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# The ups and downs of the renormalization group applied to financial time series

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## Abstract

Starting from inhomogeneous time scaling and linear decorrelation between successive price returns, Baldovin and Stella recently devised a model describing the time evolution of a financial index. We first make it fully explicit by using Student distributions instead of power law-truncated Lévy distributions; we also show that the analytic tractability of the model extends to the larger class of symmetric generalized hyperbolic distributions and provide a full computation of their multivariate characteristic functions. The Baldovin and Stella model, while mimicking well volatility relaxation phenomena such as the Omori law, fails to reproduce other stylized facts such as the leverage effect or some time reversal asymmetries. We discuss how to modify the dynamics of this process in order to reproduce real data more accurately.

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## I. HOW SCALING AND EFFICIENCY CONSTRAINS RETURN DISTRIBUTION

Finding a faithful stochastic model of price time series is still an open problem. Not only should it replicate in a unified way all the empirical statistical regularities, often called stylized facts, (cf e.g. Bouchaud and Potters [13], Cont [17]), but also easy to calibrate and analytically tractable, so as to make easier its application to derivative pricing and financial risk assessment. Up to now none of the proposed models has been able to meet all these requirements despite their variety. Recent attempts include ARCH family (Bollerslev et al. [8], Tsay [42] and references therein), stochastic volatility (Musielka and Rutkowski [35] and references therein), multifractal models (Bacry et al. [1], Borland et al. [11], Eisler and Kertész [22], Mandelbrot et al. [33] and references therein), multi-timescale models (Borland and Bouchaud [10], Zumbach [45], Zumbach et al. [47]), Lévy processes (Cont and Tankov [18] and references therein).

Recently Baldovin and Stella (B-S thereafter) proposed a new way of addressing the question. We advise the reader to refer to the original papers Baldovin and Stella [4, 5, 6] for a full description of the model as we shall only give a brief account of its main underlying principles. Using their notation let  $S(t)$  be the value of the asset under consideration at time  $t$ , the logarithmic return over the interval  $[t, t + \delta t]$  is given by  $r_{t,\delta t} = \ln S(t + \delta t) - \ln S(t)$ ; the elementary time unit is a day, i.e.,  $t = 0, 1, \dots$  and  $\delta t = 1, 2, \dots$  days. In order to accommodate for non-stationary features, the distribution of  $r_{t,\delta t}$  is denoted by  $P_{t,\delta t}(r)$  which contains an explicit dependence on  $t$ . The most impressive achievement of B-S is to build the multivariate distribution  $P_{0,1}^{(n)}(r_{0,1}, \dots, r_{n,1})$  of  $n$  consecutive daily returns starting from the univariate distribution of a single day provided that the following conditions hold:

1. No trivial arbitrage: the returns are linearly independent, i.e.  $E(r_{i,1}, r_{j,1}) = 0$  for  $i \neq j$ , with the standard condition  $E(r_{i,1}) = 0$ .
2. Possibly anomalous scaling of the return distribution with respect to the time interval  $\delta t$ , with exponent  $D$ :  $P_{0,\delta t}(r) = \frac{1}{\delta t^D} P_{0,1}(\frac{r}{\delta t^D})$ .
3. Identical form of the unconditional distributions of the daily returns up to a possible dependence of the variance on the time  $t$ , i.e.  $P_{t,1}(r) = \frac{1}{a_t} P_{0,1}(\frac{r}{a_t})$ .

As shown in the addendum of Baldovin and Stella [5] these conditions admit the solution

$$f_{0,1}^{(n)}(k_1, \dots, k_n) = \tilde{g}(\sqrt{a_1^{2D} k_1^2 + \dots + a_n^{2D} k_n^2}), \quad (1)$$

where  $f_{0,1}^{(n)}$  is the characteristic function of  $P_{0,1}^{(n)}$ ,  $\tilde{g}$  the characteristic function of  $P_{0,1}$ , and  $a_i^{2D} = i^{2D} - (i-1)^{2D}$ . In this way the full process is entirely determined by the choice of the scaling exponent  $D$  and the distribution  $P_{0,1}$ . Therefore the characteristic function of  $P_{t,\delta t}(r)$  is

$$f_{t,T}(k) = f_{0,1}^{(n)}(\underbrace{0, \dots, 0}_{t \text{ terms}}, \underbrace{k, \dots, k}_{\delta t \text{ terms}}, 0, \dots, 0) = \tilde{g}(k\sqrt{(t+\delta t)^{2D} - t^{2D}}),$$

i.e.

$$P_{t,\delta t}(r) = \frac{1}{\sqrt{(t+\delta t)^{2D} - t^{2D}}} P_{0,1}\left(\frac{r}{\sqrt{(t+\delta t)^{2D} - t^{2D}}}\right).$$

The square root in  $\tilde{g}$  in Eq. (1) introduces a dependency between the unconditional marginal distributions of the daily returns by the means of a generalized multiplication  $\otimes$  in the space of characteristic functions, i.e.,

$$f_{0,1}^{(n)}(k_1, \dots, k_n) = \tilde{g}(a_1^D k_1) \otimes_{\tilde{g}} \dots \otimes_{\tilde{g}} \tilde{g}(a_n^D k_n),$$

with  $\otimes_{\tilde{g}}$  defined by

$$x \otimes_{\tilde{g}} y = \tilde{g}(\sqrt{[\tilde{g}^{-1}(x)]^2 + [\tilde{g}^{-1}(y)]^2}). \quad (2)$$

At first sight this last equation may seem a trivial identity, but it does hide a powerful statement. Suppose indeed that instead of starting with the probability distribution  $\tilde{g}$ , one takes a general distribution with finite variance  $\sigma^2 = 2$  and characteristic function  $\tilde{p}_1$ , then Baldovin and Stella [4] show that

$$\lim_{N \rightarrow \infty} \underbrace{\tilde{p}_1\left(\frac{k}{\sqrt{N}}\right) \otimes_{\tilde{g}} \dots \otimes_{\tilde{g}} \tilde{p}_1\left(\frac{k}{\sqrt{N}}\right)}_{N \text{ terms}} = \tilde{g}(k).$$

This means that in this framework the return distribution at large scales is independent from the distribution of the returns at microscopic scales: it is completely determined by the correlation introduced by the multiplication  $\otimes_{\tilde{g}}$ , with fixed point  $\tilde{g}$ . Note that if  $\tilde{g}$  is the characteristic function of the Gaussian distribution, then  $\otimes_{\tilde{g}}$  reduces to the standard multiplication and one recovers the standard Central Theorem Limit.

As the volatility of the model shrinks in an inexorable way, Baldovin and Stella propose to restart the whole shrinking process after a critical time  $\tau_c$  long enough for the volatility autocorrelation to fall to the noise level. In this way one recovers a sort of stationarity for time series whose length is much greater than  $\tau_c$ . In this case one expects that the empirical distribution of the return  $\bar{P}_{\delta t}(r)$  over a time horizon  $\delta t \ll \tau_c$ , evaluated with a sliding window satisfies

$$\bar{P}_{\delta t}(r) = \frac{1}{\tau_c} \sum_{t=0}^{\tau_c-1} P_{t,\delta t}(r). \quad (3)$$

In the original papers no market mechanism is proposed for modeling the restart of the process; it is simply stated that the length of different runs and the starting points of the processes could be stochastic variables. In their simulations the length of the processes was fixed to  $\tau = 500$ , which corresponds to slightly more than two years of daily data.

## II. A FULLY EXPLICIT THEORY WITH STUDENT DISTRIBUTIONS

Baldovin and Stella [5] chose a power law truncated Lévy distribution to describe the returns

$$\tilde{g}(k) = \exp\left(\frac{-Bk^2}{1 + C_\alpha k^{2-\alpha}}\right).$$

Sokolov et al. [41] show that this expression is indeed the characteristic function of a probability density with power law tails whose exponent is exponent  $5 - \alpha$ . However, this choice is problematic in two respects: its inverse Fourier cannot be computed explicitly, which prevents a fully explicit theory. In addition, for equation (1) to be consistent,  $\tilde{g}(\sqrt{k_1^2 + \dots + k_n^2})$  must be the characteristic function of a multivariate probability density for all  $n$ . Baldovin and Stella [5] rely on numerical evaluation, as no other proof can be given. But as discussed in Bouchaud and Potters [13] both truncated Lévy and Student distributions yield acceptable fits of the returns on medium and small time scales. In the present context, the Student distribution, sometimes referred to as  $q$ -Gaussian in the case of non-integer degrees of freedom, is a better choice; it provides analytic tractability while fitting equally well real stock market prices, as reported by Osorio et al. [38]. The fit of the daily returns of the S&P 500 index in the period with a Student distribution

$$g_1(x) = \frac{\Gamma(\frac{\nu}{2} + \frac{1}{2})}{\pi^{1/2} \lambda \Gamma(\frac{\nu}{2})} \frac{1}{(1 + \frac{x^2}{\lambda^2})^{\frac{\nu}{2} + \frac{1}{2}}}$$

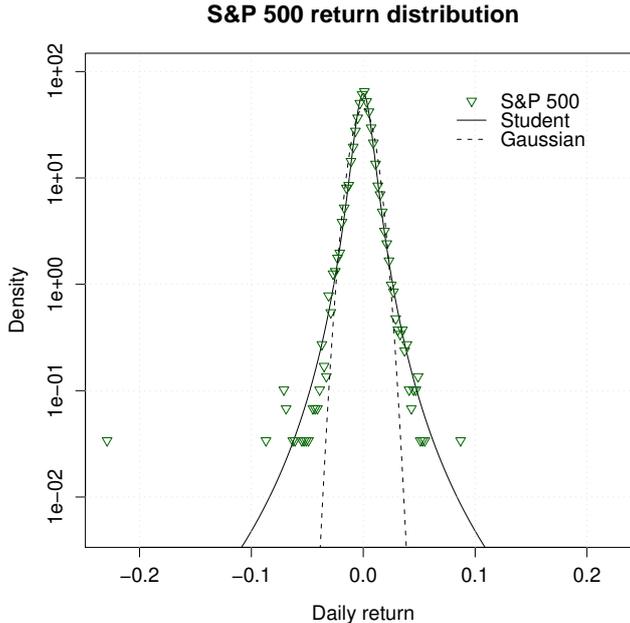


Figure 1: Centered distribution of the 14716 daily returns of the S&P 500 index (January, 3th 1950 - June, 30th 2008), and the corresponding fitting with Student ( $\nu = 3.22$ ,  $\lambda = 0.0107$ ) and Gaussian distribution ( $\sigma = 0.0088$ ).

is reported in Fig. 1[48].

The characteristic function of the Student density is

$$\tilde{g}(k) = \frac{2^{1-\frac{\nu}{2}}}{\Gamma(\frac{\nu}{2})} k^{\frac{\nu}{2}} K_{\frac{\nu}{2}}(k), \quad (4)$$

where  $K_\alpha$  is the modified Bessel function of third kind. As demonstrated in the appendix, the inverse Fourier transform of  $\tilde{g}(\sqrt{k_1^2 + \dots + k_n^2})$  for any integer  $n$  is simply the multivariate Student distribution (see also Vignat and Plastino [43]). The general form of this distribution can be written as

$$g_n^{(\nu)}(\mathbf{x}, \mathbf{\Lambda}) = \frac{\Gamma(\frac{\nu}{2} + \frac{n}{2})}{\pi^{n/2} (\det \mathbf{\Lambda})^{1/2} \Gamma(\frac{\nu}{2})} \frac{1}{(1 + \mathbf{x}^t \mathbf{\Lambda}^{-1} \mathbf{x})^{\frac{\nu}{2} + \frac{n}{2}}}, \quad (5)$$

where  $\nu > 1$  is the exponent of the power law of the tails,  $\mathcal{P}(r > R) \propto 1/R^\nu$  and  $\mathbf{\Lambda}$  is a positive definite symmetric matrix governing the variance-covariance matrix  $E(x_i, x_j) = \frac{\Lambda_{ij}}{\nu-2}$ , which does exist provided that  $\nu > 2$ .

In passing, the same properties are shared by multivariate symmetric generalized hyperbolic distributions introduced in finance by Eberlein and Keller [21](see also Bingham and

Kiesel [7]). The general case is obtained by an affine change of variable, but for sake of clarity let us restrict to

$$f(\mathbf{x}) = \frac{\alpha^{\frac{n}{2}}}{(2\pi)^{\frac{n}{2}} K_{\frac{\nu}{2}}(\alpha)} \frac{1}{(1+r^2)^{\frac{\nu}{4} + \frac{n}{4}}} K_{\frac{\nu}{2} + \frac{n}{2}}(\alpha\sqrt{1+r^2})$$

for  $\mathbf{x} \in \mathbb{R}^n$  and  $r$  the usual euclidean norm of  $\mathbf{x}$ . Student distributions are recovered in the limit  $\alpha \rightarrow 0^+$ . As shown in the appendix, its characteristic function is given for any  $n$  by

$$\tilde{f}_n(\mathbf{k}) = \frac{K_{\frac{\nu}{2}}(\sqrt{\alpha^2 + k^2})}{K_{\frac{\nu}{2}}(\alpha)} \frac{(\alpha^2 + k^2)^{\frac{\nu}{4}}}{\alpha^{\frac{\nu}{2}}}$$

with  $k = \sqrt{\sum_{i=1}^n k_i^2}$ .

In the following we restrict the discussion to the Student distributions. Hence we assume that the distribution of the return is given by Eq. (5) with characteristic function given by Eq. (4), where  $\mathbf{\Lambda}$  is a diagonal matrix

$$k = \sqrt{\mathbf{k}^t \mathbf{\Lambda} \mathbf{k}} = \lambda \sqrt{k_0^2 + (2^{2D} - 1)k_1^2 + \dots + (n^{2D} - (n-1)^{2D})k_{n-1}^2}$$

and  $\lambda^2$  governs the variance of the returns on the time scale chosen as a reference. Thanks to the fact that the diagonal elements of  $\mathbf{\Lambda}$  forms a telescoping series the process is indeed consistent for any number of discrete steps and can be generalized to the continuous time by setting, in the same consistent way,

$$\begin{aligned} &\mathcal{P}(r_{0,\Delta t_0}, r_{t_1,\Delta t_1}, \dots, r_{t_{n-1},\Delta t_{n-1}}) \\ &= g_n^{(\nu)}(r_{0,\Delta t_0}, r_{t_1,\Delta t_1}, \dots, r_{t_{n-1},\Delta t_{n-1}}, \mathbf{\Lambda} = \text{diag}(t_1^{2D}, t_2^{2D} - t_1^{2D}, \dots, t_n^{2D} - t_{n-1}^{2D})), \end{aligned} \quad (6)$$

where  $t_j = \sum_{i=0}^{j-1} \Delta t_i$ ,  $j \geq 1$  and now  $\mathbf{\Lambda} = \text{diag}(t_1^{2D}, t_2^{2D} - t_1^{2D}, \dots, t_n^{2D} - t_{n-1}^{2D})$ . The existence of the continuum process is then guaranteed by the Kolmogorov extension theorem. This process is not weakly stationary, as its variance is explicitly time-dependent. A stationary process is recovered with the choice  $D = 1/2$ . Starting from this expression a wider class of processes can be generated by suitable transformations of the time, i.e., by substituting the function  $t_i \rightarrow t_i^{2D}$  for any monotonically increasing continuous function  $t_i \rightarrow T(t_i)$ . The process followed by the price  $x(t) = \ln S(t)$  is a Student process too, with same exponent  $\nu$  and non diagonal matrix  $\Lambda_{ij} = (-1)^{i+j} T(t_{\min(i,j)})$ . It is worth to mention that a similar process has been already used in Borland [9] for option pricing. A detailed study of the this process, along with its relation with the work Heyde and Leonenko [24], is deferred to a separate paper.

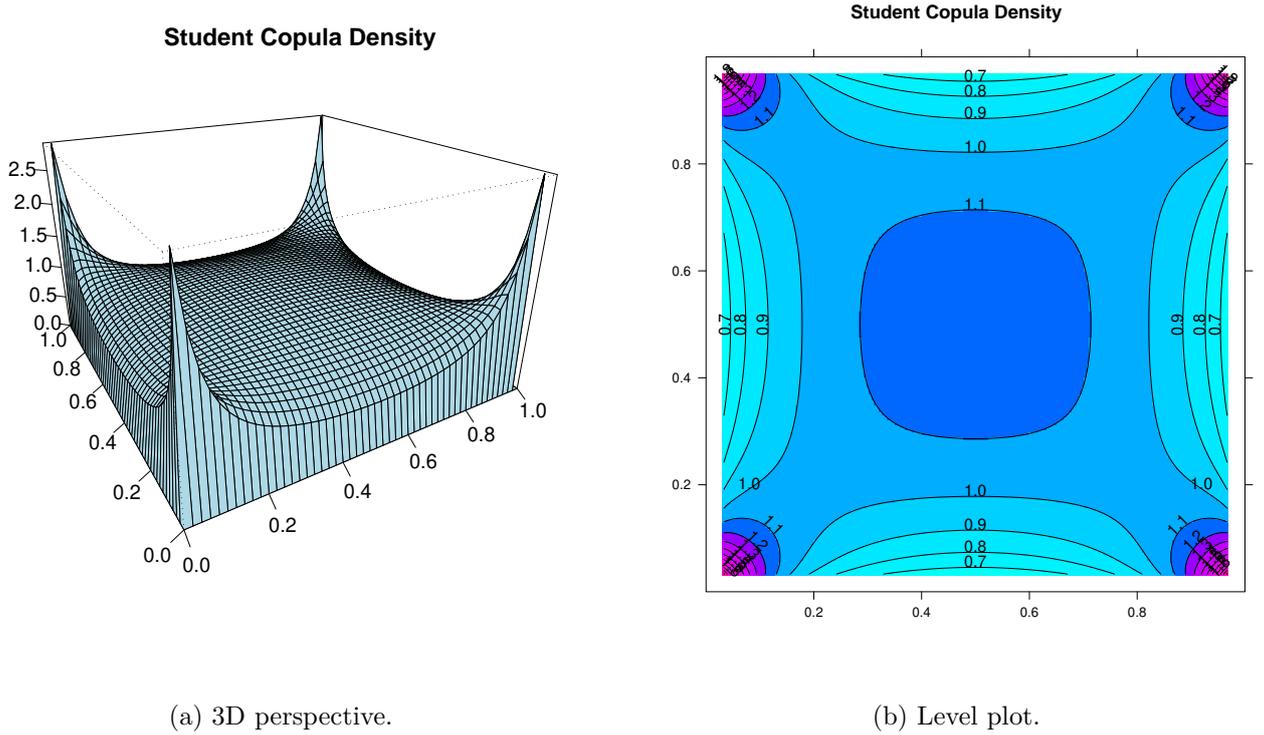


Figure 2: Student copula density with  $\nu = 3$  and trivial correlation matrix.

The Student setting makes easier to interpret the correlations induced by the point-wise non-standard product of (2) in the characteristic function space. If we consider two variables  $x_1$  and  $x_2$  distributed according to  $g_1(x)$ , the joint probability function will be  $g_2(x_1, x_2)$ . The variables  $X_i = G(x_i) = \int_{-\infty}^{x_i} dx g_1(x)$  are distributed uniformly on the interval  $[0, 1]$ ; by definition, the copula function  $c(X_1, X_2)$  (cf. e.g. Nelsen [37] for a general theory) is

$$c(X_1, X_2) = g_2(G^{-1}(X_1), G^{-1}(X_2)) \frac{dx_1 dx_2}{dX_1 dX_2} = \frac{g_2(G^{-1}(X_1), G^{-1}(X_2))}{g(G^{-1}(X_1)) g(G^{-1}(X_2))}.$$

In our case  $c$  is nothing else but the Student copula function, generally applied in finance for describing the correlation among asset prices (Cherubini et al. [16], Malevergne and Sornette [32]). A picture of this copula density with  $\nu = 3$  and  $\mathbf{\Lambda}$  the identity matrix is given in Fig. 2. Although Student and generalized hyperbolic distributions are usually adopted for modeling returns of several assets over the same time intervals, the framework proposed by Baldovin and Stella allow them to model the returns of a single asset over different time intervals.

### III. APPLICABILITY OF THIS FRAMEWORK TO REAL MARKETS

The axiomatic nature of the derivation of Baldovin and Stella is elegant and powerful: its ability to build mathematically multivariate price return distributions from a univariate distribution using only a few reasonable assumptions is impressive. Nevertheless, as stated in the introduction, a model of price dynamics must meet many requirements in order to be both relevant and useful. In this section, we examine its dynamics thoroughly.

#### A. Volatility dynamics

In Fig. 3.a we report the results of 3 simulations of the return process, each one of 500 steps and with parameters  $\nu = 3.2$  and  $D = 0.20$ . In each run the volatility decays ineluctably. Indeed by fixing the time interval  $\delta t_i = 1$ , we see from (6) that the unconditional volatility of the  $r_{t,1}$  returns is proportional to  $\sqrt{(t+1)^{2D} - t^{2D}}$ , i.e., to  $t^{D-1/2}$  for  $t \gg 1$ : the unconditional volatility decreases if  $D < 1/2$  and increases if  $D > 1/2$ , in both cases according to a power law. This appears quite clearly in Fig. 3.b, where we have computed the mean volatility decay, measured as the absolute values of the return, over 10000 process simulations. The parameters of the distributions have been chosen close to the ones representing real returns (see below).

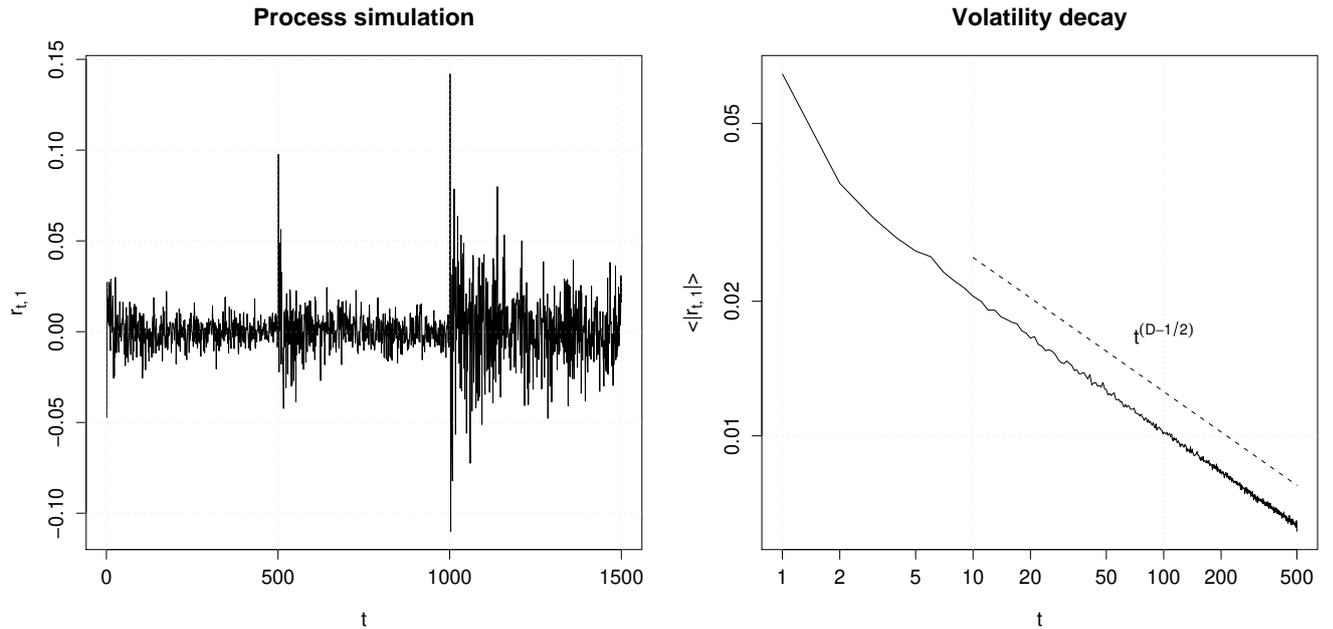
The conditional volatility can be easily computed: the distribution of the return  $r_{n,1}$  conditioned to the previous return realizations  $r_{0,1}, \dots, r_{n-1,1}$  is again a Student distribution with exponent  $\nu' = \nu + n$  and conditional variance

$$[(n+1)^{2D} - n^{2D}] \left( 1 + \sum_{i=0}^{n-1} \frac{r_{i,1}^2}{(i+1)^{2D} - i^{2D}} \right).$$

From this expression it is clear that volatility spikes in a given realisation of the process tend to be persistent (see Fig. 3.a); this is the main reason why fluctuation patterns differ much from one run to an other.

#### B. Decreasing volatility and restarts

The very first model introduced by B-S has constant volatility, which correspond to  $\mathbf{\Lambda}$  being a multiple of the identity matrix. This unfortunate feature is the main reason behind the introduction of weights, whose effect is akin to an algebraic stretching of the time, or, as



(a) 3 simulation, each 500 steps long.

(b) Decay of the volatility: average over 10000 simulation, each 500 steps long. The dashed line represents the analytic prediction.

Figure 3: Process simulation with  $\nu = 3.2$ ,  $D = 0.20$ , and  $\lambda = 0.107$ .

put forward by B-S, to a time renormalization. This in turn causes a deterministic algebraic decrease of the expectation of the volatility, as explained above and depicted in Fig. 3.b; hence the need for restarts, each attributed to an external cause.

Although this dynamics may seem quite peculiar, such restarts are found at market crashes, which are followed by periods of algebraic decaying volatility. This leads to an analogous of the Omori law for the earthquakes, as reported in Lillo and Mantegna [30] and Weber et al. [44]. The B-S model, by construction, is able to reproduce this effect in a faithfully way. In Fig. 4 the cumulative number of times the absolute value of the returns  $N(t)$  exceeds a given thresholds is depicted, for a single simulation of the process and three different value of the threshold. The fit with the prediction of the Omori law  $N(t) = K(t + t_0)^\alpha - Kt_0^\alpha$  is evident.

Crashes are good restart candidates: they provide clearly defined events that synchronize all the trader's actions. In that view, they provide an other indirect way to measure the distribution of timescales of traders, which are known to be power-law distributed (Lillo

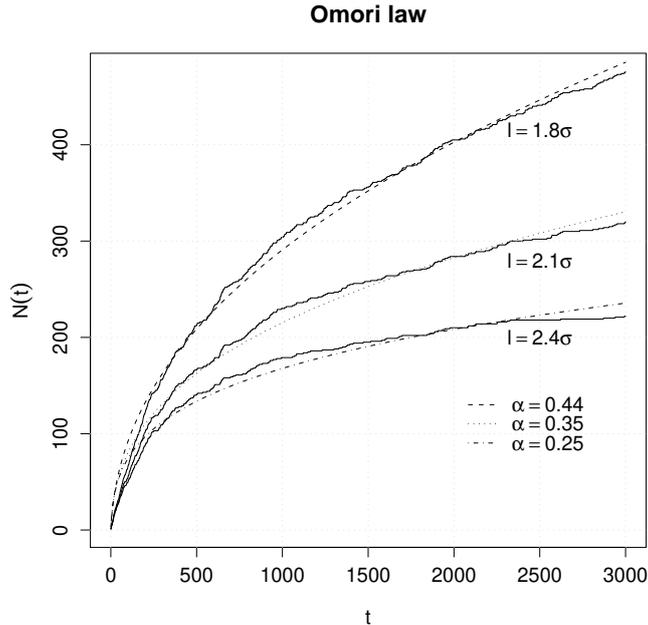
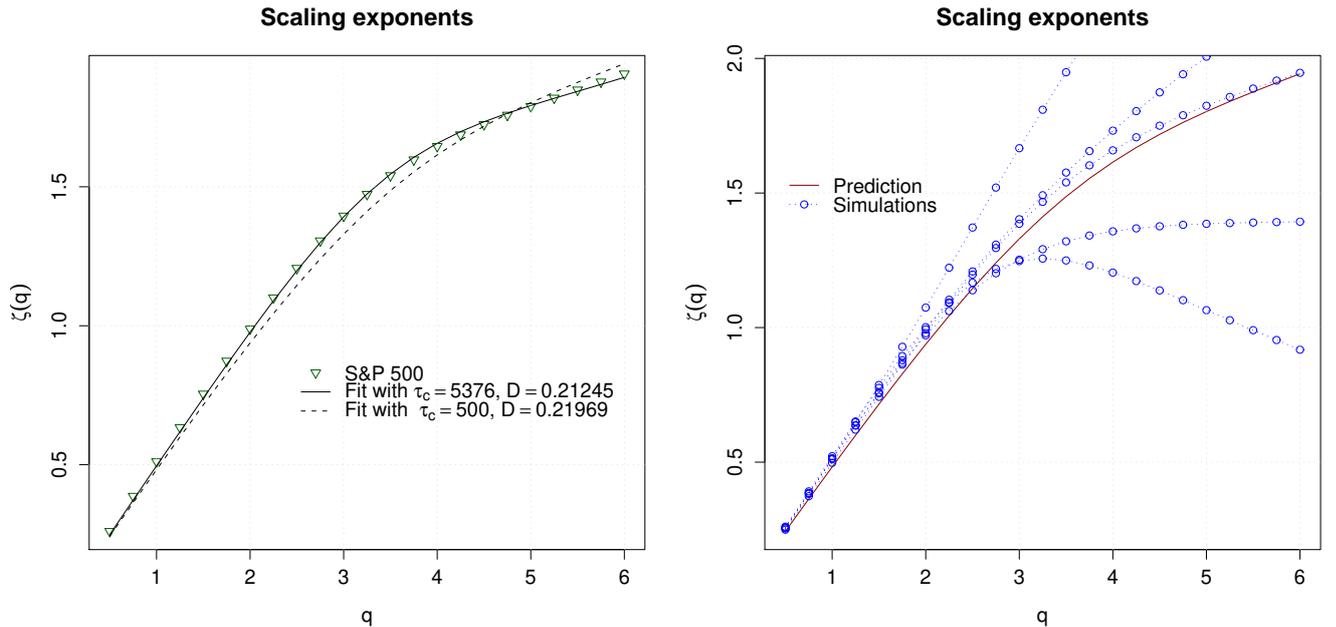


Figure 4: Omori law for a single run of the process, with  $D = 0.20$ ,  $\nu = 0.32$ .  $N(t)$  is the cumulative number the absolute value of the return exceeds a given thresholds. Three different values of the threshold  $l$  have been chosen, measured with respect to the standard deviation  $\sigma$  of the data. The dashed lines represents the fit with the Omori law  $N(t) = K(t + t_0)^\alpha - Kt_0^\alpha$ .

[29]).

Another example of algebraically decreasing volatility was recently reported by McCauley et al. [34] in foreign exchange markets in which trading is performed around the clock. Understandably, when a given market zone (Asia, Europe, America) opens, an increase of activity is seen, and vice-versa. Specifically, this work fits the decrease of activity corresponding to the afternoon trading session in the USA with a power-law and finds an algebraic decay with exponent  $\eta = 0.35$ ; this is exactly the same behavior as the one of B-S model between two restarts, with  $D = 1 - 2\eta = 0.3$ . No explanation of why the trading activity should result in this specific type of decay has been put forward in our knowledge. In this case the starting time of the volatility decay corresponds to the maximum of activity of US markets.



(a) Fitting of the empirical exponents of the real data. (b) Theoretical prediction compared to 5 simulations done with the same parameters.

Figure 5: Scaling exponents: S&P 500 data and simulations compared with theoretical prediction. All the simulations have been done with the same parameters: 30 runs of 500 steps, with  $\nu = 3.2$ ,  $D = 0.220$

### C. Apparent multifractality

The Baldovin and Stella model is able to reproduce the apparent multifractal characteristics of the real returns, i.e. the shape of  $\zeta(q)$  where  $\langle r_{\delta t} \rangle = \delta t^{\zeta(q)}$ .

The expectation is evaluated according the distribution (3), i.e. taking the mean over independent runs of the process. Hence the expectation of the  $q$ th moment in this model is

$$\langle |r|^q \rangle_{\bar{P}_{\delta t}} = \frac{\langle |r|^q \rangle_{P_{t=0, \delta t=1}}}{\tau_c} \sum_{t=0}^{\tau_c-1} [(t + \delta t)^{2D} - t^{2D}]^{q/2} \quad (7)$$

(see the addendum to Baldovin and Stella [5]). The exponents  $\zeta(q)$  are evaluated as the slopes of the linear fitting of  $\ln(\langle |r|^q \rangle_{\bar{P}_{\delta t}})$  with respect to  $\ln(\delta t)$ . Hence in our case they are determined by the expression  $\ln \sum_{t=0}^{\tau_c-1} [(t + \delta t)^{2D} - t^{2D}]^{q/2}$ , and depend only on  $D$  and  $\tau_c$ . In Fig. 5.a is depicted the fitting of the S&P 500 exponents with the model (7). The best fit is obtained with  $D = 0.212$  and  $\tau_c = 5376$ . Unfortunately a value of  $\tau_c$  that large is difficult to justify, as in the case of S&P 500 we have only 14716 daily returns, i.e. less than 3 runs of a

process with such a length. The other fit is obtained by first fixing  $\tau_c = 500$ , as in Baldwin and Stella [5] and yields  $D = 0.220$ .

The statistical significance of this approach seems anyway questionable. In Fig. 5.b we compare the theoretical expectation of the exponents with simulations. We choose the parameters  $\tau_c = 500$ ,  $D = 0.220$  both for simulations and analytic model, with  $\nu = 3.22$ . The number of restarts in the simulation is 30, in order to have a number of data points similar to the S&P 500. It is evident that the exponents evaluated from the simulated data have a really large variance.

The problem is that if the tail exponent  $\nu = 3.22$ , from an analytic perspective the moments with  $q > 3.22$  are infinite, hence, should not be taken into account in the multifractal analysis (for an analytic treatment of multifractal analysis see Jaffard [26, 27], Riedi [40]). The situation is somehow different in the case of multifractal models of asset returns (Bacry et al. [2], Mandelbrot et al. [33]), where the theoretical prediction of the tail exponents of the return distribution is relatively high (see the review of Borland et al. [11]), and the moments usually empirically measured do exist even from the analytic point of view. For attempts to reconcile the theoretical predictions of the multifractal models with the real data see Bacry et al. [3] and Muzy et al. [36].

It is worth remembering that the anomalous scaling of the empirical return moments does not imply that the return series has to be described by a multifractal model, as already pointed out some time ago in Bouchaud [12] and Bouchaud et al. [14]: the long memory of the volatility is responsible at least in part for the deviation from trivial scaling. A more detailed analysis of the real data reported in Jiang and Zhou [28] seems indeed to exclude evident multifractal properties of the price series.

#### IV. MISSING FEATURES

Since in this model the volatility is bound to decrease unless a restart occurs, it is quite clear that it does not contain all the richness of financial market price dynamics. Restarting the whole process is not entirely satisfactory, as in reality the increase of volatility is not always due to an external shock. Volatility does often gradually build up through a feedback loop that is absent from B-S mechanism. Thus, large events can also have an endogenous cause, e.g. due to the influence of traders that base their decisions on previous prices or

volatility, such as technical analysts or hedgers. This effect is completely missing from the original mechanism.

Volatility build-ups can be simulated with  $D > 1/2$ . In the particular case of foreign exchange intraday volatility patterns, the fit of an increasing part of volatility to a possibly arbitrary power-law, as performed in McCauley et al. [34] ( $\eta = 0.22$ ), corresponds indeed to chose  $D = 0.56$ . It should be noted that no equivalent of the Omori law has ever been reported for volatility build-ups: it seems that the increase of volatility either does not follow a particular and systematic law (or perhaps has not yet been the object of a thorough study).

Because of the symmetric nature of all the distributions derived above, all the odd moments are zero, hence, the skewness of real prices cannot be reproduced. This is shown well in Fig. 3 of Baldovin and Stella [6]. Another consequence is that it is impossible to replicate the *leverage effect*, i.e. the negative correlation between past returns and future volatility, carefully analyzed in Bouchaud et al. [15].

In any case, the decrease of the fluctuations in the B-S process is put by hand and results in a strong temporal asymmetry of the corresponding time series. But quite remarkably it misses the time-reversal asymmetry reported in Lynch and Zumbach [31] and Zumbach [46]. Indeed the real financial time series are not symmetric with respect to time reversal with respect to even-order moments. For instance, there is no leverage effect in foreign exchange rates, and their time series are not as skewed as indices, but they do have a time arrow. One of the indicators proposed in Lynch and Zumbach [31] is the correlation between historical volatility  $\sigma_{\delta t_h}^{(h)}(t)$  and realized volatility  $\sigma_{\delta t_r}^{(r)}(t)$ . The historical volatility series  $\sigma_{\delta t_h}^{(h)}(t)$  represents the volatility computed using the data in the past interval  $[t - \delta t_h, t]$ , and  $\sigma_{\delta t_r}^{(r)}(t)$  represents the volatility computed using the data in the future interval  $[t, t + \delta t_r]$ ; the correlation between the two series is then analyzed as a function of both  $\delta t_r$  and  $\delta t_h$ . Real financial time series present an asymmetric graph with respect to the change  $\delta t_h \leftrightarrow \delta t_r$ , with a strong indication that historical volatility at a given time scale  $\delta t_h$  is more likely correlated to realized volatility with time scale  $\delta t_r < \delta t_h$ , with peaks of correlation at time scales related to human activities. The asymmetry characteristic is absent in the Baldovin and Stella model, as showed in Fig. 6.

The strong correlation between returns guarantees the slow decay of the volatility but induces some side effects. The distribution of the returns in the model is essentially the

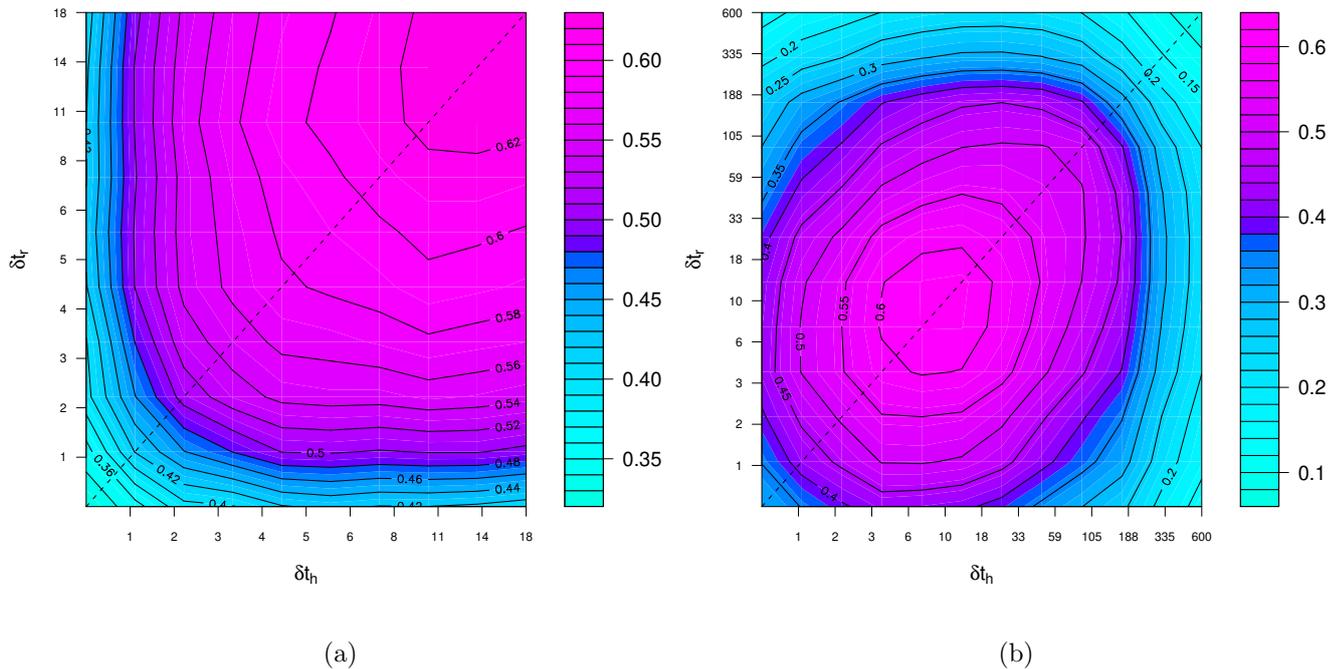


Figure 6: Correlation between historical and realized volatility of the simulated process, over different time interval  $\delta t$ . The analyzed time series was composed by 1000 runs of the basic process, each one with 200 steps, and parameter  $\nu = 3.22$ ,  $D = 0.20$ .

same with identical power law exponent for the tails. This happens independently of the time interval  $\delta t$  over which the returns are evaluated, as long as  $\delta t \ll \tau_c$ , with  $\tau_c$  of the order of hundreds days. Hence the weekly returns are distributed as the daily returns, while in the real data the tail exponent begins to increase in a remarkable way already at the intraday level (Drozd et al. [20]). The strong correlation also slows down the convergence to the Gaussian distribution of the returns when measured on larger time scale. Even if the kurtosis is not defined analytically in principle, it is possible to measure the empirical kurtosis of the returns of a simulated time series and compare with the kurtosis of the real data. In Fig. 7 we show the kurtosis of the return distribution among simulations and daily return of the S&P 500 index; the kurtosis has been computed for the returns over different interval  $\delta t$ , and the simulated processes had the same length (30 runs of 500 steps) of the real series.

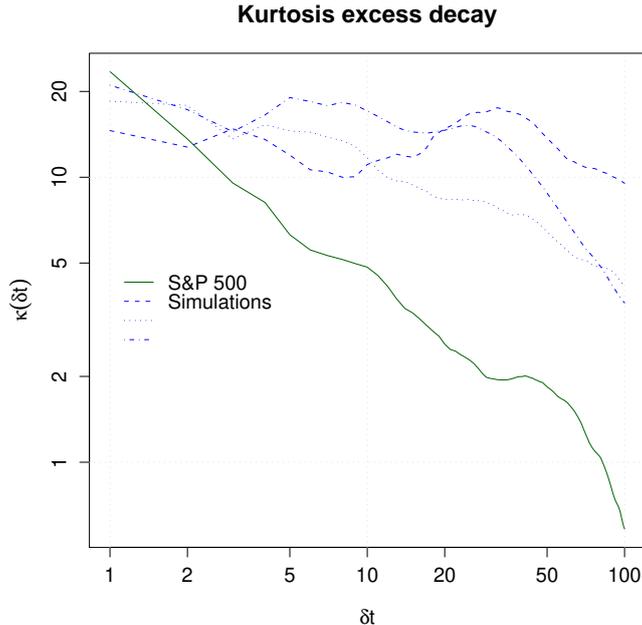


Figure 7: Comparison of the kurtosis of the returns evaluated over a time interval  $\delta t$ . Each one of the three simulations are composed by 30 runs, 500 steps long, in order to have a length comparable with that of the S&P 500 returns. The parameters are  $\nu = 3.2$ ,  $D = 0.20$ ,  $\lambda = 0.1$ .

## V. SUGGESTED IMPROVEMENTS

The main limitations of the model proposed by Baldovin and Stella are the poor volatility dynamics, the lack of skewness, some unwanted symmetry with respect to time, and the extremely slow convergence to a Gaussian. In this final section we put forward briefly some qualitative proposals of how these issues can be addressed.

The volatility dynamics can be improved by introducing an appropriate dynamics for the exponent  $D$ , i.e. introducing a dynamic  $D(t)$  controlling the diffusive process. This is equivalent to start with a model with constant volatility, i.e. with the  $\mathbf{\Lambda}$  just proportional to the identity matrix, and then introducing an appropriate evolution for the time  $t$ . This technique is employed for instance in the Multifractal Random Walk model (Bacry et al. [2]), where the time evolution is driven by a multifractal process.

The lack of skewness is a common problem of stochastic volatility models: one usually writes the return at time  $t$  as  $r_{t,\delta t} = \epsilon(t)\sigma(t)$  where  $\epsilon(t)$  is sign of the return and  $\sigma(t)$  its amplitude, a symmetric setting if the distribution of  $\epsilon(t)$  is even. One remedy found for

instance in Eisler and Kertész [22] is to bias the sign probabilities while enforcing a zero expectation; more precisely,

$$P\left(\epsilon = \pm \frac{1/\sqrt{2}}{1/2 \pm \epsilon}\right) = 1/2 \pm \epsilon.$$

The convergence to a Gaussian, or better the decay of the tail exponent of the return distribution, could be implemented by introducing different distributions for the returns at a given time scale and for modeling the non-linear correlation among them. For instance a suitable parameter  $\nu_r$  can be chosen for the daily return, and a much larger one  $\nu_c$  adopted in the copula function needed for modeling the correlation.

The Zumbach mugshot is one of the most difficult stylized fact to reproduce. To our knowledge the best results in that respect was achieved in Borland and Bouchaud [10], where a specific realization of a quadratic GARCH model is introduced, motivated by the different activity levels of the traders with different investment time horizon, which take into account the return over a large spectrum of time scales. More specifically Borland and Bouchaud use

$$\sigma_i^2 = \sigma_0^2 \left[ 1 + \sum_{\delta t=1}^{\infty} g_{\Delta t} \frac{r_{i,\delta t}^2}{\sigma_0^2 \tau \delta t} \right],$$

with  $\tau$  fixing the time scale,  $r_{t,\delta T} = \ln S(t + \delta T) - \ln S(t)$ ,  $g_{\delta t}$  measuring the impact on the volatility by the traders with time horizon  $\delta t$ , and chosen by the authors  $g_{\delta t} = g/(\delta t)^\alpha$ . This expression is rewritten also in the form

$$\sigma_i^2 = \sigma_0^2 + \sum_{j < i, k < i} \mathcal{M}(i, j, k) \frac{r_j r_k}{\tau},$$

with

$$\mathcal{M}(i, j, k) = \sum_{\Delta t = \max(i-j, i-k)}^{\infty} \frac{g_{\delta t}}{\delta t}.$$

In the present framework this would correspond to use a highly non-trivial matrix  $\mathbf{\Lambda}$ , introducing linear correlation among returns at any time lag. This means that the B-S process would not be a model of returns any more, but of stochastic volatility.

## VI. CONCLUSION

Despite its current inability to reproduce all the needed stylized facts, the new framework proposed by Baldovin and Stella is a conceptual breakthrough based on a few reasonable

first principles. Once suitably modified, it promises to bring a faithful yet workable model of financial price dynamics.

## Appendix: some useful facts about Student and symmetric generalized hyperbolic distributions

### Characteristic function of Student distributions

The standard form of univariate Student distribution is

$$g_1(x) = \frac{\Gamma(\frac{\nu}{2} + \frac{1}{2})}{\pi^{1/2}\Gamma(\frac{\nu}{2})} \frac{1}{(1+x^2)^{\frac{\nu}{2} + \frac{1}{2}}},$$

while the multivariate one is

$$g_n(\mathbf{x}) = \frac{\Gamma(\frac{\nu}{2} + \frac{n}{2})}{\pi^{n/2}\Gamma(\frac{\nu}{2})} \frac{1}{(1+r^2)^{\frac{\nu}{2} + \frac{n}{2}}}$$

with  $r = \sqrt{\sum_{i=1}^n x_i^2}$  and  $\mathcal{P}(r > R) \propto 1/R^\nu$ .

Using some standard relationships involving Bessel functions one can compute analytically the corresponding characteristic function

$$\begin{aligned} \tilde{g}_1(k_1) &= \int_{-\infty}^{+\infty} dx_1 e^{ik_1 x_1} g_1(x_1) \\ &= \frac{2\Gamma(\frac{\nu}{2} + \frac{1}{2})}{\pi^{1/2}\Gamma(\frac{\nu}{2})} k^\nu \int_0^{+\infty} dx (k^2 + x^2)^{-\frac{\nu}{2} - \frac{1}{2}} \cos(x) = \frac{2^{1-\frac{\nu}{2}}}{\Gamma(\frac{\nu}{2})} k^{\frac{\nu}{2}} K_{\frac{\nu}{2}}(k), \end{aligned}$$

with  $k = |k_1|$ ,  $K_\alpha$  the modified Bessel function of third kind, and the identity 7.12.(27) of Erdélyi [23]

$$K_\nu(z) = \frac{(2z)^\nu}{\pi^{1/2}} \Gamma(\nu + \frac{1}{2}) \int_0^\infty dt (t^2 + z^2)^{-\nu-1/2} \cos(t)$$

$$\Re(\nu) > -\frac{1}{2}, \quad |\arg(z)| < \frac{\pi}{2}.$$

For an alternative derivation we refer to Hurst [25] and to the discussion in Heyde and Leonenko [24]. An alternative expression is found in Dreier and Kotz [19].

For general  $n$  we obtain again the same expression. Indeed

$$\begin{aligned}
\tilde{g}_n(\mathbf{k}) &= \int_{\mathbb{R}^n} d^n \mathbf{x} e^{i\mathbf{k} \cdot \mathbf{x}} g_n(\mathbf{x}) \\
&= \frac{\Gamma(\frac{\nu}{2} + \frac{n}{2})}{\pi^{n/2} \Gamma(\frac{\nu}{2})} \int d^{n-2} \Omega \int_0^{+\infty} dr r^{n-1} \int_0^\pi d\phi \sin^{n-2}(\phi) e^{ikr \cos \phi} (1+r^2)^{-\frac{\nu}{2} - \frac{n}{2}} \\
&= \frac{2^{n/2} \Gamma(\frac{\nu+n}{2})}{\Gamma(\frac{\nu}{2})} k^{1-n/2} \int_0^{+\infty} dr r^{n/2} (1+r^2)^{-\frac{\nu}{2} - \frac{n}{2}} J_{n/2-1}(kr) \\
&= \frac{2^{1-\frac{\nu}{2}}}{\Gamma(\frac{\nu}{2})} k^{\frac{\nu}{2}} K_{\frac{\nu}{2}}(k),
\end{aligned}$$

with  $k = \sqrt{\sum_{i=1}^n k_i^2}$ ,  $d^{n-2} \Omega$  the surface element of the sphere  $S^{n-2}$ ,  $\phi$  the angle between  $\mathbf{k}$  and  $\mathbf{x}$  and we employed identities 7.12.(9)

$$\Gamma(\nu + \frac{1}{2}) J_\nu(z) = \frac{1}{\pi^{1/2}} \left(\frac{z}{2}\right)^\nu \int_0^\pi d\phi e^{iz \cos \phi} (\sin \phi)^{2\nu} \quad \Re(\nu) > -\frac{1}{2} \quad (8)$$

and 7.14.(51) of Erdélyi [23]

$$\int_0^\infty dt J_\mu(bt) (t^2 + z^2)^{-\nu} t^{\mu+1} = \left(\frac{b}{2}\right)^{\nu-1} \frac{z^{1+\mu-\nu}}{\Gamma(\nu)} K_{\nu-\mu-1}(bz) \quad \Re(2\nu - \frac{1}{2}) > \Re(\mu) > -1, \quad \Re(z) > 0.$$

Eventually one finds

$$\tilde{g}_n(\mathbf{k}) = \tilde{g}_1(\sqrt{k_1^2 + \dots + k_n^2}).$$

With the linear change of variables  $\mathbf{x} \rightarrow \mathbf{C}^{-1} \mathbf{x}$ , setting  $\mathbf{\Lambda}^{-1} = (\mathbf{C}^T)^{-1} \mathbf{C}^{-1}$ , i.e.  $\mathbf{\Lambda} = \mathbf{C} \mathbf{C}^T$ , one obtains the following generalizations

$$g_n(\mathbf{x}) = \frac{\Gamma(\frac{\nu}{2} + \frac{n}{2})}{\pi^{n/2} (\det \mathbf{\Lambda})^{1/2} \Gamma(\frac{\nu}{2})} \frac{1}{(1 + \mathbf{x}^t \mathbf{\Lambda}^{-1} \mathbf{x})^{\frac{\nu}{2} + \frac{n}{2}}} \quad (9)$$

with characteristic function

$$\tilde{g}_n(\mathbf{k}) = \frac{2^{1-\frac{\nu}{2}}}{\Gamma(\frac{\nu}{2})} (\mathbf{k}^t \mathbf{\Lambda} \mathbf{k})^{\frac{\nu}{4}} K_{\frac{\nu}{2}}((\mathbf{k}^t \mathbf{\Lambda} \mathbf{k})^{1/2}).$$

In the univariate case  $\mathbf{\Lambda}$  is substituted by the scalar  $\lambda^2$  and the previous expressions reduce to

$$g_1(x) = \frac{\Gamma(\frac{\nu}{2} + \frac{1}{2})}{\pi^{1/2} \lambda \Gamma(\frac{\nu}{2})} \frac{1}{(1 + \frac{x^2}{\lambda^2})^{\frac{\nu}{2} + \frac{1}{2}}} \quad (10)$$

and

$$\tilde{g}_1(k) = \frac{2^{1-\frac{\nu}{2}}}{\Gamma(\frac{\nu}{2})} (\lambda k)^{\frac{\nu}{2}} K_{\frac{\nu}{2}}(\lambda k).$$

## Moments of Student distributions

Due to the symmetry under reflection all the odd moments vanish. For the second moments we have, provided that  $\nu > 2$

$$E(x_i, x_j) = \frac{\Lambda_{ij}}{\nu - 2}.$$

The moments of order  $2n$  exist provided that  $\nu > 2n$ ; as it happens for Gaussian distribution, they can be expressed in term of the second moments

$$E(x_{j_1}, x_{j_2}, \dots, x_{j_{2n}}) = \frac{\Gamma(\frac{\nu}{2} - n)}{2^n \Gamma(\frac{\nu}{2})} \prod_{\text{all the pairings}} \Lambda_{j_{i_1} j_{i_2}} \cdots \Lambda_{j_{i_{2n-1}} j_{i_{2n}}}.$$

In the univariate case these formulas reduce to  $E(x^2) = \frac{\lambda^2}{\nu - 2}$  and

$$E(x^{2n}) = \frac{(2n - 1)!! \Gamma(\frac{\nu}{2} - n)}{2^n \Gamma(\frac{\nu}{2})} \lambda^{2n}.$$

The kurtosis is then  $\kappa = 3 \frac{\nu - 2}{\nu - 4}$ , provided that  $\nu > 4$ .

## Simulation of multivariate Student distributions

The simulation is a standard application of the technique used in the case of rotational invariance. From

$$g_n(\mathbf{x}) d^n \mathbf{x} = \frac{\Gamma(\frac{\nu}{2} + \frac{n}{2})}{\pi^{n/2} \Gamma(\frac{\nu}{2})} r^{n-1} (1 + r^2)^{\frac{1}{1-q}} d^{n-1} \Omega dr,$$

with  $r \geq 0$ , we see that the density of the angular variables is uniform, while setting  $y = \frac{r^2}{1+r^2}$ , with  $1 > y \geq 0$  and  $r = \sqrt{y/(1-y)}$ , the density of  $y$  is given by

$$\frac{1}{B(\frac{n}{2}, \frac{\nu}{2})} y^{\frac{n}{2}-1} (1-y)^{\frac{\nu}{2}-1} dy,$$

i.e. by the beta distribution with parameters  $\frac{n}{2}$  and  $\frac{\nu}{2}$ . Eventually we can simulate the multivariate  $n$  dimensional distribution by

1. Simulating  $y$  according to  $B_x(\frac{n}{2}, \frac{\nu}{2})$  and setting  $r = \sqrt{\frac{y}{1-y}}$ .
2. Simulating  $n$  i.i.d. Gaussian variables  $u_i$  and settings  $\mathbf{n} = (u_1, \dots, u_n) / \sqrt{u_1^2 + \dots + u_n^2}$ .
3. Returning  $x\mathbf{n}$ .

The more general case (9) is simulated using the same algorithm and then returning  $\mathbf{C}\mathbf{x}$ , where  $\mathbf{\Lambda}^{-1} = (\mathbf{C}^T)^{-1} \mathbf{C}^{-1}$ , i.e.  $\mathbf{\Lambda} = \mathbf{C}\mathbf{C}^T$ .

## Characteristic function of symmetric generalized hyperbolic distributions

We start from the expression

$$f_n(\mathbf{x}) = \frac{\alpha^{\frac{n}{2}}}{(2\pi)^{\frac{n}{2}} K_{\frac{\nu}{2}}(\alpha)} \frac{K_{\frac{\nu}{2}+\frac{n}{2}}(\alpha\sqrt{1+r^2})}{(1+r^2)^{\frac{\nu}{4}+\frac{n}{4}}},$$

with  $r = \sqrt{\sum_{i=1}^n x_i^2}$ ; the general case is obtained simply with an affine transformation  $\mathbf{x} \rightarrow \mu + \delta \mathbf{R}\mathbf{x}$ , with  $\mu \in \mathbb{R}^n$ ,  $\delta \geq 0$  a scale parameter, and  $\mathbf{R}$  an orthogonal transformation in  $\mathbb{R}^n$ . The central expression we need is an integral of the Sonine Gegenbauer type, cf. identity 7.14.(46) of Erdélyi [23]

$$\begin{aligned} \int_0^\infty dt J_\mu(bt) K_\nu(a\sqrt{t^2+z^2}) (t^2+z^2)^{-\frac{\nu}{2}} t^{\mu+1} \\ = b^\mu a^{-\nu} z^{\mu-\nu+1} (a^2+b^2)^{\frac{\nu}{2}-\frac{\mu}{2}-\frac{1}{2}} K_{\nu-\mu-1}(z\sqrt{a^2+b^2}) \end{aligned}$$

$\Re(\mu) > -1, \Re(z) > 0.$

For  $n = 1$ , considering that  $J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos(x)$ , we obtain

$$\begin{aligned} \tilde{f}_1(k_1) &= \int_{-\infty}^{+\infty} dx_1 e^{ik_1 x_1} f_1(x_1) = \frac{2\alpha^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}} K_{\frac{\nu}{2}}(\alpha)} \int_0^{+\infty} dx_1 \frac{K_{\frac{\nu}{2}+\frac{1}{2}}(\alpha\sqrt{1+x_1^2})}{(1+x_1^2)^{\frac{\nu}{4}+\frac{1}{4}}} \cos(k_1 x_1) \\ &= \frac{\alpha^{\frac{1}{2}} k_1^{\frac{1}{2}}}{K_{\frac{\nu}{2}}(\alpha)} \int_0^{+\infty} dx_1 J_{-\frac{1}{2}}(k_1 x_1) K_{\frac{\nu}{2}+\frac{1}{2}}(\alpha\sqrt{1+x_1^2}) (1+x_1^2)^{-\frac{\nu}{4}-\frac{1}{4}} x_1^{\frac{1}{2}} \\ &= \frac{K_{\frac{\nu}{2}}(\sqrt{\alpha^2+k_1^2})}{K_{\frac{\nu}{2}}(\alpha)} \frac{(\alpha^2+k_1^2)^{\frac{\nu}{4}}}{\alpha^{\frac{\nu}{2}}}. \end{aligned}$$

For alternative derivations in the univariate case see Hurst [25] and the references therein.

In our setting the computation is exactly the same for general  $n$ , with  $k = \sqrt{\sum_{i=1}^n k_i^2}$ ,  $d^{n-2}\Omega$  the surface element of the sphere  $S^{n-2}$ ,  $\phi$  the angle between  $\mathbf{k}$  and  $\mathbf{x}$ , using identity (8)

$$\begin{aligned} \tilde{f}_n(\mathbf{k}) &= \int_{\mathbb{R}^n} d^n \mathbf{x} e^{i\mathbf{k}\cdot\mathbf{x}} f_n(\mathbf{x}) \\ &= \frac{\alpha^{\frac{n}{2}}}{(2\pi)^{\frac{n}{2}} K_{\frac{\nu}{2}}(\alpha)} \int d^{n-2}\Omega \int_0^{+\infty} dr r^{n-1} \int_0^\pi d\phi \sin^{n-2}(\phi) e^{ikr \cos \phi} \frac{K_{\frac{\nu}{2}+\frac{n}{2}}(\alpha\sqrt{1+r^2})}{(1+r^2)^{\frac{\nu}{4}+\frac{n}{4}}} \\ &= \frac{k^{1-\frac{n}{2}} \alpha^{\frac{n}{2}}}{K_{\frac{\nu}{2}}(\alpha)} \int_0^{+\infty} dr J_{\frac{n}{2}-1}(kr) K_{\frac{\nu}{2}+\frac{n}{2}}(\alpha\sqrt{1+r^2}) (1+r^2)^{-\frac{\nu}{4}-\frac{n}{4}} r^{\frac{n}{2}} \\ &= \frac{K_{\frac{\nu}{2}}(\sqrt{\alpha^2+k^2})}{K_{\frac{\nu}{2}}(\alpha)} \frac{(\alpha^2+k^2)^{\frac{\nu}{4}}}{\alpha^{\frac{\nu}{2}}}. \end{aligned}$$

Hence the eventual result  $\tilde{f}_n(\mathbf{k}) = \tilde{f}_1(k)$ .

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- [1] E. Bacry, J. Delour, and J. F. Muzy. Modelling financial time series using multifractal random walks. *Physica A*, 299(1-2):84–92, 2001.
- [2] E. Bacry, J. Delour, and J. F. Muzy. Multifractal random walk. *Physical Review E*, 64(2):26103, 2001.
- [3] E. Bacry, A. Kozhemyak, and J. F. Muzy. Are asset return tail estimations related to volatility long-range correlations? *Physica A*, 370(1):119–126, Oct 2006.
- [4] F. Baldovin and A. L. Stella. Central limit theorem for anomalous scaling due to correlations. *Physical Review E*, 75(2):020101, 2007.
- [5] F. Baldovin and A. L. Stella. Scaling and efficiency determine the irreversible evolution of a market. *Proc. Natl. Acad. Sci. USA*, 104(50):19741–4, 2007.
- [6] F. Baldovin and A. L. Stella. Role of scaling in the statistical modeling of finance, 2008. URL <http://arxiv.org/abs/0804.0331>. Based on the Key Note lecture by A.L. Stella at the Conference on “Statistical Physics Approaches to Multi-Disciplinary Problems”, IIT Guwahati, India, 7-13 January 2008.
- [7] N. H. Bingham and R. Kiesel. Modelling asset returns with hyperbolic distributions. In J. Knight and S. Satchell, editors, *Return Distributions in Finance*, chapter 1, pages 1–20. Butterworth-Heinemann, 2001.
- [8] T. Bollerslev, R. F. Engle, and D. B. Nelson. ARCH Models. In R. F. Engle and D. L. McFadden, editors, *Handbook of Econometrics*, pages 2959–3038. Elsevier, 1994.
- [9] L. Borland. Option pricing formulas based on a non-gaussian stock price model. *Physical Review Letters*, 89(9):98701, 2002.
- [10] L. Borland and J. P. Bouchaud. On a multi-timescale statistical feedback model for volatility fluctuations. Science & Finance (CFM) working paper archive 500059, Science & Finance, Capital Fund Management, July 2005.
- [11] L. Borland, J. P. Bouchaud, J. F. Muzy, and G. O. Zumbach. The Dynamics of Financial Markets – Mandelbrot’s multifractal cascades, and beyond. Science & Finance (CFM) working paper archive 500061, Science & Finance, Capital Fund Management, January 2005.
- [12] J. P. Bouchaud. Elements for a theory of financial risks. *Physica A*, 263:415–426, February

1999.

- [13] J. P. Bouchaud and M. Potters. *Theory of financial risk and derivative pricing : from statistical physics to risk management*. Cambridge Univ. Press, second edition, 2003.
- [14] J. P. Bouchaud, M. Potters, and M. Meyer. Apparent multifractality in financial time series. *European Physical Journal B*, 13:595–599, January 2000.
- [15] J. P. Bouchaud, A. Matacz, and M. Potters. Leverage effect in financial markets: The retarded volatility model. *Physical Review Letters*, 87(22):228701, Nov 2001.
- [16] U. Cherubini, E. Luciano, and W. Vecchiato. *Copula methods in finance*. Wiley Finance. Wiley, 2004.
- [17] R. Cont. Empirical properties of asset returns: stylized facts and statistical issues. *Quantitative Finance*, 1(2):223–236, February 2001.
- [18] R. Cont and P. Tankov. *Financial Modelling with Jump Processes*, chapter 4. Financial Mathematics Series. CRC Press, 2004.
- [19] I. Dreier and S. Kotz. A note on the characteristic function of the  $t$ -distribution. *Statistics & Probability Letters*, 57(3):221–224, 2002.
- [20] S. Drozd, M. Forczek, J. Kwapien, P. Oswiecimka, and R. Rak. Stock market return distributions: From past to present. *Physica A*, 383(1):59–64, Sep 2007.
- [21] E. Eberlein and U. Keller. Hyperbolic distributions in finance. *Bernoulli*, 1(3):281–299, 1995.
- [22] Z. Eisler and J. Kertész. Multifractal model of asset returns with leverage effect. *Physica A*, 343:603–622, November 2004.
- [23] A. Erdélyi. *Higher Transcendental Functions (Vol. 2)*. McGraw–Hill Publisher, 1953.
- [24] C. C. Heyde and N. N. Leonenko. Student processes. *Advances in Applied Probability*, 37: 342–365, 2005.
- [25] S. Hurst. The characteristic function of the student  $t$  distribution. Technical Report SRR95-044, Australian National University, Centre for Mathematics and its Applications, Canberra, September 1995.
- [26] S. Jaffard. Multifractal Formalism for Functions Part I: Results Valid for All Functions. *SIAM Journal on Mathematical Analysis*, 28:944–970, 1997.
- [27] S. Jaffard. Multifractal Formalism for Functions Part II: Self-Similar Functions. *SIAM Journal on Mathematical Analysis*, 28:971–998, 1997.
- [28] Z. Q. Jiang and W. X. Zhou. Multifractality in stock indexes: Fact or fiction? *Physica A*,

387:3605–3614, June 2008.

- [29] F. Lillo. Limit order placement as an utility maximization problem and the origin of power law distribution of limit order prices. *European Physical Journal B*, 55:453–459, February 2007.
- [30] F. Lillo and R. N. Mantegna. Power-law relaxation in a complex system: Omori law after a financial market crash. *Physical Review E*, 68(1):016119, Jul 2003.
- [31] P. E. Lynch and G. O. Zumbach. Market heterogeneities and the causal structure of volatility. *Quantitative Finance*, 3(4):320–331, 2003.
- [32] Y. Malevergne and D. Sornette. *Extreme Financial Risks*. Springer, 2006.
- [33] B. Mandelbrot, A. Fisher, and L. Calvet. A multifractal model of asset returns. Cowles Foundation Discussion Papers 1164, Cowles Foundation, Yale University, September 1997.
- [34] J. L. McCauley, K. E. Bassler, and G. H. Gunaratne. Martingales, the efficient market hypothesis, and spurious stylized facts, October 2007. URL <http://arxiv.org/abs/0710.2583>.
- [35] M. Musiela and M. Rutkowski. *Martingale Methods in Financial Modelling*, chapter 7, pages 237–278. Springer Verlag, second edition, 2005.
- [36] J. F. Muzy, E. Bacry, and A. Kozhemyak. Extreme values and fat tails of multifractal fluctuations. *Physical Review E*, 73(6):066114, 2006.
- [37] R. B. Nelsen. *An introduction to copulas*. Springer Series in Statistics. Springer, second edition, 2006.
- [38] R. Osorio, L. Borland, and C. Tsallis. Distributions of high-frequency stock market observables. In M. Gell-Mann and C. Tsallis, editors, *Nonextensive entropy: interdisciplinary applications*, page 321. Oxford University Press, 2004.
- [39] Development Core Team R. *R: A Language and Environment for Statistical Computing*. R Foundation for Statistical Computing, Vienna, Austria, 2008. URL <http://www.r-project.org>.
- [40] R. H. Riedi. Multifractal processes. In P. Doukhan, G. Oppenheim, and M. S. Taqqu, editors, *Long-range Dependence: Theory and Applications*, pages 625–716. Birkhauser, 2002.
- [41] I. M. Sokolov, A. V. Chechkin, and J. Klafter. Fractional diffusion equation for a power-law-truncated Lévy process. *Physica A*, 336(3-4):245–251, May 2004.
- [42] R. S. Tsay. *Analysis of Financial Time Series*, chapter 3. John Wiley & Sons, 2002.

- [43] C. Vignat and A. Plastino. Scale invariance and related properties of  $q$ -Gaussian systems. *Physics Letters A*, 365:370–375, June 2007.
- [44] P. Weber, F. Wang, I. Vodenska-Chitkushev, S. Havlin, and H. E. Stanley. Relation between volatility correlations in financial markets and Omori processes occurring on all scales. *Physical Review E*, 76(1):016109, 2007.
- [45] G. O. Zumbach. Volatility processes and volatility forecast with long memory. *Quantitative Finance*, 4(1):70–86, 2004.
- [46] G. O. Zumbach. Time reversal invariance in finance, August 2007. URL <http://arxiv.org/abs/0708.4022>.
- [47] G. O. Zumbach, M. M. Dacorogna, J. L. Olsen, and R. B. Olsen. Measuring shock in financial markets. *International Journal of Theoretical and Applied Finance*, 3:347–355, 2000.
- [48] All the graphics and numerical calculations have been performed with R [39].