Sustainable growth: Compatibility between criterion and the initial state

Bazhanov, Andrei

Far Eastern National University, Queen's University

8 August 2008

Sustainable growth: compatibility between criterion and the initial state

Andrei V. Bazhanov

Abstract

There is a large body of research devoted to our understanding of sustainable growth in resource based economies. Some of this research is inapplicable to the real economy. This is a result of inconsistency between the commonly used criteria and the initial state of the real economy. The inconsistency can lead to either inferior, unsustainable, or nonexistent optimal paths of consumption per capita if the criterion is not linked to the initial state. We demonstrate this in a model of the Dasgupta-Heal-Solow-Stiglitz variety with the constant consumption per capita as a benchmark criterion. Our results show that the inconsistency in this case can imply Pareto inferior paths of consumption per capita.

Key words: essential nonrenewable resource, sustainable extraction, criterion inconsistency, Hartwick Rule
JEL : O13; Q32; Q38

1 Introduction

Koopmans (1965) argued that "Ignoring realities in adopting 'principles' may lead one to search for a nonexistent optimum, or to adopt an 'optimum' that is open to unanticipated objections" (p. 229). Koopmans showed for the simple model that depending on the formulation of the criterion, the optimal path could not exist. Unfortunately, the problem of consistency of a criterion with the opportunities of the specific economy to realize the optimum has not been adequately addressed in the subsequent literature. [1]

[1] There are studies that show that sustainability of consumption level and time-consistency of intertemporal consumption paths depend on the initial value of capital for the maximin programs (e.g. Leininger, 1985). A recent example of incompatibility of a criterion with the specific economy can be found by analyzing Stollery (1998). He examined the problem of a resource-extracting economy causing global

We claim in this paper that either the unsustainable[2] or the Pareto inferior path of per capita consumption can be obtained in the real ("non-optimal") economy, if the economy follows a criterion that is not linked to the initial state of the economy[3] or, in other words, if "preferences are not adjusted to opportunities" (Koopmans, 1965, p. 253). We use here a conventional model in order to construct an example showing that a "non-optimal" resource-based economy enters a Pareto inferior path of development when it tries to follow a criterion that is not linked to the initial state (implies an initial state different from the given state of the economy). Besides theoretical goals, the paper contains a "technical" result of the methodology of transition of the non-optimal economy from the given initial conditions to the initial state implied by the criterion.

We use for simplicity the constant consumption criterion. Solow (1974) used this criterion as a result of application of the maximin (Rawls, 1971) to the question of just intertemporal allocation of an essential nonrenewable resource. The constant consumption over time for the Dasgupta-Heal-Solow-Stiglitz model (DHSS) (Dasgupta and Heal, 1974; Solow, 1974; Stiglitz, 1974) is implied by the Hartwick Investment Rule (Hartwick, 1977). This result is obtained under the standard Hotelling Rule. The Rule implies that the rate of extraction $r(t)$ must always be declining, including the starting point ($dr(0)/dt = \dot{r}(0) < 0$). Moreover, the optimal initial value of $\dot{r}(0)$ is strictly defined in the framework of the DHSS model for the maximin-optimal paths.[4]

But what if we want to apply this criterion to the specific economy, whose technology and (or) the initial state are not compatible with the requirements of the criterion?[5] For example, if the elasticity of factor substitution is less than unity then the economy with a nonrenewable resource will collapse regardless of any efforts in saving (Dasgupta and Heal, 1979) and therefore this

warming and following the constant-utility optimal path. One can easily check that this criterion is not compatible with the Cobb-Douglas technology for plausible initial states by assuming constant extraction during some period. The plausible initial states imply in this framework unsustainable extraction, fast growth of temperature and collapse of the economy.

[2] We consider weak sustainability in a sense of non-decreasing per capita consumption over time.

[3] By linking a criterion to the initial state we imply that a criterion must be specified for the given technology and the given set of initial conditions. We think that a methodology of criterion specification deserves separate attention.

[4] This value depends on the initial rate of extraction $r(0)$, amount of reserve s_0 and technological parameters of the economy (formula (5)).

[5] We assume here that stickiness of both the extraction and saving prevents the economy from changing the initial conditions instantly (see Section 3) implying the necessity of a transition period for switching to the "optimal" path in a smooth way.

economy is not compatible with the criteria implying nondecreasing consumption. The unit-elasticity Cobb-Douglas economy can exhibit various patterns of declining, growing, and constant per capita consumption depending on the initial conditions and on the paths of saving and extraction. Therefore it is natural to expect that a sustainable and optimal consumption path implied by a plausible criterion can be Pareto inferior to some feasible path in this economy (combined with the specific initial conditions) if this criterion is not "linked" parametrically to the "opportunities" of the economy.

Assume that there are some phenomena in the economy (simple externalities and government interventions), whose combined effect can be expressed in terms of the tax/subsidy on the resource extraction. Due to these phenomena the resource price does not already satisfy the standard Hotelling Rule and the Rule is formulated in a modified form.[6] Assume that at the initial moment the modified Hotelling Rule implies growing rate of extraction ($\dot{r}(0) > 0$) that is consistent with the historical world oil extraction data for the last 25 years (Fig. 1a). The growing extraction is unsustainable since the limited reserve of the essential resource will eventually cause the resource exhaustion in finite time and decline of consumption to zero. In the general case of the DHSS model the economic unsustainability can be two-dimensional when an economy follows an unsustainable pattern of saving (e.g. decreasing capital to zero) in addition to unsustainable extraction. For simplicity we assume that the economy invests in the optimal way for the standard DHSS model with respect to the constant-consumption criterion, namely it follows the Hartwick Saving Rule.[7]

In our numerical examples we show that there is a sustainable path of extraction linked to the initial conditions and implying the consumption path that is Pareto superior to the one along the "optimal" path after the moment of switching. Namely, the consumption along this path follows quasi-arithmetic growth under the standard Hartwick Rule. This looks more attractive than the same pattern of growth with the positive net saving (Hamilton et al., 2006), since it does not require decreased consumption from the present generation.

[6] The most recent analysis of the reasons of distortion in the Hotelling Rule in its original form and alternative formulations of the Rule, reconciling it with the observed patterns of price and extraction, can be found in (Gaudet 2007).

[7] It is known that the world's patterns of saving are rather persistent over time despite changes in government policies. The share of saving oscillates around 20-24 percent of GDP (see e.g. IMF data). There is also evidence (e.g. Pearce and Atkinson 1993) showing that net investment, which takes into account natural capital, is around zero for some countries (Mexico, Philippines) and is mixed positive and negative for some others. Hamilton et al (2006) also obtained mixed result examining the satisfaction of the Hartwick Rule for 70 countries. Therefore our assumption about Hartwick Rule as given and independent on government policy has some justification.

We describe the model in Section 2; discuss the nature of stickiness of extraction and saving (that imply the necessity of the transition period) in Section 3; formulate the problem of smooth transition to the "optimal" path in finite time in Section 4; describe an economy-linked path in Section 5 and the transition extraction paths in Section 6; Section 7 provides theoretical results on the impossibility of switching in finite time to the "optimal" path along the "optimal" transition path; Section 8 examines the paths of consumption along the different paths of extraction; Section 9 concludes.

2 The model

We use the DHSS model with the Cobb-Douglas technology. For simplicity we consider the case with zero population growth [8], zero cost of extraction and technological progress compensating for capital depreciation. The last assumption allows us to consider the basic DHSS model for the cases with a growing economy that is important for our numerical examples. Plausible patterns of technological progress compensating for capital depreciation were examined in (Bazhanov, 2007b). All the paths such as output $q(t)$, consumption $c(t)$, produced capital $k(t)$ and so on are defined below in per capita units. Namely, output is $q = f(k, r) = k^{\alpha} r^{\beta}$, where r - current resource use, $r = -\dot{s}$, s - per capita resource stock ($\dot{s} = ds/dt$), α, $\beta \in (0, 1)$ are constants. Prices of capital and the resource are $f_k = \alpha q/k$ and $f_r = \beta q/r$, where $f_x = \partial f/\partial x$. Per capita consumption is $c = q - \dot{k}$. The Hartwick Rule implies $c = q - r f_r$ or, substituting for f_r, we have $c = q(1 - \beta)$ that means that instead of $\dot{c} = 0$ we can check $\dot{q} = 0$. Hence, in the standard DHSS model of a decentralized economy the path of extraction can be derived from the asset equilibrium condition (the standard Hotelling Rule) $\dot{f}_r/f_r = f_k$ [9] that implies $\alpha\beta q/k + \dot{r}(\beta - 1)/r = f_k = \alpha q/k$ or

$$\dot{r}/r = -\alpha q/k. \tag{1}$$

Then

$$\dot{q}/q = \alpha \dot{k}/k + \beta \dot{r}/r = \beta(\alpha q/k + \dot{r}/r) = 0 \tag{2}$$

[8] In fact, numerous literature on sustainable development starting T. Malthus work in 1798 and some recent papers, e.g., (Brander 2007) consider the population growth as the main threat to sustainability. The debates on this problem are concentrating around the estimate of the constant that could be the limit to the population growth. Hence, we can assume that the population is already stabilized on this limit.

[9] Assume that all the derivatives exist and the change of price $\dot{f}_r$ is continuous in time that is consistent with historical data (e.g. Gaudet, 2007) if we neglect the short-run oscillations.

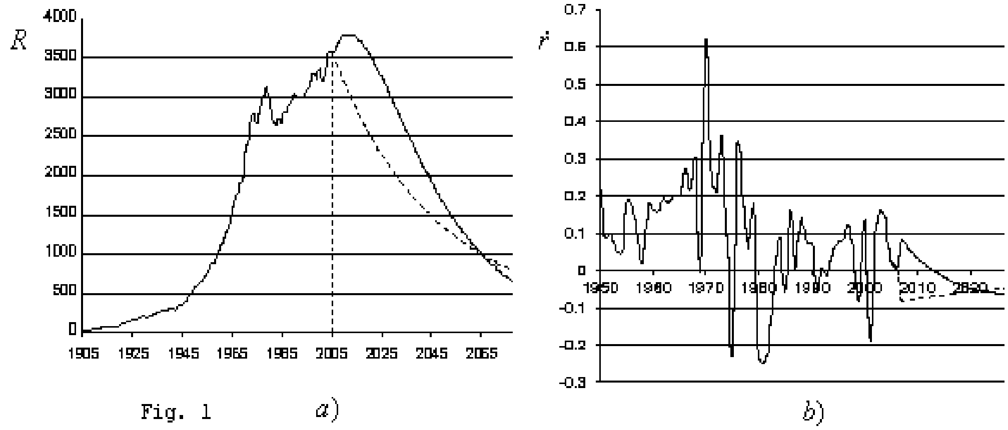

Fig. 1. World oil a) extraction (mln t/year); b) extraction changes: historical data (before 2006); "optimal" path (dotted); transition curve (solid).

that means that we really have $\dot{q} = \dot{c} = 0$ or $q = const$ (equilibrium coincides with optimum for $\dot{k} = \beta q$). Then $rf_r = \beta q = const$ and we have $\dot{k} = \beta q = const$ for deriving $k(t)$ and equation (1) for deriving $r(t)$ together with the initial conditions $r(0) = r_0$ and $s(0) = s_0$ to find constants of integration. Here s_0 is the given resource stock that must be extracted over infinite time: $s_0 = \int_0^\infty r(t)dt$. Then we have

$$r_H(t) = r_0 \left[1 + r_0 \beta t / s_0(\alpha - \beta)\right]^{-\alpha/\beta}, \tag{3}$$

where $\alpha > \beta$ (Solow condition) and

$$\dot{r}_H(t) = -\ddot{s}(t) = -\alpha r_0^2 / s_0(\alpha - \beta) \left[1 + r_0 \beta t / s_0(\alpha - \beta)\right]^{-(\alpha+\beta)/\beta}. \tag{4}$$

Since the resource is essential, the equilibrium path (3) asymptotically approaches zero (dotted line in Fig. 1a) under the optimal path of saving and the path of extraction changes $\dot{r}_H(t)$ (dotted line in Fig. 1 b)) also approaches zero, but starting from negative value

$$\dot{r}_{0H} = -\alpha r_0^2 / s_0(\alpha - \beta). \tag{5}$$

The conventional approach (e.g. Solow, 1974; Hartwick, 1977; Stollery, 1998; Asheim et al., 2007) seeks in the DHSS model the optimal pattern of saving given the initial values s_0 and k_0; the pattern of extraction is derived as the equilibrium path from the corresponding formulation of the Hotelling Rule. We assume here that the pattern of saving is optimal with respect to our criterion and it does not change over time. However, there are phenomena (including some externalities) expressed in terms of tax and modifying the Hotelling Rule. These phenomena cause the corresponding deviations in the path of extraction. We assume that there exists a one-to-one correspondence between the equilibrium path of extraction and the government's policy that is also expressed in terms of tax. Then, for simplicity, we can consider the path of extraction as a control variable implying that actually it is the equilibrium path that is controlled by the corresponding government's policy. Therefore, we will consider the resource extraction data as given instead of the given initial capital. [10]

We assume that the Hotelling Rule is modified by phenomena that cause deviation of equilibrium paths from the optimal ones. Some of these phenomena (e.g. insecure property rights, market structure etc., see (Gaudet, 2007)) cannot be changed immediately. Hence, it supports the assumption about stickiness of extraction and prevents the economy from realizing the path (3) at $t = 0$ and so we must switch to this path along some "smooth continuation" (solid line in Fig. 1a) after 2006). [11] The definitions in the following section reflect these restrictions.

3 Stickiness in the DHSS model

The constant consumption in the standard DHSS model is the result of 1) total investment of oil rent in capital ($\dot{k} = rf_r$) and 2) fulfillment of the standard Hotelling Rule ($\dot{f}_r/f_r = f_k$). For simplicity we have assumed that our specific economy deviates from the optimal one only in one "dimension of sustainability", namely, in an unsustainable pattern of extraction. This extraction is the result of influence of various phenomena (including externalities and government policy), which can be expressed in terms of tax $T(t)$ and which result in modification of the Hotelling Rule. This implies that if $p(t)$ is the "equilibrium Hotelling price" without distorting phenomena and $f_r(t) \equiv f_r[p(t), T(t)] = p(t) + T(t)$ is the observable price with distortions, then the ratio $\dot{f}_r/f_r$ is not already equal to the rate of interest. We can write it in the general form of a dynamically changing equilibrium condition:

$$\dot{f}_r/f_r = f_k + \tau(t), \tag{6}$$

[10] There is a relationship between k_0 and r_0 in the DHSS model with the given saving rule (formula (13)) that involves the initial values for extraction. The data for the oil extraction and GDP growth are more available than that for the amount of capital. Therefore, it is more convenient in our case to express k_0 in terms of extraction data.

[11] We assume that the government can use all the instruments such as taxes (Karp and Livernois, 1992), regulations (Davis and Cairns, 1999), and education (Caillaud et al., 1988; Pezzey 2002; Grimaud and Rouge 2005) that can influence the externalities in the economy and the corresponding rate of extraction. So we will concentrate on some normative and technical problems that arise during the switching in finite time to the path with desirable properties.

where $\tau(0) \neq 0$ in general case and $\tau = 0$ when $T = 0$ (standard Hotelling Rule). Assuming the existence of the derivatives and continuity of $p(t)$ implies that $T(t)$ is continuous. Also by assumption, $\dot{f}_r$ is continuous. Then (6) implies that $\tau(t)$ is continuous given that f_k is continuous in the DHSS model. We can rewrite (6) as follows: $\dot{f}_r = f_r(f_k+\tau)$ or $f_r(f_k+\tau) = (\partial f_r/\partial k)\cdot\dot{k}+(\partial f_r/\partial r)\cdot\dot{r}$. Since $f_{rk} = \alpha f_r/k$, we have $f_r(f_k+\tau-\alpha\dot{k}/k) = -f_{rr}\cdot\ddot{s}$ that gives us

$$f_r\left[p(t), T(t)\right] = m(t)\cdot\ddot{s}(t), \tag{7}$$

where $m(t) = -f_{rr}/(f_k+\tau-\alpha\dot{k}/k)$. One can see that in this formulation the Hotelling Rule is equivalent to the Second Law introduced by Newton in 1687 to account for inertial effects. Here the reason for changes in the amount of s (extraction) is the price f_r,[12] the acceleration is $\ddot{s}$, and the coefficient m shows how much effort in terms of taxes must be applied in order to change the acceleration by one unit. In other words, the Hotelling Rule gives us the exact expression for the *coefficient of inertia or stickiness of extraction* that equals the ratio of the negative slope of the demand curve $(-f_{rr} > 0)$[13] over the rate of change of the observable resource price $(f_k+\tau)$ less the product of the price elasticity of capital α and the rate of capital growth $\dot{k}/k$. The U.S. price data for the period 1870-2004 (Gaudet, 2007) shows that the rates of change of price for a number of nonrenewable resources including oil were oscillating around zero. The product $\alpha\dot{k}/k$ is also close to zero at $t = 0$ and asymptotically approaches zero with $t \to \infty$[14] that implies rather high values of m. This means, that the government should make a serious effort in terms of taxes in order to change significantly the pattern of extraction $(\dot{r})$. Another question is if it can be changed in a discontinuous way? Our assumption about continuous change of price in the long run (continuous trend), supported by historical data, implies continuity of the Hotelling Rule modifier $\tau(t)$. From this and from continuity of f_{rr}, f_k, and $\dot{k}/k$ in the DHSS model, we obtain that $m(t)$ is also continuous. Then, from equation (7) and from the continuity of f_r, we have that $\ddot{s}$ must also be continuous that justifies our assumption about the impossibility of changing $\dot{r}$ instantly.

Hence, the combination of non-optimal initial conditions $(\dot{r} > 0)$ with the stickiness cause the path of extraction that is non-optimal, at least in the small neighborhood of $t = 0$, in the sense of our criterion (constant consumption) that we consider as an example of "preferences not adjusted to opportunities".

[12] It will be profit when the cost of extraction is not zero.

[13] We can express the slope via $E_d(r)$ (the price elasticity of demand): $-f_{rr} = \left[f_r/r\right]/E_d(r)$ that implies (the other values given) that the less is the elasticity the greater is m. As is known, the price elasticity of demand for oil is significantly less than unity.

[14] For the standard DHSS model (Hartwick, 1977) $\dot{k}/k = \beta q_0/(k_0+\beta q_0 t)$ that equals to $\beta q_0/k_0$ at $t = 0$ that is less than unity.

Therefore, we will try to catch up to the "optimal" path in finite time along a transition path in the first (transition) period and we will have to find this path among non-optimal curves. We set down these assumptions below in definitions 1 and 2.

Definition 1 An *intertemporal program* $\langle f(t), c(t), k(t), r(t)\rangle_{t=0}^{\infty}$ is a set of paths $f(t)$, $c(t)$, $k(t)$, $r(t)$, $t \geq 0$ such that $f(t) = f[k(t), r(t)]$ and $c(t) = f(t) - \dot{k}(t)$.

We use below the notation $(x_1, \ldots x_n) \gg 0$ if $x_i > 0$ for all $i = \overline{1, n}$.

Definition 2 For positive initial stock of capital and resource $(k_0, s_0) \gg 0$ the set of the programs $F = \{\langle f(t), c(t), k(t), r(t)\rangle_{t=0}^{\infty}\}$ is *a feasible sheaf* at $t = 0$ and each of the paths $f(t)$, $c(t)$, $k(t)$, $r(t)$ is *a feasible path* if any program $\langle f(t), c(t), k(t), r(t)\rangle_{t=0}^{\infty}$ from F for all $t \geq 0$ satisfies the conditions:

1) $(f(t), c(t), k(t), r(t)) \gg 0$;

2) $r(t), k(t), c(t)$ are continuously differentiable and

$\sup_{t,\Delta t} |(\dot{r}(t + \Delta t) - \dot{r}(t)) / \Delta t| \leq \Delta\dot{r}_{\max} < \infty$;

3) $f(t)$ is twice continuously differentiable;

4) $\int_t^{\infty} r(t)dt \leq s(t)$;

5) $s(0) = s_0, r(0) = r_0, \dot{r}(0) = \dot{r}_0, \dot{q}(0)/q(0) = \dot{q}_0/q_0$.

Our assumption about one-to-one correspondence between government policy and the rates of extraction permits the use of the path of extraction $r(t)$ as a control variable, bearing in mind that this path is just a result of changes in the equilibrium conditions. We use regular definition of the optimal path in the problem of maximization of a welfare criterion $W[r(t)]$.

Definition 3 The feasible path $r^*(t)$ is optimal if $W[r^*(t)] \geq W[r(t)]$ for any feasible path $r(t)$.

In our example, the welfare criterion $W[r(t)] = \max_r \min_t c[r(t)]$ means constant consumption over time (Solow, 1974) if a feasible set is not restricted. In the case of the DHSS model and the given saving rule, the initial value of consumption c_0 is expressed via the initial values for extraction and for the growth of output (section 8). Therefore, definition 2 implies that c_0 is also given and we cannot change it nonsmoothly due to conditions 2) and 3). This follows nonuniqueness of the optimal path since we cannot already increase c_0 by reallocating the resource among generations (or by changing the pattern of saving as in (Solow, 1974)). Then optimal are all the paths (including unsustainable) that fill the rectangle in Cartesian plane (t, c) with the left bottom corner $(0, c_0)$ and with the right upper corner (∞, ∞) including lower side of the rectangle and excluding all other sides. This nonuniqueness makes it possible to introduce another criterion connected with sustainability. Conventional interpretation of the maximin is the constant-consumption path, therefore we again assume that the government uses this criterion as a simple example of ethical preferences that are not linked to the specific economy and that imply a sustainable path. [15]

In a normative sense, the restriction $\Delta\dot{r}_{\max} < \infty$ means that the change of the rate of extraction can be reduced without losing consumption only in continuous way with the rate not exceeding $\Delta\dot{r}_{\max}$. This value is defined e.g. by the rate of introducing substitute technology, which does not use the nonrenewable resource (e.g. solar plants). It follows directly from the production function $q = k^\alpha r^\beta$, since only an adequate instant increment of capital can compensate for a sharp fall in the resource supply. However, in a closed economy the instant additional investment means the same instant fall in consumption that makes impossible a "fair" nonsmooth switch to sustainable development. We estimate below the pattern of "additional" (foreign) investment, which can compensate for a nonsmooth fall in extraction for a simple example with a Cobb-Douglas technology. Suppose that extraction was changed at $t = 0$ in a discontinuous way:

$$\dot{r}(t) = \begin{cases} \dot{r}_0 > 0, & t = 0, \\ \dot{r}_1 < 0, & t = dt, \end{cases} \tag{8}$$

where $dt > 0$ is small enough. The question is: which pattern of $\dot{k}$ can maintain a constant level of consumption during the period $t \in [0, dt)$? Assume that the paths of extraction and current capital $(k_0 + \dot{k}_0 t)$ are linear for any $t \in [0, dt)$. Looking for such a path of capital $k_0 + k_1(t)$ that $k_1(0) = 0$ and $\Delta c = c(t) - c_0 = 0$ for any $t \in [0, dt)$, we obtain (Appendix C) that the instant increment in the rate of growth of capital must be $\dot{k}_1(0)/k_0 - \dot{k}_0/k_0 = (\beta/\alpha)(\dot{r}_0/r_0 - \dot{r}_1/r_0) > 0$.

It is interesting to examine the behavior of the economy, depending on the specific functions introducing substitute technology. However, we think that this problem needs different specification of the model and deserves special attention. For the purpose of the current paper we assume that the feasible dynamics of introducing substitute technology affects the path of extraction only at the point with maximum $|\Delta\dot{r}|$. Therefore, for the numerical examples below it is enough to estimate $\Delta\dot{r}_{\max} = 0.1$ from historical data (Fig. 1b).

[15] In comparison, say, with the positively discounted utilitarian criterion that implies the path of consumption declining to zero in the standard DHSS model for any positive rate of discount (Dasgupta and Heal, 1974 & 1979).

4 Formulation of the transition problem

For the economy $q = k^\alpha r^\beta$ given the initial reserve $s(0) = s_0$ and the initial conditions r_0, $\dot{r}_0$, $\dot{q}_0/q_0$,[16] we are going to find among the feasible paths (definition 2, Section 3) such a path of extraction $r(t)$ and such a finite moment of time $\widetilde{t}$ that $r(0) = r_0$, $\dot{r}(0) = \dot{r}_0$, and

$$\dot{r}(\widetilde{t}) = -\alpha r^2(\widetilde{t}) / \left[s(\widetilde{t})(\alpha - \beta) \right].$$

In other words, a finite moment of time $\widetilde{t}$ must be such that the change of the rate of extraction along the transition path coincides with the initial change of the rate of extraction (5) for the "optimal" path, given the current state as the initial. Besides these conditions we can require the optimality of the transition path $r_{trans}(t)$ e.g. in a sense of minimum deviation from the optimal path during the transition period. The transition paths must be such that they can be used to describe the extraction with the specific initial conditions, they must be feasible, and they must allow for sustainable consumption in the long run. Then if we show that there is a feasible path of consumption, which is Pareto superior to the path, optimal with respect to our criterion, then it will mean that our criterion is inadequate to the "opportunities" of our specific economy. An example of such a path we introduce in the following section.

5 Economy-linked growing consumption

We will consider an example of the program, which is linked to the initial set of the specific economy and which implies sustainable quasi-arithmetic growth of output and consumption under the standard Hartwick Rule. Existence of growing consumption in the DHSS model with the modified Hotelling Rule and the standard Hartwick Rule was shown e.g. in (Bazhanov, 2007c).[17]

Assume that $f(k, r) = q = q_0(1 + t\lambda_1/\lambda_0)^{1/\lambda_1}$, where q_0 is the initial output and λ_0, λ_1- parameters. Then $\dot{q} = \dot{q}_0(1 + t\lambda_1/\lambda_0)^{(1/\lambda_1 - 1)}$ that implies $\lambda_0 = q_0/\dot{q}_0$. The Hartwick Rule $\dot{k} = \beta q$ follows $c = (1 - \beta)q$ and $k = k_0 + \beta\lambda_0 \left[q(1 + t\lambda_1/\lambda_0) - q_0 \right] / (1 + \lambda_1)$. This gives us the path of extraction $r = (qk^{-\alpha})^{1/\beta}$. Below we will conditionally call this path the *economy-linked extraction*. Given r, the value of λ_1 is defined by the condition $s_0 = \int_0^\infty r dt$.

[16] These values imply the expressions for k_0, q_0, and $\dot{k}_0 = \beta q_0$ (Section 8).

[17] We think that the questions about the properties of such kind of paths, about the specific conditions of their existence, and about the path of tax require separate attention.

We will construct below the offered set of programs $\{\langle f(t), c(t), k(t), r(t)\rangle_{t=0}^{\infty}\}$ for an example with the world's oil extraction data. It is important for the goal of the paper that for some initial conditions (e.g. $\dot{q} = 0$) these paths do not exist.

6 Transition curves

For the transition period, we use the curves offered in (Bazhanov, 2007c). Assume that there exists a tax included in the term $T(t)$ (equation (7)), which can uniquely define any of these curves. The transition paths belong to the same class of rational functions as the "optimal" curve (3). The difference is in the numerator, which in the expression for the changes of extraction rate $\dot{r}$ depends on t with a negative coefficient to control "smooth breaking" in the neighborhood of $t = 0$. Namely, $\dot{r}(t)$ has the form of

$$\dot{r}_{trans}(t, b, c, d) = (\dot{r}_0 + bt)/(1 + ct)^d, \tag{9}$$

where $b < 0, c > 0, d > 1$ (for convergence $\dot{r}(t) \to -0$ with $t \to \infty$). We have $r_0 = r(0)$ to express b and then $r(t)$ has a dependence on c and d in

$$r_{trans}(t) = r_0 \left(1 + b_r t\right)/(1 + ct)^{d-1}, \tag{10}$$

where $b_r = c(d-1) + \dot{r}_0/r_0$. Coefficient $c = c(s_0)$ is expressed from the condition $s_0 = \int_0^\infty r dt$.

Hence, we have a single independent parameter d that defines the shape of the curve (including its peak) and we can use this parameter as a control variable in some selected optimization problem

$$F\left[r(t, d)\right] \to \max_d .$$

These curves are of immediate interest for our transition problem since they can imply a nonmonotonic path of consumption. This means that the consumption along some of these paths can be growing at the initial moment and constant at the moment of switching to the "optimal" path.

We have assumed that due to the phenomena modifying the Hotelling Rule, our specific economy deviates from the "optimal" only in pattern of extraction while it is following the "invest resource rent" rule. It is known that the Hartwick Rule is necessary and sufficient for keeping consumption constant in the standard DHSS model under the standard Hotelling Rule. This implies that consumption in a non-optimal economy cannot be identically constant under the standard Hartwick Rule. It can be either asymptotically constant in the long run, declining to zero, or growing with no limit. This means that we cannot use the constant consumption criterion for deriving the optimal transition path since this path does not exist. However, we can consider a generalized form of this criterion, namely, a criterion of minimum deviation from the maximum asymptote for asymptotically constant consumption, which can be obtained in the specific non-optimal economy. This criterion includes identically constant consumption for the optimal economy as a specific case. This criterion for choosing the parameter d can be formulated as follows: $F(d) = \min_d \max_t |c_{\max} - c(t,d)|$, where $c_{\max}$ is the maximum asymptote for the path with asymptotically constant consumption and $c(t,d)$ is the path of consumption along the transition curve $r(t,d)$. This criterion implies the only optimal solution in the class of transition curves, namely the curve with $d = \alpha/\beta + 2$ that is uniquely defined by the initial conditions. The optimum is unique because the consumption is asymptotically constant only along this curve while consumption along the curves with $d < \alpha/\beta + 2$ is infinitely growing and with $d > \alpha/\beta + 2$ is declining to zero (Corollary 1, (Bazhanov, 2007c, p. 190)). Indeed, for any d_1 and d_3 such that $d_1 < d_2 = \alpha/\beta + 2 < d_3$ this result implies that $F(d_1) = \infty > F(d_3) = c_{\max} \geq F(d_2) = c_{\max} - c_0$. We proceed by finding the finite moment of time for switching to the "optimal" path in the following section.

7 Switching to the "optimal" path

We define the moment of shifting to the second period $\widetilde{t}_0$ (the period of "optimal" extraction) as a solution of the "smooth switching" problem. Namely, the government introduces/changes a tax in a smooth way (the tax presumably exists and equals to zero at the moment of switching). The tax modifies the Hotelling Rule (6) and the corresponding path of extraction in such a way that the economy enters the "optimal" path when the change of the rate of extraction (acceleration) $\dot{r}$ along the transition curve equals to the initial acceleration of the "optimal" path that is being constructed at the each current moment. In order to find $\widetilde{t}_0$, the initial conditions $\widetilde{r}_0(t), \widetilde{\dot{r}}_0(t), \widetilde{s}_0(t)$ for the "optimal" curve (3) are being dynamically calculated along the transition path until accelerations coincide. Equations (5) and (9) for the accelerations imply that $\widetilde{t}_0$ must be a solution of the equation

$$(\dot{r}_0 + b\widetilde{t}_0)/(1 + c\widetilde{t}_0)^d = -\alpha r(\widetilde{t}_0)^2 / \left[\widetilde{s}_0(\widetilde{t}_0)(\alpha - \beta)\right], \tag{11}$$

where $r(\widetilde{t}_0) = \widetilde{r}_{0trans}(\widetilde{t}_0)$ is defined by the equation (10) with $d = \alpha/\beta + 2$ and the rest of the resource $\widetilde{s}_0$ at $\widetilde{t}_0$ is $\widetilde{s}_0(\widetilde{t}_0) = s_0 - \int_0^{\widetilde{t}_0} r(t)dt$.

The following propositions show that the finite solution of the equation (11) does not exist.

Proposition 1 *Equation (11) has real roots if and only if the parameter d of the transition curve (10) is such that*

$$d \leq \alpha/\beta + 2. \tag{12}$$

There are two real roots if inequality (12) is strict and one real root if it holds as an equality.

Proof (Appendix 1).

Proposition 2 *Equation (11) has only one real finite positive root if and only if $d < \alpha/\beta + 2$.*

Proof (Appendix 2).

Proposition 2 implies that the transition path, which is optimal with respect to a generalized variant of the constant-consumption criterion, does not give us the opportunity to switch to the "optimal" path in finite time. As an illustration of Proposition 2 we consider an example with the world's oil extraction data.[18] The accelerations of the transition path (10) with $d = \alpha/\beta + 2$ (left hand side of equation (11)) and dynamically constructed initial accelerations of the "optimal" curve (right hand side of equation (11)) are shown in Fig. 2 a). It can be seen that the residual of equation (11) approaches zero only asymptotically that means that our problem of "smooth switching" in finite time has no solution in this framework

However, according to Proposition 2, there is a set[19] of sub-optimal transition paths for "smooth switching" in sub-optimal finite time $\bar{t}$. Then in order to choose the unique path from this set we can apply another criterion e.g. minimum of the time of switching $\bar{t}$ subject to the constraint on the changes in the rates of extraction implied by the restricted rate of substitution between the resource and capital in the form of substitute technology. Assume,

[18] Extraction: $R_0 = 72,486.5$ [1,000 bbl/day] $\times 365 = 26,457,572$ [1,000 bbl/year] (or 3.6243 bln t/year); reserve: $S_0 = 2 \times 1,317,447,415$ [1,000 bbl] (or 2×180.47 bln t) (Oil & Gas J 2006, 104(47): 20-23.). We use coefficient 1 ton of crude oil = 7.3 barrel and we use $\dot{r}_0 = 0.04$ that is close to the average $\dot{r}$ since 1984. Methodology of estimation of historical values for $\dot{r}$ is described in (XXXX, 2006).

[19] The convergence of integral $\int_0^\infty r dt$ implies $d > 3$ therefore a set of solutions to (11) exists iff $\alpha/\beta + 2 > 3$.

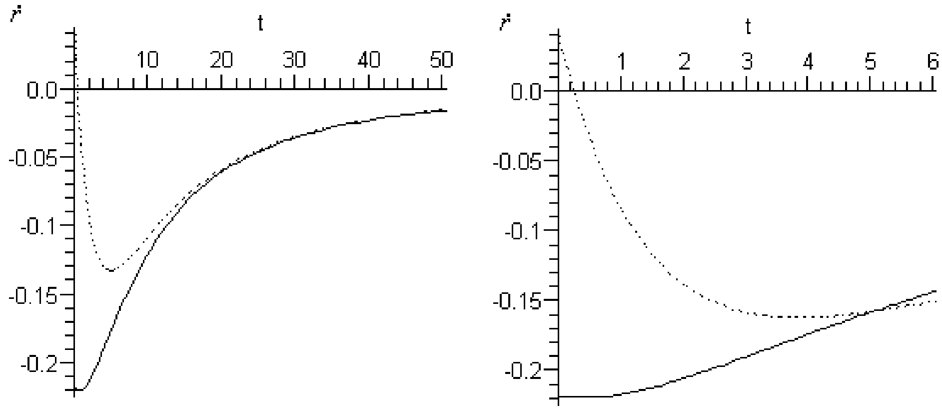

Fig. 2 a) b)

Fig. 2. Changes in extraction rates for the transition path (left hand side of equation (11), dotted line) with a) $d = \alpha/\beta + 2 = 3.2$; b) sub-optimal transition path with $d = 3.157$; the solid line for both cases is a plot of the initial accelerations for the "optimal" path (right hand side of equation (11)), constructed along the transition path.

for the sake of argument, that there exists an interior solution of this problem implying that the minimum duration time for the transition period equals, say to 5 years ($\bar{t} = 5$). This time corresponds to the transition curve with $d = \alpha/\beta + 1.957 = 3.157$ (given $\alpha = 0.3$ and $\beta = 0.25$ (Nordhaus and Boyer, 2000) [20]).

8 Comparison of consumption paths

The Hartwick Rule implies that the consumption path is $c = q - \dot{k} = (1-\beta)q = (1-\beta)k^{\alpha}r^{\beta}$. In our problem $r(t)$ is a known combination of the transition and the "optimal" paths and $k(t)$ is an unknown path of capital. We can calculate $k(t)$ from the equation for the saving rule $\dot{k} = \beta k^{\alpha} r^{\beta}$ given the estimation of k_0. From (2) we have $\dot{q}/q = \beta(\alpha q/k + \dot{r}/r)$ that implies the expression for k_0, given $r_0, \dot{r}_0$, and output percent change $(\dot{q}/q)_0$:

$$k_0 = \left\{ \left[(\dot{q}/q)_0 / \beta - \dot{r}_0/r_0 \right] / \left(\alpha r_0^{\beta} \right) \right\}^{1/(\alpha-1)}. \tag{13}$$

[20] According to (Nordhaus and Boyer, 2000, p. 43) β is calibrated on the initial interest rate that in our case equals to 0.065. Note that the obtained value of $\beta = 0.25$ is also consistent with the world's average pattern of saving.

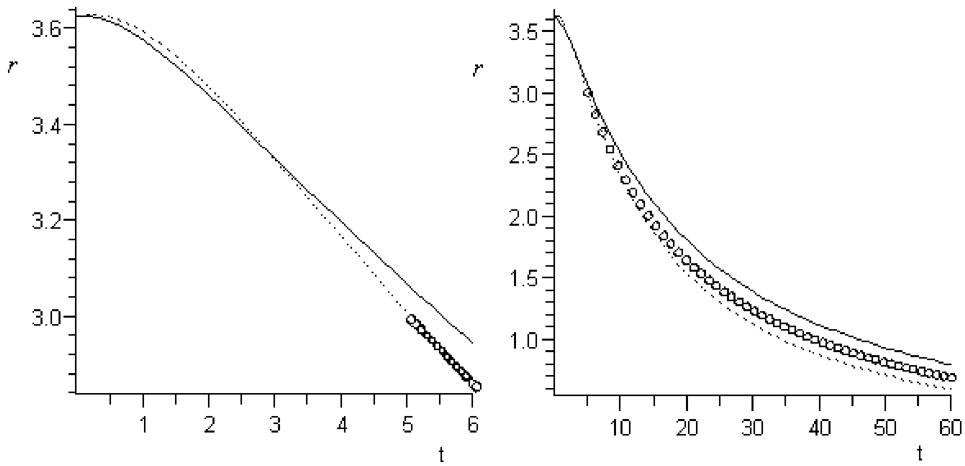

Fig. 3 a) b)

Fig. 3. Switching the extraction from the transition path (dotted line) to the "optimal" path (circles) in the short run (a) and in the long run (b); the economy-linked optimal path is depicted as a solid line.

Using $(\dot{q}/q)_0 = 0.01905$[21] and the values of $\dot{r}_0$ and r_0 for the world's oil extraction, we have $k_0 = 14.029$ and $c_0 = 2.285$ that gives us the paths of consumption along the transition and along the "optimal" paths after the moment of switching. For the economy-linked paths we obtained numerically $\lambda_1 = 60.11$. The resulting paths of extraction are depicted in Fig. 3 and the paths of consumption are in Fig. 4.

Note that consumption in the economy-linked program (solid line) is Pareto superior to both consumption paths: the transition (dotted line) and the "optimal" (circles). The economy-linked program follows the same standard Hartwick Rule as is used along the other two paths. The source of unbounded growth in this case and in the case with the transition path in the long run (Fig. 4b) is the specific combination of the phenomena (including government policies) modifying the Hotelling Rule and causing the "non-optimal" (in the sense of constant consumption) pattern of the resource extraction. The transition path, by construction, is also adjusted to the economy but unlike the economy-linked path it does not necessarily imply the quasi-arithmetic growth of consumption. It makes the transition path more flexible with respect to the initial conditions, e.g. it can be used for $\dot{q}_0 \leq 0$, while the economy-linked

[21] This value of the initial rate of growth of output implies coinciding values of capital at the moment of switching. In an arbitrary economy the value of $(\dot{q}/q)_0$ cannot, of course, be that convenient. In this case we have two-dimensional transition problem, where besides the path of extraction, the pattern of saving also must be adjusted during the transition period in order to have "optimal" amount of capital at the moment of switching. In the current paper, for the sake of argument, it is enough to consider only one-dimensional transition problem.

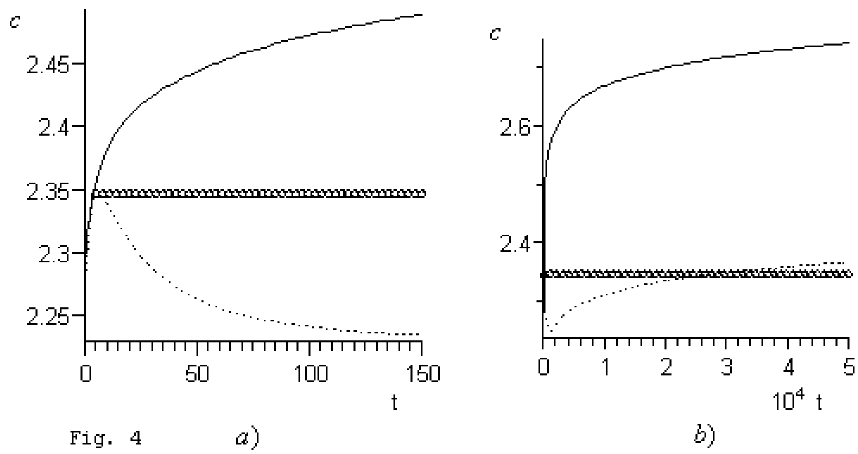

Fig. 4. Consumption along the transition path (dotted line), the "optimal" path (circles), and the economy-linked path (solid line) a) in the short run; b) in the long run.

path does not exist for $\dot{q}_0 = 0$ and implies unacceptable paths of consumption for $\dot{q}_0 < 0$. However, the possibility of nonmonotonic consumption that we used as a convenient property of the transition path is a sign of its disadvantage in comparison with the economy-linked path in the case when sustainable growth is possible. Both paths start from the same initial conditions and follow the same saving rule. The only difference between them is that the paths of the Hotelling Rule modifier are different, implying different paths of extraction (Fig. 3). This can be interpreted as less efficient government policy or stronger influence of externalities in the case with the transition path.

The example implies the natural conclusion that the tool (criterion) should be adequate to the specific problem or the preferences should be adjusted to the opportunities. We cannot choose the policy implying the economy-linked path for the transition period since the solution of this problem does not exist. In the same way, we should not use the constant-consumption criterion or some other "not-adjusted" criterion for the specific economy since we can enter a Pareto inferior path. Both these consequences can be considered as unacceptable. The conclusion, of course, does not relate to theoretical investigations developing a general methodology. It means only that the realization of such a methodology in the form of the government's policy should be made consistent with the specific situation.

9 Concluding remarks

A specific economy can enter a Pareto inferior path of consumption if the criterion of its development is "not adjusted to opportunities". In other words, if the criterion has no connections with the initial state of the economy and with its technological properties.

We have illustrated this result for the Dasgupta-Heal-Solow-Stiglitz (DHSS) model with the constant consumption over time as an example of plausible criterion for sustainable development. We assumed that the initial state of the economy was not consistent with our criterion due to some phenomena expressed in tax/subsidy and modifying the Hotelling Rule in a general form. Namely, the output and the rate of extraction were growing at the initial moment. We also assumed stickiness of extraction and saving; in other words, a restricted rate of substitution between the resource and man-made capital. The expression for the coefficient of stickiness of extraction was obtained by reformulating the Hotelling Rule into the form equivalent to the Newton's Second Law. We think that the assumption of stickiness is plausible when the part of man-made capital, compensating for the shrinking resource, is represented by new technologies (e.g. solar plants), rather than by financial capital in a fund. The stickiness implies the necessity of a transition period in order to adjust the initial state in accord with the criterion. We constructed the transition paths of extraction in a specific class of functions, which allowed the economy to switch to the "optimal" path in finite time. This "transition methodology" is a separate important result of the paper.

We introduced an example of the economy-linked path of extraction that implies sustainable quasi-arithmetic growth of consumption under the standard Hartwick Rule. The path was calibrated on the world's oil extraction data. As a result, the level of consumption along this path was Pareto superior to the consumption along the "optimal" path. Our "optimal" path, of course, can be really optimal when the economy-linked path does not exist i.e. when the specific economy is not growing at the initial moment ($\dot{q}_0 = 0$) and the initial set coincides with the one implied by this criterion.

The result specifies for a resource-based economy the claim of Koopmans (1965) that an economy can "adopt an 'optimum' that is open to unanticipated objections" if the criterion (preferences) is not linked to the particularity of the economy. The example, considered in the paper, and the patterns of optimal but unsustainable growth that can be obtained e.g. in the framework of (Stollery, 1998) raise a question of construction of such a criterion, which is consistent with the initial state of the specific economy and which implies corresponding pattern of sustainable consumption. For example, we can use an approach of linking a criterion offered in (Bazhanov, 2007c) on the example of the generalized maximin in a form of $\dot{c}^{\gamma} c^{1-\gamma} = \overline{U} = const$ that implies quasi-arithmetic growth $c(t) = c_0(1 + \varphi t)^{\gamma}$, where $\varphi = \left(\overline{U}/c_0\right)^{1/\gamma}/\gamma$. Parameters of this criterion should be consistent with the initial state and imply optimal and sustainable pattern of growth for the specific economy. There is also a question about technical possibility for an economy to adjust its initial conditions in order to catch up to a criterion implying such a pattern of sustainable growth that the economy cannot "afford" at the current moment. We think that these problems deserve a separate investigation.

10 Acknowledgments

An earlier version of the paper was presented at the 41st Annual Meeting of the Canadian Economic Association. I am grateful to the participants of this event, to John M. Hartwick, Cees Withagen, and Maurice J. Roche for very useful comments and advice.

11 Appendix A (Proof of Proposition 1)

In order to proof the main statement of Proposition 1 we will state some auxiliary facts that we will formulate as

Lemma 1 *The rational curve of extraction (10) is such that*

a) $s_0 = \frac{r_0 p_0}{c(d-2)}$;

b) the rest of the resource $s(\bar{t})$ along this curve *at* $\bar{t} \geq 0$ *is*

$$s(\bar{t}) = s_0 - \int_0^{\bar{t}} r(t)dt = \frac{r_0}{c(d-2)} \frac{p_0 + p_1 \bar{t}}{(1+c\bar{t})^{d-2}} = s_0 \frac{1 + \frac{p_1}{p_0}\bar{t}}{(1+c\bar{t})^{d-2}},$$

where $p_0 = 1 + \frac{b_r}{c(d-3)}$, $p_1 = \frac{b_r(d-2)}{d-3}$;

$r_0 = r(0)-$ *the initial rate of extraction,* s_0- *the initial stock;*

$b_r = b_r(d)$, $c = c(d)$, *and* d *are the parameters of the curve (10).*

Proof a) By the construction of $r(t)$ and since $d > 3$ we have

$$\frac{s_0}{r_0} = \int_0^{\infty} (1+ct)^{1-d} dt + b_r \int_0^{\infty} t(1+ct)^{1-d} dt = \frac{1}{c(d-2)} \left[1 + \frac{b_r}{c(d-3)} \right]$$
$$= \frac{p_0}{c(d-2)}.$$

b) By direct calculations we have

$$
\begin{aligned}
s(\bar{t}) &= s_0 - \int_0^{\bar{t}} r(t)dt \qquad (14)\\
&= s_0 - r_0 \left[\frac{1}{c(d-2)} \left\{ 1 - (1+c\bar{t})^{2-d} \right\} + b_r I(\bar{t}) \right],
\end{aligned}
$$

where

$$
\begin{aligned}
I(\bar{t}) &= \frac{1}{c^2} \left\{ \frac{1}{d-3} \left[1 - (1+c\bar{t})^{3-d} \right] - \frac{1}{d-2} \left[1 - (1+c\bar{t})^{2-d} \right] \right\} \\
&= \frac{1}{c^2\,(d-2)\,(d-3)} \\
&\quad \times \left\{ (d-2) \left[1 - (1+c\bar{t})\,(1+c\bar{t})^{2-d} \right] - (d-3) \left[1 - (1+c\bar{t})^{2-d} \right] \right\} \\
&= \frac{1}{c^2\,(d-2)\,(d-3)} \\
&\quad \times \left\{ (1+c\bar{t})^{2-d} \left[(d-3) - (d-2)\,(1+c\bar{t}) \right] + (d-2) - (d-3) \right\} \\
&= \frac{1}{c^2\,(d-2)\,(d-3)} \left\{ 1 - \frac{1+(d-2)\,c\bar{t}}{(1+c\bar{t})^{d-2}} \right\}.
\end{aligned}
$$

Then the bracket $[\cdot]$ in (14) is

$$
\begin{aligned}
[\cdot] &= \frac{1}{c(d-2)} \\
&\quad \times \left\{ \frac{(1+c\bar{t})^{d-2} - 1}{(1+c\bar{t})^{d-2}} \right\} + \frac{b_r}{c^2\,(d-2)\,(d-3)} \left\{ \frac{(1+c\bar{t})^{d-2} - 1 - (d-2)\,c\bar{t}}{(1+c\bar{t})^{d-2}} \right\} \\
&= \frac{1}{c\,(d-2)\,(1+c\bar{t})^{d-2}} \\
&\quad \times \left\{ \left(1 + \frac{b_r}{c\,(d-3)} \right) (1+c\bar{t})^{d-2} - \left(1 + \frac{b_r}{c\,(d-3)} \right) - \frac{b_r\,(d-2)}{(d-3)} \bar{t} \right\} \\
&= \frac{1}{c\,(d-2)} \left\{ p_0 - \frac{p_0 + p_1 \bar{t}}{(1+c\bar{t})^{d-2}} \right\}.
\end{aligned}
$$

Then (14) can be rewritten as follows:

$$
s(\bar{t}) = s_0 - \frac{r_0}{c\,(d-2)} \left\{ p_0 - \frac{p_0 + p_1 \bar{t}}{(1+c\bar{t})^{d-2}} \right\}.
$$

Using the result of case a) we have

$$
s(\bar{t}) = \frac{r_0}{c\,(d-2)} \frac{p_0 + p_1 \bar{t}}{(1+c\bar{t})^{d-2}} = s_0 \frac{1 + \frac{p_1}{p_0} \bar{t}}{(1+c\bar{t})^{d-2}}
$$

or the assertion of case b).

Proof of Proposition 1. We will show that the equation defining the moment $\widetilde{t}_0$ of "smooth switching" to the "optimal" curve

$$\frac{\dot{r}_0 + b\widetilde{t}_0}{(1 + c\widetilde{t}_0)^d} = -\frac{\alpha r^2(\widetilde{t}_0)}{\widetilde{s}_0(\widetilde{t}_0)(\alpha - \beta)} \tag{A.1}$$

has real roots if and only if parameter d of the rational curve (10) is such that

$$d \leq \frac{\alpha}{\beta} + 2$$

and that there are two real roots if the last inequality is strict and one real root if it holds as an equality.

Substituting for $r(\widetilde{t}_0)$ and multiplying both sides of (A.1) by $\left(1 + c\widetilde{t}_0\right)^d$ we have

$$\dot{r}_0 + b\widetilde{t}_0 = -\frac{\alpha r_0^2}{\widetilde{s}_0(\widetilde{t}_0)(\alpha - \beta)} \frac{\left(1 + b_r\widetilde{t}_0\right)^2}{\left(1 + c\widetilde{t}_0\right)^{d-2}}.$$

Applying assertion b) of Lemma 1 it can be written as

$$\dot{r}_0 + b\widetilde{t}_0 = -\frac{\alpha r_0^2}{s_0(\alpha - \beta)} \frac{\left(1 + b_r\widetilde{t}_0\right)^2}{\left(1 + \frac{p_1}{p_0}\widetilde{t}_0\right)}$$

that means that the moment of "smooth switching" $\widetilde{t}_0$ is a solution of quadratic equation

$$\left(\dot{r}_0 + b\widetilde{t}_0\right)\left(1 + \frac{p_1}{p_0}\widetilde{t}_0\right) + \frac{\alpha r_0^2}{s_0(\alpha - \beta)}\left(1 + b_r\widetilde{t}_0\right)^2 = 0$$

or

$$\lambda_2\widetilde{t}_0^2 + \lambda_1\widetilde{t}_0 + \lambda_0 = 0, \tag{A.2}$$

where $\lambda_2 = b\frac{p_1}{p_0} + \frac{b_r^2\alpha r_0^2}{s_0(\alpha-\beta)}$, $\lambda_1 = \frac{p_1}{p_0}\dot{r}_0 + b + \frac{2b_r\alpha r_0^2}{s_0(\alpha-\beta)}$, $\lambda_0 = \dot{r}_0 + \frac{\alpha r_0^2}{s_0(\alpha-\beta)}$. This equation has at least one real root (two if inequality is strict) if and only if $D = \lambda_1^2 - 4\lambda_2\lambda_0 \geq 0$, where

$$\lambda_1^2 = \frac{1}{s_0^2(\alpha-\beta)^2} \times \left\{ \left(\frac{p_1}{p_0}\dot{r}_0 + b\right)^2 s_0^2(\alpha-\beta)^2 + 4b_r\alpha r_0^2 \left(\frac{p_1}{p_0}\dot{r}_0 + b\right) s_0(\alpha-\beta) + 4b_r^2\alpha^2 r_0^4 \right\},$$

$$\lambda_2\lambda_0 = \frac{1}{s_0^2(\alpha-\beta)^2} \times \left\{ b\frac{p_1}{p_0}\dot{r}_0 s_0^2(\alpha-\beta)^2 + s_0(\alpha-\beta)\left[b_r^2\alpha r_0^2\dot{r}_0 + b\frac{p_1}{p_0}\alpha r_0^2\right] + b_r^2\alpha^2 r_0^4 \right\}.$$

Cancelling like terms and multiplying by $s_0(\alpha-\beta) > 0$ we can write our condition as $\widetilde{D} \geq 0$ where

$$\widetilde{D} = s_0(\alpha-\beta)\left[\left(\frac{p_1}{p_0}\dot{r}_0 + b\right)^2 - 4b\frac{p_1}{p_0}\dot{r}_0\right] + 4\left[b_r\alpha r_0^2\left(\frac{p_1}{p_0}\dot{r}_0 + b\right) - b_r^2\alpha r_0^2 a_0 - b\frac{p_1}{p_0}\alpha r_0^2\right].$$

Note that the first bracket $[\cdot]$ in this expression is

$$\left(\frac{p_1}{p_0}\dot{r}_0 + b\right)^2 - 4b\frac{p_1}{p_0}\dot{r}_0 = \left(\frac{p_1}{p_0}\dot{r}_0 - b\right)^2$$

and the second bracket is

$$\left[b_r\alpha r_0^2\left(\frac{p_1}{p_0}\dot{r}_0 + b\right) - b_r^2\alpha r_0^2\dot{r}_0 - b\frac{p_1}{p_0}\alpha r_0^2\right] = \alpha r_0^2\left(\frac{p_1}{p_0} - b_r\right)(b_r\dot{r}_0 - b).$$

Then the condition of the root existence is

$$\widetilde{D} = s_0(\alpha-\beta)\left(\frac{p_1}{p_0}\dot{r}_0 - b\right)^2 + 4\alpha r_0^2\left(\frac{p_1}{p_0} - b_r\right)(b_r\dot{r}_0 - b) \geq 0 \tag{A.3}$$

where

$$\frac{p_1}{p_0}\dot{r}_0 - b = c(d-2)r_0 b_r + \dot{r}_0\frac{b_r(d-2)}{d-3}\frac{c(d-3)}{c(d-3)+b_r} = 2b_r c(d-2)r_0\left[\frac{c(d-2)+\frac{\dot{r}_0}{r_0}}{2c(d-2)+\frac{\dot{r}_0}{r_0}}\right],$$

$$\frac{p_1}{p_0} - b_r = \frac{b_r(d-2)}{d-3}\frac{c(d-3)}{c(d-3)+b_r} - b_r = b_r\left[\frac{c(d-2)}{c(d-3)+b_r} - 1\right]$$
$$= -b_r\left[\frac{c(d-2)+\frac{\dot r_0}{r_0}}{2c(d-2)+\frac{\dot r_0}{r_0}}\right],$$

$$b_r\dot r_0 - b = b_r\dot r_0 + b_r c(d-2)r_0 = b_r r_0\left[c(d-2)+\frac{\dot r_0}{r_0}\right].$$

Substituting for these expressions in (A.3) we obtain

$$\widetilde{D} = s_0(\alpha-\beta)4b_r^2c^2(d-2)^2r_0^2\left[\frac{c(d-2)+\frac{\dot r_0}{r_0}}{2c(d-2)+\frac{\dot r_0}{r_0}}\right]^2$$
$$\geq 4\alpha r_0^3 b_r^2\left[\frac{c(d-2)+\frac{\dot r_0}{r_0}}{2c(d-2)+\frac{\dot r_0}{r_0}}\right]\left[c(d-2)+\frac{\dot r_0}{r_0}\right]$$

or

$$\frac{s_0(\alpha-\beta)c^2(d-2)^2}{2c(d-2)+\frac{\dot r_0}{r_0}} \geq \alpha r_0.$$

Substituting for s_0 (Lemma 1, a)) into the LHS we have

$$\frac{p_0c(d-2)}{2c(d-2)+\frac{\dot r_0}{r_0}} \geq \frac{\alpha}{\alpha-\beta}$$

and substituting for p_0 we obtain

$$\frac{(d-2)}{(d-3)}\frac{2c(d-2)+\frac{\dot r_0}{r_0}}{2c(d-2)+\frac{\dot r_0}{r_0}} \geq \frac{\alpha}{\alpha-\beta}$$

or $1-\frac{\beta}{\alpha} \geq 1-\frac{1}{d-2}$. The last expression gives us $\frac{1}{d-2} \geq \frac{\beta}{\alpha}$ or $d \leq \frac{\alpha}{\beta}+2$.

12 Appendix B (Proof of Proposition 2)

We will show that the equation defining the moment $\widetilde{t}_0$ of "smooth switching" to the "optimal" curve

$$\frac{\dot r_0 + b\widetilde{t}_0}{(1+c\widetilde{t}_0)^d} = -\frac{\alpha r^2(\widetilde{t}_0)}{\widetilde{s}_0(\widetilde{t}_0)(\alpha-\beta)} \tag{B.1}$$

has only one real finite positive root if and only if $d < \frac{\alpha}{\beta} + 2$.

It was shown in Appendix A that equation (B.1) is equivalent to the quadratic equation (A.2) that (see Lemma 1) is equivalent to equation

$$\mu_2 \tilde{t}_0^2 + \mu_1 \tilde{t}_0 + \mu_0 = 0, \tag{B.2}$$

where $\mu_2 = bp_1 + \frac{b_r^2 r_0 \alpha c(d-2)}{\alpha-\beta}$, $\mu_1 = p_1 \dot{r}_0 + bp_0 + \frac{2b_r r_0 \alpha c(d-2)}{\alpha-\beta}$, $\mu_0 = \dot{r}_0 p_0 + \frac{r_0 \alpha c(d-2)}{\alpha-\beta}$.

Substituting for b, p_0, p_1, and reorganizing, we have

$$\begin{aligned}\mu_2 &= -b_r c(d-2) r_0 \frac{b_r(d-2)}{d-3} + \frac{b_r^2 r_0 \alpha c(d-2)}{\alpha-\beta} \\ &= \frac{b_r^2 r_0 c(d-2)}{(\alpha-\beta)(d-3)} \left[\beta(d-2) - \alpha\right].\end{aligned}$$

Note that in our formulation of the problem the multiplier $b_r^2 r_0 c(d-2)/\left[(\alpha-\beta)(d-3)\right]$ in the last formula is always positive since $d > 3$, $\alpha > \beta$, $r_0 > 0$, $\dot{r}_0 > 0$ and it follows $c > 0$. Then the sign of μ_2 is defined by the sign of $\beta(d-2) - \alpha$. Namely, μ_2 is negative when $d < \frac{\alpha}{\beta} + 2$, positive when $d > \frac{\alpha}{\beta} + 2$, and zero when $d = \frac{\alpha}{\beta} + 2$.

Coefficient μ_1 is

$$\mu_1 = \frac{b_r(d-2)}{d-3} \dot{r}_0 - b_r c(d-2) r_0 \left(1 + \frac{b_r}{c(d-3)}\right) + \frac{2b_r r_0 \alpha c(d-2)}{\alpha-\beta}$$

$$= b_r(d-2) \left[\frac{\dot{r}_0 - 2r_0 c(d-2) - \dot{r}_0}{d-3} + \frac{2r_0 \alpha c}{\alpha-\beta}\right].$$

Finally we have $\mu_1 = 2b_r r_0 c(d-2) \left[\frac{\alpha}{\alpha-\beta} - \frac{d-2}{d-3}\right]$. Note that b_r is also positive in our formulation (because of the growing rate of extraction in the neighborhood of $t = 0$). Then the sign of μ_1 like the sign of μ_2 is completely defined by the same expression $\beta(d-2) - \alpha$. It can be shown that $\mu_0 > 0$ for $a_0 > 0$. The peak of parabola (B.2) is defined by equation

$$t^* = -\frac{\mu_1}{2\mu_2} = -\frac{2b_r r_0 c(d-2)\left[\beta(d-2) - \alpha\right]}{2b_r^2 r_0 c(d-2)\left[\beta(d-2) - \alpha\right]} = -\frac{1}{b_r} < 0.$$

Hence, our parabola is convex for $d < \frac{\alpha}{\beta} + 2$ and has only one positive finite root. With $d \to \frac{\alpha}{\beta} + 2 - 0$ parabola degenerates into a positive constant and the root goes to infinity.

13 Appendix C

The condition of nonsmooth change in extraction (8) without losing consumption implies for our production function

$$\Delta c = (1-\beta)\left[\left(k_0 + \dot{k}_0 t\right)^{\alpha} (r_0 + \dot{r}_0 t)^{\beta} - (k_0 + k_1(t))^{\alpha} (r_0 + \dot{r}_1 t)^{\beta}\right] = 0.$$

It follows $\left[(k_0 + k_1(t)) / \left(k_0 + \dot{k}_0 t\right)\right]^{\alpha} = \left[(r_0 + \dot{r}_0 t) / (r_0 + \dot{r}_1 t)\right]^{\beta}$ or

$\left[1 + \left(k_1(t) - \dot{k}_0 t\right) / \left(k_0 + \dot{k}_0 t\right)\right]^{\alpha} = \left[1 + (\dot{r}_0 - \dot{r}_1)\, t / (r_0 + \dot{r}_1 t)\right]^{\beta}$

that gives us $k_1(t) = \dot{k}_0 t + \left\{\left[1 + (\dot{r}_0 - \dot{r}_1)\, t / (r_0 + \dot{r}_1 t)\right]^{\beta/\alpha} - 1\right\}\left(k_0 + \dot{k}_0 t\right).$

Note, that the path of "compensating" capital is already nonlinear: $k(t) = \left(k_0 + \dot{k}_0 t\right)\left[1 + (\dot{r}_0 - \dot{r}_1)\, t / (r_0 + \dot{r}_1 t)\right]^{\beta/\alpha}$ and it dominates the pattern of capital that was before the shift in extraction since the bracket $[\cdot] > 1$ for $t \in (0, dt)$. The "new" investment rule must be

$$\begin{aligned}\dot{k}(t) &= \dot{k}_1(t) \\ &= \dot{k}_0\left[1 + (\dot{r}_0 - \dot{r}_1)\, t / (r_0 + \dot{r}_1 t)\right]^{\beta/\alpha} \\ &\quad + (\beta/\alpha)\left(k_0 + \dot{k}_0 t\right)\left[1 + (\dot{r}_0 - \dot{r}_1)\, t / (r_0 + \dot{r}_1 t)\right]^{\beta/\alpha - 1} (\dot{r}_0 - \dot{r}_1)\, r_0 / (r_0 + \dot{r}_1 t)^2\end{aligned}$$

or

$$\begin{aligned}\dot{k}_1(t) &= \left[1 + (\dot{r}_0 - \dot{r}_1)\, t / (r_0 + \dot{r}_1 t)\right]^{\beta/\alpha - 1} \\ &\quad \times \left\{\left[1 + (\dot{r}_0 - \dot{r}_1)\, t / (r_0 + \dot{r}_1 t)\right] \dot{k}_0 + (\beta/\alpha)\left(k_0 + \dot{k}_0 t\right)(\dot{r}_0 - \dot{r}_1)\, r_0 / (r_0 + \dot{r}_1 t)^2\right\}.\end{aligned}$$

This gives us that $\dot{k}_1(0) = \dot{k}_0 + (\beta/\alpha) k_0 (\dot{r}_0 - \dot{r}_1) / r_0 > \dot{k}_0$ and then $\dot{k}_1(0)/k_0 - \dot{k}_0/k_0 = (\beta/\alpha)(\dot{r}_0/r_0 - \dot{r}_1/r_0) > 0$ that is the "additional" rate of investment at $t = 0$, which must compensate for the loss of consumption due to the nonsmooth shift in extraction.

References

[1] Asheim, G.B., Buchholz, W., Hartwick J.M., Mitra T., Withagen C., 2007. Constant savings rates and quasi-arithmetic population growth under exhaustible resource constraints. J. Environ. Econ. Manage. 53, 213-229.

[2] Bazhanov, A.V., 2006. Variatsionnye printsipy modelirovaniya v resursnoj economike (Variation principles for modeling in resource economics). Vestnik

DVO RAN 6, 5-13. Available at: http://mpra.ub.uni-muenchen.de/1309/ (in Russian).

[3] Bazhanov, A.V., 2007a. The peak of oil extraction and a modified maximin principle. MPRA Paper No. 2019. In: Proceedings of the International Conference "Comparative Institution and Political Economy: Theoretical, Experimental, and Empirical Analysis", Waseda University, Tokyo, pp. 99-128. Available at: http://mpra.ub.uni-muenchen.de/2019/

[4] Bazhanov, A.V., 2007b. The peak of oil extraction and consistency of the government's short- and long-run policies. MPRA Paper No. 2507. Presented at the Seminar of School of Economics, Seoul National University, Seoul. Available at: http://mpra.ub.uni-muenchen.de/2507/

[5] Bazhanov, A.V., 2007c. The transition to an oil contraction economy. Ecol. Econ. 64, 186-193.

[6] Brander, J.A., 2007. Viewpoint: Sustainability: Malthus revisited? Can. J. Econ. 40, 1-38.

[7] Caillaud, B., Guesnerie, R., Rey, P. et al., 1988. Government intervention in production and incentives theory: a review of recent contributions. RAND J. Econ. 19, 1-26.

[8] Dasgupta, P., Heal, G., 1974. The optimal depletion of exhaustible resources. Rev. Econ. Stud. 41, 3-28.

[9] Dasgupta, P., Heal, G., 1979. Economic theory and exhaustible resources. Cambridge University Press, Cambridge.

[10] Davis, G.A., Cairns, R.D., 1999. Valuing petroleum reserves using current net price. Econ. Inquiry 37, 295-311.

[11] Gaudet, G., 2007. Natural resource economics under the rule of Hotelling. Can. J. Econ. 40, 1033-1059.

[12] Grimaud, A., Rouge, L., 2005. Polluting non-renewable resources, innovation, and growth: welfare and environmental policy. Resource Energy Econ. 27, 109-129.

[13] Hamilton, K., Ruta, G., Tajibaeva, L., 2006. Capital accumulation and resource depletion: A Hartwick Rule counterfactual. Environ. Resource Econ. 34, 517-533.

[14] Hartwick, J.M., 1977. Intergenerational equity and the investing of rents from exhaustible resources. Amer. Econ. Rev. 67, 972-974.

[15] Karp, L., Livernois, J., 1992. On efficiency-inducing taxation for a non-renewable resource monopolist. J. Public Econ. 49, 219-239.

[16] Koopmans, T.C., 1965. On the concept of optimal economic growth. In: Pontificiae Academiae Scientiarum scripta varia, 28, 225-300.

[17] Leininger, W., 1985. Rawls' maximin criterion and time-consistency: further results. Rev. Econ. Stud. 52, 505-513.

[18] Nordhaus, W.D., Boyer, J. (2000). Warming the world: Economic models of global warming. Cambridge, Mass. : MIT Press.

[19] Pearce, D.W., Atkinson, G.D., 1993. Capital theory and the measurement of sustainable development: an indicator of weak sustainability. Ecol. Econ. 8, 103-108.

[20] Pezzey, J.C.V., 2002. Sustainability policy and environmental policy. Australian National University. Economics and Environment Network Working Paper EEN0211.

[21] Rawls, J., 1971. A theory of justice. Belknap Press of Harvard University Press, Cambridge.

[22] Solow, R.M., 1974. Intergenerational equity and exhaustible resources. Rev. Econ. Stud. 41, 29-45.

[23] Stiglitz, J., 1974. Growth with exhaustible natural resources: efficient and optimal growth paths. Rev. Econ. Stud. 41, 123-137.

[24] Stollery, K.R., 1998. Constant utility paths and irreversible global warming. Can. J. Econ. 31, 730-742.