical Economics, *Economic Theory*, 1994.
31. Milnor, J.: 1965, *Topology from a Differentiable Viewpoint*, The University Press of Virginia, Charlottesville.
32. Stiglitz, J.: 1974, 'Tax Policy and the Oil Industry', mimeo, Yale University.
33. Stiglitz, J.: 1983, 'Risk Incentives and Insurance: The Pure Theory of Moral Hazard', *The Geneva Papers*, January 1983, 8:26. 4–32.
34. Stein, J.: 1992, 'Cobwebs, Rational Expectations and Futures Markets', *Review of Economics and Statistics*, **LXXIV**(1), pp. 127–134.
35. Stein, J.: 1992, 'Price Discovery Processes', *Economic Record*, pp. 34–45.
36. Wilson, Robert: 1992, 'Strategic Analysis of Auctions', *Handbook of Game Theory*, Vol. 1 (R. Aumann and S. Hart eds.), Elsevier Science Publishers, B.V.

*Program on Information and Resources,*
*Columbia University,*
*405 Low Library,*
*New York, NY 10027, U.S.A.*