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Zaman, Asad

Bilkent University

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Abstract

The Breusch-Pagan Lagrange Multiplier test for heteroskedascity is supposedly able to detect heteroskedasticity which is an arbitrary function of some set of regressors. We will show that in fact it detects only linear functions. The test is inconsistent for general alternatives, in the sense that its power does not go to 1 as the sample size increases (and in fact, can be arbitrarily low). Since in fact the Breusch-Pagan test is essentially an F test in a special model, we also give necessary and sufficient conditions for the consistency of the F test under misspecification.

1 Introduction

In a classic article, Breusch and Pagan (1979) introduced a Lagrange Multiplier test for heteroskedasticy which appears to allow for very general types of alternatives. Specifically, in a regression model $y_t = x_t'\beta + \epsilon_t$, where $\mathrm{Var}(\epsilon_t) = \sigma_t^2 = f(\gamma_0 + \gamma' z_t)$, Breusch and Pagan give a test of the null hypothesis $H_0 : \gamma = 0$ for arbitrary smooth functions f. The object of this note is to show that this apparent generality is an illusion, and the test is consistent only for $f(x) = x$, the identity function. Nonlinear functions f are tested for as alternatives *only* to the extent that they are correlated with the regressors z. In particular, for any non-zero value of γ such that

$\mathrm{Cov}(f(\gamma_0+\gamma' z_t), z_t) = 0$, the Breusch-Pagan test has no power asymptotically (i.e. is inconsistent).

As a preliminary result of independent interest, we characterize situations where the F-test is consistent in regression models. We then show that the Breusch-Pagan test is asymptotically equivalent to a certain F test, and use our characterization to get the desired result.

2 Consistency of the F test

We develop necessary and sufficient conditions for the consistency of the F test for significance of a set of regressors in a rather general setting allowing for substantial misspecification. Almost every symbol to follow will depend on the sample size T, but it will be notationally convenient to suppress this dependence.

Suppose $y = y(T)$ is a $T \times 1$ vector of observations on a dependent variable. We wish to 'explain' y by means of the regressors $\mathbf{1} = \mathbf{1}(T)$, a $T \times 1$ vector of 1's, and a $T \times K$ matrix $X = X(T)$ of observations on the independent variables. It will be convenient, and entail no loss of generality, to assume that the regressors are in the form of differences from means, so that $X'\mathbf{1} = 0$. Define $P = X(X'X)^{-1}X'$ and $Q = \mathbf{I} - \{P + (1/T)\mathbf{1}\mathbf{1}'\}$ and let $d_P = K$ be the rank of P and $d_Q = T - (K+1)$ be the rank of Q. Denoting by $\mathbf{X}$ the vector space spanned by the columns of X and by $\mathbf{Z}$ the vector space orthogonal to $\mathbf{X}$ and $\mathbf{1}$, note that Py is the projection of y onto $\mathbf{X}$ and Qy is the projection of y onto $\mathbf{Z}$.

In the linear regression model $y = \beta_0\mathbf{1} + X\beta + \epsilon$, the standard F statistic for testing the null hypothesis $\beta = 0$ can be written as

$$F = \frac{\|Py\|^2/d_P}{\|Qy\|^2/d_Q} = \frac{y'P^*y}{y'Q^*y},$$

where $P^* = P/d_P$ and $Q^* = Q/d_Q$. Effectively, the F statistic compares the average projection of y on $\mathbf{X}$ to the average projection of y on $\mathbf{Z}$. Here the word 'average' indicates that the squared length of the projection is divided by the dimension of the space on the which the projection is made.

In order to allow for misspecification, we assume that $y = \mu + \epsilon$, where $\mu = \mu(T)$ is the $T \times 1$ (nonstochasic) mean vector of the dependent variable, and the $T \times 1$ vector of errors $\epsilon = \epsilon(T)$ satisfies the following condition. There exists a constant B such that for all T and all $T \times T$ projection matrices M, the following inequality holds:

$$\mathrm{Var}(\epsilon' M \epsilon) \leq B \mathrm{tr}(M) \tag{1}$$

Hypothesis (1) is a mild assumption which will typically be satisfied by most error sequences. Lemma 1 below gives one set of sufficient conditions which ensures (1). All proofs are given in the appendix.

Lemma 1 *If a sequence of errors $\epsilon_1, \epsilon_2, \ldots$ (A) forms a martingale, and (B) for all i, j, $Var(\epsilon_i \epsilon_j) \leq B < \infty$, then it also satisfies (1).*

Intuitively speaking, the F test is designed to assess whether the regressors X have a significant relationship with μ, the mean of y. The theorem below gives necessary and sufficient conditions for the consistency of F test.

Theorem 1 *Suppose $y = \mu + \epsilon$ where ϵ satisfies the condition 1 given above. Then the F test is rejects the null with probability one if:*

$$\lim_{T \to \infty} \frac{\mu' P^* \mu}{1 + \mu' Q^* \mu} = \infty \tag{2}$$

For the converse, let $\Sigma = \Sigma(T)$ be the covariance matrix of the errors ϵ, and suppose that $0 < m < \lambda_{min}(\Sigma)$; that is, the smallest eigenvalue of the covariance matrix of the errors is bounded away from zero for all T. If the F test rejects the null with probability one then condition (2) must hold.

The theorem states that, under mild assumptions, the F test for the significance of a set of regressors X will reject the null with probability one if and only if the average projection of the mean vector μ of y on the space X is substantially larger (infinitely larger asymptotically) than its average projection on Z, the space orthogonal to X and $\mathbf{1}$.

The problem with Lagrange Multiplier tests which arises in the Breusch-Pagan case can now be illustrated in a simpler setup. Suppose that $y_t = f(\alpha + \beta' x_t) + \epsilon_t$ for $t = 1, 2, \ldots, T$, where ϵ_t is an i.i.d. sequence of errors with common distribution $N(0, \sigma^2)$. If f is any smooth function, it is easily checked that the Lagrange multiplier test of the null hypothesis that $H_0 : \beta = 0$ is equivalent to the overall F statistic for the linear regression $y_t =$

$\alpha + \beta x_t + \epsilon_t$; see Section 11.4.2 of Zaman (1996) for a derivation. Thus the Lagrange Multiplier principle suggests that the overall F statistic for a linear regression tests for the presence of *any* smooth relationship between the regressors and the dependent variable. However, our characterization of the consistency of the F-test shows that the F-test will 'detect' (i.e. be consistent for) nonlinear relationships f only to the extent that $f(\alpha + \beta' x_t)$ is linearly correlated with x_t. In particular, if a non-zero value of β is such that $\mathrm{Cov}(f(\alpha + \beta' x_t), x_t) = 0$, then the usual F test will be unable to reject the null hypothesis that $\beta = 0$ even asymptotically.

3 Inconsistency of Breusch-Pagan test

We now derive the inconsistency of the Breusch-Pagan test as a consequence of our Theorem 1. Suppose $y_t = \beta' x_t + \epsilon_t$, where ϵ_t are independent $N(0, \sigma_t^2)$. It will be convenient to adopt the following notational conventions:

- $[a_t]$ refers to a $T \times 1$ column vector with t-th element a_t.
- $[b_{ij}]$ refers to a $T \times T$ matrix with (i, j) entry b_{ij}.

Let $e_t = y_t - \hat{\beta}' x_t$ be the residuals from an OLS regression. Breusch and Pagan (1979) derived the Lagrange Multiplier (LM) test of the null hypothesis $H_0 : \gamma = 0$ given that $\sigma_t^2 = f(\alpha + \gamma' z_t)$ for some smooth function f. Koenker (1981) showed that the original LM statistic is very sensitive to the assumption of normality, while the asymptotically equivalent statistic based on the TR^2 of the (auxiliary) regression of $[e_t^2]$ on a constant and Z remains robust to non-normality. Since the overall F statistic for a regression is a monotonic transform of the TR^2, it is clear that using the overall F for the auxiliary regression will be asymptotically equivalent to the Breusch-Pagan test. Assume without loss of generality that the regressors $Z = Z(T)$ have been differenced from their means so that $Z'\mathbf{1} = 0$. Let $P = Z(Z'Z)^{-1}Z'$ be the matrix of the projection onto the column space of Z and let Q be the matrix of the projection onto the vector space orthogonal to $\mathbf{1}$ and the columns of Z.

Theorem 2 *Assume the variances σ_t^2 are bounded above: for all t, $\sigma_t^2 \leq M < \infty$. Then the Breusch-Pagan test rejects the null hypothesis of ho-*

moskedasticity with probability one if

$$\lim_{T\to\infty} \frac{[\sigma_t^2]' P^* [\sigma_t^2]}{1 + [\sigma_t^2]' Q^* [\sigma_t^2]} = \infty \tag{3}$$

The converse also holds if the variances are bounded away from zero: for all t, $\sigma_t^2 > c > 0$.

Thus consistency of the Breusch-Pagan test requires the average projection of the vector $[\sigma_t^2]$ of variances on the column space of the regressors Z to be large relative to the average projection on the orthogonal complement of this space. This shows that the Breusch-Pagan test only detects linear relationships between the variables tested for and the vector of variances $[\sigma_t^2]$. To give a simple example, suppose x_t is i.i.d. $N(0,1)$ and $y_t = a_t x_t$ where a_t is i.i.d. $N(a,1)$. This random coefficient model can be rewritten as $y_t = a x_t + \epsilon_t$, where $\sigma_t^2 = \mathrm{Var}(\epsilon_t | x_t) = x_t^2$. Letting $f(x) = x^2$, it is clear that $\sigma_t^2 = f(a + b x_t)$ with $a = 0$ and $b = 1$. Let $[e_t^2]$ be the vector of squared residuals from an OLS regression of y on x, and use the F statistic for the regression of $[e_t^2]$ on $\mathbf{1}$ and x to test for heteroskedasticity. Then our earlier results imply that this test will not reject the null, since x^2 and x are uncorrelated (because $x \sim N(0,1)$). This test is asymptotically equivalent to the Breusch-Pagan test, and hence the Breusch-Pagan will also not reject the null asymptotically. This shows clearly that the Breusch-Pagan test only detects linear relationships and is not valid for general smooth functions f.

4 Appendix

We prove Theorems 1 and 2, and also prove Lemma 1 which is useful in verifying condition (1) for error sequences.

Proof of Lemma 1: Let $\epsilon = (\epsilon_1, \ldots, \epsilon_T)$ be a sequence of random variables satisfying properties (A) and (B) of Lemma1. Let Σ be the $T \times T$ covariance matrix of the vector ϵ. We will use Σ_{ij} for the (i,j) entry of the matrix Σ. The martingale property ensures that $\Sigma_{ij} = 0$ if $i \neq j$.

Let P be any idempotent matrix. We aim to show that $\mathrm{Var}(\epsilon' P \epsilon) \leq 2B\mathrm{tr}P$ To prove this, note that

$$\mathrm{Var}(\epsilon' P \epsilon) = \mathbf{E}\left\{\mathrm{tr}(\epsilon\epsilon' - \Sigma)P\right\}^2$$

$$
\begin{aligned}
&= \sum_{i,j}\sum_{k,l}\mathbf{E}\left([\epsilon_i\epsilon_j - \sigma_{i,j}][\epsilon_k\epsilon_l - \sigma_{k,l}]P_{i,j}P_{k,l}\right) \\
&= \sum_{i,j}\mathbf{E}(\epsilon_i\epsilon_j - \sigma_{i,j})^2 P_{i,j}^2 \\
&\leq 2B\sum_{i,j}P_{i,j}^2 = 2B\mathrm{tr}(P'P) = 2B\mathrm{tr}P
\end{aligned}
$$

In this derivation, we have used the fact that if ϵ_t forms a martingale, then the terms $\mathbf{E}(\epsilon_i\epsilon_j - \sigma_{i,j})(\epsilon_k\epsilon_l - \sigma_{k,l}) = 0$ unless $i = j, k = l$ or else $i = k, j = l$.

Proof of Theorem 1: Define $\Delta = 1 + \mu' Q^* \mu$, $N = \Delta^{-1}(y'P^*y)$, and $D = \Delta^{-1}(y'Q^*y)$. It is immediate that $F = N/D$. We will show that ED converges to a strictly positive quantity and $\mathrm{Var}(D)$ goes to 0. From this it follows that convergence of F to $+\infty$ is equivalent to convergence of N to $+\infty$. Then we will show that N goes to infinity if and only the hypothesis of the theorem holds.
Step 1: $\mathbf{E}D = \Delta^{-1}(\mu'Q^*\mu + \mathrm{tr}\Sigma Q^*)$. Now $\mathrm{tr}\Sigma Q^* \leq M\mathrm{tr}Q^* = M$. It is easily deduced that ED is bounded away from $+\infty$. If the variances Σ_{tt} are greater than $m > 0$ than ED is also bounded away from 0, since $\mathrm{tr}\Sigma Q^* \geq m\mathrm{tr}Q^* = m$. Next we will show that $\mathrm{Var}(D) \to 0$.

To prove this, first note that $\mathrm{Var}(X+Y) \leq 2\mathrm{Var}(X) + 2\mathrm{Var}(Y)$. Now $\mathrm{Var}(\Delta^{-1}(y'Q^*y) = \Delta^{-2}\mathrm{Var}(2\epsilon'Q^*\mu + \epsilon'Q^*\epsilon) \leq 2\Delta^{-2}\left(2\mathrm{Var}(\epsilon'v) + \mathrm{Var}(\epsilon'Q^*\epsilon)\right)$, where $v = Q^*\mu$. It is clear that $\mathrm{Var}(\epsilon'v) \leq B\|v\|^2$ where B_2 is the upper bound on the error variances. The assumption (1) permits us to conclude that $\mathrm{Var}(\epsilon'Q^*\epsilon) = \mathrm{Var}(\epsilon'Q\epsilon)/d_Q^2 \leq B(\mathrm{tr}Q)/d_Q^2 = B_4/d_Q$. We thus conclude that

$$\mathrm{Var}(D) \leq 2\Delta^{-2}\left(2B_2\|Q^*\mu\|^2 + B_4/d_Q\right)$$

Now $\|Q^*\mu\|^2 = \mu'Q\mu/d_Q^2 = \mu'Q^*\mu/d_Q$ so that $\mathrm{Var}(D) \leq (4B/d_Q)(\mu'Q^*\mu + (1/2))/(1+\mu'Q^*\mu)^2$. This will converge to 0 as d_Q goes to infinity.
Step 2: We will now show that $EN \to +\infty$. Also, if we define $S_N = \sqrt{\mathrm{Var}(N)}$, then both $S_N \to +\infty$ $EN/S_N \to +\infty$. From these facts we can conclude that $N \to +\infty$ with probability one as follows. We wish to show that for any (large,positive) constant k, $\mathrm{P}(N > k)$ converges to one. Note that $\mathrm{P}(N > k) = \mathrm{P}((N - EN)/S_N) > (k - EN)/S_N)$. Let $X = (N - EN)/S_N$. If S_N and EN/S_N both go to $+\infty$ then $(k - EN)/S_N \to -\infty$ so the probability in question converges to $\mathrm{P}(X > -\infty)$. Since X has mean 0 and

variance 1, this probability converges to unity by, for example, Chebyshev's Inequality. It remains to show that EN and EN/S_N converge to $+\infty$.

First consider $EN = \Delta^{-1}(\mu' P^* \mu + \mathrm{tr}\Sigma P^*)$. Ignoring the term $\mathrm{tr}\Sigma P^*$ which is positive, the remaining term is assumed to converge to $+\infty$ as the main hypothesis of the theorem we are proving. Next consider $\mathrm{Var}(N) \leq \Delta^{-2}(4B/d_P)(\mu' P^* \mu + (1/2))$, following the same logic as for $\mathrm{Var}(D)$. With $S_N = \sqrt{\mathrm{Var}(N)}$ it follows that

$$EN/S_N \geq 2\sqrt{B/d_P}\frac{\mu' P^* \mu + \mathrm{tr}\Sigma P^*}{(\mu' P^* \mu + (1/2))^{1/2}}$$

It is clear that this goes to $+\infty$ provided that $\mu' P^* \mu$ does, which is entailed by the hypothesis of the theorem.

Conversely suppose the hypothesis of the theorem does not hold. It is immediate that EN fails to go to $+\infty$. Since D is bounded away from 0 and $N \geq 0$, it is immediate that F cannot converge to $+\infty$ with probability one.

Proof of Theorem 2: *Step 1:* Define $M_T = M = \left\{\sum_{t=1}^T x_t x_t'\right\}^{-1}$ A key quantity which occurs in the proof is $a_t = x_t' M_T x_t$. We will need to bound this as below. The largest possible projection of the vector $a = [a_t] = (a_1, \ldots, a_T)'$ is onto itself, so that $\|a' P a\| \leq \sum_{t=1}^T a_t^2$. To bound this, note that:

$$\sum_{t=1}^T a_t = \sum_{t=1}^T \mathrm{tr} x_t' M_T x_t = \mathrm{tr}\left(\sum_{t=1}^T x_t x_t'\right) M_T = \mathrm{tr} M_T^{-1} M_T = K$$

Since $a_t \geq 0$, it follows that $a_t \leq K$ so that

$$1 = \sum_{t=1}^T (a_t/K) \geq \sum_{t=1}^T a_t^2/K^2.$$

This implies that $\|a\|^2 \leq K^2$. From this it follows that $0 \leq a' P^* a \leq K^2/d_P$ and $0 \leq a' Q^* a \leq K^2/d_Q$.

Step 2: To prove the theorem, it suffices to show that $[e_t^2] W [e_t^2]$ behaves asymptotically similarly to $[\epsilon_t^2] W [\epsilon_t^2]$ for the matrices $W = P^*$ and $W = Q^*$, since applying Theorem 1 to the second form yields the result immediately. We will therefore analyse the difference between the two quadratic forms and show that they remain bounded asymptotically. From this the result will follow. Define $z_t = X'Wx_t$ and note that

$$e_t^2 = (\epsilon_t - x_t'WX\epsilon)^2 = (\epsilon_t - z_t'\epsilon)^2 = \epsilon_t^2 - 2\epsilon_t(z_t'\epsilon) + (z_t'\epsilon)^2.$$

Thus the difference D we wish to show is asymptotically negligible can be written as

$$\begin{aligned} D &= [e_t^2]W[e_t^2] - [\epsilon_t^2]W[\epsilon_t^2] \\ &= [(z_t'\epsilon)^2]W[(z_t'\epsilon)^2] - 4[(z_t'\epsilon)^2]W[\epsilon_t(z_t'\epsilon)] + 2[(z_t'\epsilon)^2]W[\epsilon_t^2] \\ &\quad +4[\epsilon_t(z_t'\epsilon)]W[\epsilon_t(z_t'\epsilon)] - 4[\epsilon_t(z_t'\epsilon)]W[\epsilon_t^2] \end{aligned}$$

We will show that each of the five terms in the difference converges in quadratic mean to zero asymptotically. This will prove the result.

Consider the first term $T_1 = [(z_t'\epsilon)^2]W[(z_t'\epsilon)^2]$. Note that

$$\mathbf{E}T_1 = \mathrm{tr}\mathbf{E}[(z_t'\epsilon)^2][(z_t'\epsilon)^2]'W = \mathrm{tr}[\mathbf{E}(z_i'\epsilon)^2(z_j'\epsilon)^2]W \le \sum_{t=1}^{T} \mathbf{E}(z_t'\epsilon)^4$$

The last inequality follows from Amemiya's Lemma, according to which $\mathrm{tr}AB \le (\mathrm{tr}A)\lambda_{max}(B)$ when A and B are positive semidefinite matrices. Since $W = P, Q$ are projection matrices, the largest eigenvalue is 1. Since $z_i'\epsilon$ is normal, its fourth moment is just 3 times its variance, so that

$$\mathbf{E}T_1 = 3\sum_{t=1}^{T} z_t'\Sigma z_t$$

Now Σ is a diagonal matrix with elements bounded above by $M < \infty$ and below by $m > 0$. It follows that the difference between $z_t'\Sigma z_t$ and $a_t = z_t'z_t$ is bounded, and since $\sum_t a_t = K$ as established in Step 1, we conclude that $|\mathbf{E}T_1 - K| \le C$. It follows that T_1 cannot go to infinity. We make a similar, but more complex calculation to show that the variance of T_1 is similarly bounded.

$$\begin{aligned} \mathrm{Var}(T_1) &= \mathbf{E}\left\{\mathrm{tr}\left([(z_t'\epsilon)^2][(z_t'\epsilon)^2]' - \mathbf{E}[(z_t'\epsilon)^2][(z_t'\epsilon)^2]'\right)W\right\}^2 \\ &= \sum_{i,j=1}^{T}\sum_{k,l=1}^{T} \mathrm{Cov}((z_i'\epsilon)^2(z_j'\epsilon)^2, (z_k'\epsilon)^2(z_l'\epsilon)^2)W)_{ij}W_{kl} \end{aligned}$$

To calculate the required covariance, we need the following formula. If X_1, X_2, X_3, X_4 are jointly normal, and $\sigma_{i,j} = \mathrm{Cov}(X_i, X_j)$, then

$$\begin{aligned}
\mathbf{E}X_1^2X_2^2X_3^2X_4^2 &= \sigma_{11}\sigma_{22}\sigma_{33}\sigma_{44} \\
&\quad +2\sigma_{11}\sigma_{22}\sigma_{34}^2 + 2\sigma_{11}\sigma_{23}^2\sigma_{44} + 2\sigma_{12}^2\sigma_{33}\sigma_{44} + 2\sigma_{13}^2\sigma_{22}\sigma_{44} + 2\sigma_{11}\sigma_{24}^2\sigma_{33} \\
&\quad +4\sigma_{12}^2\sigma_{34}^2 + 4\sigma_{13}^2\sigma_{24}^2 + 4\sigma_{14}^2\sigma_{23}^2 \\
&\quad +8\sigma_{11}\sigma_{23}\sigma_{24}\sigma_{34} + 8\sigma_{22}\sigma_{13}\sigma_{14}\sigma_{34} + 8\sigma_{33}\sigma_{12}\sigma_{14}\sigma_{34} + 8\sigma_{44}\sigma_{12}\sigma_{13}\sigma_{23} \\
&\quad +16\sigma_{12}\sigma_{34}\sigma_{13}\sigma_{24} + 16\sigma_{12}\sigma_{34}\sigma_{14}\sigma_{23} + 16\sigma_{13}\sigma_{24}\sigma_{14}\sigma_{23}
\end{aligned}$$

By Cauchy-Scwhartz, we have $\sigma_{ij}^2 \le \sigma_{ii}\sigma_{jj}$. It follows from the formula that $\mathbf{E}X_1^2X_2^2X_3^2X_4^2 \le 99\sigma_{11}\sigma_{22}\sigma_{33}\sigma_{44}$, so that

$$\begin{aligned}
\mathrm{Var}(T_1) &\le \sum_{i,j}\sum_{k,l}(z_i'\Sigma z_i)(z_j'\Sigma z_j)(z_k'\Sigma z_k)(z_l'\Sigma z_l)W_{ij}W_{kl} \\
&= \left(\sum_{i,j=1}^{T}(z_i'\Sigma z_i)(z_i'\Sigma z_j)W_{ij}\right)^2
\end{aligned}$$

This is bounded by the square of the mean.

To complete the proof requires showing that the other four terms in the difference of $[\epsilon_t^2]W[\epsilon_t^2]$ and $[e_t^2]W[e_t^2]$ remain bounded by a finite quantity with probability 1. These follow exactly the same procedure outlined above, and in fact are slightly easier. Thus these proofs are omitted for brevity.

Now consider the effect of replacing W by the matrix P^*. The difference between $[\epsilon_t^2]W[\epsilon_t^2]$ and $[e_t^2]W[e_t^2]$ for $W = P^*$ remains bounded by a finite quantity with probability 1 asymptotically. Thus convergence to infinity of one of the terms is equivalent to convergence to infinity of the other. For $W = Q^* = Q/d_Q$ in the denominator, the difference is again bounded by a finite quantity. Dividing by d_q which goes to infinity makes the difference go to zero asymptotically. This means that the hypothesis of the theorem applied to $[e_t^2]$ give the same results as the hypothesis applied to $[\epsilon_t^2]$. This is what we desire to prove.

References

[1] Bickel, P. J. (1978), 'Using residuals robustly I: Tests for heteroscedasticity, nonlinearity', *Annals of Statistics* **6**, 266–291.

[2] Breusch, T. & Pagan, A. (1979), 'A simple test of heteroskedasticity and random coefficient variation', *Econometrica* **47**, 1287–1294.

[3] Chesher, A. (1984), 'Testing for neglected heterogeneity', *Econometrica* **52**, 865–872.

[4] Dutta, J. & Zaman, A. (1990), What do tests for heteroskedasticity detect?, Technical Report 9022, Center for Operations Research and Econometrics, Universite Catholique de Louvain, Belgium.

[5] Koenker, R. (1981), 'A note on studentizing a test for heteroskedasticity', *Journal of Econometrics* **17**, 107–112.

[6] White, H. (1980), 'A heteroskedasticity consistent covariance matrix estimator and a direct test for heteroskedasticity', *Econometrica* **48**, 817–838.

[7] Zaman, A. (1996), *Statistical Foundations for Econometric Techniques*, Academic Press, New York.