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Bazhanov, Andrei

Far Eastern National University, Queen's University

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Sustainable growth: The extraction-saving relationship

Andrei V. Bazhanov^{a,b*}

^aDepartment of Economics, Queen's University, Kingston, ON, K7L 3N6, Canada

^bFar Eastern National University, 8 Ulitsa Sukhanova, Vladivostok, 690600, Russia

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Abstract

The paper presents two new results for the Dasgupta-Heal-Solow-Stiglitz model with an essential nonrenewable resource:

(1) the pattern of resource extraction can be more important for sustainable growth than the pattern of saving when the Hotelling Rule modifier is not small enough;

(2) the qualitative behavior of the long-run per capita consumption can be examined along any smooth enough path of extraction using the “sustainability functional” introduced in the paper.

JEL Classification: O13; O47; Q32; Q38

Keywords: sustainable growth; modified Hotelling Rule; sustainability number; Hubbert curve consumption

*Tel.: 1-(613)-533-6000 ext. 75468; fax: 1-(613)-533-6668.
E-mail: bazhanov@econ.queensu.ca

1 Introduction

Dasgupta and Heal (1979, pp. 303-306) and Hamilton et al. (2006) showed that the investments exceeding the standard Hartwick Investment Rule (Hartwick, 1977) imply sustainable unbounded growth in per capita consumption under the standard Hotelling Rule for the Dasgupta-Heal-Solow-Stiglitz (DHSS) model (Dasgupta and Heal, 1974; Solow, 1974; Stiglitz, 1974).

Stollery (1998) considered an externality (global warming) that implied modification of the Hotelling Rule and corresponding modification of the path of extraction. Combination of this extraction path with the standard Hartwick Rule resulted in sustainable bounded growth of per capita consumption in the long run. Another example was obtained in (Bazhanov, 2007b) where I examined the properties of the transition paths constructed under the assumption of the modified Hotelling Rule. This case also gave the patterns of bounded and unbounded growth of per capita consumption under the standard Hartwick Rule.

These examples raise a question about the roles of the patterns of saving and extraction for sustaining the growth of an economy in the long run. The answer to this question is the first main result of the paper (Proposition 1, Section 3). It shows that the pattern of resource extraction is more important for sustainability of growth than the pattern of saving when the Hotelling Rule modifier is not close enough to zero. The pattern of saving defines the level of consumption along the growing or declining path in this case. This result is obtained for the DHSS model with the constant-share-of-output investment rule and the modified Hotelling Rule.

The second result (Proposition 2, Section 4) allows for estimation of the sustainability of resource extraction along any smooth enough path in a sense

of nondecreasing per capita consumption in the long run. The importance of this result is connected with the long-standing problem of defining and estimating the properties of the extraction paths with the “maximum possible” and with the optimal or equilibrium rates. For example, Dasgupta and Heal (1979) showed that the equilibrium path of extraction in the problem of maximizing the discounted utilitarian criterion is unsustainable (consumption declines to zero in the long run) for any positive rate of discount. Another example is a well-known Hubbert curve of oil extraction that estimates the maximum possible rates basing on historical data. Applying the “sustainability functional” introduced in Proposition 2, I show that the long-run consumption along the Hubbert curve declines to zero regardless of the choice of the curve’s parameters and regardless of the pattern of saving. Numerical examples provided in the paper are calibrated on the current world’s oil extraction data.

2 The model

I consider the DHSS model of a decentralized economy with the technical progress exactly compensating for the capital depreciation¹ (Bazhanov, 2007a), zero population growth, zero extraction cost, and the Cobb-Douglas production function $q(t) = f(k(t), r(t)) = k^\alpha(t)r^\beta(t)$ where $\alpha, \beta \in (0, 1)$, $\alpha + \beta < 1$ are constants. Population equals to labor and the lower-case variables are in per capita units, q – output, k – produced capital, r - current resource use. Then $r = -\dot{s}$, s - per capita resource stock ($\dot{s} = ds/dt$). Prices of per capita capital and the resource are $f_k = \alpha q/k$, $f_r = \beta q/r$ where $f_x = \partial f/\partial x$. Per capita consumption is $c = q - \dot{k}$.

¹This assumption allows for correct consideration of the basic DHSS model in cases with unlimited growth in consumption.

The path of extraction $r(t)$ is assumed to be *feasible* if

- (a) $r(t) > 0$ for all $t \geq 0$;
- (b) the reserve s_0 is extracted during the infinite period $s_0 = \int_0^\infty r(t)dt$;
- (c) $r(t)$ is consistent with the initial conditions $r(0) = r_0; \dot{r}(0) = \dot{r}_0$;
- (d) $r(t)$ is twice continuously differentiable for t big enough.

I assume that

- (1) there are some phenomena in the economy such as simplest externalities, government regulations and taxes/subsidies, which modify the Hotelling Rule and which combined effect can be expressed in terms of tax $\nu(t)$. This implies that if p is the “standard equilibrium Hotelling” price then the ratio $(\dot{p} + \dot{\nu})/(p + \nu)$ is not already equal to the rate of interest $f_k(t)$ when $\nu(t) \neq 0$. Denoting the observable price by $f_r = p + \nu$, we can write it as follows:

$$\dot{f}_r(t)/f_r(t) = f_k(t) + \tau(t) \tag{1}$$

where the modifier $\tau(t) = \tau[\nu(t)] = 0$ when $\nu(t) = 0$. This generalized form of the equilibrium condition allows for various feasible scenarios of the resource extraction $r_\tau(t)$. The assumption is essential for the goals of the paper because it implies that all the examined paths of extraction are realizable. Realization of the specific extraction path depends on the concrete paths of the phenomena modifying the Hotelling Rule, including government policies. A review of these phenomena and a review of papers providing empirical evidence of distortions between the “standard Hotelling” price paths and data for various nonrenewable resources can be found in (Gaudet, 2007). I imply in this assumption that the government can use all the instruments of influence on the externalities and, by modifying the equilibrium condition (1), on the corresponding paths of extraction. For example, the government can use taxes (e.g. Karp and Livernois, 1992), regulations (e.g. Davis and

Cairns, 1999), and education (e.g. Grimaud and Rouge, 2005), in order to realize the socially optimal program. A large body of empirical research in the “oil peak” theory (e.g. Laherrere, 2000)² support plausibility of the assumption that the considered in the paper paths of extraction can be realizable. For simplicity of notation, I will omit below the subindex τ that denotes the dependence of the path of extraction on the specific combination of the phenomena modifying the standard Hotelling Rule.

(2) The economy follows an investment rule in the form of $\dot{k} = wq$ where $w \in (0, 1)$ is a constant. This rule includes the Hartwick Investment Rule as a specific case for $w = \beta$.

3 The roles of extraction and saving for growth

The modified Hotelling Rule (1) implies $\dot{f}_r/f_r = \alpha\dot{k}/k + (\beta - 1)\dot{r}/r = \alpha q/k + \tau$. After substitution of the saving rule $\dot{k} = wq$, it becomes $\alpha qw/k + (\beta - 1)\dot{r}/r = \alpha q/k + \tau$. This gives us the generalized equation in $r(t)$

$$\dot{r}/r = -(1 - w)\alpha q / [(1 - \beta)k] + \tau / (\beta - 1) \quad (2)$$

This equation can be rewritten as $\dot{r}/r = -[(1 - w)f_k + \tau] / (1 - \beta)$ or using (1) as

$$\dot{r}/r = - \left[\dot{f}_r/f_r - wf_k \right] / (1 - \beta) \quad (3)$$

Equation (3) can be derived from the production function by combining expressions for \dot{q}/q and \dot{f}_r/f_r . However the rate of extraction \dot{r}/r , as it can be seen from this equation, depends on the relationship between the rate of

²The main goal of this theory is to explain the behavior of historical extraction data by some empirical curves, and then use these paths as a forecasting tool. The patterns of consumption along some of these curves are examined in this paper.

change of the observable price \dot{f}_r/f_r and the rate of interest f_k or, in other words, on the specific modifier of the Hotelling Rule $\tau(t)$ what is explicitly expressed in (2). Equation (3) does not contain the modifier $\tau(t)$ explicitly because according to (1) the distortions caused by the phenomena associated with $\tau(t)$ are reflected in the price changes \dot{f}_r/f_r . It means that if we assume that $\tau(t)$ includes all known and unknown effects, distorting the Hotelling Rule in the real economy, then we must take for numerical examples the changes in the real market price for the term \dot{f}_r/f_r . Then, formula (3) is interpreted as follows: “*actual*” rates of extraction are defined by the rates of the “*actual*” price changes \dot{f}_r/f_r diminished by the interest rate f_k weighted by the investment coefficient w .³

Substitution of formula (2) into the expression for the output per cent change implies $\dot{q}/q = \alpha\dot{k}/k + \beta\dot{r}/r = \alpha q/k [(w - \beta)/(1 - \beta)] - \tau\beta/(1 - \beta)$ that can be rewritten as follows:

$$\dot{q}(1 - \beta)/(\beta q) = f_k(w/\beta - 1) - \tau.$$

This implies the following

Proposition 1 *In the DHSS economy, with the investment rule $\dot{k} = wq$ where $w \in (0, 1)$ and with the modified Hotelling Rule $\dot{f}_r/f_r = f_k + \tau(t)$, the sign of the change in per capita output (and consumption) is defined as follows:*

$$\dot{q} \gtrless 0 \text{ iff } \tau \lesseqgtr f_k [w/\beta - 1]$$

that means that the path of investment (defined by w) can qualitatively influence the pattern of growth only when $-f_k < \tau < f_k [1/\beta - 1]$.

³Since all variables in formula (3) are observable, it can be used for estimation of accuracy of the model for the real economy. I use the word “actual” in quotation marks because an aggregate model can reflect only qualitative behavior of a real economy with some level of inaccuracy in numbers.

The specific cases of this result are:

1. The necessity and sufficiency of the Hartwick Rule ($w = \beta$) for sustaining the constant per capita consumption in the DHSS model with the standard ($\tau(t) \equiv 0$) Hotelling Rule (Hartwick, 1977; Dixit et al, 1980).

2. The necessity and sufficiency of the modified Hartwick Rule ($w > \beta$) for sustaining per capita growth of consumption in the DHSS model with the standard Hotelling Rule ($\tau(t) \equiv 0$) (Dasgupta and Heal, 1979, p. 303-306, formula (10.33); Hamilton et al, 2006).

3. The growing per capita consumption in the DHSS economy with the modified Hotelling Rule ($\tau(t) < 0$) and the standard Hartwick Rule ($w = \beta$) (Stollery, 1998; Bazhanov, 2007b, 2008).

Note that these results were obtained under the different welfare criteria (maximin in cases 1 and 3, and utilitarian with zero discounting of utility and the specific social rates of time preference in (Dasgupta and Heal, 1979)).

Again I will substitute for $\tau(t)$ using (1) in order to express the result of Proposition 1 in terms of the rates of change of the observable market prices \dot{f}_r/f_r for our resource. The result of this substitution is formulated below as

Corollary 1. *Under the conditions of Proposition 1 the sign of change in per capita output (and consumption) is defined as follows:*

$$\dot{q} \gtrless 0 \text{ iff } \left[\dot{f}_r/f_r \right] / f_k \lesseqgtr w/\beta$$

that implies that the pattern of saving (w) can qualitatively influence the pattern of growth iff $0 < \left[\dot{f}_r/f_r \right] / f_k < 1/\beta$.

One of the practical implications of this result is that the DHSS model, given oil as the resource input, has some empirical support from the qualitative behavior of the world's economy, depending on major changes in (market) oil prices. One can recall the prosperity of the economy when the price of

oil was declining ($\dot{f}_r/f_r < 0$) or the rate of change was very small, before the spike in 1973 and the corresponding recessions after the sharp price spikes in 1973 and 1979.⁴ More recent empirical support can be found e.g. in (IEA, 2004): “World GDP would be at least half of one percent lower . . . in the year following a \$10 oil price increase.” Although, of course, the dependence of the world’s output on oil prices is much more complicated than can be described by a simple aggregate model (see e.g. IMF, 2007, p. 17; Elekdag et al, 2008).

The examples from the world’s history and a large body of empirical research on testing the Hotelling Rule⁵ support the assumption about the strong influence of different phenomena modifying the Hotelling Rule in the real economy and about relatively large absolute values of the modifier τ . Therefore, government policies with respect to extracting industries could be primary for sustainable economic development. Concentration only on the patterns of saving could not be enough. Dasgupta and Heal (1979, p. 309) wrote on this matter: “Governments of most countries . . . have in the past been concerned with the rate of investment and, more recently, with the rate of utilization of the world’s exhaustible resources.” This implies the importance of estimating the qualitative behavior of the long-run consumption along some “program” paths of extraction. The solution of this problem is in the following section.

⁴A book of D. Yergin (1991) is a good guide on the qualitative dependence of the world’s economy on oil.

⁵The review is in (Gaudet, 2007).

4 Long-run consumption for any feasible scenario of extraction

Assume that the government is going to rely on some path of extraction in the framework of the long-term energy program. This path can be obtained e.g. as a result of the application of the “oil peak” theory and/or can be recommended by an institution.⁶ Then, sustainability of growth in the economy will depend on 1) the possibility of realization of this path (reliability of the forecast) and 2) the consequences for the economy when the path is realizable. I concentrate here only on the second question, namely, on the analysis of the long-run per capita consumption along some path $r(t)$, assuming that the path is realizable.

Per capita consumption in our framework is $c = q - \dot{k} = q(1 - w)$. Then, the qualitative behavior of c along the given scenario of extraction $r(t)$ coincides with the one of q . The change of output in our model is $\dot{q} = f_k \dot{k} + f_r \dot{r} = \alpha q^2 w/k + \beta q \dot{r}/r$ that can be rewritten as $\dot{q} = (\alpha q^2 w/k) [1 + (\beta/(w\alpha))(k\dot{r}/(rq))]$. For simplicity, assume that $r(t)$ is monotone in the long run; in other words, that condition $\int_0^\infty r(t)dt = s_0$ implies that $\dot{r} < 0$ for t big enough. It follows that $\dot{q} \geq 0$ in the long run iff

$$k|\dot{r}|/(rq) \leq w\alpha/\beta \quad (4)$$

This inequality contains the unknown path of capital $k(t)$ that can be defined from the differential equation $\dot{k} = wk^\alpha r^\beta$ (the saving rule). In general case (for any feasible $r(t)$) the solution of this equation cannot be expressed in elementary functions. However, the qualitative behavior of per capita consumption (or output) in the long run can be examined by considering

⁶See the scenario of the world oil extraction by the Cambridge Energy Research Associate (CERA, 2006).

the left hand side of the inequality (4) in the limit with $t \rightarrow \infty$. Using the L'Hôpital's rule we have

$$\begin{aligned} \lim_{t \rightarrow \infty} k |\dot{r}| / (rq) &= \lim_{t \rightarrow \infty} k^{1-\alpha} r^{-1-\beta} |\dot{r}| = \infty \cdot 0 = \lim_{t \rightarrow \infty} [k^{1-\alpha}] / \{1/(r^{-1-\beta} |\dot{r}|)\} \\ &= \infty / \infty = \lim_{t \rightarrow \infty} d[\cdot] / dt / d\{\cdot\} / dt \\ &= \lim_{t \rightarrow \infty} (1 - \alpha) k^{-\alpha} \dot{k} / \{[(1 + \beta) r^\beta \dot{r} |\dot{r}| - r^{1+\beta} d|\dot{r}|/dt] / \dot{r}^2\}. \end{aligned}$$

After substitution of the saving rule $\dot{k} = wq$ for \dot{k} it becomes

$$\begin{aligned} &w(1 - \alpha) \lim_{t \rightarrow \infty} r^\beta \dot{r}^2 / [(1 + \beta) r^\beta \dot{r} |\dot{r}| + \ddot{r} r^{1+\beta}] \\ &= w(1 - \alpha) \lim_{t \rightarrow \infty} \dot{r}^2 / [\ddot{r} r - (1 + \beta) \dot{r}^2] \end{aligned}$$

since unknown function $k(t)$ cancels out and since for our case $\dot{r} \rightarrow -0$ (increasing) with $t \rightarrow \infty$ and therefore $-d|\dot{r}|/dt = \ddot{r} > 0$. This implies that condition (4) can be reformulated as follows: $\dot{q} \geq 0$ iff $\lim_{t \rightarrow \infty} \dot{r}^2 / [\ddot{r} r - (1 + \beta) \dot{r}^2] \leq \alpha / [\beta(1 - \alpha)]$. After dividing the numerator and denominator of the left hand side by \dot{r}^2 , this condition becomes: $\lim_{t \rightarrow \infty} \ddot{r} r / \dot{r}^2 \geq 1 + \beta/\alpha$. The following Proposition 2⁷ summarizes the result.

Proposition 2 *In the DHSS economy, with the investment rule $\dot{k} = wq$ where $w \in (0, 1)$ and with the modified Hotelling Rule $\dot{f}_r / f_r = f_k + \tau(t)$ the growth of output q is sustainable in the long run ($\lim_{t \rightarrow \infty} \dot{q} \geq 0$) iff*

$$\Phi [r_\tau(t)] \geq 1 + \beta/\alpha$$

where $r_\tau(t)$ is smooth enough and $\Phi [r_\tau(t)] \equiv \lim_{t \rightarrow \infty} \ddot{r}_\tau r_\tau / \dot{r}_\tau^2$.

I will call Φ the *sustainability functional for the curve of extraction* $r_\tau(t)$ and the value of $\Phi [r_\tau(t)]$ - the *sustainability number* of this curve.

⁷Proposition 2 generalizes the results obtained for specific paths of extraction in (Bazhanov, 2007b; Andreeva and Bazhanov, 2007).

Note that $\Phi[r_\tau(t)]$ does not depend on the saving coefficient w explicitly. However, this does not mean that the sign of \dot{q} does not depend on the pattern of saving at all in the long run. It would contradict the known results. Equation (2) implies that any path $r(t)$ is the result of combined influence of the pattern of investment defined by w and the path of the modifier $\tau(t)$ that depends on government interventions.

It is easy to check this result for a classical example with $\tau \equiv 0$ (the standard Hotelling Rule) and with $w = \beta$ (the standard Hartwick Rule). Then the path of extraction (see e.g. Bazhanov, 2007b) is $r_{Hart}(t) = r_0 [1 + At]^{-\alpha/\beta}$ where $A = r_0\beta / [s_0(\alpha - \beta)]$. The first derivative is $\dot{r}_{Hart}(t) = -r_0 A \alpha [1 + At]^{-\alpha/\beta - 1} / \beta$ and the second is $\ddot{r}_{Hart}(t) = -r_0 A^2 \alpha (\alpha + \beta) [1 + At]^{-\alpha/\beta - 2} / \beta^2$ that follows $\ddot{r}_{Hart} r_{Hart} / \dot{r}_{Hart}^2 \equiv 1 + \beta/\alpha$. Proposition 2 implies that per capita output and consumption are constant over time along r_{Hart} , which coincides with the well-known result of J.M. Hartwick (1977).

The path r_{Hart} is monotonically decreasing starting with $\dot{r}_{Hart}(0) = -\alpha r_0^2 / [s_0(\alpha - \beta)]$ that is not observed yet in the real economy. Further evidence is that the prices for the different kinds of nonrenewable resources do not grow exponentially, as it should be according to the standard Hotelling Rule (Gaudet, 2007). This implies that the more realistic assumption is $\tau(t) \neq 0$. The following section provides the examples of calculating the sustainability numbers for some known paths of extraction that are compatible with the data from the real economy.

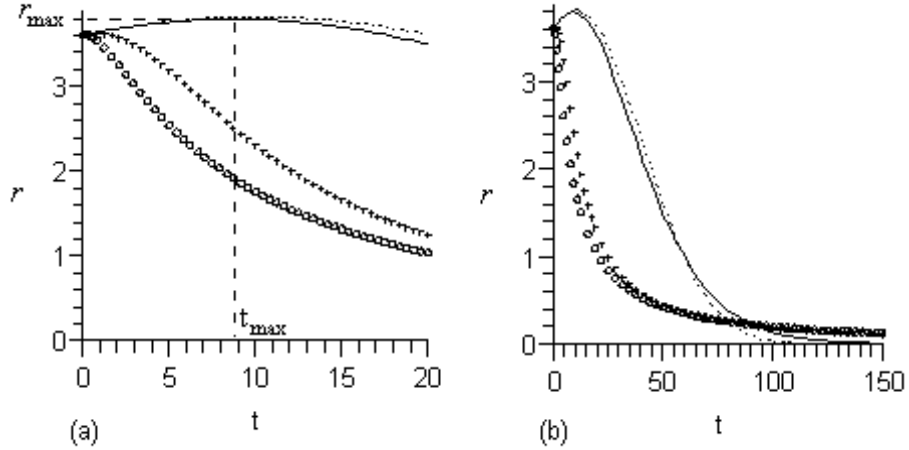


Figure 1: Scenarios of the world’s oil extraction [bln t per year; time t in years starting from 2008], (a) in the short run, (b) in the long run: the Hubbert curve (solid); the Gauss curve (dotted); the Cauchy curve (crosses); the rational curve (circles).

5 Consumption along the Hubbert and some other curves

There is a long-standing question about defining the “physical” peak of a non-renewable resource extraction. M.K. Hubbert (1956) and his followers (e.g. Laherrere, 2000) use a specific function with a single maximum (Hubbert curve) or a set of these functions, whose parameters are to be calibrated on the historical data of oil extraction and new fields discoveries. The curve(s) uniquely define the peak(s) and the rates of the future extraction. Laherrere (2000) defines the Hubbert curve as follows:

$$r_H(t) = 2r_{\max} / \{1 + \cosh [b(t - t_{\max})]\}$$

where r_{\max} is the peak of extraction in the year t_{\max} . Numerical example with the world oil extraction data gives $t_{\max} = 8.73$ and $r_{\max} = 3.7985$ (Fig. 1, solid line⁸). Parameter b defines the shape of the curve (deviation).

This curve proved to be the most accurate function for describing historical data of oil extraction in the oil-peak literature (e.g. Laherrere, 2000). That is why it is extensively used for forecasting purposes. It is known that in general, historical data of oil extraction do not follow decreasing optimal path that is implied by the model of Hotelling. This means that the modifier $\tau(t)$ was not identically equal to zero during the periods of the observations. Assume that the combination of phenomena modifying the Hotelling Rule will cause such a path of $\tau(t)$ that extraction will continue to follow the Hubbert curve in the long run as it is assumed by the oil-peak theory. Then we can estimate qualitative behavior of consumption along this path using the result of Proposition 2.

Derivatives \dot{r}_H and \ddot{r}_H are

$$\dot{r}_H = -2br_{\max} \sinh [b(t - t_{\max})] / \{1 + \cosh [b(t - t_{\max})]\}^2$$

and

$$\ddot{r}_H = 2b^2r_{\max}(\cosh [b(t - t_{\max})] - 2) / \{1 + \cosh [b(t - t_{\max})]\}^2.$$

Then,

$$\ddot{r}_H r_H / \dot{r}_H^2 = \{\cosh [b(t - t_{\max})] - 2\} / \{\cosh [b(t - t_{\max})] - 1\}.$$

This implies that sustainability number for the Hubbert curve $\Phi [r_H(t)] \equiv \lim_{t \rightarrow \infty} \ddot{r}_H r_H / \dot{r}_H^2 = 1$ what is always less than $1 + \beta/\alpha$. This means that the

⁸All the paths of extraction are calibrated on the current world's oil extraction data (World, 2007). The details of calibration are in Appendix.

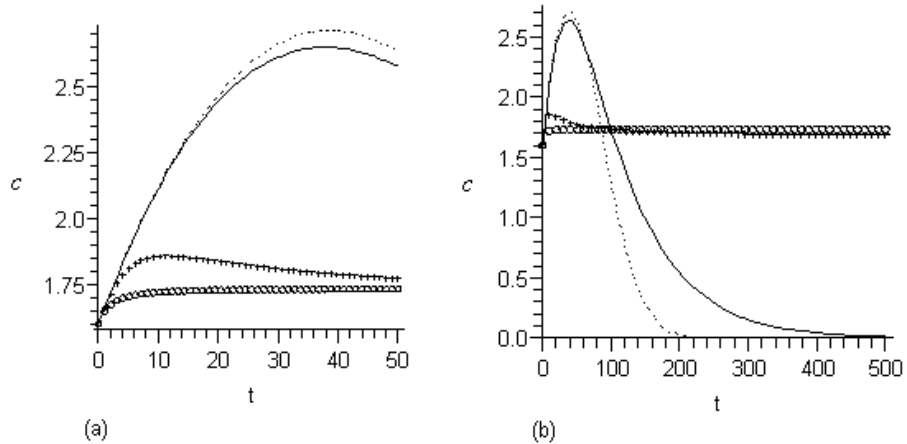


Figure 2: Paths of per capita consumption [time t in years starting from 2008], (a) in the short run, (b) in the long run along: the Hubbert curve (solid); the Gauss curve (dotted); the Cauchy curve (crosses); the rational curve (circles).

path of consumption along this curve declines to zero in the long run (Fig. 2, solid line⁹) regardless of the pattern of saving and the choice of parameters for the Hubbert curve.

Hence, the government should do its best using taxes, regulations and education in order to shift the extracting industry from following this path.

Another pattern of extraction considered in (Laherrere, 2000) is a well-known Gauss curve (Fig. 1, dotted):

$$r_G(t) = r_{\max} \exp \left[-(t_{\max} - t)^2 / 2b^2 \right],$$

⁹The paths of consumption are obtained for all cases by solving numerically the differential equation for capital with $\alpha = 0.3$ and $\beta = 0.25$. The same qualitative results were obtained for some curves in (Andreeva and Bazhanov, 2007) for $\alpha = 0.3$, $\beta = 0.05$, and $\dot{r}_0 = 0.08$. The difference was in the lower level of consumption (e.g. asymptote for the consumption path along the rational curve was 1.539 in comparison with 1.736 in the current paper).

where the roles of parameters are the same: t_{\max}, r_{\max} are the year and the amount of the maximum extraction and b describes the deviation. The derivatives are $\dot{r}_G = (t_{\max} - t)r_G/b^2$ and $\ddot{r}_G = [(t_{\max} - t)^2 - b^2]r_G/b^4$. Then the sustainability number is

$$\Phi [r_G(t)] \equiv \lim_{t \rightarrow \infty} \ddot{r}_G r_G / \dot{r}_G^2 = \lim_{t \rightarrow \infty} [(t_{\max} - t)^2 - b^2] / (t_{\max} - t)^2 = 1$$

that implies the same pessimistic outcome as for the Hubbert curve (Fig.2, dotted).

Note that another pattern of exponential extraction $r_{GR}(t) = r_0 e^{-gt}$ derived as optimal in (Grimaud and Rouge, 2005, Proposition 1, p. 115) leads to the same qualitative result in the DHSS economy as the Gauss and Hubbert curves. Indeed, $\Phi [r_{GR}(t)] \equiv 1$ that also means declining per capita consumption in the DHSS economy. However, in the framework of Grimaud and Rouge (GR) this extraction path implies exponential growth of per capita consumption. This contrast results from the difference in the models. The simplest variant of the DHSS economy (no capital decay and no technical progress) can be interpreted as a technical change exactly compensating for capital depreciation. This technical progress is asymptotically linear in some cases and “stronger” than linear in some other cases (Bazhanov, 2007a). The GR model does not contain physical capital explicitly and, of course, it does not contain capital decay. This assumption can be also reformulated as the assumption about the implied technical progress exactly compensating for capital depreciation. In addition to this implicit technical change in the GR model, there is endogenous exponential technical progress in the form of growing knowledge and there is also a specific externality caused by a polluting resource. The “extra” technical change certainly gives the GR model additional opportunities for growth in comparison with the DHSS

model. As to plausibility of the patterns of total factor productivity (TFP) or technical change, note that the implied TFP compensating for capital decay is not the most pessimistic scenario in the long run among the reported in the literature. For example, Nordhaus and Boyer use TFP in the form of $A(t) = A_0 \exp \left[\int_0^t g_0^A e^{-\delta \xi} d\xi \right]$ calling the behavior of this factor “a major uncertainty” in their DICE-99 and RICE-99 models (Nordhaus and Boyer, 2000, p. 17). This TFP is asymptotically constant ($\lim_{t \rightarrow \infty} A(t) = A_0 g_0^A / \delta$) and fast-growing in the short run.

The “optimistic” alternatives to the curves above for the DHSS model can be found among the densities of the fat-tailed distributions. These paths of extraction can be compatible with the Cobb-Douglas production function in a sense that they give the opportunity to sustain non-decreasing per capita consumption in the long run. This property is connected with the fat tail that provides more resources to the future generations. These patterns of the resource extraction make it possible to adjust capital adequately to the rate of shrinking of the essential resource. For example, the curve

$$r_C(t) = b^d r_{\max} / [b + (t_{\max} - t)^2]^d$$

is the probability density function for the Cauchy distribution for $d = 1$, where t_{\max} is the location parameter and b is the scale parameter.¹⁰ The generalizing parameter d is introduced here as a control variable for the sustainability number of this curve that is

$$\begin{aligned} \Phi [r_C(t)] &\equiv \lim_{t \rightarrow \infty} \ddot{r}_C r_C / \dot{r}_C^2 \\ &= 0.5 \lim_{t \rightarrow \infty} [(t_{\max} - t)^2 + 2d(t_{\max} - t)^2] / [d(t_{\max} - t)^2] \\ &= (1 + 2d) / 2d. \end{aligned}$$

¹⁰ t_{\max} is not the expectation because the expectation and all other higher moments do not exist for this distribution due to the divergence of the corresponding integrals.

Proposition 2 implies that in the long run $\dot{q} \geq 0$ iff $(1 + 2d)/2d \geq 1 + \alpha/\beta$ or $d \leq \alpha/(2\beta)$.¹¹ This curve with $d = \alpha/(2\beta)$ is depicted in Fig. 1 in crosses. The corresponding path of consumption is asymptotically constant as it must be for this value of d (Fig. 2, in crosses).

Another example of the extraction curve allowing for the sustainable economic development is the variant of the transition path that I called “rational” and examined in (Bazhanov, 2007b). The first derivative of this curve is $\dot{r}_R(t) = (\dot{r}_0 + bt)/(1 + ct)^d$, the curve itself is $r_R(t) = r_0(1 + b_r t)/(1 + ct)^{d-1}$, where $b_r = c(d - 1) + \dot{r}_0/r_0$ and $\ddot{r}_R(t) = [b(1 + ct) - dc(\dot{r}_0 + bt)]/(1 + ct)^{d+1}$. The initial conditions imply $b = -c(d - 2)[r_0 c(d - 1) + \dot{r}_0]$ and then the sustainability number is

$$\begin{aligned} \Phi[r_R(t)] &\equiv \lim_{t \rightarrow \infty} \ddot{r}_R r_R / \dot{r}_R^2 = r_0(1 + b_r t) [b(1 + ct) - dc(\dot{r}_0 + bt)] / (\dot{r}_0 + bt)^2 \\ &= 1 + 1/(d - 2) \end{aligned}$$

that means that the paths of consumption and production are not declining iff $d \leq \alpha/\beta + 2$. This coincides with the result of Corollary 1 (Bazhanov, 2007b). For comparison with the curve r_C , I considered r_R with $d = \alpha/\beta + 2$ that also implies asymptotically constant consumption (Fig. 1 and Fig. 2, in circles).

In the following section, I will use some of these examples in order to illustrate numerically the result of Proposition 1, namely, the roles of the Hotelling Rule modifier and the saving coefficient for the sustainability and the level of growth.

¹¹This conclusion coincides with the result obtained in (Andreeva and Bazhanov, 2007).

6 The paths of the Hotelling Rule modifier

Proposition 1 implies that the economy will follow decreasing (or increasing) path of consumption regardless of the value of the saving coefficient $w \in (0, 1)$ when the modifier $\tau(t)$ of the Hotelling Rule is not close enough to zero. This section shows how this result works for the concrete cases with the extraction curves analyzed in previous section.

It was shown that the long-run consumption declines to zero along the Hubbert and the Gauss curves for any patterns of saving. This means (Proposition 1) that in the long run the modifier $\tau(t)$ for these curves is greater than $f_k [1/\beta - 1] = \tau_{Up}$. One can see it in Fig. 3a where τ_H (τ for the Hubbert curve, solid line) asymptotically approaches a positive constant and τ_G (τ for the Gauss curve, dotted) goes to infinity while the upper (τ_{Up}) and the lower ($\tau_{Low} = -f_k$) bounds asymptotically converge to zero (dashed lines).¹² The paths of consumption, declining to zero along the Hubbert curve, are depicted in Fig. 4a for the different values of w .

The paths of $\tau(t)$ for the Cauchy and the rational curves are rather deep inside the bounds τ_{Up} and τ_{Low} (Fig. 3b, the bounds are not depicted) and they converge with the bounds to zero regardless of the value of the saving coefficient. This implies the asymptotically constant consumption for all cases (Fig. 4b). The role of w is to define the level of the asymptote for the consumption path that one can see in Fig. 4b where the paths for $w_1 = 0.05$ and $w_3 = 0.8$ are the patterns of overconsumption and overinvestment correspondingly.

The paths of consumption that must grow in the long run according

¹²The bounds τ_{Up} and τ_{Low} are depicted only for the Hubbert curve in order to not overcrowd the figure. The behavior of these values for the Gauss curve is the same with the only difference that they approach zero faster.

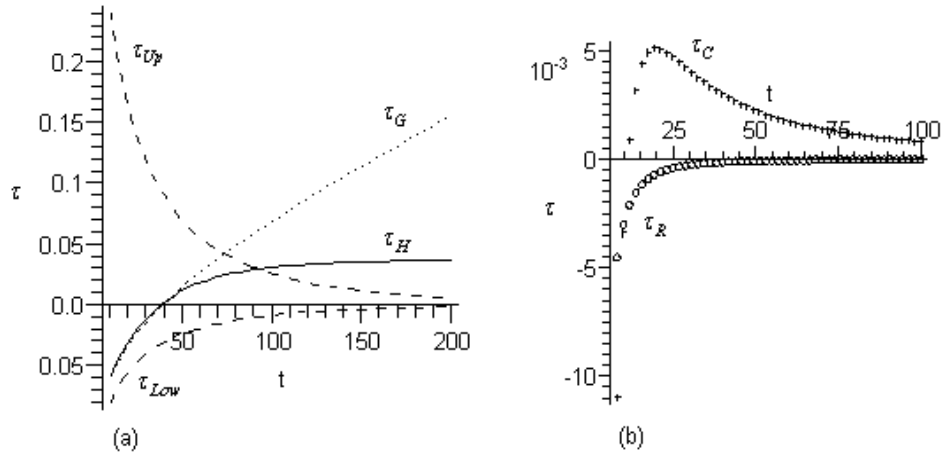


Figure 3: The paths of the Hotelling Rule modifiers for various extraction curves: (a) Hubbert τ_H (solid); Gauss τ_G (dotted); (b) Cauchy τ_C (crosses); rational τ_R (circles).

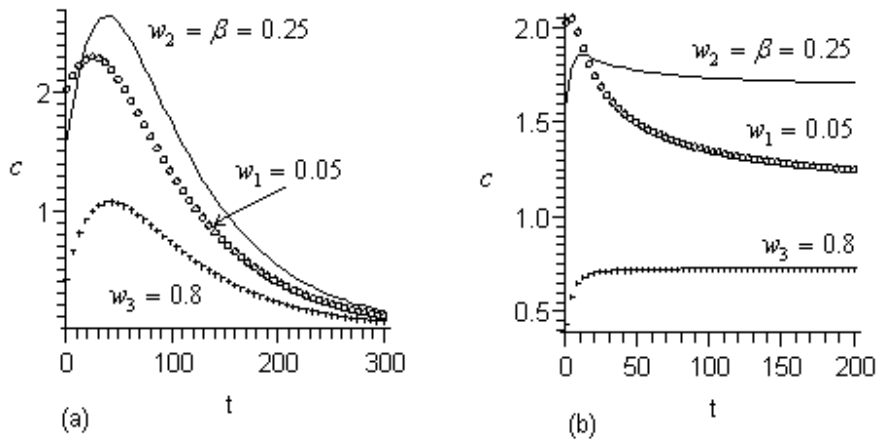


Figure 4: Paths of per capita consumption for different saving coefficients along: (a) Hubbert curve; (b) Cauchy curve with $d = \alpha/(2\beta)$.

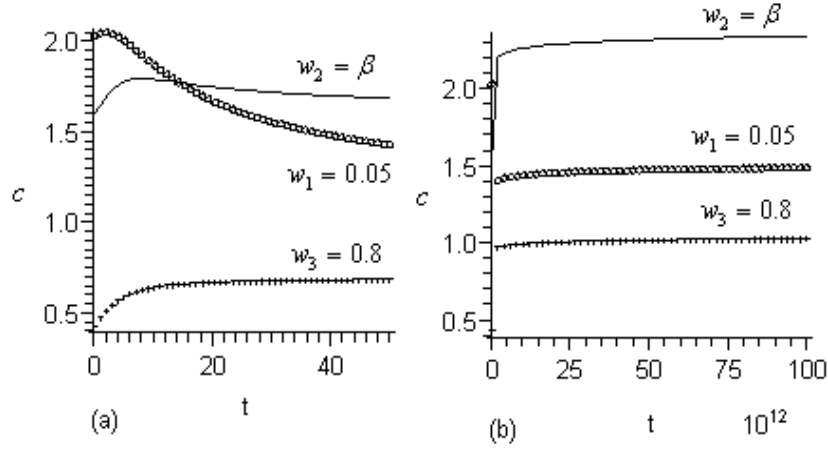


Figure 5: Paths of per capita consumption for different saving coefficients along the Cauchy curve with $d = \alpha/(2\beta) - 0.02$ (long-run unbounded growth): (a) in the short run; (b) in the long run.

to Proposition 2 are depicted in Fig. 5 for the Cauchy curve with $d = \alpha/(2\beta) - 0.02$. The properties of this curve causing the long-run growth are illustrated in Fig. 6 in terms of observable variables (Corollary 1). Even for the pattern of overconsumption ($w_1 = 0.05$) there is a moment of time ($t_{\min} \approx 5000$ years, Fig. 6) when the ratio of the change of the price over the interest rate ($[\dot{f}_r/f_r]/f_k$) becomes equal to w_1/β that corresponds to the local minimum of per capita consumption and implies rather slow but unbounded growth for $t > t_{\min}$.

7 Concluding remarks

This paper has presented two new results for the Dasgupta-Heal-Solow-Stiglitz (DHSS) model, which was extended by the modified Hotelling Rule.

- (1) Proposition 1 and Corollary 1 (Section 3) have shown that the econ-

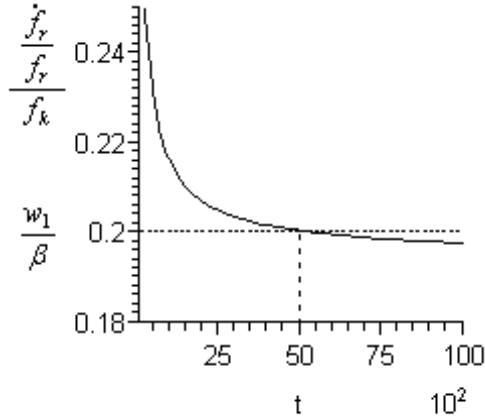


Figure 6: The path of the ratio the change-of-the-price rate over the interest rate for the Cauchy path of extraction with unbounded growth of consumption in the long run (saving coefficient $w_1 = 0.05$).

omy is growing if and only if the Hotelling Rule modifier is less than the interest rate weighted by the factor $w/\beta - 1$ where w is the saving coefficient ($\dot{k} = wq$) and β is the resource price elasticity; or, in terms of the observable variables (Corollary 1), the economy is growing if and only if the ratio of the change of the resource price over the interest rate is less than w/β . This result implies that the qualitative pattern of the economy's development (growth, stagnation, or decline) is defined by the path of the resource extraction that in turn is defined by the phenomena modifying the Hotelling Rule (including government policy). The pattern of investment (defined by w) specifies the level of consumption along the growing, constant, or declining path and defines the pattern of development in the cases when the Hotelling Rule modifier is close to zero.

(2) Proposition 2 (Section 4) provides a tool for estimating (weak) economic sustainability of some “program” paths of a nonrenewable resource ex-

traction. The easy-to-calculate “sustainability functional” offered in Proposition 2 allows for estimating the “sustainability number” for any smooth enough feasible path of the resource extraction. The sustainability number shows whether the pattern of consumption is growing, constant, or declining in the long run along this path of extraction. As an example, we¹³ have shown that the path of per capita consumption is always declining to zero in the long run along the well-known Hubbert curve regardless of the patterns of saving and the choice of parameters for this curve. This is a warning sign appealing to the government’s attention because the Hubbert curve is recognized in a large body of empirical research as the best tool for estimating the historical data of oil extraction. I offer an approach for constructing and some examples of the curves that allow for the oil-peak estimation and that are sustainable in a sense of nondecreasing consumption in the long run.

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¹³This result was obtained with Andreeva A.A in (Andreeva and Bazhanov, 2007).

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9 Appendix A. Calibration of the extraction curves

The parameters of the curves are calibrated on the current world's oil reserve and extraction data (World, 2007). The initial rate of extraction (on January 1, 2008) is $r(0) = r_0 = 3.618$ bln t /year (1 t = 7.3 barrel), the paths are assumed to satisfy the feasibility condition $\int_0^\infty r(t)dt = s_0 = 182.424$ bln t (reserve estimate on January 1, 2008), and as $\dot{r}(0)$ I took the average $\dot{r}_0 = 0.04$ since 1984 (the methodology of estimation of \dot{r}_{0_i} for historical data is in (Bazhanov, 2006). This way of calibration is essential for the feasibility of the paths of extraction while the conventional calibration on historical data usually lead to infeasible paths (Bazhanov and Vyscrebentsev, 2006).

a) *The Hubbert curve* (Andreeva and Bazhanov, 2007). The initial value r_0 implies $r_{H\max} = 0.5r_0(1 + \cosh[-b_H t_{H\max}])$. The value of $\dot{r}(0)$ gives us $t_{H\max} = (1/b_H) \ln [(b_H r_0 + \dot{r}_0)/(b_H r_0 - \dot{r}_0)]$. Coefficient $b_H = (2r_0^2 + s_0 \dot{r}_0)/(s_0 r_0)$ is obtained from the feasibility condition $\int_0^\infty r(t)dt = s_0$.

b) *The Gauss curve*. Using the condition $r(0) = r_0$ the curve can be expressed as follows: $r_G(t) = r_0 \exp[t_{G\max}^2/(2b_G^2)] \exp[-(t_{G\max} - t)^2/(2b_G^2)]$ that gives us $r_{G\max} = r_0 \exp[t_{G\max}^2/(2b_G^2)]$. Initial condition for \dot{r}_0 implies $t_{G\max} = \dot{r}_0 b_G^2/r_0$ and the feasibility condition for s_0 gives a nonlinear equation in b_G

$$(\sqrt{2\pi}/2)r_0 b_G \exp [\dot{r}_0^2 b_G^2/(2r_0^2)] \left[1 + \operatorname{erf} \left\{ \dot{r}_0 b_G/(r_0 \sqrt{2}) \right\} \right] = s_0$$

with a single relevant root that can be found numerically.

c) *The Cauchy curve*. The peak of extraction $r_{C\max} = r_0(b_C + t_{\max}^2)^d/b_C^d$ is expressed via r_0 , the initial condition for \dot{r}_0 is more convenient to use in this case for obtaining b_C that gives us $b_C = 2r_0 t_{C\max}^d/\dot{r}_0 - t_{C\max}^2$ and $t_{C\max}$ is to be found from a nonlinear equation $\int_0^\infty r_C(t, t_{C\max})dt - s_0 = 0$.

d) Calibration of *the rational curve* is in (Bazhanov, 2007b).