Is Segregation Robust?

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25 January 2007

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Abstract

This paper studies the question of how well we understand segregation. The point of departure is Schelling's spatial proximity model in one dimension. By introducing noise I show that segregation emerges as the long run prediction of neighborhood evolution, both when residents have Schelling-type threshold preferences and strict preferences for diversity. Analytical result are complemented with numerical simulations which show that within a reasonable time frame full segregation does not occur. When residents have a preference for diversity, I show that a natural perturbation away from the diversity monomorphism dramatically alters the long run prediction: integration is the unique long run prediction, even in the absence of noise.

Keywords: segregation, Markov Process, Stochastic Stability, simulations.

JEL: C72, C73, D62

1 Introduction

Residential segregation by race is a robust social phenomenon of many modern economies. Historically, in the US residential segregation was closely related to both political and economic systemic change. The abolition of slavery and the industrial advancement of the North created a demand for labor and induced mass migration of black workers from South to North. The first big wave of migration occured after World War I, while the second main wave occured post World War II (Cutler, Glaeser, and Vigdor 1999). These demographic changes in the North mapped into spatial separation of neighbourhoods by race. Spatial separation was rapid; from 1890 to 1940 the measures of segregation used by Cutler, Glaeser, and Vigdor (1999) roughly tripled. Moreover residential segregation is persistent over time. It is well documented that residential segregation leads to adverse economic outcomes (see e.g. Cutler and Glaeser (1997)). Examples are under-accumulation of human capital (Borjas 1995), higher incidence of unemployment (Topa 2001), high levels of

*I am grateful to Tilman Börgers for supervision. The paper has benefited from discussions with Erik Eyster, Sanjeev Goyal and Steffen Huck. Financial support from ESRC (grant R42200134547), ELSE, Carl Christiansen og Hustrus Legat and Konsul Axel Nielsens Mindelegat is gratefully acknowledged.

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crime activities (Glaeser, Sacerdote, and Scheinkman 1996), etc.. Moreover these deleterious economic effects tend to spill over to the rest of the economy. A theory of residential segregation should capture these two stylized facts: (i) rapidness and persistence, (ii) aggregate low welfare.

How and why do ghettos form? One of the most celebrated models of ghetto formation is Schelling's (1969, 1971, 1978)[1] spatial proximity model of segregation. Schelling's model is as stunningly simple as the results are striking. Weak incentives for residents to live with people like themselves at the micro-level can lead to remarkable order and relatively high segregation at the macro-level. Segregation emerges in Schelling's model although none of the residents have a strict preference for segregated local neighborhoods. In Schelling's model sorting of neighborhoods does not occur along socio-economic lines, but follows from individual preferences over local neighborhoods.

In Schelling's model residents live on a line. There are two types of residents A and B. Each resident is concerned about the composition of her local neighborhood. Her preferences are such that she prefers to have at least one of the two neighbors adjacent to her to be of the same type as herself. Schelling then considers an adaptive procedure through which residents are given the opportunity to revise their current choice of location. The adjustment process ends when no individual can move to a location where she will get higher utility. The process admits many stable neighborhood configurations ranging from fully integrated to fully segregated configurations. By simulation with pen-and-paper Schelling shows that when the process has converged the composition of local neighborhoods are often relatively segregated.

This paper asks if the insight of segregation is robust? To answer this question I formalize Schelling's one-dimensional model and numerically simulate it. While it is true that, starting from a randomly drawn configuration, Schelling's adaptive dynamics tend to settle down around neighborhood configurations that are relatively segregated it is not the case that only segregated configurations are selected.

The literature has suggested that the introduction of noise on top of the deterministic adaptive procedure solve the 'problem' with multiplicity of equilibria (Young 1998, Young 2001). Moreover Young (2001) shows analytically that segregated states are uniquely selected for (the set of segregated states are the only *stochastically stable* states cf. Young (1993)) even when residents have a strict preference for living in diverse neighborhoods.

To examine this claim more closely this paper formalizes Schelling's model in a perturbed Markov framework. I establish that the stochastically stable states are precisely the set of segregated states, both when players have preferences as in Schelling's model, and when they have strict preference for diversity. The case where players have a strict preference for diversity is of particular interest since segregation is the "worst" equilibrium in terms of welfare. The formalization in this paper differs from that of Young (1998, 2001) mainly in that here residents move unilaterally (as they do in Schelling), whereas in Young a pair of residents exchanges locations ('roughly' when they swap is Pareto-improving). Nevertheless

[1]Schelling (1969, 1971, 1978) consider two different models of segregation: the spatial proximity model in one and two dimensional neighborhoods, and the neighborhood tipping model. The key difference between these models is that in the former unhappy residents relocate within the residential area, whereas in the latter unhappy residents leave the area.

the analytical predictions of our models are similar, suggesting that results are robust to the micro-details of our respective modeling choices.

How do the models perform when they are numerically simulated? In particular how long is the long run, that is the wait until a segregated state is reached when the process is started in a randomly drawn starting configuration? Numerical simulations of both models show that the wait until the process hits a segregated increases rapidly in the number of residents. This is problematic for a theory of segregation. The reason is perhaps easiest to see in the model where residents have threshold preferences as in Schelling. The process will be spending most of its time in the set of recurrent classes of the unperturbed process before eventually transiting to a segregated state. The set of recurrent states of the unperturbed dynamics will be shown to be states such that all residents have at least one neighbor like themselves. Consider the case where there are two contiguous groups of each type. In order to transit to a segregated state all residents from one group, say A_1, must move to the other group, say A_2. But as A_1 shrink the selection dynamics become increasingly more likely to select a resident from A_2. I show that the implication of this bias that waiting times increase rapidly in the number of residents. Of further interest is that the insight equally applies to the models of Young (1998, 2001).

How robust are the models to perturbations to the underlying preferences? To answer this question the preferences of one player in a population of residents with a strict preference for diversity are perturbed slightly. Formally I flip one resident's 2nd and 3rd best alternative relative to the rest of the population I show that this "small" perturbation dramatically alters the predictions of the model. In fact the only recurrent states of the unperturbed dynamics are now the set of integrated states. As Young (1993) shows the stochastically stable states form a subset of set of recurrent sets of the unperturbed dynamics, so that the perturbation has eliminated the equilibrium selection 'problem'. In other words the long run prediction of segregation when residents have a strict preference for diversity is non-robust. The model with threshold preferences a la Schelling is robust to this perturbation.

Apart from Schelling's original contribution there are two more recent contributions closely related to this paper[2].

Young (1998) presents a formal analysis of a model similar to Schelling's. Again there are two types of residents, and residents have the same preferences as in Schelling. Young considers an adjustment process through which two residents can agree to swap flats, roughly they exchange locations if the swap is Pareto improving[3]. This adjustment process has many stable states. Young then introduces a small amount of noise into the system so that unfavorable trades may sometimes take place by mistake. As the noise vanishes there is strong selection among the stable states. The only stable states that survive this refinement are precisely the segregated states. Young (2001) considers a variation of Young (1998) where residents have a strict preference for integration, that is, their most preferred outcome is to live in an integrated neighborhood with one of each type. Young shows that even with

[2] Möbius (2000) considers a hybrid model of Schelling's neighborhood tipping model (Schelling 1972), allowing for local interaction. In this model instead of relocating within the residential area unhappy residents leave the area and are replaced with residents from a pool of potential residents. Möbius uses this model to explain rapid ghetto formation in Chicago.

[3] In order to establish the set of recurrent states of the unperturbed process Young also requires that a swap takes place or if at least one of the residents is better off and the other is only slightly worse off after the swap, i.e. the swap improves joint (myopic) welfare.

these preferences segregated states remain the only stochastically stable states; even though residents prefer to live in integrated neighborhoods they end up living only with their own type.

Pancs and Vriend (2003) also study Schelling's model. In their variant of Schelling's model Pancs and Vriend show that segregation occurs under quite mild assumptions about preferences over neighborhood composition. In particular segregation obtains even when residents have a strict preference for perfectly diverse neighborhoods; a stark contrast to the findings reported here. Our papers differ in our behavioral assumptions about how agents find new locations. Pancs and Vriend consider a stochastic non-noisy best-response dynamic[4], and show that the dynamics always converge to the set of segregated states. I consider a better-replies dynamics, which I find more appropriate for this low rationality environment. Incidentally, Schelling used a similar adjustment process stating that a resident moves to the closets location where she is better off. Young also considers swaps that improve joint welfare. I implicitly assume (as did Schelling) that there is some small cost of changing locations, so that a player only moves if payoff strictly increases.

One of the main innovations of this paper is to consider heterogeneous populations. In particular I show that a small amount of heterogeneity can have a dramatic effect on long run outcomes.

The plan of the paper is as follows. In section 2 I present a formalization of the model outlined by Schelling (1971), where residents have threshold preferences. I show convergence and numerically simulates the model. This section serves as a benchmark for the robustness analysis in the coming sections. In section 3 I introduce stochastic elements to the deterministic adaptive procedure developed in 2, the most important being a noisy better reply dynamics in location decisions. I consider both threshold preferences and a strict preference for diversity. In both variants the stochastically stable states are precisely the segregated states. I then numerically simulate the models, and the section ends with a discussion of the poor convergence behavior of the models. In section 4 I perturb a population of residents with a strict preference for diversity, by introducing a "social activist". Section 5 concludes.

2 Benchmark: Schelling's Model

Schelling (1969, 1971) originally modeled interaction on a line. I follow him closely except that I connect the line at both ends to form a circle. This allows me to abstract from specifying particular conditions at the two ends of the line, while leaving qualitative results intact. Since I am particularly interested in what elements of the model that drives the selection of equilibria away from diverse neighborhood structures I limit the analysis to the case where there is an equal (and even) number of residents of each type, such that a (fully) integrated stable configuration exists.

2.1 Model

There are two types of players A and B. There are $n > 1$ of each type, so that the total number of players is $2n$. Each player occupies a position on the circle, and have preferences

[4]The stochastic elements concern who gets to update and conditional on updating a resident mixes between all of her best responses.

over her local neighborhood composition. Let R be the set of players/residents.

Let L be the set of locations with typical element l_i, $i = 1, \ldots, 2n$, that is there are exactly as many locations as there are residents. The *neighborhood* of location l_i are all locations which are within one step of location l_i:

$$N_i \quad = \quad \{l_j \in L : |i - j| \leq 1\}$$

Note that l_i is itself contained in the neighborhood of l_i. I refer to the players contained in N_i (apart from the player at l_i) as the *neighbors* of l_i.

An example of a state (where $n = 8$) is shown in figure 1, where A types are white dots, and B types are black dots. The figure also shows how locations are numbered.

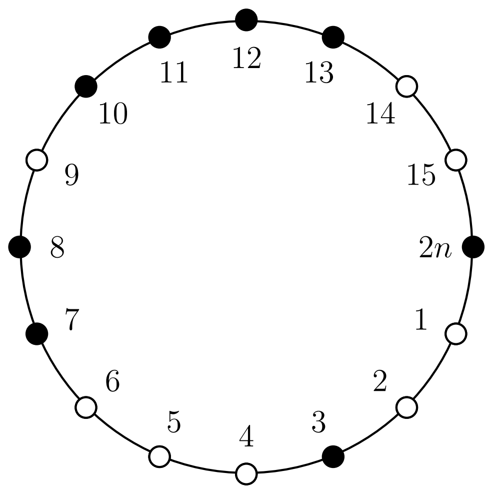


Figure 1: Example of a state with an equal number of black and white dots.

A configuration σ is an assignment of all players to a position on the circle such that no two players occupy the same position. Given a configuration σ let $|N_i^t(\sigma)|$ be the number of players of type t in the neighborhood of l_i. I sometimes omit σ when there can be no misunderstanding. Let the set of all possible configurations be Σ.

Preferences Each player cares about her local neighborhood composition. In particular a player of type t currently residing on location l_i gets the following payoff:

$$u_i^t(\sigma) \quad = \begin{cases} 0 \quad if & \frac{|N_i^t|}{3} < \frac{1}{2} \\ 1 & otherwise \end{cases}$$

Note that the player counts herself in the neighborhood. Let the set of players who have utility 0 in σ be $\underline{R}(\sigma) \subseteq R$.

Dynamics Schelling specifies the following adjustment dynamics. Let time be discrete $\tau = 0, 1, 2, \ldots$. Let the configuration at time τ be $\sigma(\tau)$. Start with an initial (randomly selected) configuration $\sigma(0)$. In each period τ pick one player in $\underline{R}(\sigma(\tau))$ by some exogenously specified procedure e.g. pick the player in $\underline{R}(\sigma(\tau))$ who has the lowest location number. Let her insert herself at a position in which she gets utility 1, such that she has to travel the fewest steps from her current position. If the player vacating l_i inserts herself at position $l_{i'}$ the new configuration $\sigma(\tau+1)$ is as follows: if the fewest number of steps from l_i and $l_{i'}$ is accomplished by moving counter-clockwise then all players who resided on locations $l_j,\ j = i', \ldots, i-1$ are moved one position clockwise so that the new location of a player who resided at location l_j is now l_{j+1}. E.g. the player living on l_{i-1} lives on l_i in the new configuration. And equivalently if the fewest number of steps can be accomplished by moving clockwise.

2.2 Analysis

As a first step in the analysis I look for patterns that are stable with respect to the adjustment process. Then in the next step I establish that from any initial configuration Schelling's model converges to an element in the set of "stable" configurations.

Stable Configurations I will say that a configuration σ is *stable* under Schelling's adjustment procedure if no player will want to change her location, given her current location and the assignment of the other players to locations on the circle. The following two definitions will be useful for what follows.

Definition 1. *A **cluster** is a contiguous group of at least two players of the same type. A cluster is* minimal *if it is of length 2.*

Definition 2. *A configuration is **integrated** if all residents belong to a minimal cluster. A configuration is **segregated** if all A players belong to the same cluster.*

Throughout I will assume that n is even. This is a necessary condition for integrated configurations to exist. Note that n even is not required for the characterization of stable configurations, nor to establish convergence.

I now characterize stable configurations:

Proposition 1. *A configuration is stable with respect to Schelling's dynamics if and only if all players have at least one neighbor of her own type.*

Proof. Suppose a player does not have any neighbors of her own type. Since the player can insert herself between any two players, independent of the configuration, she can move to a location where she has at least one neighbor of her own type.

Now suppose all players have at least one neighbor of her own type. Then she gets utility 1 and has no strict incentive to change her location. □

Let the set of stable configurations be denoted $\Sigma^* \subset \Sigma$.

Remark 1. *Note that if $n \geq 4$ and even the set of stable configurations contain the fully integrated configuration, in which all players live in a diverse neighborhood:*

$$\cdots AABBAABB \cdots$$

and the segregated configuration:

$$\cdots AAAABBBB \cdots$$

While Schelling conjectured that the dynamic process would converge, Schelling did not provide a proof. I now establish that Schelling's model with dynamic adjustment converges to some configuration in Σ^* starting from any initial configuration, $\sigma(0) \in \Sigma$.

Definition 3 (Convergence)**.** *The process has **converged** if at any period τ, when players change locations according the dynamic adjustment process, the following holds:*

$$\sigma(\tau + 1) = \sigma(\tau)$$

Given a configuration σ, let σ' be the configuration which is constructed from σ by moving player i from location σ_i to some location l: $\sigma' = \sigma_{(\sigma_i, l)}$.

Definition 4 (Pivotal)**.** *If for some $j \in N_{\sigma_i}(\sigma) \setminus \{i\}$: $u_j(\sigma) \neq u_j(\sigma')$ then player i is **ex-ante** pivotal, P^-, for player j. If $u_j(\sigma) = u_j(\sigma')$ for all j then she is ex-ante non-pivotal, P_0^-.*

*If for some $j \in N_{\sigma_i}(\sigma) \setminus \{i\}$: $u_j(\sigma) \neq u_j(\sigma')$ then player i is **ex-post** pivotal, P^+, for player j . If $u_j(\sigma) = u_j(\sigma')$ for all j then she is ex-post non-pivotal, P_0^+.*

In words, if i is *ex-ante* pivotal then the utility of at least one of i's neighbors in σ changes when i moves. If i is *ex-post* pivotal then the utility of at least one of i's neighbors in σ' changes after i's move.

I now show that Schelling's dynamic process indeed converges:

Proposition 2. *Starting from any initial configuration $\sigma \in \Sigma$ the process converges to some $\sigma \in \Sigma^*$.*

Proof. Suppose $\sigma(\tau)$ is not stable, otherwise we are done. Let the total number of players who receive utility 0 at time τ be $m(\tau)$. For any configuration we must have $0 \leq m(\tau) \leq 2n$. Let one of these players be i. i lives on location σ_i. i only has neighbors of the other type. Let her move to the location nearest to σ_i such that she has utility 1. Such a position must exist since $n > 1$, denote it l_i. By the rule of movement by inserting herself at l_i she now has one neighbor of each type. Let this new configuration be $\sigma' = \sigma_{(\sigma_i, l_i)}$.

i is either P^+ (for the player of her own type) or P_0^+. If i is P^+ for $j \in N_{l_i}(\sigma') \setminus \{i\}$ then the utility of j is now 1. i is either P^- (for either both or one of her neighbors) or P_0^-. If i is P^- for some $j \in N_{\sigma_i}(\sigma) \setminus \{i\}$ then the utility of j is now 1. Hence the lower and upper bound on the number of players with utility 0 in the next period, $m(\tau + 1)$, is:

$$m(\tau) - 4 \leq m(\tau + 1) \leq m(\tau) - 1$$

The process converges if for any τ: $m(\tau) = 0$. Thus after at most $m(0)$ steps the process has converged. □

Remark 2. *Note that the convergence result does not rely on the order in which players who have utility 0 move.*

2.3 Simulations

How does Schelling's model perform when it is simulated? One of the attractive features of Schelling's model is that it is extremely easy to do toy simulations. In this section I work out a few simulations by hand in order to get a feel for how the model works, and then in a later present results from numerical simulations.

Eye Balling I first simulate a few randomly drawn starting configurations, and see where the process ends up. I let $n = 10$. A dot over a player indicates that they player currently has utility 0. Recall that the player at the first position has neighbors at position 2 and 20.

Each successive line represents one "dotted" player starting from the left who updates her location.

$$
\begin{aligned}
&\dot{A}\dot{B}\dot{A}BBAAABBAABB\dot{A}\dot{B}AABB \\
\rightsquigarrow\ &BAABBAAABBAABB\dot{A}\dot{B}AABB \\
\rightsquigarrow\ &BAABBAAABBAABBBAAABB
\end{aligned}
$$

with 4 clusters of each type.

$$
\begin{aligned}
&B\dot{A}BB\dot{A}\dot{B}\dot{A}\dot{B}AAAA\dot{B}AABB\dot{A}BB \\
\rightsquigarrow\ &BBBAA\dot{B}\dot{A}\dot{B}AAAA\dot{B}AABB\dot{A}BB \\
\rightsquigarrow\ &BBBAAABBAAAA\dot{B}AABB\dot{A}BB \\
\rightsquigarrow\ &BBBAAABBAAAAAABBB\dot{A}BB \\
\rightsquigarrow\ &BBBAAABBAAAAAAABBBBB
\end{aligned}
$$

with 2 clusters of each type.

As can be seen the adaptive process leads to relatively segregated states, but rarely states with full segregation.

Numerical Simulations The numerically simulated model was programmed in FORTRAN 95 (Compaq Visual Fortran v6.6)[5]. In the actual implementation of the dynamic process I used the following procedure. For each simulation the starting configuration is drawn as follows. I make n draws from the set of locations, without replacement, with each location being equi-probable. Then the n players of a particular type are allocated to these locations. The n players of the other type are then allocated to the remaining available locations. The dynamic adjustment process is implemented as follows: in each period I find the first player who does not have any neighbors like herself starting from location 1. This player then moves to the location nearest to her current location where she will have at least one player like herself, ending the period. The adjustment process is repeated until all players have at least one neighbor like themselves.

A general issue when performing simulations is the relation between time in the model and time in the real phenomenon which the modeller is trying to capture. Within the present

[5] The Fortran programs are available on my website: `http://www.homepages.ucl.ac.uk/~uctpv00`.

setting, as the number of residents in the model grow, the probability that each resident gets to update her location shrinks at the same rate. In order to avoid this unattractive feature, when I report simulation results, I scale time by the number of residents ($2n$), such that a resident's probability of updating in any time period is independent of the size of the city. Thus e.g. when I report waiting times they can be thought of in terms of tenant-generations.

Table 1 gives descriptive statistics about the number of moves to convergence starting from a randomly drawn starting configuration. Mean, standard deviation, min and max of the distribution is reported. The number of residents of each type in the simulations $n = 10, 20, 50, 100$.

Table 1: Convergence Time (Schelling)

	Residents of each Type			
n	10	20	50	100
Mean	.147	.144	.1427	.1404
Std.	.0588	.0409	.0259	.0182
Max	.4	.325	.26	.215
Min	0	0	.04	.065

Note: 100.000 Observations per column. Time is measured in tenant generations.

It can be seen from the table that convergence is fast. The simulations suggest that the time to convergence is independent of the number of residents.

Table 2 shows the mean, standard deviation, the min and max number of clusters for different sizes of the interacting population. Again the number of residents is varied: $n = 10, 20, 50, 100$. Note that a configuration with cluster size $k = \frac{n}{2}$ is a (fully) integrated stable configuration, and that a configuration with cluster size equal to 1 is a (fully) segregated state. Any state with cluster size between $k = 1$ and $k = \frac{n}{2}$ is a stable configuration. In order to facilitate comparison as I vary the number of residents, the number of clusters are divided through by the maximal number of clusters possible ($n/2$), which gives us a measure in the range $(0, 1]$ independent of the number of residents.

Table 2: Number of Clusters in Stable Configurations (Schelling)

	Residents of each Type			
n	10	20	50	100
Mean	.4676	.4491	.4389	.4357
Std.	.1182	.0831	.0492	.0365
Max	1	.8	.64	.58
Min	.2	.1	.24	.28

Note: 100.000 Observations per column.

It can be seen from the table that only a subset of the set of stable configurations is selected.

In the following figure the frequency with which some stable configuration with k clusters of each type is selected under the dynamic adjustment process is graphed. It can be seen that the process selects a relative small set of the possible stable neighborhood structures.

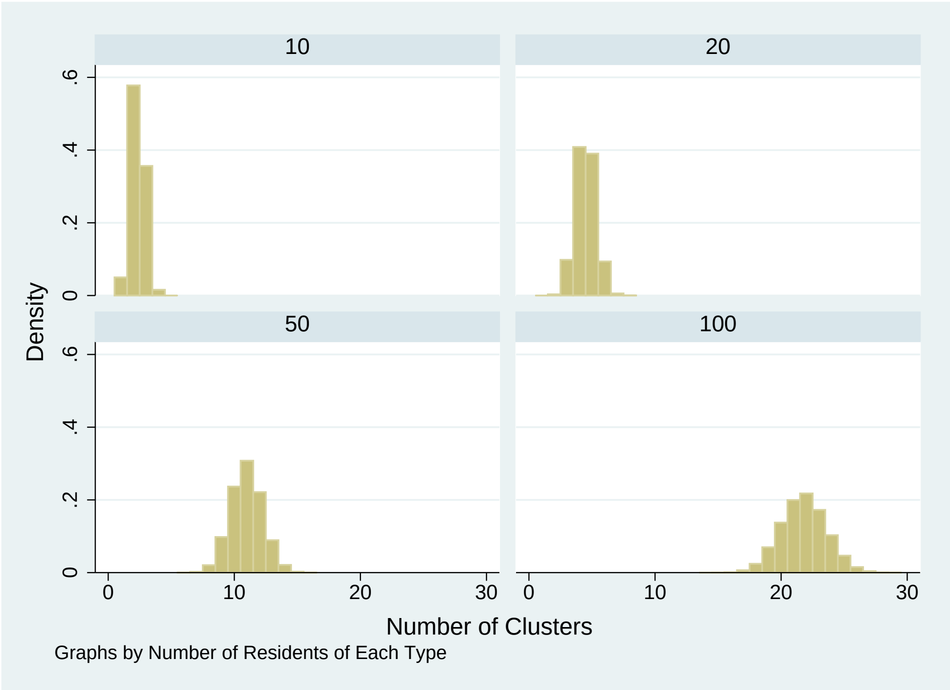


Figure 2: Selection of stable configurations under Schelling's dynamics. Number of residents of each type: $n = 10, 20, 50$ and 100. 100.000 Observations per graph.

The distribution of clusters in the stable configurations that the process selects for gives a straightforward proxy for how local neighborhoods are composed. Even more informative is the distribution of local neighborhoods. In the next table I report the median fraction of residents who have respectively 1 and 2 neighbors like themselves in their neighborhood across simulations. Note that in any stable configuration all residents must have at least one neighbor like themselves. Also note that in the fully segregated state the fraction of players with 2 neighbors like themselves is $80\%, 90\%, 96\%$ and 98% for $n = 10, 20, 50$ and 100 respectively.

Table 3: Composition of local neighborhoods (Schelling)

	Number of Residents of each Type							
	$n = 10$		$n = 20$		$n = 50$		$n = 100$	
Number Like	1	2	1	2	1	2	1	2
Median	.4	.6	.4	.6	.44	.56	.44	.56
Min	.2	0	.1	.2	.24	.36	.28	.42
Max	1	.8	.8	.9	.64	.76	.58	.72

Note: 100.000 Observations per column.

As the number of residents increase the median fraction of players who live in neighborhoods only with residents like themselves decrease. The distribution also becomes less dispersed around the median as the number of residents increase.

3 Noise

In this section I ask how noise may help to explain the emergence of segregation. Formally neighborhood evolution is modeled as a Markov process with the current configuration as the relevant state. Noise is then added on top of the deterministic updating process. As the noise vanishes the process selects states that are stochastically stable (Young 1993)[6].

I look at two different variants. First I look at a stochastic variation of Schelling's model, where the assumption about preferences, made in the previous section, are retained. In order to see how robust analytical results are to other specifications of preference orderings, I also consider a variation where residents have a strict preference for diversity. This second variant also has independent interest. Stable configurations can ranked in terms of aggregate welfare. Then I numerically simulate both models and compare them.

3.1 Noise and Schelling's Model

In this section I again look at Schelling's model, but augment with stochastic elements. I assume that players sometimes with small probability make location choices that are unexplained by the model.

3.1.1 Schelling's Model with Mistakes

Consider the following stochastic version of Schelling's model.

In each period $\tau = 0, 1, 2, \ldots$ a player is randomly selected. All players are equally likely of being chosen. Suppose player i at location l_i is chosen. i has the opportunity to move to a randomly selected location l (moving either clockwise or counter clockwise), not identical to her current location, with all locations $l \neq l_i$ being equally likely of being chosen.

The probability that player i moves to l depends on the difference between the utility at her current location and the utility she would get if she moved to l. In particular under the *unperturbed* dynamics i moves with probability one if l strictly increases her utility, and remains with probability one otherwise. Under the *perturbed* dynamics however the resident sometimes moves (stays) even though staying (moving) gives strictly higher utility. Assume that there are numbers $0 < \alpha < \beta < \gamma < \infty$. For $\epsilon \in (0, \epsilon^*]$, the decision to remain or move occur with the following state dependent probabilities:

[6] Appendix A contains a formal development of stochastic stability analysis.

1. If i's utility increases at l then she moves there with probability $1-\epsilon^\alpha$ and remains with probability ϵ^α.

2. If i is indifferent between her current location and l then she stays at her current location with probability $1-\epsilon^\beta$ and moves with probability ϵ^β.

3. If i's utility decreases at l then she stays at her current location with probability $1-\epsilon^\gamma$ and moves with probability ϵ^γ.

Note that mistake probabilities are chosen such that the greater the loss in utility from the mistake the less likely the player is to make it. Also note that we have introduced a second stochastic element relative to benchmark: residents no longer moves to the nearest location which is satisfactory, instead they become "aware" of the utility associated with a randomly drawn location.

Model Comments The fact that a resident that moves displaces other residents does not seem like a realistic assumption about how the residential market works and deserves motivation. When a resident moves she simply squeezes in and displaces the original residents. Displacement is also considered by Schelling and Pancs and Vriend.

First, one need not take displacement literally. An alternative interpretation is that there are many more locations than players, and that residents have preferences over the two nearest occupied locations to themselves, e.g. if measured in degrees the set of locations could be identified with the interval $[0, 360) \subset \mathbb{R}$. Second, from a modeling perspective the assumption ensures that players choice of neighborhood are not constrained by the overall configuration. Thus a particular stable pattern does not come about due to some arbitrary constraints on possible movements. Third, as will be showed this approach yields similar predictions about stable neighborhood structures as do the more realistic assumption of pairs of residents exchanging locations as in Young (1998). Hence, this assumption is not essential for the results.

3.1.2 Analysis

The first result in this section shows that the only recurrent states are the stable configurations.

Proposition 3. *Under the unperturbed dynamics the set of recurrent states coincides precisely with the set of stable configurations of Schelling's deterministic model, Σ^*. Moreover any recurrent state is absorbing.*

Proof. The proof can be found in appendix B.1. □

Notice that the set of recurrent states coincide with the set of recurrent states in Young (1998). This suggests that the model is robust to the alternative specifications of how players switch places in our two models. I comment on the differences to Pancs and Vriend (2003) in the next section where these differences become even more apparent.

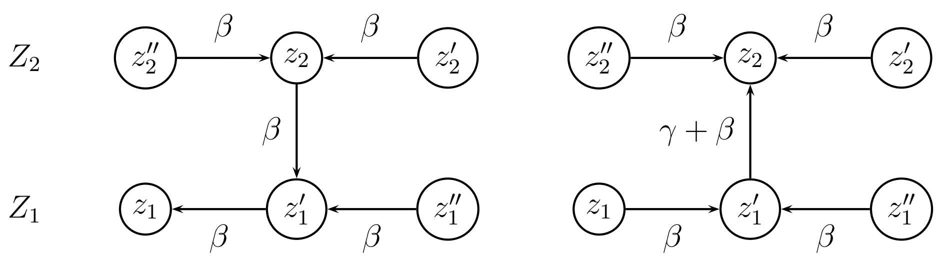
Next I show that the stochastically stable states are precisely the segregated states.

Proposition 4. *A state is stochastically stable if and only if it is segregated.*

Proof. The proof can be found in Appendix B.2. □

The intuition for the result is that it is "easier" (i.e. cheaper in terms of resistance) to reduce the number of clusters than it is to increase it. That is from any initial recurrent state, transitions which reduce the number of clusters are relatively more likely than transitions that increase the number of clusters.

Minimal cost trees are constructed using Lemmas 2 and 3. To get some intuition consider the simplest case where $K = 2$. The following picture shows two minimal cost trees, the first rooted at $z_1 \in Z_1$, i.e. a segregated state, the other rooted at $z_2 \in Z_2$ i.e. an integrated state.



Tree rooted at z_1 Tree rooted at z_2

Remark 3. *The claim of Proposition 4 holds under the weaker (but less plausible) assumption that mutation rates are independent of the magnitude of utility loss. Suppose that mistakes which leaves the player indifferent or worse off than at her current location both has cost β. Then the claim of Proposition 4 still holds since:*

$$\sum_{k<k'} \beta \quad > \quad 0$$

holds for all $k' \geq 2$.

3.2 Noise and Preferences for Diversity

In the model where residents have threshold preferences, residents are indifferent between living in integrated and segregated local neighborhoods. In this section I assume that players have a strict preference for diversity. I am interested in whether individual incentives to avoid living in a local minority are sufficiently strong to have welfare consequences, that is whether a dynamic process will select equilibria that are relatively or fully segregated.

3.2.1 A Model with Preference for Diversity

I modify the utility function[7] to reflect the assumption that players have a strict preference for diversity.

[7] I have also investigated a model where players cannot squeeze in between other players. Instead the circle has more locations than residents, such that there are empty locations. A moving player can only move to a location which is empty. In this model I show that only the fully segregated states are stochastically stable.

Preferences Given a configuration $\sigma \in \Sigma$ a player of type t who resides on location l_i has the following utility function:

$$u_i^t(\sigma) = \begin{cases} 0 & if \quad \frac{|N_i^t|}{3} \leq \frac{1}{3} \\ 1 & if \quad \frac{1}{3} < \frac{|N_i^t|}{3} \leq \frac{2}{3} \\ x & if \quad \frac{|N_i^t|}{3} > \frac{2}{3} \end{cases}$$

where $\frac{1}{2} < x < 1$.

That is players value diverse local neighborhoods, but they prefer to live in a ghetto of players like themselves, to living in a ghetto of players who are not like themselves.

Dynamics In each period $\tau = 0, 1, 2, \ldots$ one player and a location is chosen at random, with all players and all locations having positive probability of being chosen. The probability that she moves to the location depends on the utility difference between her current location and the new location.

Specifically let player i, currently living on l_i, be drawn for location revision and let l be the location she has the opportunity to move to. Assume that there are numbers: $0 < \alpha < \beta < \gamma < \delta < \psi < \infty$. For $\epsilon \in (0, \epsilon^*]$ I assume that the decision to stay at or vacate l_i for l is determined by the following procedure:

1. If i's utility increases at l then she moves there with probability $1 - \epsilon^\alpha$ and remains at l_i with probability ϵ^α.

2. If i is indifferent between her current location and l then she stays with probability $1 - \epsilon^\beta$ and moves with probability ϵ^β.

3. If i currently has utility 1 and l gives utility x then she stays with probability $1 - \epsilon^\gamma$ and moves with probability ϵ^γ.

4. If i currently has utility x and l gives utility 0, then she stays with probability $1 - \epsilon^\delta$ and moves with probability ϵ^δ.

5. If i currently has utility 1 and l gives utility 0 then she stays with probability $1 - \epsilon^\psi$ and moves with probability ϵ^ψ.

3.2.2 Analysis

I begin by characterizing the set of recurrent classes under the unperturbed dynamics, $P(0)$.

Proposition 5. *Suppose players have a preference for diversity. Under the unperturbed dynamics a configuration is recurrent if and only if each player belongs to a cluster. Moreover:*

1. *For any $\sigma, \sigma' \in \Sigma$, $\sigma \neq \sigma'$ such that both have $k < K$ clusters and all players belong to a cluster then σ and σ' are contained in the same recurrent class.*

2. *If σ is integrated, i.e. $k = K$, then it is recurrent and absorbing.*

Proof. The proof is in appendix B.3 □

The set of recurrent states are identical to the set of recurrent states in Young (2001). However in Young's model all of the recurrent states are also absorbing. The key difference here is that in the present model moves are unilateral, whereas in Young a pair of residents agree to switch places, and this occur only if the joint move is Pareto-improving ("roughly"). Clearly Pareto-improving moves must involve players of different types, but in the present model such switches would involve a player with two neighbors like herself and a player of different type living in a diverse neighborhood.

The contrast with Pancs and Vriend is stark. They assert that "*A sufficient condition* [for segregation] *on the utility function is that it implies a strict preference for perfect integration*" (Pancs and Vriend 2003, pp. 42-43) this is not the case here. In my set-up if all players strictly prefer integration to being in a minority, and thus to being in a (local) majority, then the only stable outcome is that of perfect integration. This suggest that segregation is non-robust to the behavioral assumptions. Specifically our results differ because Pancs and Vriend consider best-replies dynamics, whereas I assume that strategy revision follows better-replies. This is crucial since segregation in Pancs and Vriend is driven by the fact that a perfectly integrated location exists in any configuration (as it does in my set-up). This implies that how players order low ranked neighborhoods is inconsequential. In contrast when players follow better replies this is no longer the case.

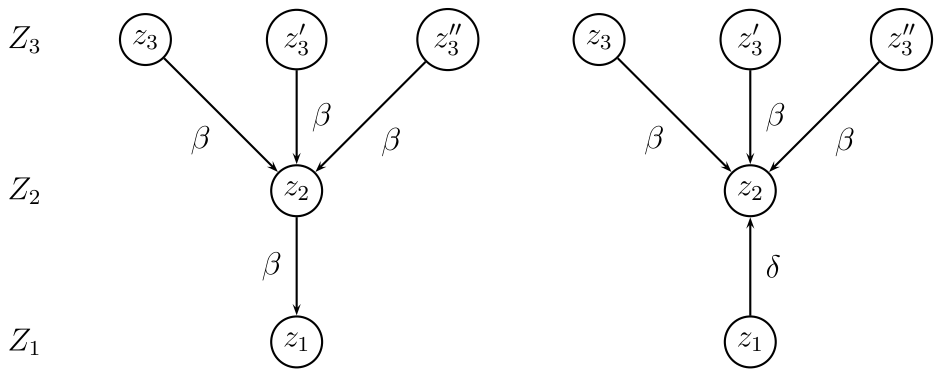
Remark 4. *Notice that the only absorbing configuration is the integrated state ($k = K$). Nevertheless configurations with $k < K$ clusters, and where all players belong to a cluster, are stable in the sense that the process will visit them infinitely often if started in that configuration. That is although players have individual micro-incentives to relocate the macro structure is stable.*

I now establish that set of segregated states are the only stochastically stable states.

Proposition 6. *A state is stochastically stable if and only if it is segregated.*

Proof. The proof can be found in Appendix B.4. □

The construction of minimal cost trees is shown in the following graph for the case $K = 3$. I show the trees for the segregated state and a state with 2 clusters.



Tree rooted at z_1 Tree rooted at z_2

3.3 Simulations

3.3.1 Stochastic Version of Schelling

In this section I present numerical simulations of the model for various parameter value of the number of residents and the noise level. The section has two main parts. In the first part I turn off the noise, i.e. I set $\epsilon = 0$. This allows us to compare how the stochastic selection procedure affects the convergence time, and clustering in the equilibria selected by the process and compare it to the deterministic version of Schelling. In the second part I examine how the model behaves when there is a positive level of noise.

No Noise Schelling's model contains two deterministic components that are given stochastic counterparts in our model. First, Schelling assumes that residents who enjoy low utility get to update their choice of location in a deterministic way. In Schelling in each round first all players that have no neighbors like themselves are marked out. Then starting from the right end of the line any player that was marked updates her location, until all marked players have either moved to a new location or they have at least one neighbor like themselves. After this a new round begins. Second, Schelling assumes that when a player moves she moves to a location closets to her current location where she has at least one neighbor like herself.

In order to see whether the stochastic counterpart of these rules play any role for convergence time and the clustering in the equilibria that are reached I simulate the model when the noise is turned off, i.e. $\epsilon = 0$.

The first table shows the convergence time to Nash.

Table 4: Convergence Time (Stochastic Schelling)

	Number of Residents of each Type			
n	10	20	50	100
Mean	2.23	3.098	4.440	5.445
Std.	1.82	1.9	2.052	2.089
Max	18.25	16.65	19.77	20.56
Min	0	0	.61	1.305

Note: $\epsilon = 0$, 10.000 Observations per column.

Comparing results to Schelling's original model, convergence is slower for the stochastic version. This is not surprising since players who are never selected under Schelling's procedure is selected for revision in the stochastic version. In particular as the configuration comes close to a recurrent state, with only a few players needing to update their locations, the probability that these players are selected decreases in the stochastic version. Thus convergence rates slow down when the configuration gets "close" to a recurrent state. Nevertheless convergence is fairly rapid, and looks roughly linear.

I now investigate whether the way players update their location choice has any bearing on what equilibria are eventually reached by the process. The next two tables gives details on this.

Table 5: Number of Clusters (Stochastic Schelling)

	Number of Residents of each Type			
n	10	20	50	100
Mean	.458	.434	.4204	.4172
Std.	.14	.101	.064	.045
Max	1	.8	.68	.58
Min	.2	.1	.2	.26

Note: $\epsilon = 0$, 10.000 Observations per column.

Compared to results from simulating Schelling's original model, the mean number of clusters in the equilibria that are reached under the stochastic version is slightly lower than in Schelling's model, but more dispersed. The next table confirms that the two models select the same set of equilibria.

Table 6: Composition of local neighborhoods (Stochastic Schelling)

	Number of Residents of each Type							
	$n = 10$		$n = 20$		$n = 50$		$n = 100$	
Number Like	1	2	1	2	1	2	1	2
Median	.4	.6	.4	.6	.44	.56	.42	.58
Min	.2	0	.1	.2	.2	.32	.26	.42
Max	1	.8	.8	.9	.68	.8	.58	.74

Note: $\epsilon = 0$, 10.000 Observations per column.

I conclude that the stochastic updating and selection of potential allocations does not have a significant impact on the distribution of equilibria that are reached. The only significant impact is on the time to convergence.

In the next part I turn on the noise.

Noise I showed analytically that in the long run the process will only visit the segregated states. However the result is silent about how long we have to wait before the long run kicks in. In this case simulations are a useful means of examining whether the selection of stochastic stability is economically meaningful. I am interested in how the model behaves with respect to three time aspects of the model: the short, medium and long run.

Definition 5. *The* ***short run*** *is the time interval:* $\{0, \ldots, \tilde{T} - 1\}$*, where* $\tilde{T} \geq 0$ *is the random time where the process hits a recurrent class for the first time. The* ***medium run*** *is the time interval:* $\{\tilde{T}, \ldots, \tilde{\tilde{T}}\}$*, where* $\tilde{\tilde{T}} \geq \tilde{T} \geq 0$ *is the random time where the process hits an element in the set of stochastically stable states for the first time. The* ***long run*** *is* $t : t > \tilde{\tilde{T}}$

For the purpose of the simulations I fix: $(\alpha, \beta, \gamma) = (1, 2, 3)$.[8]

[8]This choice is clearly somewhat arbitrary. The choice affects the rate at which mistake rates go to zero and therefore affects when the prediction of stochastic stability is valid.

The Short Run The following tables show the descriptive statistics of the time of convergence to a recurrent class.

Table 7: The Short Run (Stochastic Schelling)

n	10			20			50		
ϵ	.02	.05	.1	.02	.05	.1	.02	.05	.1
Mean	2.145	2.26	2.455	3.088	3.12	3.395	4.545	4.833	4.895
Std.	1.79	1.80	1.885	1.895	1.925	2.04	2.024	2.241	2.384
Max	12.8	11.4	10.95	11.28	12.68	15.4	15.33	16.77	16.58
Min	0	0	0	.15	.3	.1	1.08	1.10	1.01

Note: $(\alpha, \beta, \gamma) = (1, 2, 3)$, 500 Observations per column.

The Medium Run The following table shows statistics of the duration of the medium run and the fraction of time spent in states that are visited before the process hits one of the stochastically stable states for the first time.

Table 8: Duration of The Medium Run (Stochastic Schelling)

n	10			20			50		
ϵ	.02	.05	.1	.02	.05	.1	.02	.05	.1
		($\times 10^3$)			($\times 10^4$)			($\times 10^5$)	
Mean	6.725	1.07	.024	3.198	.5375	.14	1.688	.265	.0785
Std.	7.40	1.23	.255	2.775	.2075	.1048	1.202	.186	.0575
Max	7.01	7.65	.3825	22.98	2.923	.74	6.037	1.111	.394
Min	0	0	0	.0485	0	.0011	.180	.0225	.0039

Note: $(\alpha, \beta, \gamma) = (1, 2, 3)$, 500 Observations per column.

From the table above it can be seen that the duration of the medium run increases rapidly in the number of residents.

Table 9: Visited States in The Medium Run (Stochastic Schelling)

n	10			20			50		
State	No. Clusters			No. Clusters			No. Clusters		
	2	3-5	TS	2-3	4-10	TS	2-4	5-25	TS
Mean	89.5	9.0	1.6	89.8	9.6	.6	86.7	12.1	1.2
Max	100	99.6	64.6	99.9	79.7	5.1	98.5	68.9	2.7
Min	0	0	0	18.1	0	.06	30.1	.6	.7
Obs.		449			499			500	

Note: $\epsilon = .05$, $(\alpha, \beta, \gamma) = (1, 2, 3)$, TS=Transient States. Total number of observations per column: 500. Observations where process transits directly to long run not included.

Before transiting to the segregated states the process spends the majority of its time in states which are relatively segregated.

The Long Run The analytical selection result of stochastic stability is a limit result in the sense that if the noise level goes to zero then the process will spend almost all it's time in the set of stochastically stable states. In many real life situations factors which are unobserved to the modeler affect the decisions of agents. Therefore we might be unwilling to assume that the noise is vanishing. An alternative is to simulate the model starting from a stochastically stable configuration and then observing the time path of the system for a relatively long period. This allows us to numerically quantify the fraction of time that the process spends in the stochastically stable states. It also provides a way to quantify a threshold for the noise level where the prediction of stochastic stability gives a good approximation of where the system is at any time for a sufficiently long time horizon.

The table below shows the fraction of time spent in various configurations, when the process is started in a stochastically stable state. I track the process over 10^7 periods.

Table 10: Fraction of Time Spent in Stochastically Stable States (Schelling)

n	10			20			50		
ϵ	.02	.05	.1	.02	.05	.1	.02	.05	.1
Stochastically Stable States	99.99	99.79	94.9	99.98	89.1	84.7	99.96	55.1	34.3
Other Recurrent States (*)	0	0	3.4	0	10.4	11.4	0	43.6	55.8
Out of Equilibrium	.01	.21	1.8	.02	.5	3.9	.04	1.3	9.9

Note: $(\alpha, \beta, \gamma) = (1, 2, 3)$, Number of Periods: 10^7. (*) The process only spends time in recurrent states with less than 4 clusters.

It can be seen that the validity of stochastic stability as the prediction of the long run behavior of the model depends naturally on ϵ but also on the number of residents. In fact for $\epsilon = .1$ and $n = 50$ the process spends a larger fraction of time in recurrent states with cluster size 2 (39.3%) than it does in the stochastically stable states.

Fist Hitting Times How long does it take for $x\%$ of the population to become ghettoized? The following table answers this question. The table records the mean first hitting time for $x\%$ of the population to only have neighbors like themselves. Standard deviations are reported in parenthesis.

Table 11: First Hitting Times ($\times 10^3$) (Stochastic Schelling)

n		Percentage Ghettoised 50%	60%	70%	80%	90%	95%	97%	99%
10	Mean	.0081	.3265	.3265	6.68	6.68	6.68	6.68	6.68
	Std.	(.0511)	(.8675)	(.8675)	(8.085)	(8.085)	(8.085)	(8.085)	(8.085)
	Max	.6475	8.615	8.615	70.1	70.1	70.1	70.1	70.1
20	Mean	.0551	.4008	1.901	7.005	31.23	31.23	31.23	31.23
	Std.	(.2000)	(.6848)	(2.508)	(6.01)	(27.98)	(27.98)	(27.98)	(27.98)
	Max	1.417	3.465	19.91	28	229.8	229.8	229.8	229.8
50	Mean	.0045	.1958	1.151	7.878	21.80	53.53	158.0	158.0
	Std.	(.0022)	(.3830)	(1.115)	(5.079)	(13.40)	(38.36)	(106.0)	(106.0)
	Max	.0092	1.856	4.582	24.09	60.56	193.0	515.0	515.0
100	Mean	.0080	.2458	1.819	7.7	33.91	79	155.5	467.5
	Std.	(.0144)	(.3520)	(.9305)	(3.216)	(16.75)	(41.98)	(104.5)	(336.5)
	Max	.1066	1.776	4.505	16.69	106.0	220.5	700	1575

Note: $(\alpha, \beta, \gamma) = (1, 2, 3)$, 200 Observations per row for $n = 10, 20$. 50 observations for $n = 50, 100$.

Conditional Waiting Times Do waiting times depend upon where we start? The following table documents waiting times conditional upon the first recurrent class which the process hits.

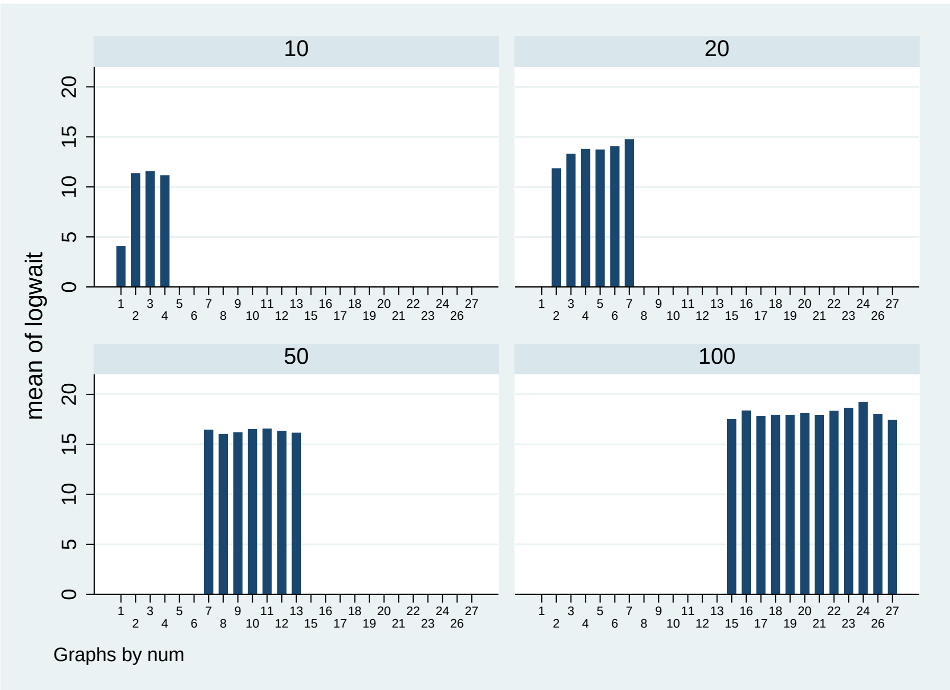


Figure 3: Conditional Mean of Log Waiting times for Stochastic Schelling. Number of clusters in starting recurrent classes on horizontal, and log time on vertical axis. For $n = 10, 20$ 200 simulations, and for $n = 50, 100$ 50 simulations were performed.

3.3.2 Preference for Diversity

In this section I present simulation results for the model analysed above. For the purpose of the simulations I have fixed: $(\alpha, \beta, \gamma, \delta, \psi) = (1, 2, \frac{5}{2}, 3, 4)$.

I begin by looking at the short run behavior of the model.

The Short Run Table 12 shows the time until the process hits a recurrent class of the unperturbed dynamics for the first time.

Table 12: Time to Convergence - The Short Run (Diversity)

n	10			20			50		
ϵ	.02	.05	.1	.02	.05	.1	.02	.05	.1
Mean	1.565	1.79	1.725	2.208	2.16	2.473	3.035	3.074	3.368
Std.	1.155	1.295	1.4	1.25	2.26	2.7	1.179	1.177	1.506
Max	7.85	6.45	8.05	7.9	8.175	7.125	8.27	7.53	11.66
Min	0	0	0	.4	.4	.5	.89	.89	.94

Note: $(\alpha, \beta, \gamma, \delta, \psi) = (1, 2, \frac{5}{2}, 3, 4)$, 200 Observations per column.

The next table details the number of clusters in the configuration when the process hits a recurrent state of the unperturbed dynamics for the first time.

Table 13: Number of Clusters (Diversity)

n	10			20			50		
ϵ	.02	.05	.1	.02	.05	.1	.02	.05	.1
Mean	.574	.586	.59	.562	.555	.558	.564	.5548	.5536
Std.	.146	.144	.14	.106	.104	.098	.0628	.0632	.0656
Max	.8	.8	.8	.8	.8	.8	.72	.76	.72
Min	.2	.4	.2	.3	.3	.3	.4	.4	.4

Note: $(\alpha, \beta, \gamma, \delta, \psi) = (1, 2, \frac{5}{2}, 3, 4)$, 200 Observations per column.

I now turn to the medium run.

The Medium Run In the simulations I track the process for a maximum of 10^7 periods. The expected wait until the process hits the set of stochastically stable states for the first time is rapidly increasing in the number of residents in the neighborhood. In particular for $n \geq 20$ the process does not hit the stochastically stable states during the period in which I track the process. Therefore the medium run behavior of the process, i.e. the states that are visited in the medium run become the most economically interesting time period. Since the long run is reached within 10^7 periods for very few observations I report the fraction of time the process spends in the recurrent classes of the unperturbed dynamics, which are not stochastically stable.

Table 14: Visited States in The Medium Run (Diversity)

n	10			20				50			
State	2	3-5	OE	2-3	4-5	6-10	OE	2-6	7-12	13-25	OE
Mean	75.40	24.49	.113	10.55	85.64	3.616	.188	0	95.01	4.54	.453
Max	99.96	80.14	.721	18.93	93.67	7.413	.223	0	98.99	10.76	.374
Min	19.69	0	.030	4.030	77.81	.888	.146	0	88.76	.602	.544

Note: $\epsilon = .05$, $(\alpha, \beta, \gamma, \delta, \psi) = (1, 2, \frac{5}{2}, 3, 4)$, OE=Out of Equilibrium. For $n > 10$ the process does not reach the set of stochastically stable states in 10^7 periods. Observations per column: 200.

The Long Run In the table below I start the process in the stochastically stable states and track it for 10^7 periods. I record the fraction of time which the process spends in the the different states.

Table 15: Fraction of Time Spent in Stochastically Stable States (Diversity)

n	10			20			50		
ϵ	.02	.05	.1	.02	.05	.1	.02	.05	.1
SSS	22.62	1.65	1.82	5.03	.491	.034	.648	.643	.107
2-3 Clusters	77.38	98.05	96.72	46.89	12.41	5.54	27.81	.815	.059
>3 Clusters	0	.207	.706	48.07	86.94	92.99	71.53	98.06	96.38
OE	.003	.091	.748	.010	.163	1.44	.013	.480	3.46

Note: $(\alpha, \beta, \gamma, \delta, \psi) = (1, 2, \frac{5}{2}, 3, 4)$, Number of Periods: 10^7. SSS=Stochastically Stable States, OE= Out of Equilibrium (transient state).

For reasonable levels of noise stochastic stability is not a valid predictor of where the process will spend most of its time, even for relatively small sizes of the residential neighborhood. Compared to the stochastic version of Schelling presented in section 3.1 the process spends much more time in states which are not stochastically stable. This is due to the change in the preferences of residents over local neighborhood composition. In particular in the stochastic version of Schelling if the process is started in a segregated state and a resident by mistake moves to a location with players different from herself then no player like herself has a strict incentive to follow her. On the other hand if residents have a preference for diversity then a mistake opens up a “beachhead” from which a new cluster can form at least temporarily. Since the new cluster is small players from the old cluster is more likely to be drawn for revision and might get the opportunity to move to a diverse neighborhood. Thus the process will spend a significant amount of time outside the segregated state.

Fist Hitting Times How long does it take for $x\%$ of the population to become ghettoized. The following table answers this question. The table records the mean first hitting time for $x\%$ of the population to only have neighbors like themselves. Standard deviations are reported in parenthesis. I only report figures for $n = 10, 20, 50$. As can be seen mean hitting times increases rapidly in n even for relatively low percentages.

Table 16: First Hitting Times ($\times 10^4$) (Diversity)

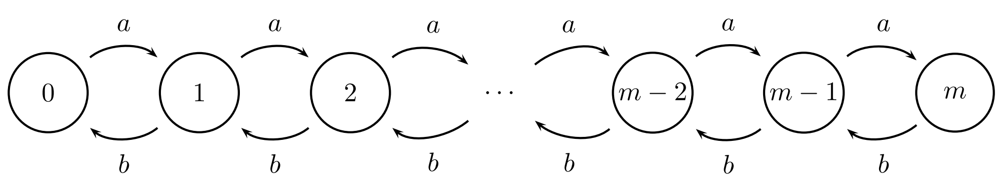
		Percentage Ghettoised							
n		50%	60%	70%	80%	90%	95%	97%	99%
	Mean	.03	.644	.644	31.00	31.00	31.00	31.00	31.00
10	Std.	(.0076)	(.9585)	(.9585)	(28.64)	(28.64)	(28.64)	(28.64)	(28.64)
	Max	.362	4.238	4.238	108.5	108.5	108.5	108.5	108.5
	Mean	.144	.9613	8.47	375.0	-	-	-	-
20	Std.	(.2503)	(.9165)	(8.03)	(40.0)	-	-	-	-
	Max	1.201	3.278	35.75	199.7	-	-	-	-
	Mean	.1175	1.402	12.60	-	-	-	-	-
50	Std.	(.1467)	(1.120)	(12.80)	-	-	-	-	-
	Max	.6566	6.232	57.30	-	-	-	-	-

Note: $(\alpha, \beta, \gamma, \delta, \psi) = (1, 2, \frac{5}{2}, 3, 4)$, 50 observations for each n. Maximum number of iterations: 10^9. "-" indicates that the process did not hit the state within maximum number of iterations.

3.4 Noise and Segregation: Comments

Why do the stochastically stable states not feature prominently within reasonable time frames? The basic intuition is as follows. Suppose the process is started at a state where there are two clusters of each type. The process hits a stochastically stable state when all players are contained on one of the two locations. Due to our assumption about mistake probabilities we need consider only the events where a player moves from one cluster by mistake to the other cluster. Therefore we can approximate the process by a random walk where the process roughly changes state with probability ϵ^β. This base probability has to be modified by how large the cluster is: residents who live in small clusters are less likely to be drawn for revision than residents who live in a larger cluster. Thus as the size of a cluster shrinks the less likely that a player from that cluster is chosen. This introduces a bias towards clusters of equal size. This behavior of this particular process is hard to characterize analytically, but we can look at a simpler random walk which preserves the main feature that the process is biased towards equal size.

Consider the following random walk on the set of integers $Z = \{0, \dots, m\}$, $m > 0$. With probability a the process moves up and with probability b the process moves down, where $a > b > 0$ and $a + b \leq 1$. Note that the process exhibits a positive drift towards m. The following figure illustrates the process:



I am interested in the conditional waiting times w_k, $0 < k \leq m$ until the process hits 0.

These can be determined from the following recursive relation:

$$w_k = a(w_{k+1}+1)+b(w_{k-1}+1)+(1-a-b)(w_k+1)$$

and the following two boundary conditions:

$$\begin{aligned} w_0 &= 0 \\ w_m &= b(w_{m-1}+1)+(1-b)(w_m+1) \end{aligned}$$

This is a second order non-homogeneous difference equation with constant coefficients, which can be solved explicitly by standard techniques (see e.g. Sydsaeter and Hammond (1995, p. 750)). Results are summarized in the following Lemma.

Lemma 1. *Consider a random walk on the integers between 0 and m. The process moves up with probability a and moves down with probability b, where $a > b > 0$ and $a + b \leq 1$. The conditional waiting time w_k, $0 < k \leq m$, until the process reaches 0 is:*

$$w_k = \frac{1-\left(\frac{b}{a}\right)^k}{(a-b)(1-\frac{b}{a})\left(\frac{b}{a}\right)^m} - \frac{1}{a-b}k$$

The waiting time until the process reaches 0 is bounded above by $-\frac{1}{a-b}m + o(\left(\frac{a}{b}\right)^m)$.

Proof. The proof is in appendix B.5. □

For the random walk with drift the waiting time until the process reaches 0 increases exponentially in m.

Consider a recurrent state of the unperturbed dynamics which contains two clusters of each type. For the purpose of illustrating the connection we need only concern ourselves with one type of players. Also for small ϵ we need only consider the possibility that β mistakes occur (recall $\gamma > \beta$ thus for small ϵ β-mistakes becomes exponentially more likely than γ-mistakes. That is the only movements that we need consider is that one player, by mistake, moves from one cluster to the other cluster. Let the two clusters be denoted c and $\tilde{c}$ respectively A stochastically stable state is reached whenever say all players from cluster c has moved to $\tilde{c}$ (or vice versa). All such moves (except the last for $r = 1$) must occur via a β mutation. More importantly the process is biased towards selecting some player from a larger cluster rather than a player from a smaller cluster. To see this note that the probability that the some player from cluster c is selected equals $\frac{n_c}{n}$, since all players are equally likely of being drawn for revision, and with remaining probability some player from cluster $\tilde{c}$ is chosen. Thus as the cluster shrinks the probability that a player from the other cluster is chosen increases. That is the probability that the cluster grows (via a player from cluster $\tilde{c}$ making a location mistake becomes increasingly higher as the cluster shrinks. Thus the process is biased towards clusters of equal size. The main simplification is that for the random walk above I have assumed that the bias does not depend upon the state of the walk, whereas in the Schelling model the closer the state moves to 0 the more unlikely is a downward jump and the more likely an upward jump.

The key observation is that the selection dynamics exhibit a drift towards clusters of equal size. Therefore the maximal waiting time until a stochastically stable state is reached increases exponentially in the number of residents. This insight is also applicable to Young's variants of Schelling.

Why are waiting times even larger when players have a preference for diversity. Consider again as state with two clusters. One minimal cluster and one containing the rest. Since epsilon is small we can ignore other mistakes apart from β mistakes. The waiting time until the minimal cluster disappears is $\frac{2}{n}\frac{2}{2n}\frac{1}{\epsilon^{\beta}}$ which is the wait until a player from small cluster is chosen and she draws a location that is diverse in the other cluster times the wait until she actually makes a β mistake. On the other hand for the minimal cluster to grow, we just need one of the ghettoized players to be drawn: $\frac{n-2-2}{n-2}$ and then she must draw an appropriate location: $\frac{2}{n}$ so the wait is roughly of order n.

There is a literature on the speed of convergence to the set of stochastically stable states (Young (1998), Ellison (1993, 2000)). The understanding of that literature is that local interaction greatly speeds up convergence to the set of stochastically stable states.

Ellison (1993) and Young (1998, chp. 6) consider local interaction when players play a two person coordination game. Both show that the waiting time until the stochastically stable state is reached is independent of the size of the system. The mechanism through which this is established differ between the two papers.

Ellison (1993) considers an adjustment dynamics a la Kandori, Mailath, and Rob (1993). That is in each period *all* players update or adjust their strategy so that their strategy is a best response to play in the previous period. With small probability players tremble. In each period a player plays a two person coordination game with all the other players. Players may weight payoffs differently from different players. In particular Ellison considers the case where players are located on a circle and a player only assign positive (equal) weight to her k nearest neighbors on either side. In the k nearest neighbor model for small enough perturbations the expected waiting time until the process reaches it's long-run steady state distribution, which puts probability mass one on the risk-dominant equilibrium, is independent of the size of the system. The intuition for the result is as follows. It is already well-known that the risk dominant equilibrium has a larger basin of attraction than the other pure equilibrium. This completely determines what will be selected for in the long run. Now suppose we start the process in the equilibrium which is not risk dominant. In order to transit to the risk dominant equilibrium it is sufficient that a suitable "small" group of players (who are connected) mutate to the risk dominant strategy. In the next period this will lead their neighbors to switch to the risk dominant strategy as well. The play of the risk dominant strategy then spreads contagiously. Since the basin of attraction of the risk dominant equilibrium is of larger size, then as ϵ becomes small the risk dominant equilibrium is selected for.

My version of Schelling's model contain no contagious element. When play has settled on an equilibrium which is not stochastically stable, all players will have at least one neighbor like themselves. A location mistake (a mutation) will lead at most one other player of the same type to revise her location. This occurs only if the mutating player belongs to a minimal cluster.

Young (1998, chp. 6) also considers selection in two person coordination games with a risk dominant equilibrium. Young considers a stochastic process in continuous time. Players update their strategies according to a Poisson process. When players update their strategy most of the time they play a noisy best response. Updates are independently and identically distributed across players. Players interact on a graph, however a given player mainly interacts within relatively small close-knit groups, loosely the group of players that a player interacts with are likely to mainly interact with each other as well. Young asks what the maximum expected wait until a large proportion $1 - p$ of the population the risk dominant equilibrium, this is called the p-inertia of the process. For sufficiently small mistake probabilities and if players interact in close-knit groups then the p-inertia of the process is bounded above, independently of the number of players.

The result relies on the following intuition. First note again that in Young's model the risk dominant equilibrium is the unique stochastically stable state. Now since all players live in close-knit groups of a given (small) size, starting from the non-risk dominant equilibrium of the coordination game, the wait until a particular group switches to the risk-dominant equilibrium is bounded above, and does not depend on the total number of players in the population. After this event if the probability that players make mistakes is sufficiently small then this group will continue playing the risk dominant for a long time. Since the process runs simultaneously for all players the waiting time until a large proportion, $1 - p$, of the population is playing the risk-dominant equilibrium is bounded.

As is apparent the interaction pattern is substantially different from the current framework.

Independent of the underlying preferences the expected wait until the process hits the set of stochastically stable is indeed very long, even for a small number of residents. Stochastic stability is quite uninformative about the behavior of the system for large time periods. Although in both models evolutionary pressures push the process towards segregated states, in the medium run individual preferences over outcomes play a significant role for local neighborhood composition. For the prediction of stochastic stability to be valid the level of noise must be significantly smaller than in the stochastic version of Schelling. This is because starting from a segregated state a location mistake now leads the process directly out of the segregated states, since the location mistake has opened up an opportunity for a player who only lived with players like herself to move to a diverse neighborhood. For a given level of noise it is relatively easier to leave the stochastically stable states.

4 Preference Heterogeneity

In this section I check robustness with respect to preference heterogeneity. I introduce a small portion of players into the population who have different preferences from the majority of the population who have a preference for diversity. These players have preferences of the following form: their most preferred neighborhood is a diverse one, but they prefer living with people different from themselves to only living with people like themselves. One interpretation is that these players are "social activists", they are willing to live isolated in order to create better outcomes.

I show that with this perturbation of population preferences only the integrated states are recurrent (and absorbing). There is no equilibrium selection problem. It might however take a very long time to reach an integrated state, and the presence of a majority of players with "normal" preferences means that evolutionary pressures tend to work against integration. I therefore ask the question to what extent the presence of a small fraction of "social activists" is able to push the population towards more integrated outcome? Our simulations suggests that local neighborhoods are remarkably diverse.

4.1 Social Activists

Assume that a small fraction of players are "social activists" in the sense that while their most preferred neighborhood is a diverse one, they prefer to live in isolated neighborhoods rather than living in ghettos with people like themselves[9]. That is technically their second and third ranked alternatives are flipped relative to the remaining population.

This leads to the following formulation of preferences for "social activists". Given a configuration σ a social activist of type t residing on location l_i, has the following utility function:

$$v_i^t(\sigma) = \begin{cases} x & if \quad \frac{|N_i^t|}{3} \leq \frac{1}{3} \\ 1 & if \quad \frac{1}{3} < \frac{|N_i^t|}{3} \leq \frac{2}{3} \\ 0 & if \quad \frac{|N_i^t|}{3} > \frac{2}{3} \end{cases}$$

where $\frac{1}{2} < x < 1$.

I make the additional assumption that all players only care about the type of the players that reside in their local neighborhood, i.e. not whether they are social activists or not.

Apart from this modification the stochastic process is left unchanged.

4.2 Analysis

The next result shows that the presence of just one social activist in a population of players with a preference for diversity has a dramatic effect on long run outcomes. In particular only integrated states are rest points of the unperturbed dynamics.

Proposition 7. *Suppose that in a population of players with a preference for diversity there is at least one* social activist. *Under the unperturbed dynamics a state is recurrent if and only if it is integrated. Moreover any integrated state is absorbing.*

Proof.
$\Leftarrow$: It is immediate that an integrated state is recurrent, since in an integrated state all players have utility 1, thus no player has strict incentive to move. This shows that an integrated state is absorbing.

$\Rightarrow$: Let a social activist of type t be denoted t^s while players with a preference for diversity are simply denoted by their type, t. We now show that the integrated states are the only recurrent states. Take any state σ which is not integrated. By the unperturbed dynamics the process can transit to a state in which all players of type t have at least one neighbour like themselves. Also by the unperturbed dynamics we can transit to a state where all type t^s players live in a diverse neighbourhood. Let this state be σ'. Suppose σ' has $1 \leq k < K$ clusters, otherwise we are done.

We now show that we can transit to a state with $k+1$ clusters of each type. Since n is even and $k < K$ there are at least two players of each type who do not live in diverse

[9] An example of what we have in mind is from the account of Anderson (1990) of the rise and dynamics of the ghetto. In a predominantly white area the Quakers bought run-down residential blocks, restored and rented them out. This offered the opportunity for socially progressive individuals to move to the "ghetto".

neighbourhoods. Thus by the unperturbed dynamics we can transit to a state with k clusters and such that there is a cluster of each type which is of length at least 4, and one of these clusters contain at least one social activist. Let the cluster of length at least 4 containing type t players be denoted k'_t, while the other is denoted $k'_{t'}$. Suppose the social activist in k'_t and does not live in a diverse neighbourhood. If she does then let one of the other players in k'_t who only have neighbours like themselves be drawn for revision and given the opportunity to move to the edge of the cluster taking up the location that the social activist had. Now let the t^s type player be drawn for revision and suppose that she picks a location contained in $k'_{t'}$, $t' \neq t$, such that at least two type t' players live on either side (this is possible since $k'_{t'}$ is of length at least 4). Since the t^s player prefers to live only with people different from herself to living only with people like herself she moves under the unperturbed dynamics. Now there is a second player in k_t who only live with people like herself. Let her be drawn for revision and let her move next to the social activist. Both these players now live in a minimal cluster and have a diverse neighbourhood. Also by their moves a new cluster of type t' players have formed. Thus we have transited to a state with $k + 1$ clusters.

Note that to reach a state with $k + 1$ clusters we only relied upon there being at least one cluster of each type of length at least 4 and that at one of these clusters contained a social activist. Hence if $k + 1 < K$ it is possible to transit to a state with $k + 1$ clusters and where there is one cluster of each type of length at least 4, so that we can repeat the procedure above until we have reached an integrated state. □

Although simple and intuitive the result is quite remarkable. The only recurrent states of the unperturbed dynamics is the integrated states. The introduction of one "social activists" solves the "problem" of multiplicity of equilibria. The only stable outcome of the process is that of "perfect" integration.

Remark 5. *Note that it follows immediately that the set of integrated states are precisely the stochastically stable states, since the set of stochastically stable states are a non-empty subset of the recurrent classes of the unperturbed dynamics.*

Slightly perturbing the population composition away from a homogeneous population where players have a strict preference for integration, results in the perfectly integrated state being the only stable configuration. This result is in stark contrast to Pancs and Vriend (2003). In their set-up such a perturbation would have no effect on their segregation result; effectively a consequence of the best reply dynamics they consider.

Remark 6. *Schelling's model in which players have threshold preferences is unaffected by the introduction of "social activists". The set of segregated states are precisely the set of stochastically stable. This is because residents have no strict preference for diversity.*

4.3 Simulations

Since the unperturbed process has a unique recurrent class the process will spend most of it's time in the integrated states for a sufficiently long time horizon, if ϵ is sufficiently small. However there might be other reason for considering positive and non-vanishing levels of noise: it might capture elements of reality which the model does not pick up, or the modeler do not wish to model. Therefore in this section I both simulate the model without noise and with noise. Simulating the model without noise gives us an estimate of the waiting time until the process hits an integrated state. Simulating the model with noise allows me to see to what extent evolutionary pressures which tend to work against integration (for the majority of players) may be countered by the introduction of some heterogeneity in preferences over local neighborhood composition.

No Noise I first simulate the model without noise. Throughout I fix the proportion of social activists to 10% of the population.

In the first table I record the time when the process first hits a state in which all players who has a preference for diversity has at least one neighbor like themselves and all social activists live in a diverse neighborhood. Such states mimic recurrent states in a model where all players have a preference for diversity. I refer to them as "pseudo-recurrent".

Table 17: Time of Convergence to Pseudo-Recurrent States (Heterogeneous Players)

n	10	20	50
Mean	3.165	6.568	21.46
Std.	2.55	5.175	15.18
Max	17.4	42.38	104.4
Min	0	.125	1.37

Note: $\epsilon = 0$, 10% of population are "social activists", 500 obs. per column.

In the next table I report the number of clusters when the process first hits a pseudo-recurrent state.

Table 18: Clustering (Heterogeneous Players)

n	10	20	50
Mean	.628	.64	.684
Std.	.1256	.0870	.6040
Max	1	.9	.84
Min	.2	.4	.44

Note: $\epsilon = 0$, 10% of population are "social activists", 500 Observations per column.

A straight forward comparison with the clustering in the diversity model with homogeneous agents shows that even in the short run the presence of actively diversity seeking players leads to significantly higher levels of integration.

In the following table I start the process at a randomly drawn configuration and then track it for 10^7 periods or until the integrated state is reached.

Table 19: Time of Convergence to Integrated State (Heterogeneous Players)

n	10 ($\times 10^3$)	20 ($\times 10^4$)	50
Mean	.64	2.483	-
Std.	.665	2.422	-
Max	5.735	14.26	-
Min	.0007	.0062	-

Note: $\epsilon = 0$, 10% of population are "social activists", 500 Observations per column. For $n = 50$ an integrated state is not hit within 10^7 periods.

The wait until the process reaches an integrated state increases rapidly as n increases.

Noise The simulations without noise showed that for reasonable time horizons the process will be out of equilibrium most of the time, at least for large n. It therefore becomes interesting to see what disequilibrium states are visited by the process. Here noise is not used for selection purposes but is introduced to capture elements of the location decision process which are not captured by the specification of preferences. For the simulations in this section I fix $\epsilon = .02$. As in the previous section I have fixed: $(\alpha, \beta, \gamma, \delta, \psi) = (1, 2, \frac{5}{2}, 3, 4)$ for all simulations. Since the majority of players have diversity preferences, evolutionary pressure tends to push neighborhoods configurations towards segregated states. At the same time a small fraction of social activists are opening up new opportunities for the formation of diverse local neighborhoods, which tends to push the evolution of the system towards more integrated states.

In the following tables I show the mean fraction of time with which the process visits different states for $n = 10, 20$ and 50 respectively. I track the process for 10^7 periods. Since I am mainly interested in how the presence of a few "social activists" affect the welfare of the majority of players who have a preference for diversity the tables indicate the fraction of time that the process spends in a state with a particular distribution of local neighborhoods for players who have a preference for diversity. E.g. in table 20 I simulate the model with $n = 10$. Since the fraction of social activists is fixed to .1 this leaves a total of 18 players who have a preference for diversity. The table is then read as follows. The time average fraction that the process spends in a state where all 18 players live in a diverse neighborhood is 33.83. That is the process spends about a third of the total time in states in which all players live in a diverse neighborhood. It spends 36.67% of the time in states in which 14 out of the 18 players have a diverse neighborhood, the remaining 4 players live in ghettos only with player like themselves.

Table 20: Visited States ($n = 10$) (Heterogeneous Players)

	No. Integrated					
No. Isolated	0-9	10-13	14	15-17	18	Sum
0	.22	12.05	36.67	12.25	33.83	95.02
1	.07	1.87	.21	2.11	0	4.26
2-18	.04	.53	.13	.01	0	.72
Sum	.34	14.45	37.01	14.37	33.83	100

Note: 10% of population are "social activists", $\epsilon = .02$, $(\alpha, \beta, \gamma, \delta, \psi) = (1, 2, \frac{5}{2}, 3, 4)$. Number of periods: 10^7

For $n = 10$ the process spends the majority of the time close to or at an integrated state. 85.2% of the time is spent in states where at least 77.8% of the players live in diverse neighborhoods.

I now increase the total number of residents to 40 players. That is 36 of the players have a preference for diversity.

Table 21: Visited States ($n = 20$) (Heterogeneous Players)

	No. Integrated				
No. Isolated	0-19	20-25	26-29	30-36	Sum
0	.24	26.83	42.41	16.62	86.10
1	.10	3.59	5.71	1.90	11.31
2-36	.03	.92	1.34	.30	2.60
Sum	.38	31.34	49.46	18.82	100

Note: 10% of population are "social activists", $\epsilon = .02$, $(\alpha, \beta, \gamma, \delta, \psi) = (1, 2, \frac{5}{2}, 3, 4)$. Number of periods: 10^7

States with full integration are now hardly reached, however the process still spends about 68.3% of its time in states where at least 72.2% of the players live in diverse neighborhoods.

When $n = 50$ there are now 90 players who have a preference for diversity.

Table 22: Visited States ($n = 50$) (Heterogeneous Players)

	No. Integrated				
No. Isolated	0-59	60-69	70-75	76-90	Sum
0	4.94	42.60	17.98	1.79	67.32
1	1.85	16.33	4.88	.42	23.49
2	.70	5.11	1.46	.09	7.36
3-90	.22	1.30	.30	.01	1.83
Sum	7.71	65.35	24.62	2.31	100

Note: 10% of population are "social activists", $\epsilon = .02$, $(\alpha, \beta, \gamma, \delta, \psi) = (1, 2, \frac{5}{2}, 3, 4)$. Number of periods: 10^7

The process spends about 92.3% of its time in states where at least 66.7% of the players live in diverse neighborhoods.

5 Concluding Remarks

In this paper people have preferences over whom they would like to have in their local neighborhood. If they do not like their current neighborhood then they can move to one which they like better. This is the basic model suggested by Schelling (1969, 1971). This paper has examined the robustness of segregation taking Schelling's original model as its point of departure. Surprisingly and counter to the understanding of the literature I have shown that whereas noise may help in sharpening the predictions of neighborhood evolution the pace at which this happens is sufficiently slow to make it unappealing as an explanation for segregation. Also the understanding of the literature has been that even when residents have a strict preference for diversity evolution tends to favor segregated outcomes. This paper has challenged this insight both at a numerical level, in terms of waiting times to reach the set of stochastically stable states, and critically by introducing a "small" preference perturbation. I showed that the preference perturbation establishes the set of integrated states as the only recurrent classes of the unperturbed dynamics. It appears that the insights of the literature are non-robust.

Much research into this area is called for. One would like to know whether our insights carry over into different geometries of neighborhoods. Whereas interaction on a circle may capture the phenomenon of segregation on streets, interaction on a two dimensional lattice appear to capture the main aspects of interaction in inner city neighborhoods. Schelling (1971) considered a version of the model analyzed in this paper in two-dimensional space.

There is research on racial preferences suggesting that whites are more biased against living with blacks than other ethnic groups (Emerson, Chai, and Yancey (2001)), even after (experimentally) controlling for socio-economic effects. In light of this research it would be interesting to explore the effects of heterogeneity in thresholds between types.

References

Anderson, E. (1990): *Streetwise - Race, Class, and Change in an Urban Community.* The University of Chicago Press.

Bergin, J., and B. L. Lipman (1996): "Evolution with State-Dependent Mutations," *Econometrica*, 64(4), 943–956.

Borjas, G. (1995): "Ethnicity, Neighborhoods, and Human-Capital Externalities," *American Economic Review*, 85(3), 365–390.

Cutler, D. M., and E. L. Glaeser (1997): "Are Ghettos Good or Bad?," *Quarterly Journal of Economics*, 112(3), 827–72.

Cutler, D. M., E. L. Glaeser, and J. L. Vigdor (1999): "The Rise and Decline of the American Ghetto," *Journal of Political Economy*, 107(3), 455–506.

Ellison, G. (1993): "Learning, Local Interaction, and Coordination," *Econometrica*, 61(5), 1047–1071.

——— (2000): "Basins of Attraction, Long-Run Stochastic Stability, and the Speed of Step-by-Step Evolution," *Review of Economic Studies*, 67(1), 17–45.

Emerson, M. O., K. J. Chai, and G. Yancey (2001): "Does Race Matter in Residential Segregation? Exploring the Preferences of White Americans," *American Sociological Review*, 66(6), 922–935.

Glaeser, E. L., B. Sacerdote, and J. A. Scheinkman (1996): "Crime and Social Interactions," *Quarterly Journal of Economics*, 111(2), 507–48.

Kandori, M., G. Mailath, and R. Rob (1993): "Learning, Mutation, and Long Run Equilibria in Games," *Econometrica*, 61, 29–56.

Möbius, M. (2000): "The Formation of Ghettos as a Local Interaction Phenomenon," Harvard University.

Pancs, R., and N. J. Vriend (2003): "Schelling's Spatial Proximity Model of Segregation Revisited," Queen Mary Working Paper No. 487.

Schelling, T. (1969): "Models of Segregation," *American Economic Review Proceedings*, 59(2), 488–493.

——— (1971): "Dynamic Models of Segregation," *Journal of Mathematical Sociology*, 1, 143–186.

——— (1972): "A process of residential segregation: Neighborhood tipping," in *Racial Discrimination in Economic Life*, ed. by A. Pascal. Lexington Books, Lexington, MA.

——— (1978): *Micromotives and Macrobehaviour*. W.W. Norton & Company.

Sydsaeter, K., and P. J. Hammond (1995): *Mathematics for Economic Analysis.* Prentice-Hall.

Topa, G. (2001): "Social Interactions, Local Spillovers and Unemployment," *Review of Economic Studies*, 68(2), 261–295.

Young, H. P. (1993): "The Evolution of Conventions," *Econometrica*, 61, 57–84.

——— (1998): *Individual Strategy and Social Structure: An Evolutionary Theory of Institutions*. Princeton University Press, Princeton, NJ.

——— (2001): "The Dynamics of Conformity," in *Social Dynamics*, ed. by S. Durlauf, and H. P. Young, chap. 5. MIT–Press.

A Stochastic Stability

This appendix gives an overview of the necessary building blocks for the theory of stochastic stability. The review largely follows the exposition in Young (1998, chap. 3.3-3.4).

Elements of Markov Theory Let $P : \Sigma \to \Sigma$ be a finite state (time-homogeneous) transition matrix. Specifically for every pair $\sigma, \sigma' \in \Sigma$, the probability of transiting at time t from σ to σ' at time $t+1$ is $P_{\sigma\sigma'}$. $P_{\sigma\sigma'} > 0$ if the process can transit from σ to σ' in one step. Otherwise $P_{\sigma\sigma'} = 0$.

Our interest is in how much time the process spend in various states. Suppose the initial state is σ^0. For each $t > 0$, $\mu^t(\sigma|\sigma^0)$ (a random variable) is the relative frequency with which state σ is visited up until period t. As $t \to \infty$, $\mu^t(\sigma|\sigma^0)$ converges almost surely to a probability distribution $\mu^\infty(\sigma|\sigma^0)$, which is the *asymptotic frequency distribution* conditional on starting the process at σ^0. $\mu^\infty(\sigma|\sigma^0)$ can be interpreted as a selection criterion since it tells us which states have support in the long run frequency distribution. If $\mu^\infty(\sigma|\sigma^0)$ is independent of σ^0 then the process is *ergodic*.

A state σ' is *accessible* from σ, denoted $\sigma \to \sigma'$ if

$$(P^t)_{\sigma\sigma'} > 0, \text{ for some integer } t > 0.$$

where P^t is the t-fold product of P. Two states σ and σ' *communicate* if both are accessible from the other, denoted $\sigma \sim \sigma'$. This relation partitions Σ into a set of equivalence classes, called communication classes. A *recurrent class* of P is a communication class such that no state not in the class is accessible from this class. Let $E_1, \ldots, E_K$ be the K distinct recurrent classes of the process. A state σ is *recurrent* if it is contained in a recurrent class, otherwise it is *transient*. Another way to understand this partitioning of states, is to see that a state σ is recurrent if conditional on starting the process in σ the probability that the state is visited infinitely often is equal to 1. If the process has only one recurrent class and this class is the entire state space, Σ, then the process is *irreducible*.

Let μ be a solution to:

$$\mu P = \mu, \text{where } \mu \geq 0 \text{ and } \sum \mu(\sigma) = 1$$

It can be shown that the solution is unique if and only if P has a unique recurrent class. μ is then referred to as the *stationary* distribution of P. If P has a unique recurrent class then μ describes the time-average asymptotic behaviour, and is independent of the initial state σ^0, that is:

$$\lim_{t\to\infty} \mu^t(\sigma|\sigma^0) = \mu^\infty(\sigma|\sigma^0) = \mu(\sigma)$$

If the system is also *aperiodic*, then for t large enough the position of the system can be approximated by μ. Let $\nu^t(\sigma|\sigma^0)$ be the probability that the state is σ at time t when it was started in σ^0. Hence we write:

$$\nu^t(\sigma|\sigma^0) = (P^t)_{\sigma^0\sigma}$$

If the process is irreducible and aperiodic then P^t converges to the matrix P^∞ in which every row equals the stationary distribution μ:

$$\lim_{t\to\infty} \nu^t(\sigma|\sigma^0) = \mu(\sigma) \text{ for all } \sigma \in \Sigma.$$

that is with probability one $\nu^t(\sigma|\sigma^0)$ and $\mu^t(\sigma|\sigma^0)$ converge to $\mu(\sigma)$.

Perturbed Markov Process This section looks at a family of perturbations of the Markov Process P, parametrized by the error rate ϵ. $P(\epsilon)$ is a regular perturbed Markov Process (of $P(0)$) if $P(\epsilon)$ is irreducible for every $\epsilon \in (0, \bar{\epsilon}]$, and for every $\sigma, \sigma' \in \Sigma$, $P(\epsilon)_{\sigma\sigma'}$ approaches $P(0)_{\sigma\sigma'}$ at an exponential rate:

$$\lim_{\epsilon\to 0} P(\epsilon)_{\sigma\sigma'} = P(0)_{\sigma\sigma'},$$

and

$$\text{if } P(\epsilon)_{\sigma\sigma'} > 0 \text{ for some } \epsilon > 0, \text{ then } 0 < \lim_{\epsilon\to 0} P(\epsilon)_{\sigma\sigma'}/\epsilon^{r(\sigma,\sigma')} < \infty \text{ for some } r(\sigma,\sigma') \geq 0.$$

$r(\sigma,\sigma')$ is the *resistance* of the transition $\sigma \to \sigma'$. Note that $r(\sigma,\sigma') = 0$ if and only if $P(0)_{\sigma\sigma'} > 0$.[10]

For any $\epsilon > 0$ the process is now irreducible, and from the results of the previous it has a unique stationary distribution, denoted μ_ϵ. Following Young (1993) a state σ is *stochastically stable* if:

$$\lim_{\epsilon\to 0} \mu_\epsilon(\sigma) > 0.$$

The main theorem of Young (1993) stated below establishes that for all σ this limit exists and that for each state this limit equals the stationary distribution of the unperturbed Markov process: $\lim_{\epsilon\to 0} \mu_\epsilon(\sigma) = \mu_0(\sigma)$. The theorem also shows how to identify stochastically stable states in terms of the stochastic potential of the recurrent classes of the unperturbed process. This is the final building block that is now introduced.

Suppose the unperturbed process has K distinct recurrent classes, $E_1, \dots, E_K$. Take pairs of distinct recurrent classes E_i and E_j, $i \neq j$. An ij-path is a sequence of states $\xi = (\sigma_1, \sigma_2, \dots, \sigma_q)$ that begins in E_i and ends in E_j. The resistance of the ij-path is equal to: $r(\xi) = r(\sigma_1, \sigma_2) + \dots + r(\sigma_{q-1}, \sigma_q)$. Let $r_{ij} = \min r(\xi)$ be the ij-path that has lowest resistance (note that $r_{ij} > 0$ since E_i and E_j are distinct recurrent classes).

Construct a complete directed graph with K vertices (that is for each vertex k there is exactly one directed edge from k to each of the $K-1$ remaining vertices). The weight of the directed edge $i \to j$ is r_{ij}. A j-tree is a set of $K-1$ edges that from every vertex different from j, has a unique directed path in the tree to j. The resistance of a tree is the sum of the resistances along it's edges. The *stochastic potential* of E_j is the minimum resistance

[10] If $P(\epsilon)_{\sigma\sigma'} = P(0)_{\sigma\sigma'} = 0$ for all $\epsilon \in (0, \bar{\epsilon}]$, then $r(\sigma,\sigma') = \infty$.

among all j-trees. Intuitively the stochastic potential of a recurrent class says something about how "easy" it is to get to the state starting from any of the other recurrent classes[11].

We now state a result from Young (1993) which characterises stochastically stable states in terms of stochastic potentials:

Theorem (Young (1993))**.** *Let $P(\epsilon)$ be a regular perturbed Markov process, and let μ_ϵ be the unique stationary distribution of $P(\epsilon)$ for each $\epsilon > 0$. Then $\lim_{\epsilon \to 0} \mu_\epsilon = \mu_0$, and μ_0 is a stationary distribution of $P(0)$. The stochastically stable states are precisely those states that are contained in the recurrent classes of $P(0)$ having minimum stochastic potential.*

Technically introducing noise allows us to use the powerful results of the previous section for *ergodic* Markov processes. The system's behaviour becomes independent of the starting conditions for sufficiently large t. Moreover the stationary distribution tells us which states are likely to be visited in the long-run.

One can think of perturbations as testing how robust equilibria are to the continual bombardment of small perturbations. Thus perturbations are a selection device, and as Young's theorem makes clear the selection is on the set of recurrent states. Bergin and Lipman (1996) show that with state-dependent mutation rates, any recurrent state (of the unperturbed process) can have support in the set of stochastically stable states. Therefore assumptions made about how fast error-rates converge to zero is in no way innocuous.

B Omitted Proofs

B.1 Proof of Proposition 3

Proof. $\Rightarrow$: Suppose $\sigma \in \Sigma^*$. Then all players have at least one neighbor like themselves, and all players have utility 1. Accordingly no player will want to vacate her current location.

$\Leftarrow$: Suppose $\sigma \notin \Sigma^*$. Therefore there must be at least one player who does not have any neighbors like herself. With positive probability this player will be drawn for revision and be matched with a location where she will have at least one neighbor like herself. Hence she will move there. Let this state be σ'. At her former location she must have had two neighbors different from herself. Therefore after she moves these player's utility must weakly increase. Also at her new location the utility of her neighbors must weakly increase: if she has two neighbors like herself then their utility is unchanged. If she has a mixed neighborhood then the utility of the neighbor different from herself remains unchanged whereas the utility of the neighbor like herself is either unchanged or has increased. Thus in σ' there are at least one player whose utility has increased (the player who has relocated) and no player whose utility has decreased. Hence either $\sigma' \in \Sigma^*$ or there is another player who does not have a neighbor like herself. But then we can apply the same procedure again. Since the number of players is finite the process arrives at stable configuration in a finite number of steps.

[11] Ellison (2000) shows how the maximal waiting time can be bounded by using the concepts of radius and co-radius. Intuitively a recurrent state is stochastically stable if it is easy to enter the basin of attraction of the state starting from any of other recurrent state of the unperturbed dynamics, and if it is relatively difficult to exit its basin of attraction when the process is found in this state.

Finally note that under $P(0)$ if the state is stable then the process will not leave this state, since no players have a strict incentive to change their location. Thus the state is absorbing. □

B.2 Proof of Proposition 4

I prove the proposition via a series of lemmas. Let Z_k, $1 \leq k \leq \frac{n}{2} \equiv K$ be the set of states with k clusters of each type, and $|Z_k|$ is the number of elements in Z_k. I first find the minimum resistance of a segregated state.

Lemma 2. *For any $1 \leq k \leq K-1$ there is a state $z \in Z_k$ such that the minimum cost of transiting from z to some $z' \in Z_{k+1}$ is $\gamma + \beta$. For any $k \geq 2$ there is a state $z \in Z_k$ such that the minimum cost of transiting from z to some $z' \in Z_{k-1}$ is β.*

Proof. The proof is in two main steps. In the first step we show that the minimum cost of transiting from some $z \in Z_k$ to a suitable state $z' \in Z_{k+1}$, $1 \leq k \leq K-1$ has minimum cost $\gamma + \beta$. In the second step we show that the minimum cost of transiting from some $z \in Z_k$ to a suitable state $z' \in Z_{k-1}$, $k \geq 2$ has cost β.

Step 1 Start from some $z \in Z_k$ with the property that for both types of residents there is a cluster which is not *minimal* and has at least 4 residents. Since $1 \leq k \leq K-1$ there must be at least one cluster which is not minimal, for all $z \in Z_k$. Furthermore since n is even either there is a cluster which contains at least 4 residents or there are at least two clusters which are not minimal. So suppose that z has a cluster of size at least 4, and let them be denoted k_A and k_B respectively.

Now let one of the residents in k_A make a mistake such that she moves to k_B and inserts herself such that she has at least two B neighbours on either side. This mistake has cost γ. Now let another resident in k_A make a location mistake such that she insert herself next to the first A who moved. This mistake has cost β. We have now arrived at some $z' \in Z_{k+1}$ at cost $\gamma + \beta$.

It remains to be shown that this cost is minimal. First observe that given a state with $1 \leq k < K$ clusters all players have utility one. Thus a new cluster can only form if a player makes a mistake. Moreover the only type of mistake which leads a new cluster to form with positive probability is that a player moves from say cluster k_A to a location where she has only players of opposite type. If she moves somewhere else then cluster k_A contains one less of type A but she is added to another pre-existing cluster of type A players. Therefore the player must make a location mistake which has cost γ (since she gets utility zero). In the ensuing state all players but one (the player who made the mistake) has utility one. If a player from cluster k_A moves next to the player who previously moved both players will have utility one. But the player had utility one before, therefore a β mistake is required for a new cluster to form.

Step 2 Start from some $z \in Z_k$ where $2 \leq k \leq K$, with the property that z has a minimal cluster for at least one of the types. There is such a state $z \in Z_k$ since $k > 1$.

Let one of the residents in the minimal cluster, call it $\tilde{k}$, move to another cluster of residents of her own type (recall $k > 1$ so there is another cluster). This has cost β. Now let the remaining player in $\tilde{k}$ be drawn for location revision and suppose she draws a new location which gives her at least one neighbour like herself. This occurs with positive probability since there are $k-1$ other clusters in z. Thus she will move at cost 0, and we have transited to a state $z' \in Z_{k-1}$. To see that this cost is minimal notice that since we started out from a state in which all players have utility one, the lower bound on the cost of transition is exactly β. □

Recall that the aim is to construct minimal cost trees for all recurrent states. Also recall that the lower bound on the minimal cost of transition is β. That is if I can prove that a transition has cost β then it is minimal. In the next lemma I show that for any $z \in Z_k$ I can construct a *path* which includes all $z' \in Z_k \setminus \{z\}$ and that this path has resistance $\beta(|Z_k|-1)$ that is the path has minimum resistance.

Lemma 3. *For any $1 \leq k \leq K$ there is a minimum resistance path ξ, which ends in $z \in Z_k$ and such that for all $z' \in Z_k$: $z' \in \xi$, and $z'' \notin Z_k$: $z' \notin \xi$ with cost $\beta(|Z_k|-1)$.*

Proof. We only show the proof for the segregated states. The proof is analogous for all other classes.

Suppose that in $z \in Z_1$ the A cluster begins at position l. Thus it ends at position $l+n-1$. We now show that the minimum cost of transiting to a state $z' \in Z_1$ such that the A cluster begins at position $l+1$ is β.

Let the player at position l be drawn for revision and suppose she has the opportunity to move to position $l+n$. Let her move counter clockwise around the circle to this location. Since her utility is the same at both locations the cost of this move is β. The A cluster now starts at position $l+1$.

Now we construct the path ξ. $\xi = (z^1, \ldots, z^{|Z_k|})$ where $z^i \in \xi: \; z^i \in Z_k, \; i = 1, \ldots, |Z_k|$ and $z = z^{|Z_k|}$. Two elements z^i and z^{i+1}, $i = 1, \ldots, |Z_k|-1$ has the property that in z^i the A cluster begins at position l and in z^{i+1} it begins at postion $l+1$. The cost of this path is the sum of the individual transitions. By the argument above the cost of ξ is $\beta(|Z_k|-1)$. □

I can now prove proposition 4:

Proof of Proposition 4. It follows from lemmas 2 and 3 that the stochastic potential of a state $z \in Z_1$ is:

$$\beta(|Z_1|-1) + \sum_{k\geq 2} \beta(|Z_k|-1) + \sum_{k\geq 2} \beta$$

where the two first parts follow from lemma 3 and the third part follows from lemma 2.

For any state $z' \in Z_{k'}$, $k' \neq 1$ the stochastic potential is:

$$\sum_{k<k'} (\beta|Z_k| + \gamma) + \beta\left(|Z_{k'}|-1\right) + \sum_{k>k'} \beta|Z_{k'}|$$

In then follows that $z \in Z_1$ has minimum stochastic potential if:

$$\sum_{k<k'} \gamma \;\; > \;\; 0$$

for all $k' \geq 2$. The claim then follows immediately since $\gamma > 0$. □

B.3 Proof of Proposition 5

Proof. I first show that the integrated state is recurrent and absorbing. Then I show that under the unperturbed dynamics any state with $k < K$ clusters of players is recurrent and any other state which also has k clusters belong to the same class. Finally I show that any other state is transient.

Step 1a Suppose σ is integrated. Then σ consist of $K \equiv \frac{n}{2}$ clusters players of each type, and each cluster is minimal, i.e. of length 2. Thus all players receive utility 1, and no player has a strict incentive to change her location.

Step 1b Take any state σ with $k < K$ clusters. Let the size of clusters in σ be $n_{1(\sigma)}, \ldots, n_{k(\sigma)}$ for some t. Suppose that $k(\sigma)$ is not minimal. I show that the process can transit to any σ' such that that $n_{k(\sigma')} = n_{k(\sigma)} - 1$ and and $n_{k'(\sigma)} = n_{k'(\sigma)} + 1$, for some $1 \leq k' < k$. Since $k(\sigma)$ is not minimal there must be a player who gets utility x. She can move to the edge of $k'(\sigma)$ where she gets utility 1, thus the move occurs with positive probability, and the process has transited to a state σ' with k clusters of size $n_{1(\sigma)}, \ldots, n_{k'(\sigma)} + 1, \ldots, n_{k(\sigma)} - 1$. Also note that the process can transit back to σ since in σ' there must be a player in $k'(\sigma')$ who gets utility x.

Notice that the process cannot transit to states with a different number of clusters of each type. For the segregated states this is obvious, so assume that σ has $1 < k < K$ clusters. By the transition dynamics the process can transit to a state where we have $K - 1$ minimal clusters, and one cluster containing the remaining players of type t. However any of the $K - 1$ minimal clusters cannot be broken up by the dynamics since all players live in integrated neighborhoods, and thus receive highest possible utility.

Step 2 Finally assume that in σ there is at least one player who does not belong to a cluster. This player has utility 0. Thus with positive probability under $P(0)$ she will be drawn for revision and move to a location where she has at least one neighbor like herself. The new configuration is $\sigma' \neq \sigma$. If there are more players who do not belong to a cluster then let them update the location choice. With positive probability the process will arrive at a state σ'' in which all players belong to a cluster. Once the process has hit σ'' it cannot go back to σ, since by step 1b above a cluster will never vanish. Moreover no player has an individual incentive to move to a location where she only has neighbors which are not like herself. This shows that σ is transient. □

B.4 Proof of Proposition 6

In order to prove the Proposition let $|Z_k|$ be the number of states with k clusters, $1 \leq k \leq K$, and recall that any two states $z, z' \in Z_k$ $z \neq z'$ and where $k < K$ z and z' are in the same recurrent class.

Lemma 4. *For $1 < k \leq K$ the minimum cost of transiting from $z \in Z_k$ to $z' \in Z_{k-1}$ is β.*

Proof. Take some state $z \in Z_k$, $1 < k \leq K$. z either contains a minimal cluster or via the unperturbed dynamics with positive probability the process can transit to a state with k clusters and at least one minimal cluster. Hence assume that z contains a minimal cluster. Let one of the residents in the minimal cluster be drawn for revision and let the location she can move to be on the edge of another cluster containing residents of her own type.

Since the resident lives in a minimal cluster she currently has utility 1, and if she moves to her new location she will also get utility 1. Thus a mistake at cost β is required. Now let the remaining resident in the minimal cluster in z be drawn for revision. Since she lived in a minimal cluster she now only has neighbors different from herself. Therefore by the unperturbed dynamics the process can transit to a state z' which contains only $k-1$ clusters. □

Lemma 5. *For $1 \leq k < K$ the minimum cost of transiting from $z \in Z_k$ to $z' \in Z_{k+1}$ is δ.*

Proof. Take some state $z \in Z_k$, $1 \leq k < K$. Since $k < K$ there is at least two players of type t who only has neighbors like themselves. Let one of these player be i and assume that she currently lives on location l. i has utility x. Moreover either z contains a cluster of type t', $t' \neq t$ players of at least length 4 or via the unperturbed dynamics the process can transit to a state with a cluster of at least length 4 of type t' players. Hence assume that z contains such a cluster and that it begins at location $l' > l$. Now let i be drawn for revision and suppose that she has the opportunity to move clockwise around the circle to $l'+2$. By construction $l'+2$ gives i a neighborhood which only contains neighbors different from herself. Thus a mistake at cost δ is required for her to move. Since z had $k < K$ clusters there is another player j of type t who only had neighbors like herself z. After i has moved this player still only has neighbors like herself. Thus via the unperturbed dynamics, with positive probability this player will be drawn for revision, and suppose she has the opportunity to move next to i. At this location she will have a diverse neighborhood, thus at no further cost she will move next to i. The cluster containing i and j is minimal thus both players enjoy their highest utility. Moreover going counter clockwise next to this cluster there is now a minimal cluster of type t' players, and going clockwise there is a cluster at least of length 2. Thus we have transited to a state $z' \in Z_{k+1}$.

It remains to be shown that this cost is minimal. That is it we need to show that any sequence of β and γ mistakes is not sufficient to transit to some $z' \in Z_{k+1}$.

First we rule out that a sequence of β mistakes. Assume that $s \geq 1$ β mistakes are sufficient for the process to transfer to some $z' \in Z_{k+1}$. Any single β cannot lead a player to insert herself in a neighborhood where she gets utility 0 unless she had utility 0 before. Players who made a β mistake and had utility 1 and x respectively must have moved to a pre-existing cluster. If a player has utility 0 it must be because all other players in the cluster she belonged to in z have moved. All of the players must have moved to a cluster that existed in z. But then no new clusters can form.

Suppose $s \geq 1$ β mistakes is the minimal cost to transfer to a state $z' \in Z_{k+1}$. Thus after $s-1$ mistakes a single β mistake is required. Let the state which is arrived at after $s-1$ mistakes be denoted z^{s-1}. Players who made a β mistake must have had either utility 1 or utility x. That is the player who moves by mistake must move from a cluster which existed in z_{s-p} next to a player who formed part of a cluster in z. That is up until $s-1$ no new clusters have been formed. Since precisely s β mistakes are required, by the hypothesis there is a β mistake which will lead to the formation of a new cluster. In z^{s-1} all players have either utility 1, x or 0. Clearly a β mistake by players with utility 1 or x does not lead a new cluster to form since they will move to a position where there is at least one player like themselves. A player with utility 0 who makes a β mistake will lead to the dissolution of previously existing cluster, and will with positive probability lead to the formation of a new cluster. However the total number of clusters cannot increase.

Now we rule out that a sequence of γ mistakes lead to some $z' \in Z_{k+1}$. After $s-1$ γ mistakes, exactly one γ mistake is required by the hypothesis. As in the argument above, γ mistakes cannot lead to the formation of new clusters only the dissolution of old clusters with positive probability. To see this note that a γ mistake leads players to vacate in integrated location for a location where they only have neighbors like themselves. Since this move must lead players to abandon a pre-existing cluster for another pre-existing cluster, no new clusters are formed with pos. prob. after $s-1$ mistakes. But after $s-1$ mistakes a player who makes a γ mistake moves to a cluster which existed in the previous state.

Finally we have to rule out that any sequence of γ and β mistakes will lead to some $z' \in Z_{k+1}$. By the argument above after any β or γ mistake no new cluster is formed without the dissolution of an old cluster. □

I can now complete the proof of Proposition 6:

Proof of Proposition 6. By Lemma 4 the stochastic potential of $z \in Z_1$ is:

$$\beta|Z_K| + \beta(K-2)$$

The stochastic potential of $z \in Z_k$, $1 < k < K$ is:

$$\beta|Z_K| + \beta(K-(k+1)) + \delta(k-1)$$

and the stochastic potential of $z \in Z_K$ is at least:

$$\beta(|Z_K|-1) + \delta(K-1)$$

since players in z enjoy the highest possible utility.

Since $\delta > \beta$ only the states in Z_1 has minimum stochastic potential. □

B.5 Proof of Lemma 1

Proof. A particular solution to the equation follows from the guess: $w_k = Dk$ for some constant D. This leads to the particular solution:

$$w_k^* = -\frac{1}{a-b}k$$

The characteristic equation for the homogeneous second order equation is:

$$ar^2 - (a+b)r + b = 0$$

which leads to roots: $r_1 = 1 \wedge r_2 = \frac{b}{a}$. So that the general solution is:

$$w_k = A + B\left(\frac{b}{a}\right)^k - \frac{1}{a-b}k$$

where A and B are constants. The result then follows from the boundary conditions.

The second observation follows from the fact that $w_m \geq w_k$, $0 \leq k < m$. □