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Abstract This paper studies endogenous entry and ex ante revenue-maximizing auctions in an independent private value setting where potential bidders have private-information entry costs. The contribution of this paper is four-fold. First, we show that any equilibrium entry can be characterized through a set of continuous and monotonic shutdown curves that separate the bidders' types into participating and nonparticipating categories. Second, the expected winning probability of a participant does not depend on his private entry cost. Furthermore, the expected winning probabilities of the participating types are given by the slopes of the shutdown curves. Third, symmetric entry equilibria (shutdown curves) implemented by the classes of ex post efficient or ex post revenue-maximizing mechanisms are completely characterized. Fourth, within these two classes of mechanisms, a modified Vickrey auction with uniform reserve price and entry subsidy is ex ante revenue-maximizing. The desired entry subsidy and reserve price are determined by the lower end of the corresponding shutdown curve.

Keyword: Auctions Design; Ex Post Efficiency; Endogenous Participation; Multidimensional Screening; Vickrey Auction.

JEL classifications: D44; D82.

1 Introduction

The impact of participation costs on bidders' entry and auctions design has been extensively studied. Samuelson (1985) has derived the symmetric entry equilibria that maximize the social welfare and the seller's expected revenue in a symmetric setting with fixed entry cost. Stegeman (1996) has further studied ex ante efficient auction without imposing symmetry on bidders. Tan and Yilankaya (forthcoming) provide a sufficient condition that guarantees the uniqueness of entry equilibrium in a standard second price auction. Kaplan and Sela (2004) instead study the efficient entry when the participation costs are private information of the bidders while their valuations are common knowledge. Green and Laffont (1984) allow both the valuations and entry costs to be private information of the bidders. They establish the existence and uniqueness of an equilibrium shutdown curve for a Vickrey auction with zero reserve price and zero entry subsidy. The shutdown curve specifies a critical value of private entry cost for each private value. A bidder participates in the auction if and only if his entry cost is lower than the critical value corresponding to his private value. Lu (2006) finds that a simple second price auction is ex ante efficient in a two-dimensional setting where the bidders have private information on both their valuations and entry costs.[1] In this paper, we further study the endogenous entry and ex ante revenue-maximizing auctions in this two-dimensional setting. Following Green and Laffont (1984), we assume that every potential bidder observes his private value and entry cost before his entry decision.[2]

Multidimensional Screening has been extensively studied by Wilson (1993), Armstrong

[1]This result extends the finding of Stegeman (1996) to a multidimensional signal setting

[2]This differs from the other branch of literature on endogenous entry and entry cost, which assumes that bidders learn their valuations after incurring the entry costs. This branch of literature includes Engelbrecht-Wiggans (1993), Levin and Smith (1994), McAfee and McMillan (1987), Tan (1992) and Ye (2004) among others.

(1996, 2000), and Rochet and Chone (1998) among others in various contexts such as price discrimination, multi-product auction and regulation. Armstrong (1996) establishes that the optimal mechanism usually involves shutdown of some lower-type agents. Rochet and Chone (1998) point out that the difficulty in multidimensional screening problem lies in the characterization of all the implementable endogenous entry equilibria. In this paper, we show that for our setting the expected winning probability of a participating bidder does not depend on his private cost at any entry equilibrium. Furthermore, the slope of any implementable shutdown curve at any inner point in the type space equals the expected winning probability of the corresponding type. This insight leads to a full characterization of the shutdown curves which are compatible with the classes of ex post efficient and ex post revenue-maximizing mechanisms. Here and hereafter, "Ex post efficient" refers to allocating the object to the participant with highest private value, provided that this value is higher than the seller's valuation. "Ex post revenue-maximizing" refers to allocating the object to the participant with highest virtual value, provided that this value is higher than the seller's valuation. For these classes of mechanisms, we fully identified the implementable shutdown curves. The closed form shutdown curves are explicitly provided for a 2-bidder case.

For any shutdown curve that is compatible with any of these two classes of mechanisms, we find that a modified Vickrey auction with appropriately set reservation price and participation subsidy implements the corresponding entry and maximizes the seller's expected revenue within the class of mechanisms. The desired entry subsidy and reserve price are determined by the lower end of the corresponding shutdown curve. Thus the search for the ex ante revenue-maximizing auction within these classes boils down to the search for the optimal shutdown curve that is compatible with these classes of mechanisms. Therefore, it must be a modified Vickrey auction that is ex ante revenue-maximizing within each class.

The paper is organized as follows. Section 2 sets up the model and provides the necessary and sufficient conditions for a feasible mechanism-entry combination, when bidders have two private signals, namely their private values and entry costs. Section 3 then characterizes the entry equilibria that are compatible with the classes of ex post efficient and ex post revenue-maximizing mechanisms. Section 4 shows that a modified Vickrey auction is ex ante revenue-maximizing within each of these classes. Section 5 concludes.

2 The Model and Preliminaries

In this section, we will first introduce the model and the concepts of mechanism and shutdown curve. Second, we will characterize the sufficient and necessary conditions for a feasible mechanism-entry combination. Third, we show a version of revenue equivalence theorem, which says that the seller's expected revenue from a mechanism that implements any given untrivial endogenous entry equilibrium is completely determined by the winning probabilities.[3]

2.1 The Setting

A seller wants to sell an indivisible object to N potential bidders through an auction. Denote the set of potential bidders by $\mathcal{N} = \{1, 2, ..., N\}$. The seller's value of the object is v_0, which is public information. The ith bidder's private value for the object is $v_i,\ i \in \mathcal{N}$. The values $v_i,\ i \in \mathcal{N}$ are independently distributed on interval $[\underline{v}, \overline{v}]$ with cumulative distribution function $F(\cdot)$ and density function $f(\cdot)(> 0)$. The virtual value function is defined as $J(v) = v - \frac{1-F(v)}{f(v)}$. Each potential bidder i has a private participation cost

[3]Here, an untrivial endogenous entry equilibrium refers to the situation where the measure of the participating types belongs to $(0, 1)$.

$c_i,\ i \in \mathcal{N}$. The private participation costs $c_i,\ i \in \mathcal{N}$ are independently distributed on the interval $[\underline{c}, \overline{c}]$ with cumulative distribution function $G(\cdot)$ and density function $g(\cdot)$. $F(\cdot)$ and $G(\cdot)$ are public information. In this paper, we consider the case where all c_i and $v_j,\ i, j \in \mathcal{N}$ are mutually independent. Every bidder observes his private value and private participation cost before he makes his participation decision. The seller and the bidders are risk neutral. The timing of the game is as follows.

Time 0: Nature reveals the set of potential bidders $\mathcal{N}$, the seller's private value v_0 and distributions $F(\cdot)$ and $G(\cdot)$, which are public information. Every bidder i observes his private value v_i and participation cost c_i, $i \in \mathcal{N}$.

Time 1: The seller announces the rule of the mechanism.

Time 2: The bidders simultaneously and confidentially make their entry decisions. If bidder i participates, he incurs the entry cost c_i and then submits a message. If a bidder does not participate, he takes the outside opportunity and receives 0.

Time 3: The payoff of the seller and the participating bidders are determined according to the announced rule at time 1.

A mechanism that implements endogenous entry naturally cannot require nonparticipants to submit messages. Following Stegeman (1996), we introduce a null message $\emptyset$ to denote the signal of a nonparticipant.

In a direct semirevelation mechanism, the message space is $\mathcal{M} = M^N$ where $M = \{[\underline{c}, \overline{c}] \times [\underline{v}, \overline{v}]\} \cup \{\emptyset\}$, the outcome functions accommodate all participation possibilities in the following form: payment function $x_i(\mathbf{m})$ and winning probability function $p_i(\mathbf{m})$ for bidder i, $\forall i \in \mathcal{N}$, where $\mathbf{m} = (m_1, m_2, ..., m_N)$ is the message vector and $m_i \in M$ is the message of bidder i. We denote the above mechanism by $(\mathbf{p}, \mathbf{x})$, where $\mathbf{p} = (p_1(\mathbf{m}), p_2(\mathbf{m}), ..., p_N(\mathbf{m}))$ and $\mathbf{x} = (x_1(\mathbf{m}), x_2(\mathbf{m}), ..., x_N(\mathbf{m}))$. It is clear that $\mathbf{p}$ should satisfy the following "feasibility" restrictions: $p_i(\mathbf{m}) \geq 0,\ \forall i \in \mathcal{N},\ \forall \mathbf{m} \in \mathcal{M}$, and $\sum_{i=1}^{N} p_i(\mathbf{m}) \leq 1,\ \forall \mathbf{m} \in \mathcal{M}$. In addition, we naturally assume that nonparticipants have no chance of winning the object and their payments to the seller are zero, i.e, $p_i(\mathbf{m}) = x_i(\mathbf{m}) = 0 \ if \ m_i = \emptyset, \ \forall i \in \mathcal{N}$.[4]

Stegeman (1996) provides a semirevelation principle that accommodates nonparticipation of bidders.[5] Although this principle is proved for the case of fixed entry cost, clearly it also applies to our setting where entry costs are private information of bidders.[6] Based on this semirevelation principle, there is no loss of generality to characterize the implementable entry equilibria and ex ante revenue-maximizing mechanisms by considering only the **truthful** direct semirevelation mechanisms. For a **truthful** direct semirevelation mechanism, every participant reveals truthfully his type, the nonparticipants submit a null message.

2.2 Characterizing Feasible Mechanism-Entry Combinations

The following Lemma 1 partially characterizes the implementable entry equilibria. It states that at any entry equilibrium, the participating and nonparticipating types of any bidder must be divided by a nondecreasing and continuous shutdown curve.

Lemma 1: *Any equilibrium entry can be described through a set of nondecreasing and continuous shutdown curves $C_i(v_i) : [\underline{v}, \overline{v}] \to [\underline{c}, \overline{c}], \ \forall i \in \mathcal{N}$, which satisfies the following property: For bidder $i, \ i \in \mathcal{N}$ with type (c_i, v_i), he participates if $c_i < C_i(v_i)$, and he does not participate if $c_i > C_i(v_i)$.*

Proof: see appendix.

If $C_i(\cdot)$ is not always equal to $\underline{c}$ or $\overline{c}$, let (v_i^ℓ, v_i^u) to denote the interval on which $C_i(v_i)$ falls into $(\underline{c}, \overline{c})$. For convenience, if $C_i(v_i) \equiv \underline{c}$, we define $v_i^\ell = v_i^u = \overline{v}$; and if $C_i(v_i) \equiv \overline{c}$,

[4]This assumption is consistent with the **no passive reassignment** (NPR) assumption adopted by Stegeman (1996).

[5]Please refer to Lemma 1 of Stegeman (1996) for details.

[6]Detailed proof will be provided upon request.

we define $v_i^\ell = v_i^u = \underline{v}$. Note that $C_i(v_i) = \underline{c}$ if $v_i < v_i^\ell$, and $C_i(v_i) = \overline{c}$ if $v_i > v_i^\ell$.

For those types on the shutdown curves, their expected payoff if they participate is as follows. **First**, if $v_i^\ell < v_i^u$, the expected payoff of types $(C_i(v_i), v_i)$ where $v_i \in [v_i^\ell, v_i^u]$ is exactly zero when participating. On one hand, it can not be bigger than zero. Otherwise, bidder i with type $(C_i(v_i) + \epsilon, v_i)$ where ϵ is a small positive number, will get strictly positive expected payoff if he participates and mimics the message of $(C_i(v_i), v_i)$. This conflicts with the definition of $C_i(v_i)$. On the other hand, the expected payoff of types $(C_i(v_i), v_i)$ where $v_i \in [v_i^\ell, v_i^u]$ can not be smaller than zero. If he participates and mimics the message of type $(C_i(v_i) - \epsilon, v_i)$, he will at least get $-\epsilon$. Since ϵ can be arbitrarily close to zero, the best expected payoff of type $(C_i(v_i), v_i)$ cannot be negative if he participates and submits the optimal message. Similar arguments lead to that the expected payoff of types $(C_i(v_i), v_i)$ where $v_i \in [\underline{v}, v_i^\ell)$ is no bigger than zero when participating; the expected payoff of bidder i with types $(C_i(v_i), v_i)$ where $v_i \in (v_i^u, \overline{v}]$ is no smaller than zero when participating. **Second**, if $C_i(v_i) \equiv \underline{c}$, the expected payoff of bidder i with types $(\underline{c}, v_i)$ where $v_i \in [\underline{v}, \overline{v}]$ is no bigger than zero if he participates. **Third**, if $C_i(v_i) \equiv \overline{c}$, the expected payoff of bidder i with types $(\overline{c}, v_i)$ where $v_i \in [\underline{v}, \overline{v}]$ is no smaller than zero if he participates.

Based on the results in the previous paragraph, we can specify the participation of bidders whose types are on the shutdown curves as follows: If $v_i^\ell < v_i^u$, we assume that bidder i with types $(C_i(v_i), v_i)$ where $v_i \geq v_i^\ell$ participates, and bidder i with types $(C_i(v_i), v_i)$ where $v_i < v_i^\ell$ does not participate. If $C_i(v_i) \equiv \overline{c}$, we assume all types of bidder i participate; If $C_i(v_i) \equiv \underline{c}$, we assume no type of bidder i participates. Because the measure of all involved types on the shutdown curves is zero, this specification does not affect the participation and bidding strategies of other types of bidders. Therefore, the seller's expected revenue is not affected. Denote the set of all participating types of bidder i corresponding to $C_i(\cdot)$ by $S_i^e(C_i)$, $i \in \mathcal{N}$. Note that $S_i^e(C_i)$ is empty if $C_i(v_i) \equiv \underline{c}$, and

$S_i^e(C_i)$ is $[\underline{c}, \overline{c}] \times [\underline{v}, \overline{v}]$ if $C_i(v_i) \equiv \overline{c}$.

$S_i^e(C_i)$ can be equivalently described through a nondecreasing and continuous function $V_i(c_i)$ on $[\underline{c}, \overline{c}]$, which is defined as follows. If $S_i^e(C_i)$ is empty, we define $V_i(c_i) \equiv \overline{v}$. Otherwise, $V_i(c_i) = \inf\{v_i | (c_i, v_i) \in S_i^e(C_i)\}$ when $c_i \le c_i^u = C_i(v_i^u)$, and $V_i(c_i) = \overline{v}$, if $c_i > c_i^u$. Clearly, $V_i(\cdot)$ satisfies the following properties: For bidder i, $i \in \mathcal{N}$ with type (c_i, v_i), he participates if $v_i > V_i(c_i)$, and he does not participate if $v_i < V_i(c_i)$.

Denote the type of bidder i by $t_i = (c_i, v_i)$. For a continuous and nondecreasing shutdown curve $C_i(\cdot)$ for bidder i, define $m_i(t_i) = t_i$ if $t_i \in S_i^e(C_i)$, and $m_i(t_i) = \emptyset$ if $t_i \notin S_i^e(C_i)$. $\forall\ t_i, t_i' \in [\underline{c}, \overline{c}] \times [\underline{v}, \overline{v}]$, define

$$U_i(\mathbf{p}, \mathbf{x}; t_i, t_i') = E_{\mathbf{t}_{-i}}\{v_i p_i(t_i', \mathbf{m}_{-i}(\mathbf{t}_{-i})) - x_i(t_i', \mathbf{m}_{-i}(\mathbf{t}_{-i}))\} - c_i, \tag{1}$$

where $\mathbf{t}_{-i} = (t_1, ..., t_{i-1}, t_{i+1}, ..., t_N)$, and $\mathbf{m}_{-i}(\mathbf{t}_{-i}) = (m_1(t_1), ..., m_{i-1}(t_{i-1}), m_{i+1}(t_{i+1}), ..., m_N(t_N))$. The support of $\mathbf{t}_{-i}$ is $\mathcal{T}_{-i} = \{[\underline{c}, \overline{c}] \times [\underline{v}, \overline{v}]\}^{N-1}$. If $(\mathbf{p}, \mathbf{x})$ is a truthful direct semirevelation mechanism which implements $\mathcal{C} = (C_i(\cdot))$, then $U_i(\mathbf{p}, \mathbf{x}; t_i, t_i')$ is bidder i's interim expected utility when he submits t_i' given his private signal is t_i.

Using $U_i(\mathbf{p}, \mathbf{x}; t_i, t_i)$, the necessary and sufficient conditions for a direct auction mechanism $(\mathbf{p}, \mathbf{x})$ to be a **truthful** direct semirevelation mechanism that implements given $\mathcal{C}$ can be written as follows.

$$(i)\ U_i(\mathbf{p}, \mathbf{x}; t_i, t_i) \ge 0,\ \forall\ t_i \in S_i^e(C_i),\ \forall\ i \in \mathcal{N}, \tag{2}$$

$$(ii)\ U_i(\mathbf{p}, \mathbf{x}; t_i, t_i) \ge U_i(\mathbf{p}, \mathbf{x}; t_i, t_i'),\ \forall\ t_i, t_i' \in S_i^e(C_i),\ \forall\ i \in \mathcal{N}, \tag{3}$$

$$(iii)\ U_i(\mathbf{p}, \mathbf{x}; t_i, t_i) \ge U_i(\mathbf{p}, \mathbf{x}; t_i, t_i'),\ \forall\ t_i \in S_i^e(C_i),\ t_i' \notin S_i^e(C_i),\ \forall\ i \in \mathcal{N}, \tag{4}$$

$$(iv)\ U_i(\mathbf{p}, \mathbf{x}; t_i, t_i') \le 0,\ \forall\ t_i \notin S_i^e(C_i), t_i' \in [\underline{c}, \overline{c}] \times [\underline{v}, \overline{v}],\ \forall\ i \in \mathcal{N}, \tag{5}$$

$$(v)\ p_i(\mathbf{m}) = x_i(\mathbf{m}) = 0\ if\ m_i = \emptyset,\ p_i(\mathbf{m}) \ge 0,\ \forall i \in \mathcal{N},\ \sum_{i=1}^{N} p_i(\mathbf{m}) \le 1,\ \forall \mathbf{m} \in \mathcal{M}. \tag{6}$$

(5) guarantees that low-type bidders do not participate while (2) guarantees that high-type bidders do participate. (3) and (4) guarantee that high-type bidders reveal truthfully their types when participating.

Define

$$Q_i(\mathbf{p}, t_i) = E_{\mathbf{t}_{-i}} p_i(\mathbf{m}(\mathbf{t})), \tag{7}$$

where $\mathbf{m}(\mathbf{t}) = (m_1(t_1), m_2(t_2), ..., m_N(t_N))$, $\mathbf{t} = (t_1, t_2, ..., t_N)$. If $(\mathbf{p}, \mathbf{x})$ is a truthful direct semirevelation mechanism that implements shutdown $\mathcal{C}$, then $Q_i(\mathbf{p}, t_i)$ is the conditional expected probability that bidder i with type t_i gets the object.

Using $U_i(\mathbf{p}, \mathbf{x}; t_i, t_i)$ and $Q_i(\mathbf{p}, t_i)$, the following Lemma reinterprets the necessary and sufficient conditions (2) to (6) for a direct semirevelation mechanism $(\mathbf{p}, \mathbf{x})$ to be a **truthful** direct semirevelation mechanism that implements $\mathcal{C}$.

Proposition 1: *A direct semirevelation mechanism* $(\mathbf{p}, \mathbf{x})$ *is a* **truthful** *direct semirevelation mechanism that implements* $\mathcal{C}$ *where* $C_i(\cdot)$, $\forall i \in \mathcal{N}$ *is not always equal to* $\underline{c}$ *or* $\overline{c}$*, if and only if the following conditions and (4), (5) and (6) hold.*

$$Q_i(\mathbf{p}, t_i) \geq Q_i(\mathbf{p}, t_i'),\ \forall\ t_i = (c_i, v_i), t_i' = (c_i, v_i') \in S_i^e(C_i), v_i \geq v_i', \forall i \in \mathcal{N}, \tag{8}$$

$$\frac{\partial U_i(\mathbf{p}, \mathbf{x}; t_i, t_i)}{\partial v_i} = Q_i(\mathbf{p}, t_i),\ \forall t_i = (c_i, v_i) \in S_i^e(C_i),\ \forall i \in \mathcal{N}, \tag{9}$$

$$\frac{\partial U_i(\mathbf{p}, \mathbf{x}; t_i, t_i)}{\partial v_i} = C_i'(v_i),\ \forall t_i = (c_i, v_i) \in S_i^e(C_i),\ \textit{where } v_i \in [v_i^\ell, v_i^u],\ \forall i \in \mathcal{N}, \tag{10}$$

$$\frac{\partial U_i(\mathbf{p}, \mathbf{x}; t_i, t_i)}{\partial c_i} = -1,\ \forall t_i = (c_i, v_i) \in S_i^e(C_i),\ \forall i \in \mathcal{N}, \tag{11}$$

$$U_i(\mathbf{p}, \mathbf{x}; t_i, t_i) = 0,\ \forall\ t_i = (c_i, v_i),\ \textit{where } c_i = C_i(v_i),\ v_i \in [v_i^\ell, v_i^u],\ \forall i \in \mathcal{N}. \tag{12}$$

Proof: see appendix.

From (9) and (10), we have that $Q_i(\mathbf{p}, t_i)$ is independent of c_i for given $v_i \in [v_i^\ell, v_i^u]$, and $C_i(v_i)$ must be a nondecreasing function. If $v_i^u < \overline{v}$, it is also true that $Q_i(\mathbf{p}, t_i)$ is independent of c_i if $v_i \in (v_i^u, \overline{v}]$. The reasons are the following. If $v_i^u < \overline{v}$, then (11) implies that the difference in the expected payoff of bidder i with types (c_i, v_i) and (c_i, v_i') in $S_i^e(C_i)$ does not depend the entry cost c_i. Thus from (9), $Q_i(\mathbf{p}, t_i) = \frac{\partial U_i(\mathbf{p}, \mathbf{x}; t_i, t_i)}{\partial v_i}$ does not depend on c_i. Combining (8), we further have that a participant wins in a higher probability if and only if his value is higher. These results are formally stated in the following Corollary.

Corollary 1: *(i) The expected winning probabilities of the participating bidders do not depend on their entry costs. (ii) A participant wins in a higher probability if and only if his value is higher.*

2.3 The Seller's Expected Revenue

For any truthful direct semirevelation mechanism $(\mathbf{p}, \mathbf{x})$ that implements given $\mathcal{C}$, the seller's expected utility is given by

$$R_0(\mathbf{p}, \mathbf{x}) = E_{\mathbf{t}}\Big\{v_0[1 - \sum_{i=1}^{N} p_i(\mathbf{m}(\mathbf{t}))] + \sum_{i=1}^{N} x_i(\mathbf{m}(\mathbf{t}))\Big\}. \tag{13}$$

The support of $\mathbf{t}$ is $\mathcal{T} = \{[\underline{c}, \overline{c}] \times [\underline{v}, \overline{v}]\}^N$.

Proposition 1 does not consider the cases of $C_i(\cdot) \equiv \underline{c}$ and $C_i(\cdot) \equiv \overline{c}$, $\forall i \in \mathcal{N}$. When $C_i(\cdot) \equiv \underline{c}$ or $C_i(\cdot) \equiv \overline{c}$, $\forall i \in \mathcal{N}$, only (12) in Proposition 1 may not hold. However, even when $C_i(\cdot) \equiv \underline{c}$ or $C_i(\cdot) \equiv \overline{c}$, $\forall i \in \mathcal{N}$, there is no loss of generality to derive the ex ante revenue-maximizing auction by simply assuming (12) holds for the following reasons. First, when $C_i(\cdot) \equiv \underline{c}$, (12) does not change the entry decision of bidder i, we still have $S_i^e(C_i) = \emptyset$. Second, when $C_i(\cdot) \equiv \overline{c}$, (12) may not hold because the left hand side can be positive. In this case, an entry fee equal to the expected payoff of type $(\overline{c}, \underline{v})$ restores (12) while increases the seller's expected revenue. With these observations, we can assume the conditions in Proposition 1 for revenue-maximization.

Using Proposition 1, the seller's expected utility from a truthful direct semirevelation mechanism $(\mathbf{p}, \mathbf{x})$ that implements $\mathcal{C}$ can be written as in Lemma 2. The seller's problem is to maximize $R_0(\mathbf{p}, \mathbf{x})$ subject to the restrictions on $(\mathbf{p}, \mathbf{x})$ and $\mathcal{C}$, which are specified in Proposition 1.

Lemma 2: *For any truthful direct semirevelation mechanism* $(\mathbf{p}, \mathbf{x})$ *that implements* $\mathcal{C}$*,* *we have*

$$R_0(\mathbf{p},\mathbf{x}) = v_0 - \sum_{i=1}^{N}\int_{\underline{c}}^{\overline{c}}\int_{V_i(c_i)}^{\overline{v}} c_i f(v_i)g(c_i)dv_i dc_i - \sum_{i=1}^{N}\int_{\underline{c}}^{c_i^{\ell}}(c_i^{\ell}-c_i)g(c_i)dc_i$$
$$+E_{\mathbf{t}}\Big\{\sum_{i=1}^{N} p_i(\mathbf{m}(\mathbf{t}))(J(v_i)-v_0)\Big\}. \tag{14}$$

Proof: see appendix.

Note that the expected revenue of the seller does not depend on the payment functions $\mathbf{x}$. This is due to the property (12) for this two-dimensional screening problem.

From Proposition 1 and Lemma 2, we have the following **revenue equivalence theorem** with endogenous entry.

Corollary 2: *For a mechanism that implements endogenous entry $\mathcal{C}$ where $C_i(\cdot)$, $\forall i \in \mathcal{N}$ is not always equal to $\overline{c}$, the seller and bidders' expected payoffs are completely determined by the shutdown curves $\mathcal{C}$ and the bidders' winning probabilities for all $\mathbf{t} \in \mathcal{T}$.*

proof: For a truthful direct semirevelation mechanism that implements endogenous entry $\mathcal{C}$, the result follows immediately from Proposition 1 and Lemma 2. For a general mechanism that implements endogenous entry $\mathcal{C}$, the result holds due to the semirevelation principle. $\square$

If $C_i(\cdot) \equiv \overline{c}$, then the seller and bidders' expected payoffs also depend on the payoff of the lowest participating type $(\overline{c}, \underline{v})$.

3 Some Classes of Implementable Entry Equilibria

Now we are ready to characterize the entry equilibria implemented by two classes of mechanisms: the ex post efficient mechanisms and ex post revenue-maximizing mechanisms. Due to the complexity of the problem, we focus on symmetric entry across bidders. In other words, all bidders participate following a common shutdown curve $C(\cdot)$.

Definition 1: *A mechanism is* **ex post efficient** *if it allocates the object to the player (the seller or a participating bidder) with the highest valuation.*

Definition 2: *A mechanism is* **ex post revenue-maximizing** *if it allocates the object to the player (the seller or a participating bidder) with the highest virtual valuation.*

Note that an ex post revenue-maximizing mechanism maximizes the seller's ex post expected revenue for any given participating bidders. When the virtual value function $J(v)$ increases v, the ex post revenue-maximization implies allocating the object to the participating bidder with the highest valuation given it is higher than $J^{-1}(v_0)$. The optimality of setting a reserve price of $J^{-1}(v_0)$ for the seller's ex post revenue is clear from Myerson (1981).

3.1 Entry with Ex Post Efficient Mechanisms

We first characterize the symmetric shutdown curve implemented by any ex post efficient mechanism.

Proposition 2: *Suppose that* $(\mathbf{p}, \mathbf{x})$ *is a truthful direct semirevelation mechanism that implements* $C(\cdot)$*. If* $(\mathbf{p}, \mathbf{x})$ *is ex post efficient, the symmetric shutdown curve* $C(\cdot)$ *must satisfy*

$$C'(v) = \begin{cases} \left(A_0 + \int_{v_s}^{v} G(C(v'))f(v')dv'\right)^{N-1}, & \text{if} \quad v \in [\max\{v_0, v_s\}, v_u], \\ 0, & \text{if} \quad v \in [\underline{v}, \max\{v_0, v_s\}), \end{cases} \tag{15}$$

where $A_0 = \int_{\underline{v}}^{\overline{v}} \int_{C(v)}^{\overline{c}} g(c)f(v)dcdv$*, and* $v_u = \sup_{\{C(v)<\overline{c}\}} v$*. The initial condition is given by* $C(v_s) = c_s$*, where* (c_s, v_s) *can be any point on the left or bottom boundary of* $[\underline{c}, \overline{c}] \times [\underline{v}, \overline{v}]$*.*

Proof: see appendix.

Proposition 2 implies that for any ex post efficient mechanism, the implemented $C(v)$ must belong to the family characterized by (15). As the lower end of the shutdown curve moves from $(\underline{c}, \overline{v})$ to $(\overline{c}, \underline{v})$ along the left and bottom boundaries of the domain of (c, v), we obtain all the shutdown curves which are compatible with ex post efficiency.

3.2 Entry with Ex Post Revenue-Maximizing Mechanisms

We now characterize the symmetric shutdown curve implemented by a mechanism with ex post optimality. When $J(\cdot)$ is an increasing function, the ex post optimality in terms of the seller's revenue means allocating the object to the participating bidder with the highest valuation if it is higher than $J^{-1}(v_0)$. Similar to Proposition 2, we have

Proposition 3: *Suppose that* $(\mathbf{p}, \mathbf{x})$ *is a truthful direct semirevelation mechanism that implements* $C(\cdot)$*. If* $(\mathbf{p}, \mathbf{x})$ *is ex post revenue-maximizing, the symmetric shutdown curve* $C(\cdot)$ *must satisfy*

$$C'(v) = \begin{cases} \left(A_0 + \int_{v_s}^{v} G(C(v'))f(v')dv'\right)^{N-1}, & \text{if} \quad v \in [\max\{J^{-1}(v_0), v_s\}, v_u], \\ 0 & \text{if} \quad v \in [\underline{v}, \max\{J^{-1}(v_0), v_s\}), \end{cases} \tag{16}$$

where $A_0 = \int_{\underline{v}}^{\overline{v}} \int_{C(v)}^{\overline{c}} g(c)f(v)dcdv$*, and* $v_u = \sup_{\{C(v)<\overline{c}\}} v$*. The initial condition is given by* $C(v_s) = c_s$*, where* (c_s, v_s) *is a point on the left or bottom boundary of* $[\underline{c}, \overline{c}] \times [\underline{v}, \overline{v}]$*.*

Proposition 3 implies that for any ex post revenue-maximizing mechanism, the implemented $C(v)$ must belong to the family characterized by (16). As the lower end of the shutdown curve moves from $(\underline{c}, \overline{v})$ to $(\overline{c}, \underline{v})$ along the left and bottom boundaries of the domain of (c, v), we obtain all the shutdown curves which are compatible with ex post optimality.

3.3 Examples of Closed Form Shutdown Curves

Suppose (c_i, v_i), $\forall i \in \mathcal{N}$ are independently and uniformly distributed on $[0, 1] \times [0, 1]$. Assume $v_0 = 0$. We have from (15)

$$C'(v) = \left(A_0 + \int_{v_s}^{v} C(v')dv'\right)^{N-1}, \forall v \in [v_s, v_u]. \tag{17}$$

This leads to

$$\frac{C''(v)}{C(v)} = (N-1)C'(v)^{\frac{N-2}{N-1}}, \forall v \in [v_s, v_u]. \tag{18}$$

This ordinary differential equation combined with initial condition $C(v_s) = c_s$ pins down a particular solution for $C(v)$. While (c_s, v_s) moves from $(0,1)$ to $(1,0)$ along the left and bottom boundary of $[0,1] \times [0,1]$, all implementable shutdown curves are obtained.

In particular, when $N = 2$, we can obtain the closed form shutdown curves that are implemented by the ex post efficient mechanisms. When $c_s = 0, v_s \in [0,1)$, $C(v)$ is given by

$$C(v) = \begin{cases} \frac{e^{-v_s}}{e^{1-v_s}+e^{v_s-1}} e^{v} - \frac{e^{v_s}}{e^{1-v_s}+e^{v_s-1}} e^{-v}, & \text{if} \quad v \in [v_s, 1], \\ 0, & \text{if} \quad v \in [0, v_s]. \end{cases} \tag{19}$$

When $c_s \in [0, e^{-1}],\ v_s = 0$, $C(v)$ is given by

$$C(v) = \frac{1 + c_s e^{-1}}{e + e^{-1}} e^{v} - \frac{1 - c_s e}{e + e^{-1}} e^{-v}, \quad v \in [0,1]. \tag{20}$$

When $c_s \in [e^{-1}, 1],\ v_s = 0$, $C(v)$ is given by

$$C(v) = \begin{cases} \frac{1+v_u}{2} e^{v-v_u} + \frac{1-v_u}{2} e^{v_u - v}, & \text{if} \quad v \in [0, v_u], \\ 1 & \text{if} \quad v \in [v_u, 1], \end{cases} \tag{21}$$

where $v_u \in [0,1]$ and $c_s = C(0)$.

Moreover, the closed form shutdown curves that are implemented by the ex post revenue-maximizing mechanisms are the following. When $c_s = 0, v_s \in [J^{-1}(0), 1)$ where $J^{-1}(0) = 1/2$, $C(v)$ is still given by (19). When $c_s \in [0, e^{-1/2}],\ v_s = 0$, $C(v)$ is given by

$$C(v) = \begin{cases} \frac{1+c_s e^{-1/2}}{1+e} e^{v} - \frac{1-c_s e^{1/2}}{1+e^{-1}} e^{-v}, & \text{if} \quad v \in [1/2, 1], \\ c_s & \text{if} \quad v \in [0, 1/2]. \end{cases} \tag{22}$$

When $c_s \in [e^{-1/2}, 1],\ v_s = 0$, $C(v)$ is given by

$$C(v) = \begin{cases} 1 & \text{if} \quad v \in [v_u, 1], \\ \frac{1+v_u}{2} e^{v-v_u} + \frac{1-v_u}{2} e^{v_u - v}, & \text{if} \quad v \in [1/2, v_u], \\ c_s & \text{if} \quad v \in [0, 1/2], \end{cases} \tag{23}$$

where $v_u \in [1/2, 1]$ and $c_s = C(\frac{1}{2})$.

4 Ex Ante Revenue-Maximizing Auctions

This section provides the ex ante revenue-maximizing mechanisms among each of the two classes studied in section 3. We first establish the following Lemma.

Lemma 3: *Suppose $C(\cdot)$ is not always equal to $\overline{c}$. (i) Any ex post efficient mechanism implementing a shutdown curve $C(\cdot)$ of the Proposition 2 class achieves the same expected revenue for the seller. (ii) Any ex post revenue-maximizing mechanism implementing a shutdown curve $C(\cdot)$ of the Proposition 3 class achieves the same expected revenue for the seller.*

Proof: The results hold from Corollary 2. Corollary 2 shows that the seller's expected revenue from a mechanism implementing a given entry equilibrium $C(\cdot)$ which is not always equal to $\overline{c}$, is completely determined by the winning probabilities of the bidders. Note that no matter for ex post efficient mechanism or ex post revenue-maximizing mechanism, the winning probabilities have already been fixed provided that the shutdown curve has been given. □

In the case that $C(\cdot) \equiv \overline{c}$, we can use an entry fee to extract all the expected surplus of the lowest type $(\overline{c}, \underline{v})$. Therefore, when $C(\cdot) \equiv \overline{c}$ Lemma 3 still holds for mechanisms that provides the lowest type $(\overline{c}, \underline{v})$ zero payoff.

4.1 Revenue-Maximizing Auctions with Proposition 2 Entry

The following proposition gives some results on the revenue-maximizing auction that implements a shutdown curve $C(v)$ of the Proposition 2 family.

Proposition 4: *Suppose $C(v)$ is a solution of (15) and $v_\ell = \inf_{\{C(v)>\underline{c}\}} v$, $c_\ell = C(v_\ell)$. (i) A modified Vickrey auction with a reserve price of $\max\{v_0, v_\ell\}$ and an entry subsidy c_ℓ to the participating bidders is revenue-maximizing among all ex post efficient auctions that implement $C(v)$. (ii) Under the regularity condition that $J(v)$ increases wrt. v, if*

$J(v_\ell) \geq v_0$, the above-defined auction is also revenue-maximizing among all auctions that implement $C(v)$.

Proof: see appendix.

Based on Proposition 4, the following Corollary 3 establishes that a modified Vickrey auction with a uniform reserve price and a uniform participation subsidy is the revenue-maximizing auction that allocates the object to the participant with the highest private value provided that it is higher than the seller's valuation.

Corollary 3: *A modified Vickrey auction with an optimal uniform reserve price and a uniform participation subsidy is the revenue-maximizing ex post efficient auction. Among those who participate in the auction, the highest bidder gets the object if his valuation is higher than the seller's valuation. The winner pays the second highest bid or the reserve price, whichever is higher. Moreover, every participating bidder gets the participation subsidy. While the optimal participation subsidy equals the entry cost at the lower end of the optimal shutdown curve, the optimal reserve price equals the seller's valuation or the valuation at the lower end of the optimal shutdown curve, whichever is higher.*

Proof: From Proposition 2, for a mechanism selling the object to the bidder with the highest private value among the participants, the shutdown curve $C(v)$ must be the solution of (15). Thus Proposition 4 applies. The problem of finding the revenue-maximizing ex post efficient auction reduces to selecting the best shutdown curve among the family defined in Proposition 2. No matter which shutdown curve is optimal,[7] a modified Vickrey auction with an optimal reservation price and participation subsidy is the revenue-maximizing ex post efficient mechanism. According to Proposition 4, the optimal participation subsidy equals the entry cost at the lower end of the optimal shutdown curve, the optimal reserve price equals the seller's valuation or the valuation at the lower end of the optimal

[7]Since the set of all equicontinuous and monotonic shutdown curve is compact, the optimal shutdown curve exists. The equicontinuity and monotonicity of the shutdown curve come from Proposition 1.

shutdown curve, whichever is higher. □

4.2 Revenue-Maximizing Auctions with Proposition 3 Entry

The following proposition gives the counterpart results on revenue-maximizing auction that implements an entry of the Proposition 3 family.

Proposition 5: *Suppose $C(v)$ is a solution of (16) and $v_\ell = \inf_{\{C(v)>\underline{c}\}} v$, $c_\ell = C(v_\ell)$. Under the regularity condition that $J(v)$ increases wrt. v, a modified Vickrey auction with a reserve price of $\max\{J^{-1}(v_0), v_\ell\}$ and an entry subsidy c_ℓ to the participating bidders is revenue-maximizing among all mechanisms that implement $C(v)$.*

Proof: The Proof is similar to that of Proposition 4. □

Based on Proposition 5, the following Corollary 4 establishes that a modified Vickrey auction with a uniform reserve price and a uniform participation subsidy is the revenue-maximizing auction satisfying ex post revenue-maximization.

Corollary 4: *A modified Vickrey auction with optimally set uniform reserve price and entry subsidy is revenue-maximizing among all auctions that satisfy ex post revenue-maximization. Among those who participate in the auction, the highest bidder gets the object and pays the second highest bid or the reserve price, whichever is higher. Moreover, every participating bidder gets the entry subsidy. While the optimal participation subsidy equals the entry cost at the lower end of the optimal shutdown curve, the optimal reserve price equals $J^{-1}(v_0)$ or the valuation at the lower end of the optimal shutdown curve, whichever is higher.*

Proof: The Proof is similar to that of Corollary 3. □

5 Concluding Remarks

This paper considers the endogenous entry and auctions design when bidders have a two-dimensional private signals, i.e., their participation costs and private values. We find that for any implementable entry equilibrium, the expected winning probability of a participant must not depend on his private cost. Moreover, the expected winning probabilities of the participants are given by the slopes of their shutdown curves. Based on this insight, the families of implementable shutdown curves by the classes of ex post efficient mechanisms and ex post revenue-maximizing mechanisms are fully characterized.

We further establish that in each of these classes of mechanisms, a modified Vickrey auction with a uniform reserve price and a uniform participation subsidy is ex ante revenue-maximizing. The optimal entry subsidy and reserve price are determined by the lower end of the respective optimal shutdown curve. For the ex nate revenue-maximizing auction within the ex post efficient class, the optimal reserve price equals the seller's valuation or the valuation at the lower end of the optimal shutdown curve, whichever is higher. For the ex ante revenue-maximizing among the ex post revenue-maximizing class, the optimal reserve price equals the usual optimal reserve price $J^{-1}(v_0)$ or the valuation at the lower end of the optimal shutdown curve, whichever is higher.

Appendix

Proof of Lemma 1: First, we show the existence of shutdown curves. For a given mechanism, consider any type $(c_i, v_i) \in [\underline{c}, \overline{c}] \times [\underline{v}, \overline{v}]$. If this type of bidder i participates with a positive probability, bidder i with types (c_i', v_i) where $c_i' < c_i$ must participate with probability 1. The arguments are the following. If bidder i with (c_i', v_i) where $c_i' < c_i$ participates with probability 1 and mimics the message of type (c_i, v_i) when participating, he gets strictly positive expected payoff since his entry cost is lower. This implies that bidder i with (c_i', v_i) where $c_i' < c_i$ must gain strictly positive payoff when he participates and submits optimal message. Thus bidder i with (c_i', v_i) where $c_i' < c_i$ participates with probability of 1. Equivalently, if bidder i with (c_i, v_i) participates with probability 0, bidder i with types (c_i', v_i) where $c_i' > c_i$ must participate with probability 0. Based on this observation, for each $v_i \in [\underline{v}, \overline{v}]$, we have a critical value $C_i(v_i) \in [\underline{c}, \overline{c}]$ so that bidder i with types (c_i, v_i) where $c_i < C_i(v_i)$ must participate with probability 1, and bidder i with types (c_i, v_i) where $c_i > C_i(v_i)$ must not participate at all. Note that there is no possibility of stochastic participation unless for types $(v_i, C_i(v_i)), \forall v_i \in [\underline{v}, \overline{v}]$.

Second, we consider the monotonicity of these shutdown curves. We claim that $C_i(v_i) \geq C_i(v_i')$, if $v_i > v_i'$. We show this by contradiction. Suppose $C_i(v_i) < C_i(v_i')$ for $\overline{v} \geq v_i > v_i' \geq \underline{v}$. Consider bidder i with type (c_i, v_i) where $c_i \in (C_i(v_i), C_i(v_i'))$. If he participates and mimics the message of type (c_i, v_i'), his expected payoff is at least equal to that of type (c_i, v_i'), which is strictly positive. This leads to that bidder i with type (c_i, v_i) must participate with probability of 1. This conflicts with the assumption that bidder i with type (c_i, v_i) where $c_i \in (C_i(v_i), C_i(v_i'))$ does not participate.

Third, we show that $C_i(v_i)$ is continuous. We show this by contradiction. Suppose $C_i(\cdot)$ is not continuous at $v_i \in (\underline{v}, \overline{v})$ without loss of generality. Then we must have $\lim_{v \to v_i -} C_i(v) < C_i(v_i)$ since $C_i(\cdot)$ is nondecreasing. Note that $C_i(\cdot)$ is a bounded nondecreasing function, so we have $\lim_{v \to v_i -} C_i(v)$ exists. Consider bidder i with type (c_i, v_i) where $c_i \in (\lim_{v \to v_i -} C_i(v), C_i(v_i))$. The expected payoff of bidder i with type (c_i, v_i) must be strictly positive. Then bidder i with type $(c_i, \tilde{v}_i)$ where $\tilde{v}_i$ is slightly smaller than v_i also get strictly positive expected payoff if he mimics the message of type (c_i, v_i). This result conflicts with the assumption that $(c_i, \tilde{v}_i)$ does not participate. $\square$

Proof of Proposition 1: From the arguments that follow Lemma 1, we have (12) must hold.

From (1) and (7), we have that

$$U_i(\mathbf{p},\mathbf{x};t_i,t_i') = U_i(\mathbf{p},\mathbf{x},t_i',t_i') + (v_i - v_i')Q_i(\mathbf{p},t_i') + (c_i' - c_i),$$
$$\forall\ t_i = (c_i,v_i), t_i' = (c_i',v_i') \in S_i^e(C_i), \forall i \in \mathcal{N}. \tag{A.1}$$

From (3) and (A.1), we have

$$U_i(\mathbf{p},\mathbf{x};t_i,t_i) \geq U_i(\mathbf{p},\mathbf{x},t_i',t_i') + (v_i - v_i')Q_i(\mathbf{p},t_i') + (c_i' - c_i),$$
$$\forall t_i = (c_i,v_i), t_i' = (c_i',v_i') \in S_i^e(C_i), \forall i \in \mathcal{N}. \tag{A.2}$$

Thus (3) is equivalent to (A.2). Using (A.2) twice, we have that

$$(v_i - v_i')Q_i(\mathbf{p},t_i') \leq U_i(\mathbf{p},\mathbf{x};t_i,t_i) - U_i(\mathbf{p},\mathbf{x},t_i',t_i') \leq (v_i - {v_i}')Q_i(\mathbf{p},t_i),$$
$$\forall\ t_i = (c_i,v_i), t_i' = (c_i,v_i') \in S_i^e(C_i), v_i \geq v_i', \forall\ i \in \mathcal{N}. \tag{A.3}$$

Equation (A.3) implies (8). From (A.3), we have that

$$Q_i(\mathbf{p},t_i)\delta \leq U_i(\mathbf{p},\mathbf{x},t_i',t_i') - U_i(\mathbf{p},\mathbf{x};t_i,t_i) \leq Q_i(\mathbf{p},t_i')\delta,$$
$$\forall\ t_i' = (c_i,v_i+\delta), t_i = (c_i,v_i) \in S_i^e(C_i),\ \forall\ i \in \mathcal{N}. \tag{A.4}$$

Since $Q_i(\mathbf{p},t_i)$ is increasing in v_i, (A.4) implies

$$\frac{\partial U_i(\mathbf{p},\mathbf{x};t_i,t_i)}{\partial v_i} = Q_i(\mathbf{p},t_i),\ \forall i \in \mathcal{N},\ \forall t_i = (c_i,v_i) \in S_i^e(C_i), \forall\ i \in \mathcal{N}. \tag{A.5}$$

Adopting similar procedure for deriving (A.5), we have

$$\frac{\partial U_i(\mathbf{p},\mathbf{x};t_i,t_i)}{\partial c_i} = -1,\ \forall i \in \mathcal{N},\ \forall t_i = (c_i,v_i) \in S_i^e(C_i), \forall\ i \in \mathcal{N}. \tag{A.6}$$

(A.5) and (A.6) give (9) and (11).

Consider any four types in $S_i^e(C_i)$: (c_i,v_i), $(C_i(v_i),v_i)$, (c_i,v_i') and $(C_i(v_i'),v_i)$ where $v_i' = v_i + dv_i$ with $dv_i > 0$ and $v_i,\ v_i' \in [v_i^\ell, v_i^u]$. From (A.5), we have

$$\begin{aligned} &U_i(\mathbf{p},\mathbf{x};(c_i,v_i'),(c_i,v_i')) - U_i(\mathbf{p},\mathbf{x};(c_i,v_i),(c_i,v_i)) \\ &= \frac{\partial U_i(\mathbf{p},\mathbf{x};(c_i,v_i),(c_i,v_i))}{\partial v_i} dv_i + O((dv_i)^2). \end{aligned} \tag{A.7}$$

From (12) and (A.6), we have

$$U_i(\mathbf{p},\mathbf{x};(c_i,v_i'),(c_i,v_i')) - U_i(\mathbf{p},\mathbf{x};(c_i,v_i),(c_i,v_i)) = C_i'(v_i)dv_i + O((dv_i)^2). \tag{A.8}$$

(A.7) and (A.8) imply that

$$C_i'(v_i) = \frac{\partial U_i(\mathbf{p},\mathbf{x};t_i,t_i)}{\partial v_i},\ \forall t_i=(c_i,v_i)\in S_i^e(C_i),\ where\ v_i\in[v_i^\ell, v_i^u],\ \forall i\in\mathcal{N}. \tag{A.9}$$

(A.9) gives (10).

We have shown (8)-(12) from (2), (3) and (5). Now we need to show (2) and (3) from (8)-(12), (5) and (6). (2) is directly from (12), (9) and (11). In order to show (3), we only need to show (A.2).

$\forall t_i=(c_i,v_i), t_i'=(c_i',v_i')\in S_i^e(C_i), \forall i\in\mathcal{N}$, where $v_i>v_i'$, (8), (9) and (11) imply

$$\begin{aligned}
&U_i(\mathbf{p},\mathbf{x};t_i,t_i) - U_i(\mathbf{p},\mathbf{x},t_i',t_i')\\
&= [U_i(\mathbf{p},\mathbf{x};t_i,t_i) - U_i(\mathbf{p},\mathbf{x},t_i'',t_i'')] + [U_i(\mathbf{p},\mathbf{x},t_i'',t_i'') - U_i(\mathbf{p},\mathbf{x},t_i',t_i')]\\
&= (c_i'-c_i) + \int_{v_i'}^{v_i} Q_i(\mathbf{p},s_i)dv \ge (v_i-v_i')Q_i(\mathbf{p},t_i') + (c_i'-c_i),
\end{aligned} \tag{A.10}$$

where $t_i''=(c_i',v_i)$ and $s_i=(c_i',v)$. Similarly, $\forall t_i=(c_i,v_i), t_i'=(c_i',v_i')\in S_i^e(C_i), \forall i\in\mathcal{N}$, where $v_i<v_i'$, the same result holds.

Thus we have (A.2), i.e., (3) is shown. □

Proof of Lemma 2: From (1),

$$\begin{aligned}
&\int_{\underline{c}}^{\overline{c}}\int_{\underline{v}}^{\overline{v}} U_i(\mathbf{p},\mathbf{x};t_i,m_i(t_i))f(v_i)g(c_i)dv_idc_i\\
&= \int_{\underline{c}}^{\overline{c}}\int_{V_i(c_i)}^{\overline{v}} \left\{\int_{\mathcal{T}_{-i}}[v_ip_i(\mathbf{m}(\mathbf{t})) - x_i(\mathbf{m}(\mathbf{t}))]\mathbf{f}_{-i}(\mathbf{t}_{-i})d\mathbf{t}_{-i} - c_i\right\}f(v_i)g(c_i)dv_idc_i\\
&= \int_{\mathcal{T}}[v_ip_i(\mathbf{m}(\mathbf{t})) - x_i(\mathbf{m}(\mathbf{t}))]\mathbf{f}(\mathbf{t})d\mathbf{t} - \int_{\underline{c}}^{\overline{c}}\int_{V_i(c_i)}^{\overline{v}} c_if(v_i)g(c_i)dv_idc_i,
\end{aligned} \tag{A.11}$$

where $\mathbf{f}_{-i}(\cdot)$ is the density function of $\mathbf{t}_{-i}$, and $\mathbf{f}(\cdot)$ is the density function of $\mathbf{t}$. We use $U_i(\mathbf{p},\mathbf{x};t_i,\emptyset)$ to denote the expected payoff of bidder i if he does not participate. Note that $U_i(\mathbf{p},\mathbf{x};t_i,\emptyset)=0,\ \forall t_i,\ \forall i\in\mathcal{N}$.

From (A.11), we have

$$\sum_{i=1}^{N}\int_{\underline{c}}^{\overline{c}}\int_{\underline{v}}^{\overline{v}} U_i(\mathbf{p},\mathbf{x};t_i,m_i(t_i))f(v_i)g(c_i)dv_i dc_i$$
$$= \int_{\mathcal{T}}\sum_{i=1}^{N}[v_i p_i(\mathbf{m}(\mathbf{t})) - x_i(\mathbf{m}(\mathbf{t}))]\mathbf{f}(\mathbf{t})d\mathbf{t} - \sum_{i=1}^{N}\int_{\underline{c}}^{\overline{c}}\int_{V_i(c_i)}^{\overline{v}} c_i f(v_i)g(c_i)dv_i dc_i. \quad \text{(A.12)}$$

From (13) and (A.12),

$$R_0(\mathbf{p},\mathbf{x}) = v_0 - \sum_{i=1}^{N}\int_{\underline{c}}^{\overline{c}}\int_{\underline{v}}^{\overline{v}} U_i(\mathbf{p},\mathbf{x};t_i,m_i(t_i))f(v_i)g(c_i)dv_i dc_i - \sum_{i=1}^{N}\int_{\underline{c}}^{\overline{c}}\int_{V_i(c_i)}^{\overline{v}} c_i f(v_i)g(c_i)dv_i dc_i$$
$$+ \int_{\mathcal{T}}\sum_{i=1}^{N} p_i(\mathbf{m}(\mathbf{t}))(v_i - v_0)\mathbf{f}(\mathbf{t})d\mathbf{t}. \quad \text{(A.13)}$$

From (9), (11) and (12), we have

$$\int_{\underline{c}}^{\overline{c}}\int_{\underline{v}}^{\overline{v}} U_i(\mathbf{p},\mathbf{x};t_i,m_i(t_i))f(v_i)g(c_i)dv_i dc_i$$
$$= \int_{\underline{c}}^{\overline{c}}\int_{V_i(c_i)}^{\overline{v}} U_i(\mathbf{p},\mathbf{x};t_i,t_i)f(v_i)g(c_i)dv_i dc_i$$
$$= \int_{\underline{c}}^{\overline{c}}\Big(\int_{V_i(c_i)}^{\overline{v}}\big(\int_{V_i(c_i)}^{v_i} Q_i(\mathbf{p},(c_i,s_i))ds_i\big)f(v_i)dv_i\Big)g(c_i)dc_i$$
$$+ \int_{\underline{c}}^{\overline{c}}\int_{V_i(c_i)}^{\overline{v}} U_i(\mathbf{p},\mathbf{x},(c_i,V_i(c_i)),(c_i,V_i(c_i)))f(v_i)g(c_i)dv_i dc_i$$
$$= \int_{\underline{c}}^{\overline{c}}\int_{V_i(c_i)}^{\overline{v}} Q_i(\mathbf{p},(c_i,v_i))(1-F(v_i))g(c_i)dv_i dc_i + \int_{\underline{c}}^{c_i^{\ell}}\Big(\int_{\underline{v}}^{\overline{v}}(c_i^{\ell} - c_i)f(v_i)dv_i\Big)g(c_i)dc_i$$
$$= \int_{\underline{c}}^{\overline{c}}\int_{V_i(c_i)}^{\overline{v}} Q_i(\mathbf{p},(c_i,v_i))\frac{1-F(v_i)}{f(v_i)}f(v_i)g(c_i)dv_i dc_i + \int_{\underline{c}}^{c_i^{\ell}}(c_i^{\ell} - c_i)g(c_i)dc_i. \quad \text{(A.14)}$$

Note that the type $(c_i, V_i(c_i))$ where $c_i \le c_i^{\ell}$ enjoys a surplus of $c_i^{\ell} - c_i$.

From (7), we have

$$\int_{\underline{c}}^{\overline{c}}\int_{V_i(c_i)}^{\overline{v}} Q_i(\mathbf{p},(c_i,v_i))\frac{1-F(v_i)}{f(v_i)}f(v_i)g(c_i)dv_i dc_i = \int_{\mathcal{T}} p_i(\mathbf{m}(\mathbf{t}))\frac{1-F(v_i)}{f(v_i)}\mathbf{f}(\mathbf{t})d\mathbf{t}. \quad \text{(A.15)}$$

From (A.14) and (A.15), we have

$$\sum_{i=1}^{N}\int_{\underline{c}}^{\overline{c}}\int_{\underline{v}}^{\overline{v}} U_i(\mathbf{p},\mathbf{x};t_i,m_i(t_i))f(v_i)g(c_i)dv_i dc_i$$
$$= \int_{\mathcal{T}}\sum_{i=1}^{N}[p_i(\mathbf{m}(\mathbf{t}))\frac{1-F(v_i)}{f(v_i)}]\mathbf{f}(\mathbf{t})\mathbf{dt} + \int_{\underline{c}}^{c_i^{\ell}}(c_i^{\ell} - c_i)g(c_i)dc_i. \quad \text{(A.16)}$$

From (A.13) and (A.16), we have

$$R_0(\mathbf{p},\mathbf{x}) = v_0 - \sum_{i=1}^{N}\int_{\underline{c}}^{\overline{c}}\int_{V_i(c_i)}^{\overline{v}} c_i f(v_i) g(c_i) dv_i dc_i - \sum_{i=1}^{N}\int_{\underline{c}}^{c_i^\ell}(c_i^\ell - c_i) g(c_i) dc_i$$
$$+ \int_{\mathcal{T}}\Big\{\sum_{i=1}^{N} p_i(\mathbf{m}(\mathbf{t}))(J(v_i) - v_0)\Big\}\mathbf{f}(\mathbf{t})d\mathbf{t}. \tag{A.17}$$

□

Proof of Proposition 2: From (10) and (9), $C'(v)$ is the winning probability of participant i with value $v_i = v$. For an ex post efficient auction, the winning probability of participant i with $v_i < v_0$ is zero; the winning probability of participant i with $v_i \geq v_0$ is the probability of all the other bidders do not participate or their valuations are lower when they participate. □

Proof of Proposition 4: Since the mentioned auction is a modified second price auction, it is a weakly dominant strategy of the participants to bid their true values when participating. Therefore, the allocation of the auction must be ex post efficient.

Next, we show the auction implements $C(\cdot)$. We consider the case where $C(\cdot)$ is not always equal to $\underline{c}$ or $\overline{c}$ without loss of generality. We use $\tilde{C}(\cdot)$ to denote the symmetric shutdown curve implemented by the auction. According to Lemma 1, $\tilde{C}(\cdot)$ must be nondecreasing and continuous. Clearly, $\tilde{C}(\cdot)$ cannot be always equal to $\underline{c}$ or $\overline{c}$. We define $\tilde{v}_\ell = \inf_{\{\tilde{C}(v) > \underline{c}\}} v$, $\tilde{c}_\ell = \tilde{C}(v_\ell)$ and $\tilde{v}_u = \sup_{\{\tilde{C}(v) < \overline{c}\}} v$. Note that we must have $\tilde{v}_\ell \geq v_0$.

Denote the probability of nonparticipation of a bidder by $\tilde{A}_0$. Since the auction is ex post efficient, we have that the winning probability of bidder i is $\tilde{Q}_i(c_i, v_i) = 0,\ \forall v_i \in [\underline{v}, \tilde{v}_\ell)$. $\tilde{Q}_i(c_i, v_i) = \big(\tilde{A}_0 + \int_{\tilde{v}_\ell}^{v_i} G(\tilde{C}(v)) f(v) dv\big)^{N-1},\ \forall v_i \in [\tilde{v}_\ell, \tilde{v}_u]$. From Proposition 1, we have $\tilde{C}'(v_i) = \big(\tilde{A}_0 + \int_{\tilde{v}_\ell}^{v_i} G(\tilde{C}(v)) f(v) dv\big)^{N-1}, \forall v_i \in [\tilde{v}_\ell, \tilde{v}_u],\ c_i \leq \tilde{C}(v_i)$. This means that $\tilde{C}(\cdot)$ belongs to the Proposition 2 class.

To show that $\tilde{C}(\cdot)$ is same as $C(\cdot)$, we only need to show that $(\tilde{c}_\ell, \tilde{v}_\ell) = (c_\ell, v_\ell)$. The type $(\tilde{c}_\ell, \tilde{v}_\ell)$ is indifferent between participation and nonparticipation by construct of $\tilde{C}(\cdot)$. On the other hand, the type (c_ℓ, v_ℓ) is clearly indifferent between participation and nonparticipation in the mentioned auction. If $(\tilde{c}_\ell, \tilde{v}_\ell)$ is different from (c_ℓ, v_ℓ), then either $v_\ell < \tilde{v}_\ell, c_\ell \geq \tilde{c}_\ell$ or $v_\ell > \tilde{v}_\ell, c_\ell \leq \tilde{c}_\ell$ as both $(\tilde{c}_\ell, \tilde{v}_\ell)$ and (c_ℓ, v_ℓ) are on the left or bottom boundary of the type space. However, in either case, it is impossible that both types $(\tilde{c}_\ell, \tilde{v}_\ell)$ and (c_ℓ, v_ℓ) are indifferent between participation and nonparticipation in the mentioned auction. This contradiction implies that $(\tilde{c}_\ell, \tilde{v}_\ell) = (c_\ell, v_\ell)$ must hold.

Based on Lemma 3, the mentioned auction is then the revenue-maximizing auction among all ex post efficient auctions that implement $C(\cdot)$. Moreover, under the regularity condition that $J(v)$ increases *wrt.* v, the above-defined auction is the revenue-maximizing auction mechanism implementing shutdown $C(v)$ if $J(v_\ell) \geq v_0$. This is clear from (14). $\square$

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