Comparative Advante and Efficient Advertising in the Attention Economy

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Abstract

We analyze the problem that enterprises face when having to decide on the most effective way to advertise several items belonging to their inventories within the company's webpages. We show that the ability to arbitrarily partition a website among items leads to a comparative advantage among webpages which can be exploited so as to maximize the total utility of the enterprise. This result, which also applies to the case of several competitive providers, is then extended to dynamical scenarios where both the advertising allocation and the exposure levels vary with time.

1 Introduction

A distinguishing feature of the information era is the saliency of people's attention as a scarce resource. Unlike an earlier time when information was not ubiquitous and thus a valuable resource, its easy availability has shifted the focus to the limited bandwidth that people can devote to ubiquitous media and news. This explains the new predominance of intangibles like style and design, as opposed to the more physical content of products, when attempting to capture the limited attention of consumers [14].

A glaring and old example of the competition for attention is advertisement, which has always exploited the prevailing technology to reach audiences effectively. As insights and methods from fields as disparate as psychology and operations research became available, advertisers used them to both target more effectively their audience and to decide how to optimally allocate financial resources to given media [9, 19]. More recently, the advent of the web has made possible to target advertising in an even more effective and dynamic way, to the point that only those searching or using for a particular piece of information are potentially exposed to adverts relevant to that piece.

While the subject of limited attention has been the focus of research in psychology [12], neuroscience [22] and — to a lesser extent — behavioral economics [8] for a number of years, it is only recently that it has been analyzed in terms of the new behavioral and communication modalities it creates [7, 10, 11, 13, 14, 15, 23].

From the point of view of economic theory, Falkinger has recently produced a model based on psychological evidence that describes the competition among producers for the attention of users [5]. By focusing on the sender side of the problem rather than the receiver, he showed how changes in either the method of broadcast generation or the ability to reach large numbers of people on a global scale affect both attention levels and the number of viable receivers of a given message. Interestingly, he also demonstrates how a wider diffusion of signals among a population can diminish the equilibrium number of broadcasters in spite of the fact that each receiver has access to a larger variety of broadcaster than before.

In this paper we also analyze the problem of allocating resources to given messages from the sender point of view, with a focus on the problem of placing adverts for many products in many vehicles or websites. A consideration of this problem leads to another interesting effect that takes place in attention economies. This is best seen when considering producers having to decide on the most effective way to advertise several items belonging to their inventories within a number of the company's webpages.

Specifically, consider a producer having two products, 1 and 2, and two webpages belonging to his own firm where they can be advertised. As is the case in many situations we can typically assume that each page tends to advertise a different family of products. We say that webpage 1 has an *absolute advantage* over webpage 2 in one product if the return to the producer on the cost of advertising that product in website 1 is higher than the return of advertising that product in website 2.

A more careful analysis of the utility incurred in such process however, shows than in many cases website 2 should only advertise the product in which it has a *comparative advantage* to website 1. We say that website 2 has comparative advantage over website 1 in advertising a given product if the relative returns to website 2 of advertising that product over the others is higher than the relative returns from advertising that product in website 1.

This result can at times seem paradoxical, for it leads to situations whereby although website 2 can advertise product 1 twice more effectively than product 2, it should only advertise product 2 in order to maximize the total utility to the producer.

Thus comparative advantage, a well known principle in trade economics [2], is also present in attention economies along with the same apparent paradoxes that makes the principle so misunderstood in spite of its age and exposure. Moreover, comparative advantage is not restricted to the case where the products belong to the same producer. As we also show, it shows up in the general competitive case analyzed by Falkinger [5].

In what follows we consider the problem of resource allocation for advertising *many* products in several websites (which in what follows will stand for vehicles as well) taking into account exposure levels and utility functions. We use methods that have been developed for optimally placing advertisements for single products in a number of vehicles [3, 6, 24]. We then show how one can determine the optimal allocation of resources into several websites and how comparative advantage appears and is characterized, both in the case of single providers and many competitive ones. Next we consider the dynamical case where both the advertising allocation and the exposure levels vary with time. Using a dynamic programming approach we show how the continuous time optimization problem can be mapped to a one period problem of the type studied by Srinivasan[24]. Finally we study the stable limit, characterized by having both the relative price ratios and allocations constant, and show how it coincides with the static solution found in the earlier part of the paper.

2 Resource allocation and advertising

2.1 The model

Consider an enterprise that advertises its m products on n different *vehicles*, which may be websites or the many webpages under the company's main one, a selection of keywords in a keyword auction, or different TV channels. Each of these websites can be used to advertise either one product exclusively, or be divided into fractions to advertise more than one product at the same time. In what follows we will refer to websites but the analysis carries over to any other vehicle.

In order to maximize its *total utility* from advertising, the enterprise can adjust its *advertising allocation* at any time, so as to achieve the optimal *advertising exposure level* [17]. In what follows we will formally define how resource allocation within a website affects the exposure levels of its various advertising targets, and how these exposure levels in turn determine the company's utility.

Let u be an $n \times m$ nonnegative matrix whose entry u_{ij} represents the fraction of website i is allocated to advertise product j. This fraction can be either in absolute value or a relative one. Let U be the feasible allocation set. For example, a company might host n subsites under its main website. If it allocates a proportion u_{ij} of its i'th subsite to advertise product j, the feasible set is then

$$U = \left\{ (u_{ij})_{n\times m} : u_{ij} \geq 0, \ \sum_j u_{ij} \leq 1 \right\}. \tag{1}$$

Let x be a non-negative m-vector whose i'th component x_i measures the advertising exposure level of product i, which is in turn determined by the advertising allocation u. This exposure level can be measured by the total number of clicks on a given product over a given time interval. We assume a simple linear relation between u and x (this assumption will be relaxed later in Section 3):

$$x_j = b_{1j}u_{1j} + \cdots + b_{nj}u_{nj}, \quad j = 1, \ldots, m, \tag{2}$$

where $b_{ij} > 0$ since advertising always increases exposure level. Thus b_{ij} measures the effectiveness of website i at advertising product j, which is again measured by the number of clicks for that product.

The company's utility is defined as the gain obtained by advertisement minus its cost:

$$V = g(x) - c(u). \tag{3}$$

In situations where the cost is a constant (as is the case with websites) one can also write $V = g(x)$, where g is a utility function which is nondecreasing in x.

Thus the company seeks to solve

$$\max\ g(x) \quad \text{s.t. } u \in U. \tag{4}$$

2.2 Comparative advantage

2.2.1 Two websites and two products

Let us start with the simplest case: there are only two websites and two products ($m = n = 2$). The social objective is to maximize the company's utility, or equivalently

$$\begin{array}{ll} \max & g(x_1, x_2) = g(b_{11}u_{11} + b_{21}u_{21}, b_{12}u_{12} + b_{22}u_{22}) \\ \text{s.t.} & u \geq 0,\ u_{11} + u_{12} \leq 1,\ u_{21} + u_{22} \leq 1. \end{array} \tag{5}$$

We say that vehicle 1 has *comparative advantage* for advertising product 1 if

$$\frac{b_{11}}{b_{21}} > \frac{b_{12}}{b_{22}}. \tag{6}$$

Clearly, under this definition vehicle 2 has a comparative advantage over website 1 for advertising product 2.

If Eq. (6) holds one can show that either $u_{12} = 0$ or $u_{21} = 0$ in the optimal allocation. Suppose otherwise that both $u_{12} > 0$ and $u_{21} > 0$. Consider the following small change in u:

$$\Delta u_{11} = -\frac{b_{21}}{b_{11}}\Delta u_{21} = -\Delta u_{12} = \frac{b_{22}}{b_{12}}\Delta u_{22} > 0. \tag{7}$$

When the change is small we can keep $u_{12} > 0$ and $u_{21} > 0$. The value of g will not be affected since x_1 and x_2 remain unchanged. It is easy to check that while the first constraint in Eq. (5) is binding after the change, the second constraint cannot be satisfied, i.e.:

$$\Delta u_{21} + \Delta u_{22} = -\frac{b_{22}}{b_{12}}\left(\frac{b_{11}}{b_{21}} - \frac{b_{12}}{b_{22}}\right)\Delta u_{12} < 0. \tag{8}$$

Thus one can increase both x_1 and x_2 without violating the constraints, but doing so will cause an increase in g and contradict optimality. Therefore, it cannot be that both $u_{12} > 0$ and $u_{21} > 0$; one of them has to be zero.

When $u_{12} = 0$ website 1 advertises only product 1, so $u_{11} > 0$, thus product 2 has to be advertised by website 2, for otherwise the utility is zero. Thus $u_{22} > 0$. When $u_{21} = 0$ a similar argument leads to the same conclusion, i.e. $u_{11} > 0$ and $u_{22} > 0$. This means that if a website has comparative advantage in advertising a product then it should always advertise that product (it may or may not advertise the other product). We emphasize that this result does not depend on the explicit form of g.

Example 1. Consider a Cobb-Douglas utility with the two products equally important: $g(x) = x_1^{1/2} x_2^{1/2}$. This is equivalent to $U(x) = x_1 x_2$ after a monotonic transformation.

Solution. We list without proof the optimal solution for three possible cases, neglecting degeneracy:

Case 1. $\frac{b_{11}}{b_{21}} > \frac{b_{12}}{b_{22}} > 1$.

$$u_{11} = \frac{b_{11}}{b_{12}} \frac{b_{12} + b_{22}}{2}, \quad u_{12} = \frac{b_{12} - b_{22}}{2}, \quad u_{21} = 0, \quad u_{22} = b_{22}. \tag{9}$$

Our result says that if vehicle 1 has *absolute advantage* over vehicle 2 in both productss, then vehicle 2 should only process advertise the product in which it has comparative advantage. This result can be perplexing in some cases. For example, consider the case where $b_{11} = 5$, $b_{12} = b_{21} = 2$, and $b_{22} = 1$. Although website 2 can advertise product 1 two times more effectively than product 2, it should only advertise product 2.

Case 2. $1 > \frac{b_{11}}{b_{21}} > \frac{b_{12}}{b_{22}}$.

$$u_{11} = b_{11}, \quad u_{12} = 0, \quad u_{21} = \frac{b_{21} - b_{11}}{2}, \quad u_{22} = \frac{b_{22}}{b_{21}} \frac{b_{11} + b_{21}}{2}. \tag{10}$$

Similar to Case 1.

Case 3. $\frac{b_{11}}{b_{21}} > 1 > \frac{b_{12}}{b_{22}}$.

$$u_{11} = b_{11}, \quad u_{12} = b_{21} = 0, \quad u_{22} = b_{22}. \tag{11}$$

In words, both websites should specialize if and only if each website has absolute advantage in advertising one product. □

2.2.2 The comparative advantage characterization

The result of Section 2.2.1 can be generalized to the case of more than two websites and more than two products. Assume that

$$\frac{b_{i_1 j_1}}{b_{i_2 j_1}} > \frac{b_{i_1 j_2}}{b_{i_2 j_2}} \tag{12}$$

for websites i_1, i_2 and products j_1, j_2. Then one of $u_{i_1 j_2}$ and $u_{i_2 j_1}$ must be zero.

2.2.3 Two websites and m products

Without loss of generality we can order the websites by comparative advantage, so that website 1 has comparative advantage in advertising products with smaller labels:

$$\frac{b_{11}}{b_{21}} > \cdots > \frac{b_{1m}}{b_{2m}}. \tag{13}$$

By the comparative advantage characterization, for any $1 \le j < k \le m$ it must be that either $u_{2j} = 0$ or $u_{1k} = 0$. Therefore there must exist some J such that

$$\begin{array}{lll} u_{1j} > 0, & u_{2j} = 0 & \text{for } 1 \le j < J; \\ u_{1j} = 0, & u_{2j} > 0 & \text{for } J < j \le m. \end{array} \tag{14}$$

In words, website 1 should advertise products $1, \ldots, J-1$ and possibly J, and vehicle 2 should advertise products $J+1, \ldots, m$ and possibly J.

Example 2. (Leontief utility [18]) $g(x) = \min\{x_1, \ldots, x_m\}$. The social objective is to maximize the least gain of all products.

Proof. By introducing an auxiliary variable $z = g(x)$, the problem can be written as a linear program:

$$\begin{array}{ll} \max & z \\ \text{s.t.} & \sum_i b_{ij} u_{ij} \ge z \text{ for } j = 1, \ldots, m; \\ & \sum_j u_{ij} \le 1 \text{ for } i = 1, 2. \end{array} \tag{15}$$

Using a Lagrangian multiplier it is not hard to show that Eq. (15) is equivalent to

$$\max \sum_{i,j} p_j b_{ij} u_{ij} \quad \text{s.t.} \sum_j u_{ij} \le 1 \quad \text{for } i = 1, 2, \tag{16}$$

where $\{p_j\}$ is a set of *shadow prices* [1]. The Lagrangian of Eq. (16) is

$$\begin{aligned} L(y_{ij}; w_i) &= \sum_{i,j} p_j b_{ij} u_{ij} - \sum_i w_i \left(\sum_j u_{ij} - 1 \right) \\ &= \sum_i w_i + \sum_{i,j} u_{ij} \left(p_j b_{ij} - w_i \right), \end{aligned} \tag{17}$$

where $\{w_i\}$ is a set of *shadow wages.* We can now easily write out the dual problem of Eq. (16) as:

$$\min_w \sum_i w_i \quad \text{s.t. } w_i \ge p_j b_{ij} \quad \text{for } i = 1, 2;\ j = 1, \ldots, m. \tag{18}$$

The constraints in Eq. (18) hold for $u_{ij} > 0$. Thus we have

$$\begin{array}{lll} w_1 = b_{1j} p_j, & w_2 < b_{2j} p_j & \text{for } 1 \le j < J; \\ w_1 < b_{1j} p_j, & w_2 = b_{2j} p_j & \text{for } J < j \le m. \end{array} \tag{19}$$

In words, product j should be advertised by website 1 wholly if $w_1/w_2 > b_{1j}/b_{2j}$, and by website 2 if $w_1/w_2 < b_{1j}/b_{2j}$.

Linear programming duality implies that

$$w_i = \sum_j p_j b_{ij} u_{ij}, \tag{20}$$

so a website's wage is just the total price of its resource allocation. Observe that Eq. (16) is separable. This implies that if each website i maximizes its own wage subject to its capacity constraint:

$$\max \sum_j p_j b_{ij} u_{ij} \quad \text{s.t.} \quad \sum_j u_{ij} \leq 1, \tag{21}$$

then the solution will be exactly the same as Eq. (16). Thus if we interpret shadow prices as real prices, the corresponding competitive market will lead to an efficient allocation.

Due to the comparative advantage characterization (Eq. (14)) the optimal allocation can be determined by solving one simple equation:

$$b_{11}^{-1} + \cdots + b_{1,J-1}^{-1} + \theta b_{1J}^{-1} = (1-\theta) b_{2J}^{-1} + b_{2,J+1}^{-1} + \cdots + b_{2m}^{-1}, \tag{22}$$

where it is required that $\theta \in [0, 1)$ and $1 \leq J \leq m$. □

2.2.4 n websites and two products

Once again we label the websites in decreasing order of their comparative advantage:

$$\frac{b_{11}}{b_{12}} > \cdots > \frac{b_{n1}}{b_{n2}}. \tag{23}$$

Like before, the solution has a simple form

$$\begin{array}{lll} u_{i1} > 0, & u_{i2} = 0 & \text{for } 1 \leq i < I; \\ u_{i1} = 0, & u_{i2} > 0 & \text{for } I < i \leq n. \end{array} \tag{24}$$

In words, websites $1, \ldots, I-1$ should advertise product 1 and websites $I + 1, \ldots, n$ should advertise product 2. Website I may advertise both products.

Example 3. (Cobb-Douglas utility) $g(x) = x_1^{\alpha} x_2^{1-\alpha}$, where $0 < \alpha < 1$.

Proof. From the comparative advantage characterization, the optimal advertising levels have the simple form

$$x_1 = b_{11} + \cdots b_{I-1,1} + \theta b_{I1}, \tag{25}$$

$$x_2 = (1-\theta) b_{I2} + b_{I+1,2} + \cdots + b_{n2}, \tag{26}$$

where $0 \leq \theta < 1$. Ignoring degeneracy for the moment (i.e. assuming that $0 < \theta < 1$), the optimal θ must satisfy the first order condition

$$\frac{\partial \log g(x)}{\partial \theta} = \frac{\alpha}{x_1} b_{I1} - \frac{1-\alpha}{x_2} b_{I2} = 0, \tag{27}$$

or

$$\frac{\alpha\, b_{I1}}{(1-\alpha)b_{I2}} = \frac{b_{11} + \cdots + b_{I-1,1} + \theta b_{I1}}{(1-\theta)b_{I2} + b_{I+1,2} + \cdots + b_{n2}}. \tag{28}$$

This equation has a solution since the left side decreases with I and the right side increases with I.

If we define two shadow prices

$$p_1 = \frac{\partial \log g(x)}{\partial x_1} = \frac{\alpha}{x_1}, \quad p_2 = \frac{\partial \log g(x)}{\partial x_2} = \frac{1-\alpha}{x_2}, \tag{29}$$

Eq. (27) can be also written as $p_1 b_{I1} = p_2 b_{I2}$, so website I is indifferent to advertising products 1 or 2. □

Note that in the general case one can no longer sort the websites or products by comparative advantage, and has to solve the full optimization problem. The comparative advantage characterization still holds though.

3 Competing for attention

In the last two sections we demonstrated how several websites or other media belonging to the same advertiser can maximize the total utility of the producer. There are situations however, where several producers or advertisers might compete for the same target audience, as is customary in the world of traditional advertising or other types of attention seeking activities. In order to address it we resort to Falkinger's model of an attention economy in which each sender seeks to maximize its own utility while attempting to capture a fraction of the finite amount of attention available [5]. As we now show, in this case there is also a comparative advantage among senders.

To see this, we follow Falkinger in assuming that the impact of a sender on a receiver decreases as the receiver's attention capacity decreases. Thus senders have to compete for the finite total capacity of the receives. In our notation this means that the impact, v_{ij}, that a given media or website i has on the target of product j is no longer linear in its signal strength $b_{ij}u_{ij}$. Instead, we choose v_{ij} to be proportional to the fraction of signal strength among all websites:

$$v_{ij} = \frac{b_{ij}u_{ij}}{\sum_k b_{kj}u_{kj}}. \tag{30}$$

The total impact of website i on its target is just $v_i = \sum_j v_{ij}$. The margin of website i when advertising product j is then given by

$$\frac{\partial v_i}{\partial u_{ij}} = \frac{b_{ij}}{\sum_k b_{kj}u_{kj}} - \frac{b_{ij}^2 u_{ij}}{(\sum_k b_{kj}u_{kj})^2}. \tag{31}$$

Since in the continuum limit the second term is negligible, we have

$$\frac{\partial v_i}{\partial u_{ij}} = \frac{b_{ij}}{\sum_k b_{kj} u_{kj}} \equiv b_{ij} p_j, \tag{32}$$

where $p_j = (\sum_k b_{kj} u_{kj})^{-1}$ is the shadow price. We thus see that in equilibrium the comparative advantage characterization we described in earlier section again holds in this more general scenario. For example, in situations where $b_{i1}/b_{i2} > p_2/p_1$ website i should not advertise product 2.

4 Dynamical resource allocation

We now generalize our model by allowing both the advertising allocation and the exposure levels to change with time. Let $u(t)$ be an $n \times m$ nonnegative matrix whose entry $u_{ij}(t)$ represents how much resource of website i is allocated to advertise product j at time t.[1] The feasible allocation set U_t may or may not depend on time. Let $x(t)$ be a nonnegative m-vector whose i'th component $x_i(t)$ denotes the advertising exposure level of product i at time t. The exposure levels at time 0 is assumed to be given. These levels will go down if the company makes no effort in advertising (forgetting effect), but can also go up as some websites are used to advertise. Based on such considerations we write down the dynamical equation for x_t:

$$\dot{x}_t = A_t x_t + B_t(u_t), \quad t \in [0, T]. \tag{33}$$

where A_t is an $m \times m$ *decay matrix* describing how the exposure levels decay with time and how they mutually affect each other. For example, if $A_t = \text{diag}(-\beta_1, \ldots, -\beta_n)$ each $x_i(t)$ decays exponentially fast. B_t is a *response function* that maps an $n \times m$ matrix to an m-vector and it describes how an advertising allocation affects the exposure levels. For simplicity we assume that B_t is linear as before:

$$B_j(t) = b_{1j}(t) u_{1j}(t) + \cdots + b_{nj}(t) u_{nj}(t), \quad j = 1, \ldots, m. \tag{34}$$

Each $b_{ij}(t) \geq 0$ since advertising increases exposure level.

The company's utility is defined as the gain from advertisement minus the cost. Because time is continuous, it is appropriate to define a gain flow with density $g_t(x_t)$, and a cost flow with density $c_t(u_t)$. At the end time T the company receives a lump-sum final gain of $G(x_T)$ which can be regarded as the total gain after T. The only assumption imposed on g_t and G is that they are both nondecreasing and concave. The company's total utility is thus

$$V = G(x_T) + \int_0^T [g_t(x_t) - c_t(u_t)]\, dt. \tag{35}$$

[1]Notations like $u(t)$ and u_t will be used interchangeably from now on for easier readability.

Thus, the company seeks to

$$\begin{aligned} \max \quad & V = G(x_T) + \int_0^T [g_t(x_t) - c_t(u_t)]\, dt \\ \text{s.t.} \quad & \dot{x}_t = A_t x_t + B_t(u_t) \quad \text{for } t \in [0,T], \\ & u_t \in U_t \quad \text{for } t \in [0,T], \\ & x_0 \text{ given.} \end{aligned} \tag{36}$$

Our model closely follows Srinivasan [24], but with two important differences. First, in our case time is continuous rather than discrete, which makes it easier to analyze the system's dynamical behavior. The second important difference is that in our model we allow for the company to have the extra freedom to fine-tune its resources so as to be able to advertise more than one product per website. In other words, $u(t)$ is an $n \times m$ matrix, and not an n-vector, something which is not possible in Srinivasan's model. As we show in the next section, it is from these additional degrees of freedom that the comparative advantage in advertising emerges.

4.1 Mapping dynamics to one period

We follow the standard Pontryagin approach to solve the dynamic programming problem (36). We thus introduce the Hamiltonian

$$H_t(x_t, u_t, p_t) = g_t(x_t) - c_t(u_t) + p_t[A_t x_t + B_t(u_t)], \tag{37}$$

where p_t is the adjoint function (row vector with m components) satisfying the adjoint equation

$$\dot{p}_t = -\nabla_x H_t(x_t^*, u_t^*, p_t) = -p_t A_t - \nabla g_t(x_t^*) \tag{38}$$

with the boundary condition

$$p_T = \nabla G(x_T^*). \tag{39}$$

The optimal control u_t^* satisfies

$$H_t(x_t^*, u_t^*, p_t) = \max_{u_t \in U_t} H_t(x_t^*, u_t, p_t), \tag{40}$$

or

$$p_t B_t(u_t^*) - c_t(u_t^*) = \max_{u_t \in U_t} [p_t B_t(u_t) - c_t(u_t)]. \tag{41}$$

We next verify that (u^*, x^*) is indeed an optimal solution. Let (u, x) be any other feasible solution, and V be the corresponding total utility. We then have

$$\begin{aligned}
V^* - V &= G(x_T^*) - G(x_T) + \int_0^T [g_t(x_t^*) - g_t(x_t) - c_t(u_t^*) + c_t(u_t)]\,dt \\
&\geq \nabla G(x_T^*)(x_T^* - x_T) + \int_0^T [\nabla g_t(x_t^*)(x_t^* - x_t) - c_t(u_t^*) + c_t(u_t)]\,dt \\
&= p_T(x_T^* - x_T) + \int_0^T [(-\dot{p}_t - p_t A_t)(x_t^* - x_t) - c_t(u_t^*) + c_t(u_t)]\,dt \\
&= \int_0^T [p_t(\dot{x}_t^* - \dot{x}_t) - p_t A_t(x_t^* - x_t) - c_t(u_t^*) + c_t(u_t)]\,dt \\
&= \int_0^T [p_t(F(u_t^*) - F(u_t)) - c_t(u_t^*) + c_t(u_t)]\,dt \\
&\geq 0, \qquad (42)
\end{aligned}$$

where the first "$\geq$" follows from the concavity of both G and g, the third "$=$" is integration by parts, and the last "$\geq$" is Eq. (41).

Eq. (41) shows that the optimal strategy u_t^* must solve

$$\max\,[p_t B_t(u_t) - c_t(u_t)] \quad \text{s.t. } u_t \in U_t. \qquad (43)$$

Thus, the continuous time optimization problem reduces to a one period problem.

Assuming that cost is constant and U_t is the same as before, Eq. (43) can be written explicitly as

$$\max \sum_{i,j} p_j(t) b_{ij}(t) u_{ij}(t) \quad \text{s.t. } \sum_j u_{ij}(t) \leq 1 \text{ for } i = 1, \ldots, n. \qquad (44)$$

For fixed t this equation has the same functional form as Eq. (16), so we again obtain the comparative advantage property, along with a shadow wage characterization.

4.2 The stable solution

In the dynamic setting the adjoint price function, $p(t)$, may change with time, so the optimal advertising allocation $u(t)$ may also change. We are interested in finding the stable solution in which both the relative price ratio and the allocation stay constant. We now consider a concrete example to keep the calculations transparent.

Example 4. Suppose there are n websites and two prices ($m = 2$). The decay matrix is assumed to be homogeneous and constant: $A(t) = -\beta I$. The utility kernel is Cobb-Douglas: $g_t(x) = x_1^{\alpha_1} x_2^{\alpha_2}$, and the final gain is zero: $G = 0$. Also assume that $b_{ij}(t)$ and $c_t(u_t)$ are constants.

Proof. If the price ratio eventually stabilizes, we have

$$\frac{d}{dt}\left(\frac{p_1(t)}{p_2(t)}\right) = 0, \tag{45}$$

or equivalently

$$\frac{\dot{p}_1(t)}{\dot{p}_2(t)} = \frac{p_1(t)}{p_2(t)} = \frac{\partial g(x)/\partial x_1}{\partial g(x)/\partial x_2} = \frac{\alpha x_2}{(1-\alpha)x_1} = \text{constant}, \tag{46}$$

where the second equality follows from Eq. (38). This equation is consistent with Eq. (29). Therefore the optimal allocation in the stable limit is again given by Eq. (28), which coincides with the optimal allocation of the static model. □

5 Discussion

In this paper we analyzed the problem faced by enterprises when having to decide on the most effective way to advertise several items belonging to their inventories within the company's website. This is an important problem given the extraordinary role that advertising plays in the information society and the need to design effective ways of reaching people with a limited amount of attention. Thus, an efficient way of dynamically placing adverts within websites can have an important effect both on the awareness that people have of given issues and products and on the financial viability of producers and retailers.

A consideration of this problem leads to an interesting and at times seemingly paradoxical effect. As we showed, the ability to arbitrarily partition a website among items leads to a comparative advantage among webpages which can be exploited so as to maximize the total utility of the enterprise. This result, which also applies to the case of several competitive providers analyzed by Falkinger, was then extended to dynamical scenarios where both the advertising allocation and the exposure levels vary with time. The extension to continuous time adjustments is extremely relevant to the problem of Internet keyword auctions, as bidding for words takes place continuously.

While our analysis was performed within the context of the web, there is nothing that prevents our results to be extended to other vehicles such as television advertising or other media. Provided one can partition a vehicle among several offerings, a comparative advantage can arise which can be suitably exploited to maximize returns to the advertisers. Moreover, the present popularity of Internet advertising through keyword auctions suggests the use of this mechanism for deciding on the best stratetigies to advertise. While the simplest and logical way to bid for given keywords is dictated by the perceived competitive advantage and popularity of given search items, our work suggest a more nuanced investigation of the returns accrued by placing adverts under different search queries.

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