The Allocation of Economic Capital in Opaque Financial Conglomerates

Mierzejewski, Fernando

Katholieke Universiteit Leuven

3 July 2008

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Fernando Mierzejewski[†]

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[†]*Faculty of Economy and Applied Economy, Katholieke Universiteit Leuven, Naamsestraat 69, 3000 Leuven, Belgium*
Email: `Fernando.Mierzejewski@wis.kuleuven.be`

Abstract

The capital structure of firms that face restrictions on liquidity (i.e. that cannot hedge continuously) is affected by the agency costs and moral-hazard implicit in the contracts they establish with stockholders and customers. It is demonstrated in this paper that then an optimal level of capital exists, which is characterised in terms of the actuarial prices of the involved agreements. The capital principle so obtained explicitly depends on risk and expectations and it can be naturally applied to allocate balances inside multidivisional corporations. In particular, an optimal decentralised mechanism is defined, which stimulates the exchange of information between central and divisional administrations. A novel model of capital is thus formulated, which extends the classic theoretical framework (sustained by the well-known proposition of Modigliani and Miller and the model of deposit insurance of Robert Merton) and integrates the financial and actuarial theoretical settings.

Key words: Economic Capital; Capital Allocation; Deposit Insurance; Distorted Risk principle; Value-at-Risk.
JEL-Classification: G10, G20, G30, G32.

1 Introduction

Financial intermediaries establish contractual liabilities with customers to attract funds that are spent on securities. The managers of these companies *create value* to shareholders when the market value (X) of *net assets*, equal to assets (A) minus liabilities (D), is greater than zero (i.e. when $X = A - D > 0$). Some companies that fear they can obtain negative balances at the end of the investment period maintain cash provisions, in the form of *capital*, to guarantee that all outstanding liabilities will be honoured. Several *types* of capital are distinguished in the literature, depending on the purpose it serves to the institution and on the criterion employed to fix its level. Thus, on the one hand, *cash* capital represents a balance required to execute transactions, while *working* capital additionally includes operational expenses.[1] On the other hand, *regulatory* capital is defined according to some accounting standard (as in Basel, 2004), while *equity* corresponds to the portion of reserves

[1]See, for example, Williams et al., 2002, and also Howells and Bain, 2005.

provided by shareholders. Finally, many authors speak of *economic* and *risk* capital when some particular criterion, based on economic or statistical considerations, is proposed.

In lines with Merton (1974, 1977), the demand for capital will be corresponded to a demand for *deposit insurance* in this paper.[2] Accordingly, the terms *economic* and *risk* capital will be indistinctly used to refer to the smallest amount required *to insure the value of net assets against a loss in value relative to the risk-free investment of those net assets.* In this context, the difference between *economic* and *equity* capitals represents a balance that is supplied by managers in attention to some solvency requirement. Three main components of the *capital structure* can thus be distinguished: a net liability contracted with *customers*, an amount of equity supplied by shareholders and a cash balance provided by managers. Since the economic capital is equal to the sum of the last two components, the problem of capital allocation can be roughly corresponded to the determination of the proportions of the portfolio of assets that are funded by means of *internal* and *external* debt.

Holding capital imposes an *opportunity* cost on firms because these funds could be alternatively employed on profitable investments. Such costs lead financial institutions to prefer (external) debt and accordingly demand *as less capital as possible.* In fact, in a seminal paper, Modigliani and Miller (1958) claim that, if at any moment firms can borrow and lend *any* amount of capital at a single interest rate, they can adjust their balance sheets whenever is needed and hence cash provisions impose a cost without any benefit. Then *rational* decision-makers (who maximise value) should demand no capital at all.

More specifically, Merton (1997) states that the presence of *credit-sensitive* customers obliges *opaque* institutions (whose investment activities are not fully observed by outsiders) to rely on a third-party guarantor, who agrees to honour the outstanding liabilities when bankruptcy is declared.[3] Then the market values of equity and debt can be respectively corresponded to the values of a *call* and a *put* option on the value of assets, with exercise price equal to the value of debt, which implies that the market value of the firm (or the market value of the assets' portfolio) is independent of the capital structure, as predicted by the MM-proposition (see also Miller, 1998). A fundamental assumption for this mechanism to work is that capital and financial securities are continuously traded in competitive markets.

The correspondence of capital to deposit insurance implies that the hypothesis of *perfect hedging* can be reestablished by imposing that a *unique* price exists for the insurance of any single claim. Under such circumstances, managers are indifferent between hedging and insurance and are certainly indifferent about the amount of economic capital. However, hedging and insurance cannot be regarded as equivalent tools in practice. As a matter of fact, although competitive forces in capital and security markets lead transactions to be always produced at a unique price, this is not always the case of insurance markets, where *non-standarised* policies are transacted (see Goovaerts et al., 2005, and Venter, 1991).

Hence, liquidity restrictions may arise from two different sources. In the first place, firms are not always able to trade continuously in capital and security markets. Then the moral-hazard arising due to the opaqueness of financial intermediaries induces the appearance of premiums over the market cost of capital, which should be established on an actuarial basis.

[2]The meaning and scope of this interpretation will be clarified later in *Sections* 2, 3 and 5.

[3]The presence of *credit-sensitive* customers increases external controls and monitoring due to the *moral-hazard* implicit in the administration of deposits. See also Ross (1989).

Secondly, the buyers and sellers of insurance can maintain different perceptions about risks and can accordingly assign different prices to their corresponding guarantees.

A model will be presented in this paper to characterise the demand for reserves of financial institutions that access to capital markets where *hedging* is restricted to some extent and individuals differ on their *expectations* about the riskiness of the transacted securities. This model extends the classic theoretical framework that assumes *perfect* competition. It can in fact be demonstrated that the capital structure does affect the value of firms under conditions of *imperfect* competition. The *optimal* cash balance can then be determined, in such a way that the market value of the firm, defined as the difference between the actuarial prices of equity and the default claim, plus the return offered by a non-risky bond, is maximised. A precise description of the conditions under which *the capital structure matters* is provided in this way, and the limits of the Modigliani and Miller invariance propositions are then clearly stated. This is the main contribution of this paper.[4]

Within a multi-business environment, differences in expectations between central and divisional managers lead to discrepancies about the optimal levels maintained by subsidiaries. Inefficiencies then arise in the form of *under-* and *over-* investment, respectively corresponded to the cases when *too much* and *too few* capital is demanded. This is a most important issue in corporate finance, for in the former case favourable investment opportunities can be lost, while in the later case firms can damage their credit quality (as perceived by customers) if *too much* risk is assumed. Since the optimal capital principle proposed in this paper explicitly depends on risk and expectations, it proves to be a convenient tool to describe such interactions inside corporations. In fact, two optimal *allocation principles* can be defined on this basis, respectively related to *centralised* and *decentralised* administrations. This is the second main contribution of this paper.[5]

At the empirical level, the capital and allocation principles derived from the extended setting show important advantages over other principles found in the literature (see Albrecht, 2004). Thus, in the first place, they are founded on economic fundamentals. Secondly, since they are expressed in terms of the *quantile* function of underlying risk, they can be applied to any kind of probability distributions and hence, they are suitable both to finance and insurance applications. The quantile function is actually well-known by researchers and practitioners and it has been actually recommended by the Basel Committee on Banking Supervision (2004). Finally, again at the theoretical level, the allocation rule proposed by Merton and Perold (1993) — which is based on the market price of deposit insurance with unrestricted hedging — can be obtained as a particular case of the optimal capital principle (*Section* 10). The most relevant facts of the classic theory of capital are thus extended.

2 Agency Costs and Moral-Hazard in Opaque Financial Institutions

In the model of Modigliani and Miller (1958), the set of financial securities is divided into *equivalent* classes, in such a way that the returns offered by any two assets in a same class are

[4]The optimal capital principle is presented in the first part of the paper, in *Sections* 2, 3, 4 and 5.
[5]The optimal allocation principles are presented in *Sections* 6, 7, 8 and 9.

proportional to each other — or equivalently, every two assets belonging to the same class can be regarded as perfect substitutes for each other. Firms fund their assets' portfolios with own capital (K) or issue bonds offering a constant yield per unit of time (delivering some cash flow D at maturity). Then the market price of portfolios containing assets belonging to the same class can at most differ in a scaling factor α:

$$V = A = K + D = \alpha \cdot r$$

Both assets and bonds are supposed to be traded in *perfect* markets, which means that both the transactions of securities and bonds can be performed at any moment and without restrictions. Therefore, the proportions of capital and debt can be modified at any moment without affecting the value of the firm (which is always equal to $A = \alpha \cdot r$) and hence the market valorisations of firms are always independent of the underlying capital structure.

In subsequent papers, Modigliani and Miller prove that the irrelevance of the capital structure can still be maintained in the presence of taxes[6] and dividends deliver to shareholders (Modigliani and Miller, 1961). The critical assumption supporting these results is that managers and stockholders share expectations about the future payoffs of the net assets' portfolio. Informational asymmetries between managers and stockholders lead to the appearance of *agency costs* inside institutions. The cost of equity might depend on the level of capital under such circumstances.

Informational asymmetries are also present in the relationship between managers and customers. Indeed, as pointed out by Merton (1997), financial companies tend to be *opaque* institutions, for their investment activities are not fully observed by outsiders. As a consequence, it is difficult for customers to assess the risk assumed by intermediaries and for the later to effectively reflect their solvency state. On the other hand, since the benefits of financial intermediation (such as economies of scale and reduced transaction costs) can be effectively transmitted to customers only if intermediaries can convincingly assure that their liabilities are free of default, both the availability and the price of capital are dependent on the perceptions of customers regarding the credit quality of the borrowing institutions. Such services are said to be *credit-sensitive* by Merton. In other words, because of the *moral-hazard* implicit in the relationship between managers and customers, firms are obliged to fix their capital structures in order to increase their *honourability* at the eyes of investors (see also Cummins and Sommer, 1996, and Myers and Read, 2001).

On these grounds, Merton states that the practice of financial intermediation requires a third-party *guarantor* (whose willingness and capability to meet obligations are beyond question) to assure the safety of deposits. This role might be assumed by another financial company, as well as by some governmental division. The terms of the contract are the following. When bankruptcy is declared (i.e. when $X = A - D < 0$), the firm defaults its assets to the guarantor and the guarantor meets the bondholders' claims. In this case, nothing is left to shareholders. A second contract is simultaneously established with shareholders, promising to pay the value of net assets when this balance is positive (i.e. when $X = A - D > 0$) in exchange of a certain amount of equity capital. Consequently, shareholders and bondholders

[6] In fact, tax regimes tend to favour the use of debt, the only exception being some regimes of double taxation. See also Modigliani and Miller, 1963, and Miller, 1977.

respectively receive the payments $\max(0, A - D)$ and D at the maturity date, while the guarantor has to afford the cost $\min(0, A - D)$.

In other words, while shareholders own the right to *buy* the portfolio of assets at the price D at maturity, the firm owns the right to *sell* the portfolio of assets to the guarantor at the price D. Equivalently, we can say that shareholders are the owners of a *European call option* $C(A, D)$ on the value of assets with exercise price equal to the value of debt, while managers are the owners of a *European put option* $P(A, D)$ on the value of assets with exercise price equal to the value of debt. Bondholders, on the other hand, are the owners of the sure stream D. Thus, information asymmetries impose on managers the obligation of combining three kind of securities: the firm itself (or the portfolio of assets held by the firm), a promise to pay at any event its outstanding liabilities, and a particular security guaranteeing the payment to creditors when bankruptcy is declared. Such a portfolio can be built by simultaneously hiring debt in the form of a zero coupon bond, selling a call option to shareholders and buying a put option to the guarantor (Merton, 1974 and 1977).

In fact, since the value of the call option $C(A, D)$ corresponds to a cash flow payed by shareholders in exchange of the surplus accrued by the portfolio of net assets, it can then be regarded as the market value of *equity*. In a similar way, the term $D \cdot e^{-rT} - P(A, D)$ can be regarded as the market value of the *guaranty*, for it represents the net stream received by the firm from the guarantor. From the *Put-Call parity theorem* (see Black and Scholes, 1973, Merton, 1974, and also Cummins and Sommer, 1996) we obtain that:

$$V = A = C(A, D) + D \cdot e^{-rT} - P(A, D) \tag{1}$$

Therefore, though both the price of equity and the price of the guarantee depend on the underlying capital structure, the market price of the firm is always equal to the value of assets, whatever the chosen funding strategy (as predicted by the MM-proposition, see Miller, 1998).

Consequently, even in the presence of informational asymmetries, the capital structure does not affect the market valorisations of firms, as long as *trading in assets and bonds takes place continuously in time.* However, this cannot be always said of real markets. The major hypothesis of this paper thus states that financial institutions *cannot hedge fully* and accordingly, that hedging cannot be regarded as a perfect *substitute* for insurance. This means, specifically, that deposit insurance cannot be corresponded to an option claim and that *Equation* 1 cannot be invoked to support the MM-proposition. Instead, an actuarial approach should be introduced to determine the *fair* price of the claims $(A-D)_+ := \max(0, A-D)$ and $(A-D)_- := -\min(0, A-D)$ respectively held and afforded by shareholders and guarantors at the end of the investment period. This will be formally accomplished in *Sections* 3, 4 and 5.

3 The Optimal Capital Structure

Both asset and liability claims will be regarded as random variables in the following, while the market value of net assets will be expressed as the product of the level of investment I and some random pertubation X:

$$A - D = I \cdot X \quad \Leftrightarrow \quad X := \frac{A - D}{I} \tag{2}$$

Then the level of investment can be regarded as the *principal* of the net portfolio. The level of capital will be also represented as a proportion of the level of investment:

$$K = I \cdot k \quad \Leftrightarrow \quad k := \frac{K}{I} \tag{3}$$

The ratio k represents a *capital-to-investment* or a *cash-to-risk* ratio. It determines the proportion of *internal* financing of the firm. In this context, the determination of the capital structure involves choosing the proportions of *internal* and *external* financing when building *solvent* or *equilibrated* portfolios (satisfying $A - D > K$).

Let us analyse the payments received at the maturity date by some firm that maintains insurance and equity contracts in the terms stated in the previous section. Thus, on the one hand, one agreement is celebrated between the firm and a guarantor, which obliges the later to honour the total capital loss $I \cdot (X + k)_{-} = I \cdot min(0, X + k)$ in exchange of a certain premium payed by the former at the beginning of the investment period. Simultaneously, shareholders pay a certain *price* to managers at the beginning of the investment period, in exchange of receiving the random capital profit $I \cdot (X - k)_{+} = I \cdot max(0, X - k)$ at the end. The capital K is invested in a banking account to obtain the risk-free return r_0 (in other words, it is converted to a risk-free zero-coupon bond with internal return r_0). The payments received by the firm at maturity are then given by:

$$\begin{array}{ccc} I \cdot (X - k) + r_0 \cdot K & if & X \geq k \\ I \cdot (X + k) + r_0 \cdot K & if & X \leq -k \\ 0 & if & -k < X < k \end{array} \tag{4}$$

Accordingly, when $X \geq k$ the firm can afford its debt and pay a surplus to shareholders, i.e. the firm is *solvent* in this case. By contrast, when $X \leq -k$ the capital K is deliver to the guarantor who has to afford the residual loss $I \cdot (X + k)$. Shareholders receive nothing in this case. Finally, when $-k < X < k$ the firm cannot return the total amount of capital to shareholders, although the total debt attracted from customers can be honoured and the guaranty is not invoked. Stockholders might decide to sell their shares or to call for portfolio restructuring under such circumstances.

We have already pointed out that risk can be completely suppressed through hedging if cash and securities can be traded *to any desired extent* in the market. Indeed, under such circumstances, the prices (per unit of investment) of the contracts established with shareholders and guarantors are respectively given by the prices of a call and a put option on the value of the random capital return X with exercise price equal to the cash-to-risk ratio k, in such a way that the put-call parity can be invoked to obtain (as in *Equation* 1):

$$X = C(X, k) + k \cdot e^{-r_0 T} - P(X, k)$$

where T denotes the time to maturity. Thus, the value of the firm does not depend on the cash-to-risk ratio k as long as continuos rebalancing of portfolios is allowed. When this is not possible due to liquidity restrictions, the prices of the contracts established with stockholders and guarantors should be determined on an actuarial basis. Accordingly, the *price of equity* and the *cost of bankruptcy* will be respectively corresponded to the terms $E\left[(X-k)_+\right]$ and $E\left[(X+k)_-\right]$ in the following, for these terms represent the *fair* or *actuarial* prices of the underlying exposures (see Goovaerts et al., 1984). Hence the value of the firm at the end of the investment period (as perceived by managers) is given by:

$$V \;=\; \left(E\left[(X-k)_+\right] - E\left[(X+k)_-\right]\right) \cdot I + r_0 \cdot K$$

Within this context, the cash-to-risk ratio k affects the net return on investment and hence the value of the firm.

Given some fixed level of investment I, every *rational* manager must choose the capital structure that maximises the firm's value per unit of investment V/I:[7]

$$\max_k \; E\left[(X-k)_+\right] - E\left[(X+k)_-\right] + r_0 \cdot k \tag{5}$$

The solution to the maximisation problem can be determined by applying Lagrange optimisation. The first-order condition actually leads to:

$$\frac{d}{dk}E\left[(X-k^*)_+\right] - \frac{d}{dk}E\left[(X+k^*)_-\right] + r_0 \;=\; 0$$

The mathematical expectations of the corresponding excess return terms are defined as:

$$\begin{aligned} E\left[(X-k)_+\right] &= \textstyle\int_k^{\infty}(x-k)\cdot dF_X(x) = \int_k^{\infty}(x-k)\cdot f_X(x)dx \\ E\left[(X+k)_-\right] &= -\textstyle\int_{-\infty}^{-k}(x+k)\cdot dF_X(x) = -\int_{-\infty}^{-k}(x+k)\cdot f_X(x)dx \end{aligned} \tag{6}$$

where F_X and f_X respectively denote the *cumulative* and the *density* probability functions of the random variable X, which are defined as (see, for example, De Finetti, 1975):

$$F_X(x) = P\{X \leq x\} = \int_{-\infty}^{x} dF_X(x) \quad with \quad f_X(x) = \frac{dF_X}{dx}(x) > 0 \quad \forall\ x$$

Hence, every random variable is uniquely determined by its corresponding probability distribution F_X. Besides, an equivalent characterisation is provided by the *cumulative* or *tail* (also known as *survival*) probability distribution $T_X(x) = 1 - F_X(x)$, $\forall\ x$, defined as:

$$T_X(x) = P\{X > x\} = \int_{x}^{\infty} dF_X(x) \quad with \quad \frac{dT_X}{dx}(x) < 0 \quad \forall\ x$$

As the derivative of the expected excess return requires to derive an integral operator with respect to a variable affecting its limits of integration, the *Leibnitz's rule* can be applied:

$$\frac{\partial}{\partial z}\int_{u(z)}^{v(z)} \phi(z,x)\ dx = \int_{u(z)}^{v(z)} \frac{\partial \phi(z,x)}{\partial z}dx + \phi\left(z, v(z)\right)\cdot v'(z) - \phi\left(z, u(z)\right)\cdot u'(z) \tag{7}$$

[7] A similar approach is proposed by Froot et al., 1993. In this model, the fundamental motive for the existence of an optimal balance is that output variability is *undesirable* when investment presents *diminishing* marginal returns — i.e. when output is expressed as a *concave* function of the level of investment — because funding and investment plans can be affected in a *costly* way under such circumstances. See also Froot and Stein, 1998, and Froot, 2007.

Accordingly,

$$\frac{dE\left[(X-k)_+\right]}{dk} = -P\left\{X > k\right\} = -T_X(k) \quad \Longrightarrow \quad \frac{d^2E\left[(X-k)_+\right]}{dk^2} = -\frac{dT_X(k)}{dk} \tag{8}$$

and also, since $P\left\{X \leq -k\right\} = P\left\{-X > k\right\}$:

$$\frac{dE\left[(X+k)_-\right]}{dk} = -P\left\{-X > k\right\} = -T_{-X}(k) \quad \Longrightarrow \quad \frac{d^2E\left[(X+k)_-\right]}{dk^2} = -\frac{dT_{-X}(k)}{dk} \tag{9}$$

Therefore, the first-order condition leads to the following equality in terms of the negative and positive tail probability functions:

$$T_{-X}(k^*) + r_0 \;=\; T_X(k^*) \tag{10}$$

or equivalently, in terms of the cumulative probability functions:

$$1 - F_{-X}(k^*) + r_0 \;=\; 1 - F_X(k^*)$$

Since the term $E\left[(X-k)_+\right]$ represents the expected excess over capital, the term $T_X(k) = -dE\left[(X-k)_+\right]/dk$ corresponds to the magnitude of the reduction in free-cash-flow produced when adding an additional unit of cash to the stock of capital — instead of investing it in the portfolio of assets. Similarly, the terms $E\left[(X+k)_-\right]$ and $T_{-X}(k) = -dE\left[(X+k)_-\right]/dk$ respectively represent the *cost of bankruptcy* and the marginal gain in value obtained due to the reduction in the cost of bankruptcy (and hence in the price of the guarantee) when an additional unit of investment is added to the stock of cash (see *Equations* 8 and 9). Therefore, according to *Equation* 10, the optimal level of capital is determined at the point where the marginal gain equals the marginal loss in value due to allocating one additional unit of investment to the stock of reserves instead of spending it on assets. In other words, capital is demanded up to the point where the marginal return on capital (to the left-hand side of *Equation* 10) is equal to the marginal return on investment (to the right-hand side of *Equation* 10).

The existence of a solution to the optimisation problem of *Equation* 5 can be mathematically assured as long as the second derivative of the objective function is lower than zero, i.e. as long as (from *Equations* 8 and 9):

$$\frac{d^2E\left[(X-k^*)_+\right]}{dk^2} - \frac{d^2E\left[(X+k^*)_-\right]}{dk^2} \;<\; 0 \quad \Leftrightarrow \quad \frac{dT_{-X}(k^*)}{dk} \;<\; \frac{dT_X(k^*)}{dk}$$

and since $T_X = 1 - F_X$, and also $dF_X(x)/dx = f_X(x) = P\{X = x\}\ \forall x$, we finally obtain that the second-order condition can be expressed in terms of the mass probability densities:

$$P\left\{X = k^*\right\} \;<\; P\left\{X = -k^*\right\} \tag{11}$$

We thus arrive to the (reasonable) conclusion that liquidity provisions provide a benefit to firms only when the probability that attaining a capital loss of a certain magnitude is greater than the probability of obtaining a capital gain of the same magnitude. In particular, *symmetric* probability distributions (around the expected value) satisfy:

$$P\{X = E[X] - x\} \;=\; P\{X = E[X] + x\} \quad \forall x$$

Therefore, within the class of symmetric probability distributions, the objective function of the maximisation problem is concave when $E[X] < 0$ but convex when $E[X] > 0$, and hence a maximum is attained only when $E[X] < 0$. Consequently, capital is beneficial to financial institutions only when $E[X] < 0$.

4 Liquidity Restrictions in Capital Markets

As demonstrated in the previous section, the capital structure may well affect the market value of financial institutions in the presence of liquidity restrictions. Liquidity restrictions arise, in the first place, because the portion of capital provided by stockholders is determined in a regular frequency (such as yearly, quarterly or monthly) and cannot be modified until the end of the period. Since the level of equity and the frequency of revisions are the result of *negotiations* between managers and shareholders, changing some agreement is *costly* and can reduce the market valorisation of the firm. On the other hand, choosing some *risk-based* capital principle (such as the one defined by *Equations* 5 and 10) implies that the amount of economic capital must be subject to constant revisions if the riskiness of the series of the value of the net-assets' portfolio is varying — i.e. if the series of capital $P\&L$ of the net-assests' portfolio is *non-stationary*. This means that managers are obliged to rely on some market of cash balances (or inter-bank loans) in order to maintain a total level of capital that is consistent with the underlying exposure.

Albeit preferring external debt reduces the controls imposed by shareholders, this strategy also raises the costs associated to moral-hazard (on the side of customers) and bankruptcy.[8] Hence the controls and monitoring established by customers and regulators are expected to increase in this situation. Consequently, when deciding their capital structures, firms have to face a trade off between paying high spreads because of opaqueness and signaling costs on the one hand, and sacrificing potential competitive advantages when maintaining idle balances on the other.[9] As established in *Equation* 10, the optimal cash-to-risk ratio k^* must be determined at the point where the *opportunity* cost of capital, equal to the reduction of the excess of return $T_X(k^*) = -dE\left[(X - k^*)_+\right]/dk$, just offsets its marginal benefit, equal to the reduction in the cost of bankruptcy plus the risk-free interest rate, i.e. $T_{-X}(k^*) + r_0 = -dE\left[(X + k^*)_-\right]/dk + r_0$.

Another kind of liquidity restrictions arises due to the fact that the *opportunity* cost of capital, that is perceived by managers as the reduction in the price of equity induced when certain level of capital is maintained, is not necessarily equal to the return r they have to pay to borrow in the market of cash balances (i.e. in general $T_X(k^*) \neq r$). Indeed, while the cost of equity reflects the *agency* costs between managers and stockholders, the market capital cost r is determined according to the capacity and willingness to pay of the borrower and it

[8] This is especially true in highly leveraged firms, where managers have strong incentives to take risk. See Jensen, 1986. The role of bankruptcy costs in the determination of the capital structure has been already mentioned by Stiglitz, 1972 (see also Stiglitz, 1988).

[9] In the words of Stephen Ross (1989), *firms and institutions are monitored and controlled through a complex set of implicit and explicit contractual relations*. See also Fama, 1980.

then reflects the moral-hazard in the relationship with customers. It depends, in particular, on the capital structure of the borrower institution, i.e. (as it is normally established in the corporate finance literature, see Williams et al. 2002) on the *leverage* ratio of the borrower institution, which is normally defined as K/D and also $D/I = (I-K)/I$, or equivalently, on its *cash-to-risk* ratio $k = K/I$.

As a matter of fact, to determine the price of loans, the creditors of opaque organisations rely on their own research and monitoring, as well as on the information published in the media and the risk categorisations provided by specialised (private and governmental) institutions. The credit ratings observed in practice normally include a finite number of categories. Within each class, every firm is supposed to face the same risk of default and hence every firm is allowed to borrow at the same interest rate (as in the MM-proposition), in such a way that the more the concerns of creditors about the credit capacity of firms in a certain class, the higher the level of the corresponding cost of capital and vice-versa. This means that lenders cannot discriminate *perfectly* and that borrowers can remain in the same class as long as they do not drastically modify their capital structures. In other words, as long as firms do not drastically vary their cash-to-risk ratios, they can regard the market capital cost as a constant.

In order to explicitly introduce the cost of capital r in the model, let us consider a firm that belongs to a certain class determined by the capital cost r and maintains the cash-to-risk ratio k. The cost of equity of this kind of firms is given by:

$$E[(X-k)_+] \;=\; E[X_+] - r\cdot k \quad\Leftrightarrow\quad \frac{E[X_+]-E[(X-k)_+]}{k} \;=\; r \tag{12}$$

Jensen (1986) has noticed that internal monitoring is more intense when positive balances are obtained at the end of the investment period and cash is at disposal in excess of what is required to fund every ongoing (solvent) investment project. In this case, it is said that a firm owns *free-cash-flow*. Accordingly, although the agency costs inside financial firms can be certainly reduced by diminishing the amount of equity, there is also a pressure to raise these costs since higher amounts of free-cash-flow are obtained in this case. In *Equation* 12, such effect is completely transferred to the moral-hazard effect, for reductions in the cash-to-risk ratio must be necessarily followed by increments in the market capital cost.

However, if (as previously suggested) borrowers are grouped in a finite number of categories, *Equation* 12 is likely to be violated and differences are expected between the internal and external estimations of the cost of capital:

$$\Delta \;=\; \frac{E\left[X_+\right]-E\left[(X-k)_+\right]}{k} - r \;\neq\; 0$$

Accordingly, capital is regarded as *too expensive* for those firms that obtain $\Delta < 0$, for in this case maintaining the cash-to-risk ratio k produces a loss that is lower than the alternative cost of borrowing the same balance in the market. These firms prefer to demand reserves instead of relying on external finance. Conversely, capital is *cheap* for those firms that obtain $\Delta > 0$, for they have to incur in a higher loss if they maintain some cash balance. These firms prefer to demand no capital at all.

The case $\Delta > 0$ actually represents the situation of firms that obtain gains over the level of capital and dispose of *free-cash-flow*. In the model of Jensen (1986), frictions and mismanagement are specially severe within firms disposing of *high* amounts of free-cash-flow, resulting from the competition, between managers and shareholders, to take control of the profits generated by the company.[10] The level of capital demanded by such firms can be described by means of the second-order condition. Indeed, given any cash-to-risk ratio k, we expect the probability $P\{X = k\}$ to be higher for firms that obtain higher amounts of free-cash-flow, and that eventually this probability exceeds $P\{X = -k\}$ when a certain capital threshold is surpassed. As we have already noticed, such turning point is found at $k = E[X]$ in the case of symmetric probability distributions. In any case, there will always be institutions with sufficiently high cash in excess for whom to maintain a single unit of capital induces a net loss in value. Those firms prefer to demand no capital at all and do not consequently internalise the cost of bankruptcy. Accordingly, not only agency costs are expected to be high inside firms with abundant free-cash-flow, also the incentives to provide capital cushions to guarantee their outstanding liabilities are non-existant in these firms, a situation that may aggravate the moral-hazard in the relationship between managers and customers.

In conclusion, short-term stickiness inherent in the equity contracts established with stockholders, as well as in the credit categorisations determined by lenders in the markets of cash balances, prevent financial institutions from *continuously* adjusting their capital structures. Thus, on the one hand, demanding additional equity from shareholders may well increase the agency costs inside the institution. But on the other hand, raising the amount of external debt may raise the bankruptcy costs faced by the institution and the moral-hazard inherent in their relationship with creditors. More explicitly, diminishing the cash-to-risk ratio *always* leads the bankruptcy costs term $E[(X + k)_-]$ to rise, and *sometimes*, when the magnitude of the variation in k is high enough to produce the firm to transit from one risk category to another, also the level of the market capital cost r may increase. Conversely, raising k *always* reduces the actuarial price of the bankruptcy claim and *sometimes* also reduces the cost of capital. An optimal *capital principle* will be derived in the next section, based on an optimal compromise of bankruptcy costs and the market price of external debt.

5 Economic Capital as the Optimal Deductible

The value, per unit of investment, of firms that can borrow at the interest rate r (whose cost of equity is given by *Equation* 12) is equal to:

$$\frac{V}{I} = E[X_+] - E[(X + k)_-] - (r - r_0) \cdot k$$

Maximising value is then equivalent to minimise the total burden of default, equal to the price of insurance (represented by the term $E\left[(X + k)_-\right]$) plus the net benefit to be obtained when investing capital at the interest rate r instead of maintaining it at the low (non-risky) rate r_0:

[10]Competition in product and factor markets should push utilities to a minimum level — eventually to zero. Then only those activities generating substantial economic rents are able to generate substantial amounts of free cash flow. Such activities are corresponded to product and factor markets where market forces are *weak* and where monitoring is more important than ever (Jensen, 1986).

$$\min_{k}\ E\left[(X+k)_{-}\right]+(r-r_0)\cdot k \tag{13}$$

This problem has been already used to derive a rule of capital allocation by Dhaene et al. (2003, 2008), Goovaerts et al. (2005) and also Laeven and Goovaerts (2004). They regard its solution as an optimal *solvency margin*, which establishes a compromise between the cost of capital on the one hand and the solvency requirements on the other. When justifying the implementation of this rule, they emphasise that arbitrage opportunities are difficult to exploit in insurance markets.

The first and second order conditions can be obtained by combining *Equations* 6 and 7:

$$\begin{aligned} \tfrac{d}{dk}E\left[(X+k^*)_{-}\right]+r-r_0 \;&=\; -T_{-X}(k^*)+r-r_0 \;=\; 0 \\ \tfrac{d^2}{\partial k^2}E\left[(X+k^*)_{-}\right] \;&=\; \tfrac{d}{dk}F_{-X}(k^*) \;=\; f_{-X}(k^*) \;=\; P\left\{X=-k^*\right\} \;>\; 0 \end{aligned} \tag{14}$$

Therefore, as long as some capital loss is produced with non-zero probability, a range exists where the term $E\left[(X+k)_{-}\right]$ is *convex* in k, and if additionally the marginal benefit of adding the first unit of capital is greater than the net investment premium (i.e. if additionally $T_{-X}(0)>r-r_0$), then a level of capital exists that minimises the criterion of *Equation* 13. Under such circumstances, the optimal capital demand is determined by the *quantile* function of the probability distribution of the series of capital losses of the underlying risk:

$$k^* \;=\; T_{-X}^{-1}(r-r_0) \;=\; F_{-X}^{-1}(1-r+r_0) \tag{15}$$

The optimal level of surplus is then expressed as the *Value-at-Risk* (or VaR) for the *confidence probability level* $\nu=r-r_0$.[11] The fact that the confidence level in the definition of VaR is replaced by a net premium in *Equation* 15 is a consequence of the first-order condition, which determines an exchange between a sure flow and a flow of probability (see *Equation* 14). Thus, the higher the liquidity premium ν (i.e. the more the free-cash-flow at disposal), the more expensive is to maintain a cash balance and hence the less capital is demanded. Conversely, the lower this premium, the cheaper the capital and hence the more the demanded quantity of this resource. The minimum and the maximum levels are respectively chosen when $\nu\geq 1$ and when $\nu\leq 0$.

From the actuarial viewpoint, the expected excess loss $E\left[(X+k)_{-}\right]$ represents the *fair* price of a special insuring contract (sometimes called *layer*) that obliges the insurer to pay to the policyholder the excess of the loss over the level k, when such a loss is produced (see Goovaerts, 1984). In this context, the amount k represents a guarantee provided by the policyholder in order to assure the insurer (up to some extent) that every reasonable care will be taken to reduce the underlying exposure. In other words, the guarantee k, which is known as the *deductible* or *retention* in the literature, is introduced in insurance contracts as a means of reducing the costs derived from *moral-hazard*. Within this framework, the optimal level of capital defined in *Equation* 13 corresponds to the *optimal deductible* or *optimal retention* of the related insuring liability contract.

[11] The reader not familiar with the concept of VaR can find a good survey in Hull (2000). The VaR has been recommended for the implementation of good risk management practices by the Basel Committee on Banking Supervision (2004).

Notice, however, that *full-coverage* is implicitly assumed in the model, because the actuarial prices of equity and insurance have been expressed in terms of mathematical expectations that consider *unlimited* losses and gains over the level of capital (see *Equation* 5). But insurance contracts always specify a maximum payment in practice and full-coverage does not actually exists in real markets. The question then arises of who does eventually bear the risk of deposits. According to the terms of the guaranteeing contracts previously defined, we can say that risk-bearing is roughly distributed in the following way: any loss up to the retention level k is payed by the firm (recall that shareholders only endure the *equity* component of the economic capital, $k^{EQU} \leq k$); losses that are higher than the retention level are payed by the *guarantor* or *insurer*, as long as these losses do not surpasses a maximum *disaster* level M^{DIS}; finally, some companies can look for additional protection by establishing a contract with some *reinsurance* institution that agrees to pay any loss greater than the disaster level M^{DIS} but lower than some *catastrophe* level M^{CAT}. Thus, in the case of *catastrophic* events, it is the society as a whole who has to afford the losses — through governmental divisions, private creditors, companies and householders. This explains why it is in the interest of regulators to define *good practices* and regulatory requirements that can induce financial intermediaries to seek for protection according to the risk borne.

From the economic point of view, the existence of an *optimal* level of capital implies that choosing a different level necessarily leads to over- or under-investment. Indeed, *idle* money, that could be assigned to profitable investments, is maintained in excess when more capital than the optimal is demanded. By contrast, when the stock of capital is lower than the optimal level, risk is taken in excess, a fact that might eventually increase the frequency of losses (as well as disaster and catastrophic events) and induce investors to raise their concerns about the credit quality of the firm. The price at which the firm can attract debt in the market might increase under such circumstances. Therefore, independently of whether managers consider or not any of the optimisation problems established in *Equations* 5 or 13, their capital preferences should approach to the solutions of these problems, for only following this strategy the market value of firms can be maximised (or the burden of bankruptcy can be minimised). On these grounds, the capital principles defined in *Equations* 10 and 15 can be regarded as *decision* variables, which provide a basis for the determination of the aggregate behaviour of markets or multidivisional corporations. Thus, in particular, such rules will be used later in *Sections* 8 and 9 to determine allocation mechanisms to be applied inside centralised and decentralised organisations.

Having defined a principle to determine the optimal cash balance, the problem of distributing it inside multidivisional organisations will be addressed in the rest of the paper. As explained later in *Section* 6, the main issues to be considered are informational asymmetries (producing agency costs) inside institutions and how the decentralisation of capital decisions can be used in the interest of the conglomerate as a whole. Specifically, an optimal centralised allocation principle (depending on the beliefs of the central administration about the risks taken by subsidiaries) will be defined in *Section* 8, and an optimal decentralised mechanism will be proposed in *Section* 9, which leads to the centralised allocation at the time that forces divisional managers to *reveal their informational type*. But then some mathematical framework should be adopted allowing the *risk* or *insurance* principles $E\left[(X-k)_+\right]$ and $E\left[(X+k)_-\right]$ to be distorted according to the preferences of decision-makers. As will be shown next in *Section* 7, the *distorted-probability* principle (which modify the probabilities in the expectation operator) provides a characterisation that leads to an explicit separation of the effects of risk and preferences in the capital structure. Such separation will allow for a clear representation of the informational asymmetries between central and divisional managers, as well as the costs derived from decentralisation.

6 The Allocation of Capital within Multidivisional Corporations

As established in the previous sections, the prices of securities that produce random outcomes are completely determined by risk. Then the insurance price of portfolios containing such instruments are also dependent on risk and hence, every capital *allocation principle* — distributing capital among the outstanding divisions of some multidivisional corporation — should be strictly based on it (see, among others, Albrecht, 2004, and Goovaerts et al., 2005). On these grounds, we will respectively denote as $X_1, \ldots, X_n$ and $K[X_1], \ldots, K[X_n]$ the series of random $P\&L$ produced by divisions and the *stand-alone* capital requirements at the divisional level, where the function $K[\cdot]$ is corresponded to some economic criterion. In particular, the amount of capital demanded at the corporate level is given by $K[X]$, where (as before) the random variable $X = \sum_{i=1}^{n} X_i$ denotes the aggregated $P\&L$.

It is a well-known theoretical fact that when the variations of the portfolios held by subsidiaries are not *perfectly correlated*, the losses suffered by some divisions can be at least partially compensated by the gains obtained in others — i.e. divisions can partially *hedge* or *insure* each other. It is then claimed that a benefit arises due to *diversification*, which implies that the capital required at the aggregate level is generally lower than the sum of the levels maintained by divisions when acting as independent units. In fact, provided that the operator $K[\cdot]$ is *convex*, the *Jensen's inequality* can be used to prove that (see Merton and Perold, 1993):

$$K[X] \;\leq\; \sum_{i=1}^{n} K[X_i] \tag{16}$$

Thus, in particular, some businesses that would be unprofitable on a stand-alone basis (due to high capital requirements) might be profitable within a firm holding businesses with offsetting risks. Under such conditions, the decentralisation of capital decisions might produce *under-investment* and hence *full-allocation* is frequently mentioned as a desirable property:

$$\sum_{i=1}^{n} K_i = K[X] \quad , \quad with \;\; K_i \geq 0 \;\; \forall\, i = 1, \ldots, n \tag{17}$$

where $K_1, \ldots, K_n$ denote the levels of capital determined according to some *centralised* criterion. In this context, the *covariance* allocation principle is introduced (see Albrecht, 2004):

$$K_i = E\left[X_i\right] + \frac{Cov\left(X, X_i\right)}{Var[X]} \cdot \left(K - E[X]\right) \quad with \quad X = \sum_{i=1}^{n} X_i$$

Full-allocation is obtained as a consequence of the following properties of the expectation and covariance operators:

$$E[X] = \sum_{i=1}^{n} E[X_i] \quad and \quad Var[X] = \sum_{i=1}^{n} Cov\,(X, X_i)$$

Then the covariance principle satisfies the three properties just mentioned: it is *risk-based*, it explicitly introduces covariances and finally, it fully allocates capital.

However, Merton and Perold (1993) have pointed out that full-allocation can also lead to under-investment under certain circumstances and consequently, that such strategy does not always correspond to the best practice. This claim can be understood by considering a specific approach of allocating capital. Indeed, let us define the *incremental* capital requirement of some division as the amount of capital required at the corporate level $K[X]$ minus the capital required by the conglomerate when the particular business unit is eliminated:

$$I_K[X_i] = K[X] - K\,[X - X_i] \quad with \quad X = \sum_{i=1}^{n} X_i$$

Thus, the incremental risk capital represents the *minimum* cash balance required to support a certain business unit. It can be easily verified that for every $n \geq 2$ the following identity holds:

$$(n-1) \cdot X \;=\; \sum_{i=1}^{n} \sum_{j \neq i} X_j \;=\; \sum_{i=1}^{n} (X - X_i)$$

We can apply again the Jensen's inequality (see the technical appendix in Merton and Perold, 1993) to obtain that:

$$(n-1) \cdot K[X] \;\leq\; \sum_{i=1}^{n} K[X - X_i] \quad \Longrightarrow \quad \sum_{i=1}^{n} I_K[X_i] \;\leq\; K[X] \tag{18}$$

Hence the sum of the incremental capitals is lower than the capital required at the aggregate level and then full-allocation overstates the marginal capital requirements. In other words, while the stand-alone allocation induces a loss in efficiency to the conglomerate (whenever a *convex* principle $K[\cdot]$ is introduced), full-allocation can induce divisions to incur in a loss of efficiency, for the sum of the minimum additional (or incremental) capital requirements is lower than the level determined (according to the same principle) to the whole conglomerate. On these grounds, Merton and Perold recommend do not fully allocate capital.[12]

Therefore, the main issue when specifying a capital allocation principle is how to simultaneously incorporate the preferences of central and divisional administrations. Discrepancies inside institutions can be explained on the grounds of differing attitudes towards risk, as

[12]The principle can be naturally extended by considering the *marginal-capital* requirement, defined by Myers and Read (2001) as the variation in the capital requirement in response to a marginal increment in the exposition to risk, i.e. $M_K[X_i] = dK[X]/dX_i$. But then full-allocation can be guaranteed only if some conditions on the valuation function $K[X]$ are satisfied (see also Mildenhall, 2004, and Grundl and Schmeiser, 2007).

well as on differing *expectations* sustained by information and knowledge. Under such conditions, the distribution of capital reflects the *competition* between divisional and central managers on the one hand, and stockholders on the other, to take control of the funds generated by the company. Denault (2001) has shown that the theory of *coalitional games* can be used to build a model that explicitly characterises such competition. In the model, subsidiaries choose between working as stand-alone entities and forming holdings (or *coalitions*) with other divisions. In this setting, the *sub-additivity* property (*Equation* 16) implies that subsidiaries have incentives to form coalitions, since in this way they obtain a benefit due to the reduction of the aggregate cost of bankruptcy. However, at the same time, the *super-additivity* property (which in particular satisfies the incremental allocation principle, see *Equation* 18) implies that the subsidiaries forming a certain coalition have incentives to abandon it, as long as in this case the aggregate capital requirement is higher than the sum of the optimal surpluses determined on a stand-alone basis. Within this theoretical framework, Denault demonstrates that any *coherent* allocation principle should actually satisfy full-allocation.

As will be shown later in *Sections* 8 and 9, the capital principle of *Equation* 22 can be naturally adapted to characterise the divisional capital requirements in multi-businesses corporations. Thus, an optimal centralised allocation principle is defined in *Section* 8, in the sense that it minimises the sum of the insurance prices of the divisional excess losses. Later in *Section* 9, an optimal mechanism is described that is based on the stand-alone optimal levels of reserves, but that leads to the same aggregate requirements as in the centralised allocation. The instrument of this mechanism is the *internal* cost of capital. We will first digress from this topic in the next section, in order to provide a mathematical description of the effect of *expectations* over actuarial prices, on which basis the allocation principles of *Sections* 8 and 9 will be built.

7 The Distorted Demand for Capital

Within the framework of the well-known *utility-theory of choice* under risk, *preferences* are uniquely corresponded to *utility functions* and economic decisions are based on the expected utility:

$$E_u[X] = \int u(x) \cdot dF_X(x)$$

In fact, for any utility function $u(\cdot)$, the expected utility operator $E_u[\cdot]$ always provides an *order* of risks. On these grounds, the process of decision-making can be described. Although this framework has been already used for obtaining a demand for capital (or more generally, a demand for liquidity, see Tobin, 1958, Holmstrom and Tirole, 2000, and Choi and Oh, 2003), the formulas obtained in this way are not really tractable and the description in terms of preferences is dependent on the specification of the utility function. Besides, we prefer extensions of the capital principle defined in *Equations* 13 and 15 that maintain its principal features; in this context, we prefer to distort probabilities instead of payments, as long as in this way the fundamental prescription given in *Equation* 15, relating the optimal level of capital to the quantile of the underlying exposure, is preserved. In other words, we would like the distorted capital to be equal to the quantile of a *distorted probability* distribution. According to this view, preferences directly affect the perceptions of risk maintained by decision-makers.

Yaari (1987) has demonstrated that preferences can be alternatively corresponded to a *distortion* function $\varphi : [0,1] \rightarrow [0,1]$ affecting the underlying probability distributions. On this basis, the distorted expectation operator is defined:

$$\mathrm{E}_\varphi[X] = \int x \; dF_{\varphi,X}(x) = \int T_{\varphi,X}(x) \; dx \quad \text{with} \;\; T_{\varphi,X}(x) = \varphi\left(T_X(x)\right) \;\; \forall x \tag{19}$$

where $F_{\varphi,X} = P_\varphi\{X \leq x\}$ and $T_{\varphi,X} = P_\varphi\{X > x\}$ respectively denote the *distorted cumulative* and the *distorted tail* probability functions representing the underlying risk. Wang et al. (1997) have proved that the risk-principle introduced in *Equation* 19 satisfy a set of good properties for the pricing of insurance claims. In particular, it preserves a class of *stochastic* orders, meaning that it strictly depends on risk — i.e. more risky claims are given higher prices and vice-versa. Besides, *risk-lover* and *averse-to-risk* attitudes are respectively characterised by convex and concave distortion functions, for in these cases the expectation operator is respectively under and over-estimated. *Risk-neutrality* is represented by the identity function $\varphi(x) = x \; \forall x$ (see also Wang and Young, 1998).

Although the family of *acceptable* distortions (in the terms established by Yaari and Wang) is broad, we will consider in the following the particular class of *proportional-hazards* transforms, defined as $\varphi_\theta(p) = p^{\frac{1}{\theta}} \quad \forall p \in [0,1]$, in such a way that the corresponding risk-principle is now expressed as (see Wang, 1995):

$$\mathrm{E}_\theta[X] = \int x \; dF_{\theta,X}(x) = \int T_{\theta,X}(x) \; dx \quad \text{with} \;\; T_{\theta,X}(x) = T_X(x)^{\frac{1}{\theta}} \;\; \forall x \tag{20}$$

Consequently, the probability beliefs and hence the price of risk are amplified when $\theta > 1$ and then this range of the distortion parameter θ accounts for the behaviour of *averse-to-risk* individuals. Besides, the higher the magnitude of θ, the more the price of risk is over-estimated and then the more the aversion-to-risk. Similarly, the price of risk is under-estimated when $\theta < 1$, so that in this way the behaviour of *risk-lovers* is characterised. Thus, the lower the magnitude of θ, the lower the price of risk and then the more the preference for risk. *Risk-neutral* decision-makers are characterised by $\theta = 1$, in which case the distorted probability principle is equal to the traditional expectation operator.

The capital principle defined in *Equations* 13 and 15 can be naturally extended to the risk pricing framework based on the distorted-probability principle defined in *Equation* 20:

$$\min_k \;\; E_\theta\left[(X+k)_-\right] + (r - r_0) \cdot k \tag{21}$$

with $E_\theta\left[(X+k)_-\right] = -\int_{-\infty}^{-k}(x+k) \cdot dF_{\theta,X}(x)$ (see *Equation* 6). The Leibnitz's integral rule can be invoked once again to derive the first-order condition, as in the non-distorted case (see *Equation* 14), in such a way that the *optimal* demand for capital is given by the inverse function of the distorted tail-probability:

$$k_{\theta,X}(r - r_0) \;=\; T^{-1}_{\theta,-X}(r - r_0) \;=\; F^{-1}_{\theta,-X}(1 - r + r_0) \tag{22}$$

or equivalently:

$$k_{\theta,X}(r-r_0) \;=\; T_{-X}^{-1}\left((r-r_0)^{\theta}\right) \;=\; F_{-X}^{-1}(1-(r-r_0)^{\theta})$$

Discrepancies relative to preferred cash-balances can thus be explained on the basis of the underlying risks, preferences and the opportunity cost of capital. Notice that although in *Equation* 22 the parameter θ eventually affects the cost of capital, from the mathematical point of view it is not this variable but rather the underlying probability distribution what is distorted, a fact that has been stressed by choosing the notation $F_{\theta,X}$ and $T_{\theta,X}$. The reader should not confuse this setting, where expectations distort the *risk perceptions* of decision-makers, with that of heterogeneous estimations of the cost of capital.

8 An Optimal Risk-Based Allocation Rule for Centralised Organisations

Let X and K respectively denote the aggregate exposure and the level of capital maintained by some financial conglomerate, and let $X_1,\ldots,X_n$ and $K_1,\ldots,K_n$ respectively denote the series of capital returns of the portfolios held by its subsidiaries and their corresponding cash balances. Consistently with the capital principle defined in *Equations* 21 and 22, we look for a rule based on the residual risks of subsidiaries. A criterion is thus required to compare the sum of the subsidiaries' residual risks, equal to $\sum_{i=1}^{n}(I_i\cdot X_i+K_i)_-$, with the conglomerate's residual exposure $(I\cdot X+K)_-$, where $I_1,\ldots,I_n$ and $I=\sum_{i=1}^{n} I_i$ respectively denote the levels of investment maintained by subsidiaries and the conglomerate, with $I\cdot X=\sum_{i=1}^{n} I_i\cdot X_i$.

The comparison of random variables can be carried out with the help of the concept of *stochastic precedence*. Indeed, a random variable X is said to *precede* another random variable Y (or equivalently, a random variable Y is said to *dominate* X) in the *first stochastic order* if and only if at every point it accumulates less probability in the *tails* of the distribution function (see Goovaerts et al., 1984):

$$X\leq Y \;\Leftrightarrow\; P\left\{X>k\right\}\leq P\left\{Y>k\right\}\;\forall k \;\Leftrightarrow\; T_X(k)\leq T_Y(k)\;\forall k \tag{23}$$

This is a convenient order indeed, for the optimal level of capital is expressed in *Equations* 15 and 22 in terms of the tail probability. Moreover, it can be demonstrated that the following inequality holds with probability one for the first stochastic-order:

$$(I\cdot X+K)_- \;\leq\; \sum_{i=1}^{n}(I_i\cdot X_i+K_i)_- \quad \textit{where} \quad \sum_{i=1}^{n} K_i\leq K$$

Recall that, as stated by Denault, the property of *super-additivity* (which states that $\sum_{i=1}^{n} K_i\leq K$) implies that there are incentives for divisions to leave the conglomerate and to form independent firms. From the inequality above we obtain that a benefit exists, resulting from the reduction in the riskiness of the conglomerate's portfolio with respect to the sum of the divisional portfolios, that can be shared with divisions to put incentives on them to remain in the conglomerate.

It can be proved as well that the first stochastic order is preserved by the distorted-probability principle (defined in *Equations* 19 and 20), in such a way that, for any informational type θ:

$$\begin{array}{cc} & E_\theta\left[(I \cdot X + K)_-\right] \;\leq\; \sum_{i=1}^n E_\theta\left[(I_i \cdot X_i + K_i)_-\right] \\ where & \\ & \sum_{i=1}^n K_i \leq K \end{array}$$

Given some level of aggregate investment $I > 0$, define the following *weights*:

$$\omega_i = \frac{I_i}{I} \;\; \forall i \quad \Longrightarrow \quad \sum_{i=1}^n \omega_i = 1$$

Hence, letting $k = K/I$ and $k_i = K_i/I_i$, with $i = 1, \ldots, n$, respectively denote the cash-to-risk ratios maintained by the conglomerate and subsidiaries, we obtain that:

$$\begin{array}{cc} & E_\theta\left[(X + k)_-\right] \;\leq\; \sum_{i=1}^n \omega_i \cdot E_\theta\left[(X_i + k_i)_-\right] \\ where & \\ & \sum_{i=1}^n \omega_i \cdot k_i \leq k \end{array}$$

An allocation principle can then be naturally introduced, which minimises the weighted sum of the expectations of the residual risks:

$$\begin{array}{cc} & \min_{k_1,\ldots,k_n} \;\sum_{i=1}^n \omega_i \cdot E_\theta\left[(X_i + k_i)_-\right] \\ such\ that & \\ & \sum_{i=1}^n \omega_i \cdot k_i = k \end{array} \tag{24}$$

Such *allocation principle* ensures the sum of the expected insured returns of subsidiaries is as close as possible to the lower bound $E_\theta\left[(X + k)_-\right]$.

As in the previous sections, the solution to the problem of *Equation* 24 can be obtained by applying Lagrange optimisation. In fact, the first-order conditions are now written as:

$$\begin{array}{c} \frac{\partial}{\partial k_i} \sum_{i=1}^n \omega_i \cdot E_\theta\left[(X_i + k_i^*)_-\right] + \lambda \cdot \omega_i = \left(-T_{\theta,-X_i}\left(k_i^*\right) + \lambda\right) \cdot \omega_i \;=\; 0 \quad \forall\, i = 1, \ldots, n \\ \sum_{i=1}^n \omega_i \cdot k_i^* - k \;=\; 0 \end{array} \tag{25}$$

Combining both equations leads the Lagrange multiplier λ to be determined by:

$$\sum_{i=1}^n \omega_i \cdot k_i^* \;=\; \sum_{i=1}^n \omega_i \cdot T_{\theta,-X_i}^{-1}(\lambda) \;=\; k$$

Besides, since $T_{\theta,-\omega_i \cdot X_i}^{-1} = \omega_i \cdot T_{\theta,-X_i}^{-1} \;\forall\, \omega_i > 0$, we obtain:

$$\sum_{i=1}^n T_{\theta,-\omega_i \cdot X_i}^{-1}(\lambda) \;=\; k \quad with \quad X = \sum_{i=1}^n \omega_i \cdot X_i$$

The question then arises of whether there is some dependence structure for which the sum of the quantiles is expressed as the quantile of the sum of the individual exposures, for in this case it would be possible to obtain a close expression for the multiplier λ.

In fact, as demonstrated by Dhaene et al. (2002), the property of the sum of the quantiles mathematically characterises the *comonotonic* dependence structure, where *comonotonicity* represents a case of extreme dependence, when no diversification effect can be attained by pooling risks together. In fact, given a random vector $(X_1, \ldots, X_n)$ with marginal cumulative distribution functions $(F_{X_1}, \ldots, F_{X_n})$, the *comonotonic* random vector $(X_1^c, \ldots, X_n^c)$ is mathematically defined in such a way that if U denotes the random variable *uniformly* distributed in the interval $[0,1]$, such that $F_U(u) = u \;\forall\; u \in [0,1]$, $F_U(u) = 0 \;\forall\; u < 0$ and $F_U(u) = 1 \;\forall\; u > 1$, the following identity holds in distributions:

$$(X_1^c, \ldots, X_n^c) = \left(F_{X_1}^{-1}(U), \ldots, F_{X_n}^{-1}(U)\right)$$

In this way, the realisation of a single event (related to the uniform random variable U) simultaneously determines all the components of any *comonotonic* random vector. Moreover, since the functions $\left(F_{X_1}^{-1}, \ldots, F_{X_n}^{-1}\right)$ are all *non-decreasing*, all the components of the vector $(X_1^c, \ldots, X_n^c)$ *move in the same direction*. Hence, as already stated, the quantile function of the sum of the components of any comonotonic random vector is equal to the sum of the component quantile functions, a fact that supports the use of the *comonotonic dependence structure* to characterise the aggregate demand for cash balances.

Let T_{θ,X^c} denote the tail-probability of the *comonotonic* sum $X^c = \omega_1 \cdot X_1^c + \ldots + \omega_n \cdot X_n^c$, where $(X_1^c, \ldots, X_n^c)$ represents the *comonotonic* random vector with the same *marginal* distributions as $(X_1, \ldots, X_n)$. Then, from the first-order conditions written in *Equation* 25 we obtain that:

$$T_{\theta,-X^c}^{-1}(\lambda) \;=\; \sum_{i=1}^{n} T_{\theta,-\omega_i \cdot X_i^c}^{-1}(\lambda) \;=\; \sum_{i=1}^{n} T_{\theta,-\omega_i \cdot X_i}^{-1}(\lambda) \;=\; k \quad with \quad X^c = \sum_{i=1}^{n} \omega_i \cdot X_i^c$$

where $T_{\theta,-X^c} = \left(\sum_{i=1}^{n} T_{\theta,-\omega_i \cdot X_i}^{-1}\right)^{-1}$ denotes the distribution function of the comonotonic sum. Hence the Lagrange multiplier is equal to:

$$\lambda \;=\; T_{\theta,-X^c}(k) \tag{26}$$

while the optimal allocation rule is given by:

$$k_i^* \;=\; T_{\theta,-X_i}^{-1}(\lambda) \;=\; T_{\theta,-X_i}^{-1}\left(T_{\theta,-X^c}(k)\right) \quad \forall\; i = 1, \ldots, n \tag{27}$$

Thus, from the mathematical point of view, the optimal levels of capital are corresponded to the *projections* over the subspaces determined by the individual risks. This interpretation is consistent with the fact that the problem of *Equation* 24 actually minimises a distance, as stated by Goovaerts et al. (2004) and also Laeven and Goovaerts (2004).

The optimal allocation determined in *Equation* 27 depends on the level of capital k chosen for the conglomerate. This capital is supposed to depend on the market conditions, i.e. on the market price of capital. When central managers choose the level that minimises the criterion of *Equation* 21, the optimal level of capital for the conglomerate depends on the net return $r - r_0$ as in *Equation* 22:

$$k_{\theta,X^c} \;=\; T_{\theta,-X^c}^{-1}(r - r_0) \tag{28}$$

Replacing the optimal level k^* in *Equation* 26 leads the Lagrange multiplier to be given by:

$$\lambda \;=\; T_{\theta,-X^c}\left(T^{-1}_{\theta,-X^c}(r-r_0)\right) \;=\; r-r_0 \tag{29}$$

Thus the Lagrange multiplier is equal to the net return on capital — a fact that is consistent with the interpretation of the Lagrange multiplier as a *shadow price*. Replacing λ in *Equation* 27 leads to the optimal allocation rule:

$$k_{\theta,X_i} \;=\; T^{-1}_{\theta,-X_i}(r-r_0) \quad \forall\, i=1,\ldots,n \tag{30}$$

Accordingly, the optimal levels of capital correspond to the levels that central managers would choose for the divisional risks on the grounds of their net return on capital $r-r_0$ and their expectations (represented by the informational type θ). On these grounds, it is regarded as a *centralised* allocation.

9 Capital Allocation as an Optimal Decentralised Mechanism

Subsidiaries that are allowed to choose their capital structures on their own interest solve the problem of *Equation* 21:

$$\min_{k_i}\; E_{\theta_i}\left[(X_i+k_i)_-\right] + (r-r_0)\cdot k_i \tag{31}$$

The *stand-alone* allocation of capital is thus determined according to *Equation* 22:

$$k_{\theta_i,X_i} \;=\; T^{-1}_{\theta_i,-X_i}(r-r_0) \quad \forall\, i=1,\ldots,n \tag{32}$$

Comparing *Equations* 30 and 32, we notice that the centralised and stand-alone allocations differ due to differing expectations between central and divisional managers, i.e. due to differences in the parameters θ and θ_i. A loss of efficiency is then produced at the corporate level. Indeed, *under-investment* occurs at the corporate level when $\sum_{i=1}^n k_{\theta_i,X_i} > k_{\theta,X^c}$ and *over-investment* when $\sum_{i=1}^n k_{\theta_i,X_i} < k_{\theta,X^c}$. Similarly, preferring the centralised allocation respectively leads to under- and over-investment at the divisional level when $k_{\theta,X_i} > k_{\theta_i,X_i}$ and when $k_{\theta,X_i} > k_{\theta_i,X_i}$.

Therefore, deciding between the centralised and the stand-alone allocations of capital is just a matter of *preferences*. Those central administrations that are confident in their own estimations will prefer to rely on the centralised allocation. Others will try to incorporate, at least partially, the view of divisional managers. The principle of *Equation* 32 explicitly serves this purpose. Indeed, let $k_1,\ldots,k_n$ denote the surpluses prefer by divisional managers, in such a way that, from *Equation* 32 we obtain:

$$r-r_0 = T_{\theta_i,-Xi}(k_i) = T_{-Xi}(k_i)^{\frac{1}{\theta_i}} \quad\Longrightarrow\quad \theta_i = \frac{\log T_{-Xi}(k_i)}{\log\,(r-r_0)} \quad \forall\, i=1,\ldots,n \tag{33}$$

Hence, given any risk X_i and a level of capital k_i, the liquidity premium $r-r_0$ and the informational type θ are related to each other. Accordingly, as long as the central administration force subsidiaries to invest their balances at some internal return ρ (instead of the risk-free interest rate r_0), they are force to act on the interest of the conglomerate at the time that their types are *revealed*.

Such a mechanism can be explicitly defined in the form of an *optimal contract*, as suggested by Diamond and Verrecchia (1982):

$$\min_{k,\rho} \; E_\theta\left[(X+k)_-\right] + (r-r_0)\cdot k$$

subject to

$$k_i = \mathbf{arg}\min_{k_i} \; E_{\theta_i}\left[(X_i+k_i)_-\right] + (r-\rho)\cdot k_i \quad \forall\; i=1,\ldots,n$$

$$\textstyle\sum_{i=1}^n \omega_i \cdot k_i = k$$

As long as the optimal stand-alone allocations are given by *Equation* 32, the contract can be equivalently established in the following way:

$$\begin{array}{c} \min_{k,\rho} \; E_\theta\left[(X+k)_-\right] + (r-r_0)\cdot k \\ \textit{subject to} \\ \sum_{i=1}^n \omega_i \cdot k_{\theta_i,X_i}(r-\rho) = k \end{array} \tag{34}$$

Letting λ denote the Lagrange multiplier, we obtain that the optimal level of capital at the corporate level k^* and the optimal internal return on capital ρ^* are given by the first-order conditions:

$$\tfrac{d}{dk}E_\theta\left[(X+k^*)_-\right] + (r-r_0) - \lambda = -T_{\theta,X}\left(k^*\right) + (r-r_0) - \lambda = 0$$

$$\lambda\cdot\textstyle\sum_{i=1}^n \omega_i \cdot k'_{\theta_i,X_i}(r-\rho^*)\cdot(-1) = 0 \implies \lambda = 0$$

$$\textstyle\sum_{i=1}^n \omega_i \cdot k_{\theta_i,X_i}(r-\rho^*) = \sum_{i=1}^n \omega_i \cdot T^{-1}_{\theta_i,-X_i}(r-\rho^*) = \sum_{i=1}^n T^{-1}_{\theta_i,-\omega_i\cdot X_i}(r-\rho^*) = k^*$$

Therefore, if $T_{\theta_1,\ldots,\theta_n,X^c}$ denotes the tail-probability function of the comonotonic sum with marginal distributions $(T_{\theta_n,-\omega_1\cdot X_1},\ldots,T_{\theta_n,-\omega_n\cdot X_n})$, i.e. $T_{\theta_1,\ldots,\theta_n,X^c} = \left(\sum_{i=1}^n T^{-1}_{\theta_i,-\omega_i\cdot X_i}\right)^{-1}$, we obtain:

$$\rho^* = r - T_{\theta_1,\ldots,\theta_n,X^c}\left(k^*\right) = r - T_{\theta_1,\ldots,\theta_n,X^c}\left(T^{-1}_{\theta,-X^c}(r-r_0)\right)$$

with:

$$k^* = T^{-1}_{\theta,-X^c}(r-r_0) \tag{35}$$

But the parameters $\theta_1,\ldots,\theta_n$ are actually not observed by central managers, which means that this allocation cannot be implemented in practice. As a solution, we can assume that first central managers have estimations $\hat\theta_1,\ldots,\hat\theta_n$ of the types of divisional managers — which can be determined on the basis of experience or previous allocations. Then the optimal internal return on capital should be defined as:

$$\rho^* = r - T_{\hat\theta_1,\ldots,\hat\theta_n,X^c}\left(T^{-1}_{\theta,-X^c}(r-r_0)\right) \tag{36}$$

Replacing $r_0 = \rho^*$ in *Equation* 32, we obtain that the *decentralised* optimal allocation of capital is attained at the levels:

$$k_i^* \;=\; T^{-1}_{\theta_i,-Xi}(r-\rho^*) \;=\; T^{-1}_{\theta_i,-Xi}\left(T_{\hat{\theta}_1,\ldots,\hat{\theta}_n,X^c}\left(T^{-1}_{\theta,-X^c}(r-r_0)\right)\right) \quad \forall\, i=1,\ldots,n \tag{37}$$

Notice that the optimal internal cost of capital is equal to the external cost of capital only when central managers access to the private information of subsidiaries (and as long as they agree to incorporating it into decision-making), at the time that the proper comonotonic portfolio characterises the aggregated exposure. Then the liquidity premium, equal to the difference between the external and the optimal internal costs of capital, explicitly measures the disagreement between central and divisional managers.

The optimal mechanism can then be implemented in the following way. In *Stage* 1, based on their experience and information, central managers determine estimations of the informational types of subsidiaries, $\hat{\theta}_1,\ldots,\hat{\theta}_n$. In case that only poor information is available, they can assume that divisional managers are risk-neutrals and accordingly choose $\hat{\theta}_1 = \cdots = \hat{\theta}_n = 1$. In *Stage* 2, divisions are asked to freely decide their levels of reserves. The conditions of the contract are such that although subsidiaries can invest in the market to obtain the return r, they can only invest their cash reserves in a special agreement with the central administration to obtain the return ρ^*. In the final *Stage* 3, central managers use the allocations of *Stage* 2 to renew their estimations of the types of subsidiaries from *Equation* 33. In this way, divisional managers *reveal their types*.

In the model proposed by Stoughton and Zechner (2007), divisional managers are characterised by their attitude-towards-risk and the effort they expend to acquire information. Divisional cash flows are increased when more information is accessed. An informational parameter is then supposed to augment the expected value of the investment in a multiplicative fashion. Risk is described by *volatility* and decisions are affected by the *net* cost of capital. As in the model of Merton and Perold (1993), the economic capital is proportional to volatility. The *internal* cost of capital (fixed by central managers) is used as a tool to give incentives to subsidiaries to act in the general interest of the firm. Thus, capital allocation is justified as part of a general mechanism that stimulates the exchange of information between central and divisional managers inside the institution. As a result, sometimes subsidiaries voluntarily reduce their exposure to risk and assign their cash excesses to less productive investments. So distortions can be present in the model in the form of under- and over-investment.

The interpretation of Stoughton and Zechner naturally applies to the mechanism stated above. Besides, the alternative expressions provided for the *external* and *internal* costs of capital can be easily implemented and are naturally corresponded to economic concepts.[13] Finally, as demonstrated in the following section, in the particular case when risks are described by the Gaussian probability distribution, the optimal capital principle of *Equation* 22 extends the capital rule proposed by Merton and Perold (1993). This result is consistent with the fact that this principle has been established as an extension of the model of Merton and Perold, applicable to the case when decision makers cannot hedge fully and are obliged to rely on insurance and reinsurance to carry on with their business activities.

[13]Thus, while the external capital cost is related to the actuarial price of excess-cash-flow, the internal cost of capital is related to an optimal agreement between central and divisional managers.

Table 1: The Optimal Capital Principle under Different Risk Parametrisations

	Tail Probability	**Optimal Capital Principle**
Gaussian	$T(x) = 1 - \Phi\left(\frac{\mu+x}{\sigma}\right) \quad \forall\ x$ with $\quad \Phi(x) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{x} exp\left(-\frac{y^2}{2}\right) dy \quad \forall x$	$k(\nu) = \sigma \cdot \Phi^{-1}\left(1 - \nu^{\theta}\right) - \mu$ with $\quad \nu = r - r_0$
Log-Normal	$T(x) = 1 - \Phi\left(\frac{\mu+ln(x)}{\sigma}\right) \quad \forall\ x$	$k(\nu) = exp\left(\sigma \cdot \Phi^{-1}\left(1 - \nu^{\theta}\right) - \mu\right)$
Exponential	$T(x) = exp\left(\frac{C-x}{\beta}\right) \quad \forall\ x \geq 0$	$k(\nu) = C - (\theta \cdot \beta) \cdot ln(\nu)$
Paretian	$T(x) = \left(\frac{x}{B}\right)^{\frac{1}{\alpha}} \quad \forall\ 0 < x < B$	$k(\nu) = B \cdot \nu^{\theta \cdot \alpha}$

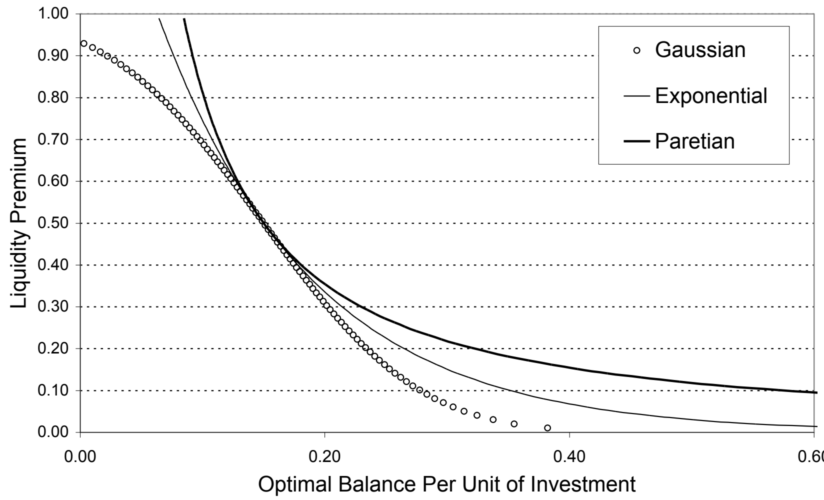
10 The Optimal Capital Principle for Some Well-Known Probability Distributions

An appealing feature of the capital principle defined in this paper is its adaptability to any family of probability distributions. As a consequence, the capital requirements of different *types* of risks can be described on the same basis and hence, the model can be also implemented in institutions that hold securities exposed to non-homogeneous risks. This is particularly the case of insurance companies, that simultaneously deal with highly standardised policies, such as car and fire insurance, as well as some individual contracts involving high payments depending on events of low probability. This is also the case of some financial conglomerates that hold standard financial securities (transacted in highly liquid markets), as well as non-liquid derivatives and claims contingent on disaster and catastrophic events.

Explicit analytic expressions are obtained for a wide class of well-known probability distributions. In $Table$ 1, some of these expressions are presented in terms of the liquidity premium $\nu = r - r_0$. In $Figure$ 1, the optimal capital requirements are depicted for some of the risk classes presented in $Table$ 1. Notice that, given any level of the liquidity premium, the optimal cash balance under Paretian risks is always higher than the cash balance demanded under Exponential risks, which in turn is always higher than the optimal capital under Gaussian risks. In fact, Paretian tails are uniformly greater than Exponential, which in turn are greater than Gaussian tails. This means that, according to the first-stochastic order, Gaussian risks are *dominated* by Exponential risks, which in turn are *dominated* by Paretian risks (see $Equation$ 23 and the related discussion in $Section$ 7). Therefore, the optimal capital principle consistently assigns higher surpluses to riskier claims and hence, it is *strictly risk-based*.

As depicted in $Figure$ 2, the Gaussian optimal principle (obtained when the underlying

Figure 1: The Optimal Capital Principle of Neutral Decision Makers



exposure follows a Gaussian probability distribution) follows a straight line in the plane of cash-to-risk ratios and standard deviations (used as estimators of volatilities). This is also the case of the allocation rule of Merton and Perold, because (as stated in the technical appendix of Merton and Perold, 1993) the formula of risk capital (per unit invested on net assets) can be approximated by:

$$k^{MP} \;=\; 0.4 \cdot \sigma \sqrt{T}$$

where T and σ respectively denote the time to maturity and the standard deviation of the series of $P\&L$ of the net assets' portfolio. In $Figure$ 2, the value $T = 1$ has been assumed.

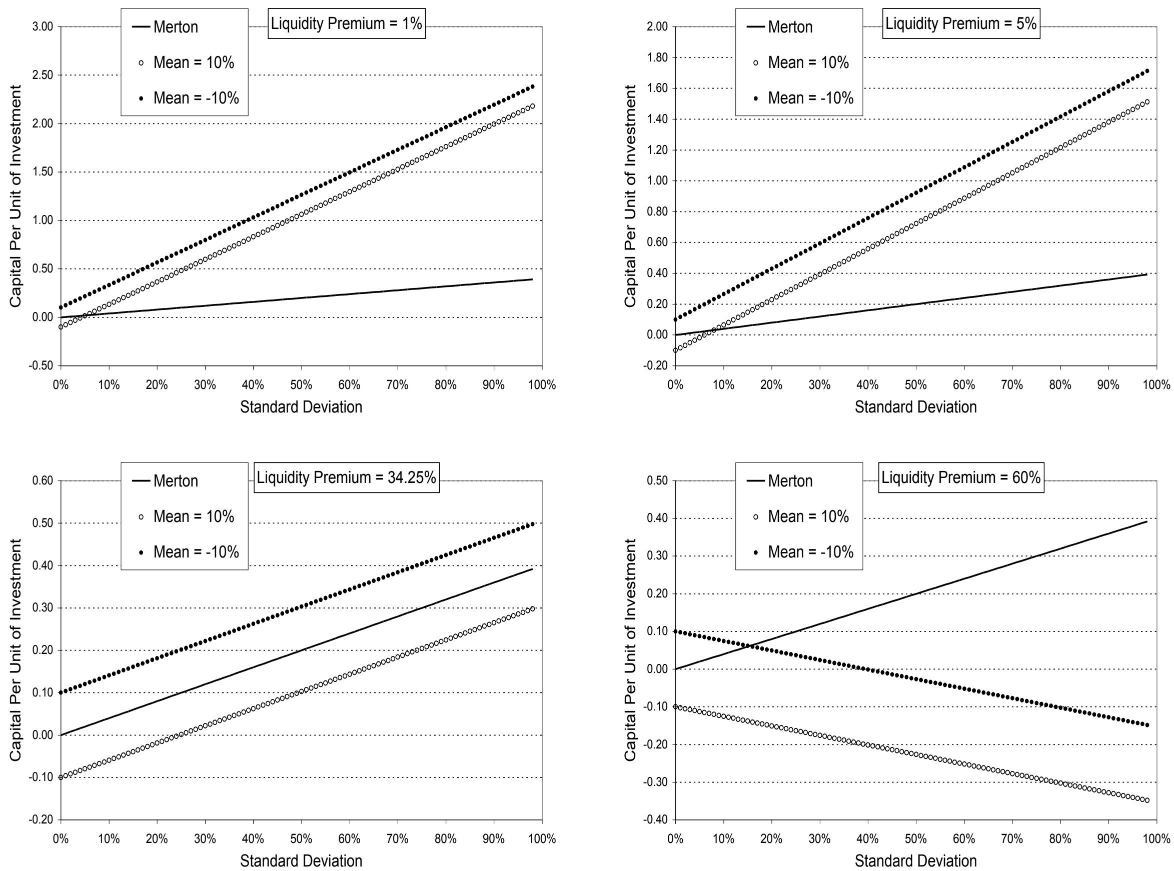
Therefore, while the capital principle of Merton and Perold intersects the capital axe at the origin and has a constant slope (equal to 0.4), the optimal principle intersects at the inverse of the mean return μ of net assets and its slope depends on the liquidity premium ν. This means, in particular, that at *low* volatilities firms that obtain capital profits (with $\mu > 0$) prefer to lend all their balances and do not maintain reserves at all. In fact, they only demand capital when:

$$\sigma \cdot \Phi^{-1}(1 - \nu^{\theta}) - \mu > 0 \quad \Leftrightarrow \quad \sigma > \frac{\mu}{\Phi^{-1}(1 - \nu^{\theta})}$$

This result is consistent with the behaviour of firms that own free-cash-flow (see *Section* 2). On the other hand, firms that obtain capital losses (with $\mu < 0$), choose a level of reserves equal to the mean loss of net assets, i.e. $k = -\mu$, when $\sigma = 0$.

On the other hand, the slope $\Phi^{-1}(1 - \nu^{\theta})$ of the optimal capital line depends on the distorted liquidity premium. *Lower* liquidity premiums imply that firms hold *less* cash in excess and hence that they are willing to exchange *more* capital by any unit of incremented volatility (in particular, the slope tends to $+\infty$ when $\nu \to 0$). Averse-to-risk and risk-lover decision-makers (respectively characterised by $\theta > 1$ and $\theta < 1$, see *Section* 7) respectively under- and over-estimates the premium for liquidity. Moreover, since $\Phi^{-1}(1 - \nu^{\theta}) < 0$ when

Figure 2: The Optimal Capital Line and the Allocation Rule of Merton and Perold (1993)



$\nu > 0.5$, the slope of the capital line turns negative in this case. Accordingly, and *contrary* to the common intuition, the capital requirements may *decrease* with the level of volatility if the liquidity premium is sufficiently high (see the graph to the right lower corner in $Figure$ 2). Eventually, when the volatility surpasses a certain level (depending on the expected return μ and the premium $\nu > 0.5$) firms prefer to lend all their balances.

In conclusion, the optimal capital principle determines an *optimal capital line* in the Gaussian case, which relates the optimal proportion of reserves in terms of the mean return and the volatility of the underlying risk, as well as the premium for liquidity offered in the market. The Merton's principle is obtained as a particular case of the optimal capital line when risks are Gaussian with $\mu = 0$ and $\Phi^{-1}(1 - \nu^{\theta}) = 0.4$, i.e. when $\nu^{\theta} = 1 - \Phi(0.4) \approx 34.25\%$ (see the graph to the left lower corner in $Figure$ 2). This situation can be naturally corresponded to a *competitive* environment, where firms obtain no capital $P\&L$ ($\mu = 0$) and the distorted liquidity premium is *high enough* ($\nu^{\theta} \approx 34.25\%$). Recall that the optimal capital principle has been obtained as an extension of the option-based Merton's principle — both are based on the net assets' claim, whose payments at maturity are given in $Equation$ 4. The optimal capital principle defined in this paper is thus *meaningful* from the economic point of view.

11 Conclusions

Firms that continuously trade capital and securities demand no cash reserves, for they can fit their balances at any moment through borrowing and lending (Modigliani and Miller, 1958). In fact, as proved by Merton (1974, 1997), although the market prices of equity and deposit insurance (which are the main components of the capital structure) actually depend on the level of reserves, the value of firms does not depend on it (see also Miller, 1998). This result is a consequence of the fact that continuous hedging suppresses risk.

However, when firms face restrictions on liquidity (in other words, when borrowing and lending may change the price of capital if the transacted amounts break on through certain thresholds) the capital structure determines the agency costs and the moral-hazard implicit in the contracts that managers respectively establish with stockholders and customers. Hence the market prices of equity and deposit insurance should be determined on an actuarial basis. As demonstrated in this paper, an *optimal* cash balance then exists, which leads to an optimal *capital principle* that is consistent with economic fundamentals and is easy to implement for a wide class of probability distributions. Moreover, since the level of capital is explicitly related to the *deductible* or *retention* of the corresponding insurance contract, it explicitly represents the moral-hazard on the side of customers.

In particular, when the underlying risk follows a Gaussian probability distribution, an optimal *capital line* is obtained relating the optimal proportion of capital to the standard deviation. This principle naturally extends the capital allocation rule proposed by Merton and Perold (1993).

Finally, the allocation of capital to the subsidiaries of multidivisional corporations is determined by the differences in skills, information and aversion-to-risk of central and divisional managers. This means that central and local administrations do not always agree on which combinations of debt and capital lead to under- and over-investment.

The capital principle defined in this paper can be extended in such a way that a single *informational type* modify expectations. The differences in expectations can be explicitly measured in this way. Two different *allocation principles* can be defined on this basis, the one of which is applicable to *centralised* and the other to *decentralised* organisations. In fact, an *optimal mechanism* can be established for the distribution of capital, which allows subsidiaries to independently choose their capital structures, at the time that leads the conglomerate to collect the same aggregate balance as under the centralised allocation. The implementation of the mechanism forces subsidiaries to *reveal their types*. In this context, the optimal mechanism can be used as a tool to stimulate the exchange of information between central and divisional administrations.

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