Sperner Lemma, Fixed Point Theorems, and the Existence of Equilibrium

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3 May 2020

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May 3, 2020

Abstract

In characterizing the existence of general equilibrium, existing studies mainly draw on Brouwer and Kakutani fixed point theorems and, to some extent, Gale-Nikaido-Debreu lemma. In this paper, we show that Sperner lemma can play a role as an alternative powerful tool for the same purpose. Specifically, Sperner lemma can be used to prove those theorems as well as the lemma. Additionally, Kakutani theorem is shown as a corollary of Gale-Nikaido-Debreu lemma. For a demonstration of the use of Sperner lemma to prove general equilibrium existence, we consider two competitive economies marked either by production goods or financial assets. In each case, we successfully provide another proof on the existence of a general equilibrium using only Sperner lemma and without a need to call on the fixed point theorems or the lemma.

JEL Classification: C60, C62, D5.

Keywords: Sperner lemma, Simplex, Subdivision, Fixed Point Theorem, Gale-Nikaido-Debreu Lemma, General Equilibrium.

1 Introduction

In economics, general equilibrium models have been one of the most powerful tools for analyzing the dynamics of the economy. They also help evaluate and quantify the economic impact of different policies. While the theory goes as far back as the 19th Century with Walras (1877), the mathematical and computational modeling has its foundation in the 1950s with pioneering works of Arrow and Debreu (1954) and Debreu (1959). This school of thought focuses on gaining an insight into the

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interactions between markets and agents as well as the determination of prices and quantities through the market mechanism. Studying the economy as a whole, general equilibrium models encompass a multitude of different goods markets in which a change in one market may have a follow-on impact on other markets.

For any general equilibrium model, having a thorough examination of the existence of an equilibrium is obviously of the first-order importance. In doing so, existing studies make use of several fixed point theorems, including Brouwer and Kakutani fixed point theorems (from now and henceforth Brouwer and Kakutani theorems for short).[1] Looking back to history, Arrow and Debreu (1954) use Eilenberg-Montgomery fixed point theorem which is more general than Kakutani Theorem. In the Theory of Value of Debreu (1959), he uses Gale-Nikaido-Debreu (GND) lemma to prove the existence of general equilibrium. It is noteworthy that there are several versions of GND lemma and those proofs require Kakutani theorem or Knaster-Kuratowski-Mazurkiewicz theorem.[2] McKenzie (1959) uses the Brouwer fixed point theorem to prove the existence of a competitive equilibrium.[3]

The above summary indicates that Kakutani and Brouwer theorems as well as GND lemma have played a central role in establishing the existence of general equilibrium in competitive economies.[4] While the continued usage of the fixed point theorem for this particular purpose is certainly fine, this high path dependence raises the need for an alternative innovative approach to the issue. In this paper, we attempt to answer the following question: Is there another direct way of proving the existence of a general equilibrium that is equally as good as the conventional method used by the literature? In working out for the answer, we arrive at Sperner lemma, obtained by Sperner (1928), as a dispensable toolkit. Notice that Sperner lemma is a purely combinatorial result concerning with labeling the vertices of subsimplices. Our main contribution to the existing literature is to present a new method of proving the existence of a general equilibrium in competitive economies where Sperner lemma is a useful tool instead of the fixed point theorems.

As a demonstration, we consider two hypothetical cases: an economy with production and a two-period stochastic economy with incomplete financial markets. In each case, we establish the existence of general equilibrium by only using Sperner

[1] Brouwer theorem simply states that every continuous mapping f of an n-dimensional ball to itself has a fixed point x, i.e., $f(x) = x$. It was separately proved by Brouwer and Hadamard in 1910 (Hadamard, 1910; Brouwer, 1911). Kakutani theorem obtained by Kakutani (1941) is a generalization of Brouwer theorem to the case of correspondence.

[2] When proving this lemma, Debreu (1959) and Nikaido (1956) use Kakutani theorem while Gale (1955) uses Knaster-Kuratowski-Mazurkiewicz and Kakutani theorems. See Florenzano (2009) for an excellent review of this issue.

[3] Note that McKenzie assumes the total production set is a cone. This assumption is crucial to prove this fixed point actually corresponds to an equilibrium price.

[4] Another important lemma in the general equilibrium theory is Gale and Mas-Colell's lemma introduced and proved by Gale and Mas-Colell (1975, 1979). Their proof makes use of Kakutani theorem and Michael (1956)'s selection theorem.

lemma. Moreover, we also prove Brouwer and Kakutani theorems as well as different versions of GND lemma by using Sperner lemma. Upon these results, we additionally emphasize that Kakutani theorem can be obtained as a corollary of GND lemma. This is proved by adapting the argument of Uzawa (1962) for continuous mapping.[5]

Some authors have attempted to prove Kakutani Theorem by using Sperner lemma. Indeed, Sondjaja (2008) provides a proof by using Sperner lemma but he needs to make use of von Neumann (1937)'s approximation lemma. This makes the proof a bit more cumbersome. Tanaka (2012) proves the so-called *hyperplane labeling lemma*, generalizing Sperner's original lemma. He then combines this result and the approximate minimax theorem to prove Kakutani Theorem. Our proof seems to be more straightforward and direct as it only uses the core notions of combinatorial topology.[6]

The paper proceeds as follows. In Section 2, we review some basic concepts such as the notion of subsimplex, simplicial subdivision, Sperner lemma, and the maximum theorem. In Section 3, we use Sperner lemma to prove Brouwer and Kakutani fixed point theorems as well as GND lemma. We then separately consider two economies with either production goods or financial assets. In each case, we demonstrate that using Sperner lemma is sufficient for characterizing equilibrium existence. Finally, Section 4 concludes the paper.

2 Preliminaries

In this section, we introduce basic terminologies and necessary background for our work. First, we present definitions from combinatorial topology based on which we state the Sperner lemma. After that, we provide a brief overview of correspondences and the maximum theorem which are extensively used for proving the existence of a general equilibrium.

2.1 On Sperner lemma

Consider the Euclidean space $\mathbb{R}^n$. Let $e^1 = (1, 0, 0, \ldots, 0), e^2 = (0, 1, 0, \ldots, 0), \ldots,$ and $e^n = (0, 0, \ldots, 0, 1)$ denote the n unit vectors of $\mathbb{R}^n$. The unit-simplex Δ of $\mathbb{R}^n$ is the convex hull of $\{e^1, e^2, \ldots, e^n\}$. A simplex of Δ, denoted by $[[x^1, x^2, \ldots, x^n]]$, is the convex hull of $\{x^1, x^2, \ldots, x^n\}$ where $x^i \in \Delta$ for any $i = 1, \ldots, n$, and the

[5] Recall that Uzawa (1962) only proves the equivalence between Brouwer theorem and Walras' existence theorem.

[6] In comparison, the Sperner lemma and the stated fixed points theorems are roughly equivalent. For instance, Knaster, Kuratowski, and Mazurkiewicz (1929) use Sperner lemma to prove Knaster-Kuratowski-Mazurkiewicz theorem which implies Brouwer theorem. Meanwhile, Yoseloff (1974) and Park and Jeong (2003) prove Sperner lemma by using Brouwer theorem. See Park (1999) for an excellent survey about fixed point theorems.

vectors $(x^1 - x^2, x^1 - x^3, \ldots, x^1 - x^n)$ are linearly independent.[7] Given a simplex $[[x^1, x^2, \ldots, x^n]]$, a face of this simplex is the convex hull $[[x^{i_1}, x^{i_2}, \ldots, x^{i_m}]]$ with $m < n$, and $\{i_1, i_2, \ldots, i_m\} \subset \{1, 2, \ldots, n\}$.

We now define the notions of simplicial subdivision (or triangulation) and labeling (see Border (1985) and Su (1999) for a general treatment) before stating the Sperner lemma.

Definition 1. *T is a simplicial subdivision of Δ if it is a finite collection of simplices and their faces $\Delta_i,\ i = 1, \ldots, p$ such that*

- $\Delta = \cup_{i=1}^{p} \Delta_i$,
- $ri(\Delta_i) \cap ri(\Delta_j) = \emptyset, \forall i \neq j$.

Recall that if $\Delta_i = [[x^{i_1}, x^{i_2}, \ldots, x^{i_m}]]$, then $ri(\Delta_i) \equiv \{x \mid x = \sum_{k=1}^{m} \alpha_k x^k(i);\ \sum_k \alpha_k = 1;$ and $\forall k : \alpha(k) > 0\}$.

Simplicial subdivision simply partitions an n-dimensional simplex into small simplices such that any two simplices are either disjoint or share a full face of a certain dimension.

Remark 1. *For any positive integer K, there is a simplicial subdivision $T^K = \{\Delta_1^K, \ldots, \Delta_{p(K)}^K\}$ of Δ such that $Mesh(T^K) \equiv \max_{i \in \{1, \ldots, p(K)\}} \sup_{x,y} \{\|x - y\| : x, y \in \Delta_i^K\} < 1/K$. For example, we can take equilateral subdivisions or barycentric subdivisions.*

We focus on the labeling of these subdivisions with certain restrictions.

Definition 2. *Consider a simplicial subdivision of Δ. Let V denote the set of vertices of all the subsimplices of Δ. A labeling R is a function from V into $\{1, 2, \ldots, n\}$. A labeling R satisfies the* ***Sperner condition*** *if:*

$$x \in\ ri[[e^{i_1}, e^{i_2}, \ldots, e^{i_m}]] \Rightarrow R(x) \in \{i_1, i_2, \ldots, i_m\}.$$

In particular, $R(e^i) = i, \forall i$.

Note that Sperner condition implies that all vertices of the simplex are labeled distinctly. Moreover, the label of any vertex on the edge between the vertices of the original simplex matches with another label of these vertices. With these in mind, we can now state Sperner lemma.

Lemma 1. *(Sperner) Let $T = \{\Delta_1, \ldots, \Delta_p\}$ be a simplicial subdivision of Δ. Let R be a labeling which satisfies the Sperner condition. Then there exists a subsimplex $\Delta_i \in T$ which is completely labeled, i.e. $\Delta_i = [[x^1(i), \ldots, x^n(i)]]$ with $R(x^l(i)) = l, \forall l = 1, \ldots, n$.*

[7] We can verify that the vectors $(x^1 - x^2, x^1 - x^3, \ldots, x^1 - x^n)$ are linearly independent if and only if the vectors $(x^1, x^2, \ldots, x^n)$ are affinely independent (i.e., if $\sum_{i=1}^{n} \lambda_i x_i = 0$ and $\sum_{i=1}^{n} \lambda_i = 0$ imply that $\lambda_i = 0\ \forall i$).

Sperner lemma guarantees the existence of a completely labeled subsimplex for any simplicially subdivided simplex in accordance with the Sperner condition. A proof of this lemma can be found in several text books or papers such as Sperner (1928), Berge (1959), Scarf and Hansen (1973), Le Van (1982). In particular, the original proof uses an inductive argument based on a complete enumeration of all completely labeled simplices for a series of lower dimensional problems. Meanwhile, proofs using constructive arguments date back to Cohen (1967) and Kuhn (1968) (see Scarf (1982) for a demonstration of the constructive proof).

2.2 On correspondences and the maximum theorem

Let $X \subset \mathbb{R}^l, Y \subset \mathbb{R}^m$. A correspondence Γ from X into Y is a mapping from X into the set of subsets of Y. The graph of Γ is the set graph$\Gamma = \{(x, y) \in X \times Y : y \in \Gamma(x)\}$. A correspondence $\Gamma : X \to Y$ is *closed* if its graph is closed.

Definition 3. *A correspondence $\Gamma : X \to Y$ is lower semicontinuous at point x if for any $y \in \Gamma(x)$ and for any sequence $\{x^n\} \subset X$ converging to x, there exists a subsequence $\{y^{n_k}\}$ with $y^{n_k} \in \Gamma(x^{n_k}), \forall k$, such that $\{y^{n_k}\}$ converges to y when k converges to $+\infty$. Γ is lower semicontinuous on X if it is lower semicontinuous everywhere on X.*

Definition 4. *A correspondence $\Gamma : X \to Y$ is upper semicontinuous at point x if (i) $\Gamma(x)$ is compact, non-empty, and (ii) for any sequence $\{x_n\}$ converging to x, for any sequence $\{y_n\}$ with $y_n \in \Gamma(x_n), \forall n$, there exists a subsequence $\{y_{n_k}\}$ which converges to $y \in \Gamma(x)$.*

A correspondence is *continuous* if it is both lower semicontinuous and upper semicontinuous. Note that if X is compact then Γ is upper semicontinuous if and only if Γ is closed. It is also clear that if Γ is upper semicontinuous and $K \subset X$ is compact, then $\Gamma(K)$ is compact. Recall that if Γ is single valued, the notions of continuity, upper semicontinuity, and the lower semicontinuity turn out to be equivalent.

We can now state the Theorem of the Maximum, the proof of which was first given by Berge (1959).

Theorem 1. *(Berge, 1959) Let $X \subset \mathbb{R}^l, Y \subset \mathbb{R}^m$, $\Gamma : X \to Y$, and $\phi : Y \to \mathbb{R}$. Assume ϕ is continuous and Γ is also a continuous correspondence. Then the function h defined by*

$$h(x) = \max\{\phi(y) : y \in \Gamma(x)\}$$

is continuous. Moreover, the correspondence, called argmax, ζ defined by

$$\zeta(x) = \{y \in \Gamma(x) : h(x) = \phi(y)\}$$

is upper semicontinuous.

3 Main results

3.1 Using Sperner lemma to prove fixed point theorems

Brouwer fixed point theorem is considered as one of the most fundamental results in topology. Kakutani fixed point theorem is a generalization of Brouwer theorem for the case of set-valued functions. These two theorems have a wide application across different fields of mathematics and economics

We now formally state Kakutani theorem and use Sperner lemma to prove it.

Theorem 2. *(Kakutani) Let ζ be an upper semi continuous correspondence, with non empty convex compact values from a non-empty convex, compact set $V \subset \mathbb{R}^N$ into itself. Then there exists a fixed point x, i.e. $x \in \zeta(x)$.*

Proof. Since any convex compact set in $\mathbb{R}^N$ is homeomorphic to a simplex, we present here a proof for the case where the set V is the unit-simplex Δ of $\mathbb{R}^N$.

Let $\epsilon > 0$ be given. Since Δ is compact, there exists a finite covering of Δ with a finite family of open balls $\left(\tilde{B}\left(x^i(\epsilon), \epsilon\right)\right)_{i=1,\dots,I(\epsilon)}$. Take a partition of unity subordinate to the family $\left(\tilde{B}\left(x^i(\epsilon), \epsilon\right)\right)_{i=1,\dots,I(\epsilon)}$, i.e. a family of continuous non-negative real functions $(\alpha_i)_{i=1,\dots,I(\epsilon)}$ from Δ in $\mathbb{R}_+$ such that $Supp(\alpha_i) \subset B\left(x^i(\epsilon), \epsilon\right), \forall i$ and $\sum_{i=1}^{I(\epsilon)} \alpha_i(x) = 1, \forall x \in \Delta$.[8]

Take $y^i(\epsilon) \in \zeta(x^i(\epsilon)), \forall i$ and define the function $f^\epsilon : \Delta \to \Delta$ by $f^\epsilon(x) = \sum_{i=1}^{I(\epsilon)} \alpha_i(x) y^i(\epsilon)$. This function is continuous.

Let $K > 0$ be an integer and consider a simplicial subdivision T^K such that $Mesh(T^K) < 1/K$ (see Remark 1). We define a labeling R as follows:[9]

$$\text{for } x \in \Delta, R(x) = l, \text{ if } x_l \geq f_l^\epsilon(x). \tag{1}$$

Such a labeling is well defined because $\sum_l x_l = \sum_l f_l^\epsilon(x) = 1$. Moreover, this labeling satisfies the Sperner condition. Indeed, take $x \in \mathrm{ri}[[e^{i_1}, \dots, e^{i_r}]]$ (recall that $(e^i)_i$ are the unit-vectors of $\mathbb{R}^N$.) We claim that $R(x) \in \{i_1, \dots, i_r\}$. If not, $x_l < f_l^\epsilon(x), \forall l \in \{i_1, \dots, i_r\}$ and we get a contradiction:

$$1 = \sum_{l \in \{i_1, \dots, i_r\}} x_l < \sum_{l \in \{i_1, \dots, i_r\}} f_l^\epsilon(x) \leq 1.$$

According to Sperner lemma, there exists a completely labeled subsimplex $S^K = [[x^{K,1}, \dots, x^{K,N}]]$, with $x_l^{K,l} \geq f_l^\epsilon(x^{K,l})\ \forall l = 1, \dots, N$.

Let $K \to +\infty$, there exists a subsequence $(K_t)_{t \geq 1}$ such that $x^{K_t,l}$ converges to x^l for any $l = 1, \dots, N$. Since $Mesh(T^K)$ tends to zero, we must have $x^1 = x^2 = \cdots = x^N$.

[8]For the notion of partition of unity, see, for instance, Aliprantis and Border (2006)'s Section 2.19.

[9]This labeling is the same as in Scarf (1967) and Border (1985).

Let $x^*(\epsilon)$ be this point. By continuity, we have $f^\epsilon(x^{K_t,l}) \to f^\epsilon(x^*(\epsilon))\ \forall l$. Since $x^*_l(\epsilon) \geq f^\epsilon_l(x^*(\epsilon))\ \forall l$, we get $x^*(\epsilon) = f^\epsilon(x^*(\epsilon))$.

Since $\left(\tilde{B}\left(x^i(\epsilon), \epsilon\right)\right)_{i=1,\dots,I(\epsilon)}$ is a covering of Δ, we have $x^*(\epsilon) \in \cap_{i\in J(\epsilon)} \tilde{B}\left(x^i(\epsilon), \epsilon\right)$, where $J(\epsilon) \subset \{1, \dots, I(\epsilon)\}$. Hence

$$x^*(\epsilon) = f^\epsilon(x^*(\epsilon)) = \sum_{i\in J(\epsilon)} \alpha^i(x^*(\epsilon)) y^i(\epsilon) \tag{2a}$$

$$\text{with} \sum_{i\in J(\epsilon)} \alpha^i(x^*(\epsilon)) = 1, y^i(\epsilon) \in \zeta(x^i(\epsilon)), \forall i \in J(\epsilon). \tag{2b}$$

Observe that $\forall i \in J(\epsilon), x^i(\epsilon) \in B(x^*(\epsilon), \epsilon) \subset \mathbb{R}^N$. Therefore, $y^i(\epsilon) \in \zeta\big(B(x^*(\epsilon), \epsilon)\big)$ and $f^\epsilon(x^*(\epsilon)) \in co\Big(\zeta\big(B(x^*(\epsilon), \epsilon)\big)\Big)$.

From Carathéodory's convexity theorem,[10] we have a decomposition

$$f^\epsilon(x^*(\epsilon)) = \sum_{i=1}^{N+1} \beta_i(x^*(\epsilon)) \tilde{y}^i(x^*(\epsilon)) \tag{3}$$

with $\tilde{y}^i(x^*(\epsilon)) \in \zeta\big(B(x^*(\epsilon), \epsilon)\big)$, $\beta_i(x^*(\epsilon)) \geq 0$, $\sum_{i=1}^{N+1} \beta_i(x^*(\epsilon)) = 1$.

Let $\epsilon \to 0$. Without loss of generality, we can assume $x^*(\epsilon) \to \bar{x} \in \Delta$, $\beta_i(x^*(\epsilon)) \to \bar{\beta}_i \geq 0$, $\sum_{i=1}^{N+1} \bar{\beta}_i = 1$, and $\tilde{y}^i(x^*(\epsilon)) \to \bar{y}^i \in \zeta(\bar{x}), \forall i = 1, \dots, N+1$. This implies $\bar{x} = \sum_{i=1}^{N+1} \bar{\beta}_i \bar{y}^i$. Since $\zeta(\bar{x})$ is convex, we get $\bar{x} \in \zeta(\bar{x})$. The proof of Kakutani theorem is, therefore, over. $\square$

Brouwer fixed point theorem, stated below, is a corollary of Kakutani fixed point theorem when ζ is a single valued mapping.

Corollary 1. *(Brouwer) Let ϕ be a continuous mapping from a non-empty convex compact set into itself. Then there exists a fixed point x, i.e. $x = \phi(x)$.*

Remark 2. *In the literature, Brouwer theorem has been used to prove Kakutani theorem. Indeed, the original proof of Kakutani theorem in Kakutani (1941) relies on the application of Brouwer theorem to single-valued mappings approximating the given set-valued mapping.*[11]

[10] Carathéodory (1907)'s convexity Theorem states that: In an n-dimensional vector space, every vector in the convex hull of a nonempty set can be written as a convex combination using no more than $n+1$ vectors from the set. For a simple proof, see Florenzano and Le Van (2001)'s Proposition 1.1.2 or Aliprantis and Border (2006)'s Theorem 5.32.

[11] For a pedagogical purpose, we summarize here the proof of Kakutani. Let S^n be the n-th barycentric simplicial subdivision of Δ. For each vertex x^n of S^n, take an arbitrary point $y^n \in \zeta(x^n)$. This mapping can be extended linearly to a continuous point-to-point mapping $x \to \phi_n(x)$ of Δ to itself. Applying Brouwer theorem, there exists $x_n \in \Delta$ such that $x_n = \phi_n(x_n)$. Let n tend to infinity, there is a subsequence of (x_n) converging to a point x^* which is actually a fixed-point of ζ.

Florenzano (1981) also makes use Brouwer theorem to prove Kakutani theorem.[12] *More precisely, for any $\epsilon > 0$, Florenzano considers a covering of Δ by a finite family of open balls and defines the function f^ϵ as in our above proof. By applying Brouwer theorem, f^ϵ has a fixed point x^ϵ. Let $\epsilon \to 0$, then $x^\epsilon \to \bar{x}$. To prove that $\bar{x} \in \zeta(\bar{x})$, assume that this is not a case, then apply the Separation Theorem to the sets $\{\bar{x}\}$ and $\zeta(\bar{x})$ to get a contradiction.*

We proceed as in Florenzano (1981) but use Sperner lemma to get a fixed point x^ϵ of the function f^ϵ. Let $\epsilon \to 0$, then $x^\epsilon \to \bar{x}$. To prove that $\bar{x} \in \zeta(\bar{x})$, we proceed differently. More precisely, we apply Carathéodory's convexity theorem to get a decomposition (3) of $f^\epsilon(x^*(\epsilon))$. When $\epsilon \to 0$, x can be expressed as a convex combination of elements which belong $\zeta(\bar{x})$. So, $\bar{x} \in \zeta(\bar{x})$.

3.2 Using Sperner lemma to prove Gale-Nikaido-Debreu lemma

The customary proofs of the existence of a general equilibrium also make use of either GND lemma (Debreu, 1959; Gale, 1955; Nikaido, 1956) or Gale and Mas-Colell lemma (Gale and Mas-Colell, 1975, 1979) whose proofs, in turn, require Kakutani or Brouwer theorems.[13] In what follows, we show that GND lemma can be proven by using only Sperner lemma.

Lemma 2. *(Gale-Nikaido-Debreu lemma) Let Δ be the unit-simplex of $\mathbb{R}^N$. Let ζ be a continuous mapping from Δ into $\mathbb{R}^L$. Suppose ζ satisfies the condition*

$$\forall p \in \Delta,\ p \cdot \zeta(p) \leq 0.$$

Then there exists $\bar{p} \in \Delta$ such that $\zeta(\bar{p}) \leq 0$.

Proof. Let $K > 0$ be an integer and consider a simplicial subdivision T^K of the unit-simplex Δ of $\mathbb{R}^N$ such that $Mesh(T^K) < 1/K$. With any vertex p^i of T^K, we associate $\zeta(p^i)$. We have $p^i \cdot \zeta(p^i) \leq 0$. We consider the following labeling:

$$\text{For } p \in \Delta,\ R(p) = i \text{ if } \zeta_i(p) \leq 0.$$

Such a labeling is well defined. Indeed, if not, $\zeta_i(p) > 0$ for all i and

$$0 \geq \sum_i p_i \zeta_i(p) > 0$$

leads to a contradiction. Note that this labeling satisfies the Sperner condition. Indeed, take $p \in \text{ ri}[[e^{i_1}, \ldots, e^{i_m}]], m < N$. Then $R(p) \in \{i_1, \ldots, i_m\}$. If not, $\zeta_i(p) > 0, \forall i \in \{i_1, \ldots, i_m\}$ and $0 \geq p \cdot \zeta(p) = \sum_{i \in \{i_1, \ldots, i_m\}} p_i \zeta_i(p) > 0$. That is a contradiction.

Now, from Sperner lemma, for any K, there exists a completely labeled subsimplex $[[p^{i_1}, \ldots, p^{i_N}]]$ which satisfies for any $l = 1, \ldots, N$, $\zeta_l(p^{i_l}) \leq 0$. Let $K \to +\infty$. Then, $\forall l, p^{i_l} \to \bar{p}$ and $\zeta(p^{i_l}) \to \zeta(\bar{p})$. Obviously, $\zeta_l(\bar{p}) \leq 0, \forall l = 1, \ldots, N$. In other words, $\zeta(\bar{p}) \leq 0$. □

[12] See Proposition 2 in Florenzano (1981).

[13] See Florenzano (1982), Florenzano and Le Van (1986) for other versions of GND lemma.

We can also consider GND lemma in its strong and alternative formulations. Importantly, we are able to show that Sperner lemma can be used for an effective proof of each formulation.

Lemma 3. *(Gale-Nikaido-Debreu lemma: strong version) Let Δ be the unit-simplex of $\mathbb{R}^N$. Let ζ be an upper semi-continuous correspondence with non-empty, compact, convex values from Δ into $\mathbb{R}^N$. Suppose ζ satisfies the following condition:*

$$\forall p \in \Delta,\ \forall z \in \zeta(p), p \cdot z \leq 0. \tag{4}$$

Then there exists $\bar{p} \in \Delta$ such that $\zeta(\bar{p}) \cap \mathbb{R}^N_- \neq \emptyset$.

Proof. Let $A = \max\{\|z\|_1 : z \in \zeta(\Delta)\}$. Let $\epsilon \in (0,1)$. Since Δ is compact, there exists a finite covering of Δ with a finite family of open balls $\left(\tilde{B}\left(x^i(\epsilon), \epsilon\right)\right)_{i=1,\ldots,I(\epsilon)}$. Take a partition of unity subordinate to the family $\left(\tilde{B}\left(x^i(\epsilon), \epsilon\right)\right)_{i=1,\ldots,I(\epsilon)}$, i.e. a family of continuous non negative real functions $(\alpha_i)_{i=1,\ldots,I(\epsilon)}$ from Δ in $\mathbb{R}_+$ such that Supp $\alpha_i \subset B\left(x^i(\epsilon), \epsilon\right), \forall i$ and $\sum_{i=1}^{I(\epsilon)} \alpha_i(x) = 1, \forall x \in \Delta$. Take $y^i(\epsilon) \in \zeta(x^i(\epsilon)), \forall i$ and define the function $f^\epsilon(x) = \sum_{i=1}^{I(\epsilon)} \alpha_i(x) y^i(\epsilon) \in \Delta$. This function is continuous.

Given $x \in \Delta$, there exists a set $J(x) \subset \{1, \ldots, I(\epsilon)\}$ such that $x \in \cap_{i \in J(x)} \tilde{B}\left(x^i(\epsilon), \epsilon\right)$. We have $f^\epsilon(x) = \sum_{i \in J(x)} \alpha_i(x) y^i(\epsilon)$ with $\sum_{i \in J(x)} \alpha_i(x) = 1$. We have

$$\forall i \in J(x), x^i(\epsilon) = x + \epsilon u^i(x), \text{ with some } u^i(x) \in B(0,1)$$

which implies that: $\forall i \in J(x), y^i(\epsilon) \in \zeta(x^i(\epsilon)) = \zeta(x + \epsilon u^i(x)) \subset \zeta\big(B(x,\epsilon)\big)$. By consequence, $f^\epsilon(x) \in co\Big(\zeta\big(B(x,\epsilon)\big)\Big)$. According to Carathéodory's convexity theorem, we have a decomposition

$$f^\epsilon(x) = \sum_{i=1}^{N+1} \beta_i(x,\epsilon) \tilde{y}^i(x,\epsilon)$$

with $\tilde{y}^i(x,\epsilon) \in \zeta(x + \epsilon u^i)$ where $u^i \in B(0,1)$, $\beta_i(x,\epsilon) \geq 0$, $\sum_{i=1}^{N+1} \beta_i(x,\epsilon) = 1$. From this, we have

$$\begin{aligned} x \cdot f^\epsilon(x) &= \sum_{i=1}^{N+1} \beta_i(x,\epsilon)(x + \epsilon u^i) \cdot \tilde{y}^i(x,\epsilon) - \epsilon \sum_{i=1}^{N+1} \beta_i(x,\epsilon) u^i \cdot \tilde{y}^i \\ &\leq \epsilon \sum_{i=1}^{N+1} \beta_i(x,\epsilon) \|u^i\| \cdot \|\tilde{y}^i\| \leq \epsilon A \sum_{i=1}^{N+1} \beta_i(x,\epsilon) = \epsilon A \end{aligned}$$

since $(x + \epsilon u^i) \cdot \tilde{y}^i(x,\epsilon) \leq 0$ (see condition (4)), $\|u^i\| \leq 1$ and $\|\tilde{y}^i\| \leq A$. Therefore, we get that

$$\forall x \in \Delta, \exists i,\ f_i^\epsilon(x) \leq \epsilon A. \tag{5}$$

Indeed, if $\forall i, f_i^\epsilon(x) > \epsilon A$, then $\epsilon A < \sum_i x_i f_i^\epsilon(x) \le \epsilon A$ which is a contradiction.

Let $K > 0$ be an integer and consider a simplicial subdivision T^K of the unit-simplex Δ of $\mathbb{R}^N$ such that $Mesh(T^K) < 1/K$ and define the labeling R as follows:

$$\forall x \in \Delta,\ R(x) = i,\ \text{if}\ f_i^\epsilon(x) \le \epsilon A.$$

According to (5), this labeling is well-defined. It also satisfies the Sperner condition

$$x \in [[e^{i_1}, \dots, e^{i_m}]] \Rightarrow R(x) = i \in \{i_1, \dots, i_m\}$$

Indeed, if $f_i^\epsilon(x) > \epsilon A, \forall i \in \{i_1, \dots, i_m\}$, then $\epsilon A \ge x \cdot f^\epsilon(x) = \sum_{i \in \{i_1,\dots,i_m\}} x_i f_i^\epsilon(x) > \epsilon \sum_{i \in \{i_1,\dots,i_m\}} x_i = A\epsilon$, which is a contradiction.

Sperner lemma implies that there exists a completely labeled subsimplex $[[x^{K,1}, \dots, x^{K,N}]]$ with $R(x^{K,l}) = l, \forall l = 1, \dots, N$, i.e., $f_l^\epsilon(x^{K,l}) \le \epsilon A, \forall l = 1, \dots, N$.

Let $K \to +\infty$, there is a subsequence (K_t) such that

$$\forall l, x^{K_t,l} \to x^\epsilon \in \Delta, \quad f^\epsilon(x^{K_t,l}) \to f^\epsilon(x^\epsilon)$$
$$\text{and, therefore, } f_l^\epsilon(x^\epsilon) \le \epsilon A, \forall l = 1, \dots, N.$$

Since $\left(\tilde{B}\left(x^i(\epsilon), \epsilon\right)\right)_{i=1,\dots,I(\epsilon)}$ is a covering of Δ, there exists a set $J(x^\epsilon) \subset \{1, \dots, I(\epsilon)\}$ such that $x \in \cap_{i \in J(x^\epsilon)} \tilde{B}\left(x^i(\epsilon), \epsilon\right)$. We have $f^\epsilon(x^\epsilon) = \sum_{i \in J(x^\epsilon)} \alpha_i(x^\epsilon) y^i(x^\epsilon)$ with $\sum_{i \in J(x^\epsilon)} \alpha_i(x^\epsilon) = 1$. Use the same argument as in our proof of Kakutani theorem, we get

$$f^\epsilon(x^\epsilon) = \sum_{i=1}^{N+1} \beta_i(x^\epsilon) \tilde{y}^i(x^\epsilon)$$

with $\tilde{y}^i(x^\epsilon) \in \zeta\big(B(x^\epsilon, \epsilon)\big)$.

Let $\epsilon \to 0$, we get that

$$\begin{aligned}
&x^\epsilon \to \bar{x} \in \Delta, \quad \beta_i(x^\epsilon) \to \bar{\beta}_i \ge 0, \sum_{i=1}^{N+1} \bar{\beta}_i = 1, \\
&\tilde{y}^i(x\epsilon) \to \bar{y}^i \in \zeta(\bar{x}), \forall i = 1, \dots, N+1, \\
&f^\epsilon(x^\epsilon) \to \bar{z} = \sum_{i=1}^{N+1} \bar{\beta}_i \bar{y}^i \in \zeta(\bar{x}),\ \text{since}\ \zeta(\bar{x})\ \text{is convex} \\
&f_l^\epsilon(x^\epsilon) \le \epsilon A, \forall l = 1, \dots, N \Rightarrow \bar{z}_l \le 0, \forall l = 1, \dots, N.
\end{aligned}$$

This means that $\bar{z} \in \zeta(\bar{x}) \cap \mathbb{R}^N_-$. The proof is over.

□

From Lemma 3, we can additionally derive two stronger versions of GND lemma. Each of them is stated and proved below.

Lemma 4. *Let Δ be the unit-simplex of $\mathbb{R}^N$. Let ζ be an upper semicontinuous correspondence with non-empty, compact, convex values from Δ into $\mathbb{R}^N$. Suppose ζ satisfies the condition*

$$\forall p \in \Delta,\ \forall z \in \zeta(p), p \cdot z = 0.$$

Then there exist $\bar{p}$, $\bar{z} \in \zeta(\bar{p})$ such that (1) $\bar{z} \leq 0$, and (2) $\forall i = 1, \ldots, N, \bar{p}_i \neq 0 \Rightarrow \bar{z}_i = 0$.

Proof. Since "$\forall p \in \Delta,\ \forall z \in \zeta(p), p \cdot z = 0$" $\Rightarrow$ "$\forall p \in \Delta,\ \forall z \in \zeta(p), p \cdot z \leq 0$", from Lemma 3, there exist $\bar{p}$ and $\bar{z} \in \zeta(\bar{p})$ such that $\bar{z} \leq 0$. Since $\bar{p} \cdot \bar{z} = 0$, the conclusion is immediate. □

Lemma 5. *Let Δ be the unit-simplex of $\mathbb{R}^N$. Let ζ be an upper semicontinuous correspondence with non-empty, compact, convex values from Δ into $\mathbb{R}^N$. Suppose ζ satisfies the condition*

$$\forall p \in \Delta,\ \exists z \in \zeta(p), p \cdot z \leq 0.$$

Then there exists $\bar{p} \in \Delta$ such that $\zeta(\bar{p}) \cap \mathbb{R}^N_- \neq \emptyset$.

Proof. For $p \in \Delta$, let $\tilde{\zeta}(p) = \{z \in \zeta(p) : z \cdot p \leq 0\}$. The correspondence $\tilde{\zeta}$ is upper semicontinuous, convex, and compact valued from Δ into $\mathbb{R}^N$. It satisfies the assumptions of Lemma 3. Hence there exist $\bar{p}$ and $\bar{z} \in \tilde{\zeta}(\bar{p}) \subset \zeta(\bar{p})$, such that $\bar{z} \leq 0$. □

We now consider an *alternative statement* of GND lemma, the proof of which directly follows from Lemma 2.

Lemma 6. *(Gale-Nikaido-Debreu) Let S denote the unit-sphere, for the norm $\|\cdot\|_2$ of $\mathbb{R}^N$. Let ζ be an upper semicontinuous correspondence from $S \cap \mathbb{R}^N_+$ in $\mathbb{R}^N$ which satisfies*

$$\forall q \in S \cap \mathbb{R}^N_+,\ \forall z \in \zeta(q),\ q \cdot z \leq 0.$$

Then,

$$\exists \bar{q} \in S \cap \mathbb{R}^N_+,\ \text{such that } \zeta(\bar{q}) \cap \mathbb{R}^N_- \neq \emptyset.$$

Proof. For $p \in \Delta$ define $\mu(p) = \frac{1}{\sqrt{p_1^2 + \ldots + p_N^2}}$, and for $q \in S \cap \mathbb{R}^N_+$ define $\lambda(q) = \sum_{i=1}^N q_i$. We have, if $q \in S \cap \mathbb{R}^N_+$ then $p = \frac{q}{\lambda(q)} \in \Delta$ and if $p \in \Delta$ then $q = \mu(p)p \in S \cap \mathbb{R}^N_+$. Define also for $p \in \Delta$, $\eta(p) = \zeta(\mu(p)p)$. Obviously, η is upper semicontinuous with convex and compact values. We have

$$\forall p \in \Delta, \forall z \in \eta(p) = \zeta\left(\mu(p)p\right),\ \mu(p)p \cdot z \leq 0 \Leftrightarrow p \cdot z \leq 0.$$

From Lemma 2, there exist $\bar{p} \in \Delta$ and $\bar{z} \in \eta(\bar{p})$ such that $\bar{z} \leq 0$ or, equivalently, there exist $\bar{q} = \mu(\bar{p})\bar{p},\ \bar{z} \in \eta(\bar{p}) = \zeta(\mu(\bar{p})\bar{p}) = \zeta(\bar{q})$ such that $\bar{z} \leq 0$. □

Remark 3 (Kakutani theorem and Gale-Nikaido-Debreu lemma)**.** *We emphasize that Kakutani theorem can be obtained as a corollary of GND lemma. We prove this by adapting the argument of Uzawa (1962) for continuous mapping.*[14] *Let ζ be an upper semicontinuous correspondence, with non-empty convex compact values from Δ into itself. Define, for $p \in \Delta$,*

$$\psi(p) = \left\{ y : y = z - \frac{p \cdot z}{\sum_{i=1}^{N} p_i^2} p, \text{ with } z \in \zeta(p) \right\}$$

One can check that ψ is upper semicontinuous and convex valued. Moreover, for any $p \in \Delta$, any $y \in \psi(p)$, we have $p \cdot y = 0$. Hence, from Lemma 4, there exist $\bar{p} \in \Delta$ and $\bar{y} \in \psi(\bar{p})$ which satisfy $\bar{y} \leq 0$, and $\forall i = 1, \dots, N, \bar{p}_i \neq 0 \Rightarrow \bar{y}_i = 0$. In other words, there exist $\bar{p} \in \Delta$ and $\bar{z} \in \zeta(\bar{p})$ satisfying two conditions:

$$\forall i = 1, \dots, N, \ \bar{z}_i \leq \frac{\bar{p} \cdot \bar{z}}{\sum_{i=1}^{N} \bar{p}_i^2} \bar{p}_i$$
$$\forall i = 1, \dots, N, \ \bar{p}_i \neq 0 \Rightarrow \bar{z}_i = \frac{\bar{p} \cdot \bar{z}}{\sum_{i=1}^{N} \bar{p}_i^2} \bar{p}_i.$$

Hence, if $\bar{p}_i = 0$, we have $0 \leq \bar{z}_i \leq 0$ which in turn implies that $\bar{z}_i = 0$. Let $\mu = \frac{\bar{p} \cdot \bar{z}}{\sum_{i=1}^{N} \bar{p}_i^2}$. We obtain that $\bar{z}_i = \mu \bar{p}_i$ for any $i = 1, \dots, N$. Since $\bar{z} \in \Delta, \bar{p} \in \Delta$, we have $\mu = 1$. Hence, $\bar{p} = \bar{z} \in \zeta(\bar{p})$.

3.3 Using Sperner lemma to prove the existence of general equilibrium

We consider two hypothetical cases: an economy with production and an two-period stochastic economy with incomplete financial markets. Without recourse to the fixed-point theorems or GND lemma, we are successful in establishing the results. Our proofs are novel as they only make use of Sperner lemma.

3.3.1 Equilibrium existence in an economy with production

Consider an economy with L consumption goods, K input goods which may be capital or labor, I consumers, and J firms. Each consumer i has an initial endowment of consumption goods $\omega^i \in \mathbb{R}_+^L$, an initial endowment of inputs $y_0^i \in \mathbb{R}_+^K$, and a utility function u^i depending on her/his consumptions $x^i \in \mathbb{R}_+^L$. The firms produce consumption goods. Firm j has production functions $F^j = (F_1^j, \dots, F_L^j)$ and uses a vector of inputs $(y_1^j, \dots, y_K^j) \in \mathbb{R}_+^K$. The production functions satisfy $F_l^j \geq 0$, and $F^j \neq 0$. We do not exclude that $F_l^j = 0$ for some l (e.g., firm j does not produce good l).

[14] Florenzano (1982) also proves Kakutani theorem from GND lemma but she considers for the unit ball instead of the simplex Δ.

We adopt the following set of standard assumptions concerning the specifications of an economy with production.

Assumption 1. *(i) Each utility function is strictly concave, continuous, and strictly increasing.*
(ii) The endowments of consumption goods satisfy $\forall i,\ \omega^i \in \mathbb{R}^L_{++}$.
(iii) The endowments of inputs satisfy $\forall i,\ y_0^i \in \mathbb{R}^K_{++}$.
(iv) For any l, $F_l^j(0) = 0$, *and if* $F_l^j \neq 0$ *then it is strictly concave, strictly increasing.*
(v) The firms distribute their profits among consumers. The share coefficients θ^{ij}, $i = 1, \dots, I$ *and* $j = 1, \dots, J$ *are positive and satisfy* $\sum_i \theta^{ij} = 1, \forall j$.

In this economy, each firm j maximizes its profit given the prices p of outputs and the prices q of inputs. Let

$$\Pi^j(p,q) = \max_{y \in \mathbb{R}^K_+} \{p \cdot F^j(y) - q \cdot y\}.$$

We observe that for any (p,q), $\Pi^j(p,q) \geq p \cdot F^j(0) - q \cdot 0 = 0$.

On the other hand, given the prices p of outputs and the prices q of inputs, each consumer i solves the problem

$$\max u^i(x^i) \text{ subject to } x^i \in \mathbb{R}^L_+ \text{ and } p \cdot x^i \leq p \cdot \omega^i + \sum_j \theta^{ij} \Pi^j(p,q) + q \cdot y_0^i.$$

We now introduce the definitions of equilibrium and feasible allocation for such an economy with production.

Definition 5. *An equilibrium is a list* $((x^{i*})_{i=1,\dots,I}, (y^{j*})_{j=1,\dots,J}, p^*, q^*)$ *satisfying (i)* $p^* \gg 0, q^* \gg 0,$[15] *(ii) given prices, households and firms maximize their utility and profit respectively, (iii) all markets clear.*

Definition 6. *An allocation* $((x^i)_i, (y^j)_j)$ *is feasible if*

(i) $x^i \in \mathbb{R}^L_+$ *for any* $i = 1, \dots, I$, $y^j \in \mathbb{R}^K_+$, *for any* $j = 1, \dots, J$,

(ii) $\sum_{i=1}^I x^i \leq \sum_{i=1}^I \omega^i + \sum_{j=1}^J F^j(y^j)$,

(iii) $\sum_{j=1}^J y^j \leq \sum_{i=1}^I y_0^i$.

The set of feasible allocations is denoted by $\mathcal{F}$. It is convex and compact. We denote by X^i the set of allocations x^i such that there exist $(x^{-i}) \in (\mathbb{R}^L_+)^{I-1}$ and (y^j) which satisfy $((x^i, x^{-i}), (y^j)) \in \mathcal{F}$. We denote by Y^j the set of inputs (y^j) such that there exist allocations (x^i) which satisfy $((x^i), (y^j)) \in \mathcal{F}$. Note that all of these sets are convex, compact, and nonempty.

[15] For $x \in \mathbb{R}^L_+$, $x \gg 0$ means that $x_l > 0\ \forall l = 1, \dots, L$.

Let X be a closed ball of $\mathbb{R}^L_+$ that contains all the X^i in its interior. Also, let Y be a closed ball of $\mathbb{R}^K_+$ that contains all the sets Y^j in its interior.

We will consider an *intermediate economy* in which the consumption sets equal to X and the inputs sets equal to Y. In this economy, given prices p and q, the behavior of each firm j can be recast as: $\max_{y^j \in Y}\{p \cdot F^j(y^j) - q \cdot y^j\}$. Accordingly, the behavior of each consumer i can be recast as

$$\max u^i(x^i) \text{ subject to } x^i \in X \text{ and } p \cdot x^i \le p \cdot \omega^i + \sum_j \theta^{ij}\Pi^j(p,q) + q \cdot y_0^i.$$

Definition 7. *An equilibrium of the intermediate economy is a list* $((x^{i*})_{i=1,\ldots,I}, (y^{j*})_{j=1,\ldots,J}, p^*, q^*)$ *that satisfies*

(i) $p^* \gg 0, q^* \gg 0$,

(ii) For any i, $x^{i*} \in X$ *and* $p^* \cdot x^{i*} = p^* \cdot \omega^i + \sum_j \theta^{ij}\Pi^j(p^*,q^*) + q^* \cdot y_0^i$,

(iii) For any i, $x^i \in X$, $p^* \cdot x^i \le p^* \cdot \omega^i + \sum_j \theta^{ij}\Pi^j(p^*,q^*) + q^* \cdot y_0^i \Rightarrow u^i(x^i) \le u^i(x^{i*})$,

(iv) For any j, $y^{j*} \in Y$ *and* $\Pi^j(p^*,q^*) = p^* \cdot F^j(y^{j*}) - q^* \cdot y^{j*}$,

(v) $\sum_{i=1}^I x^{i*} = \sum_{i=1}^I \omega^i + \sum_{j=1}^J F^j(y^{j*})$ *and* $\sum_{j=1}^J y^{j*} = \sum_{i=1}^I y_0^i$.

Since the utility functions and the production functions are strictly increasing, an equivalent definition can be reached by refining condition (v) in Definition 7. More precisely, an equilibrium in this intermediate economy is a list $((x^{i*})_{i=1,\ldots,I}, (y^{j*})_{j=1,\ldots,J}, p^*, q^*)$ that satisfies the conditions (i-iv) in Definition 7 together with

(vi') For any $l = 1, \ldots, L$, $\sum_{i=1}^I x_l^{i*} - \left(\sum_{i=1}^I \omega_l^i + \sum_{j=1}^J F_l^j(y^{j*})\right) \le 0$,

(vii') For any $k = 1, \ldots, K$, $\sum_{j=1}^J y_k^{j*} - \sum_{i=1}^I y_{0,k}^i \le 0$,

(viii') For any $l = 1, \ldots, L$, $p_l^* \left(\sum_{i=1}^I x_l^{i*} - \left(\sum_{i=1}^I \omega_l^i + \sum_{j=1}^J F_l^j(y^{j*})\right)\right) = 0$,

(viv') For any $k = 1, \ldots, K$, $q_k^* \left(\sum_{j=1}^J y_k^{j*} - \sum_{i=1}^I y_{0,k}^i\right) = 0$.

The following remark is important for the analysis of the equilibrium existence.

Remark 4. *If* (x^*, y^*) *solves the problems of the consumers and the firms, then* (x^*, y^*) *satisfies* ***Weak Walras Law****:*

$$p \cdot \left(\sum_i (x^{*i} - \omega^i) - \sum_j F^j(y^*)\right) + q \cdot \left(\sum_j y^{*j} - \sum_i y_0^i\right) \le 0. \tag{6}$$

*However, if $\sum_i (x^{*i} - \omega^i) - \sum_j F^j(y^*) \leq 0$ and $\sum_j y^{*j} - \sum_i y_0^i \leq 0$, i.e., $(x^*, y^*) \in \mathcal{F}$, since the utility functions are strictly increasing and the feasible set $\mathcal{F}$ is in the interior of $X \times Y$, the allocation (x^*, y^*) satisfies **Walras Law**:*

$$p \cdot \Big(\sum_i (x^{*i} - \omega^i) - \sum_j F^j(y^*) \Big) + q \cdot \Big(\sum_j y^{*j} - \sum_i y_0^i \Big) = 0. \tag{7}$$

We now use Sperner lemma to prove the existence of an equilibrium for the intermediate economy. We will show that it is actually an equilibrium for the initial economy.

Proposition 1. *Under above assumptions, there exists an equilibrium in the intermediate economy.*

Proof. Let $\alpha > 0$ and consider the following transformed problem of the producer:

$$\Pi^{j,\alpha}(p,q) = \max\{p \cdot F^j(y^j) - q \cdot y^j : y^j \in Y \text{ and } q \cdot y^j - p \cdot F^j(y^j) \leq \alpha\}.$$

It is obvious that the set $C^{j,\alpha}(p,q) = \{y \in Y : q \cdot y^j - p \cdot F^j(y^j) \leq \alpha\}$ is convex, compact, nonempty, and has a nonempty interior. Therefore, the correspondence $C^{j,\alpha}$ is continuous.

Let $\eta^{j,\alpha}(p,q) = \{y^j \in Y : \Pi^{j,\alpha}(p,q) = p \cdot F^j(y^j) - q \cdot y^j\}$. It can be deduced from the theorem of the maximum (see Theorem 1) that $\Pi^{j,\alpha}$ is a continuous function and, since the production function is strictly concave, $\eta^{j,\alpha}$ is a continuous mapping.

Consider also the transformed problem of the consumer:

$$\max u^i(x^i) \text{ subject to } x^i \in X,\ p \cdot x^i \leq p \cdot \omega^i + \sum_j \theta^{ij} \Pi^{j,\alpha}(p,q) + q \cdot y_0^i.$$

It is easy to see that the set $D^{i,\alpha}(p,q) = \{x^i : x^i \in X,\ p \cdot x^i \leq p \cdot \omega^i + \sum_j \theta^{ij} \Pi^{j,\alpha}(p,q) + q \cdot y_0^i\}$ is convex, compact. Moreover, it has a nonempty interior.[16] Hence, $D^{i,\alpha}$ is a continuous correspondence.

Denote $\Delta = \{(x_1, \ldots, x_{L+K}) \geq 0 : \sum_{i=1}^{L+K} x_i = 1\}$.

For $(p,q) \in \Delta$ and $i = 1, \ldots, I$, we define

$$\xi^i(p,q) = \{x^i \in X : u^i(x^i) \geq u^i(x'),\ \text{if } p \cdot x' \leq p \cdot \omega^i + \sum_j \theta^{ij} \Pi^{j,\alpha}(p,q) + q \cdot y_0^i\}.$$

Since the correspondence $D^{j,\alpha}$ is continuous and the utility functions are strictly concave, the Maximum Theorem implies that ξ^i is continuous for any i.

[16] Indeed, observe that $\Pi^{j,\alpha}(p,q) \geq 0$. If $p = 0$ then $q > 0$ and $q \cdot y_0^i > 0$. We have $0 < \sum_j \theta^{ij} \Pi^{j,\alpha}(p,q) + q \cdot y_0^i$. If $p \neq 0$, choose x^i close to ω^i and $x^i \ll \omega^i$. Then $p \cdot (x^i - \omega^i) < 0 \leq \sum_j \theta^{ij} \Pi^{j,\alpha}(p,q) + q \cdot y_0^i$.

We adopt that $N = L+K$, $\pi = (p,q) \in \Delta$ and define the excess demand mappings

$$\begin{aligned}
\xi(\pi) &= \sum_{i=1}^{I}(\xi^i(\pi) - \omega^i) - \sum_{j=1}^{J} F^j(\eta^{j,\alpha}(\pi)) \\
\eta^\alpha(\pi) &= \sum_{j=1}^{J} \eta^{j,\alpha}(\pi) - \sum_{i=1}^{I} y_0^i \\
\zeta(\pi) &= (\xi(\pi), \eta^\alpha(\pi)).
\end{aligned}$$

Note that the mapping ζ is continuous.

Let $K > 0$ be an integer and consider a simplicial subdivision T^K of the unit-simplex Δ of $\mathbb{R}^N$ such that $Mesh(T^K) < 1/K$. We define a labeling R as follows:

$$\text{For } \pi \in \Delta,\ R(\pi) = i, \text{ where } i \text{ satisfies } \zeta_i(\pi) \leq 0.$$

This labeling is well-defined since, by Weak Walras Law, for any $\pi \in \Delta$, we have $\sum_{i=1}^N \pi_i \zeta_i(\pi) \leq 0$, there must be some i with $\zeta_i(\pi) \leq 0$ (if not, $\pi \cdot \zeta(\pi) > 0$). Observe that $\zeta(e^i) \leq 0$, where e^i is i-vertex of Δ. Indeed, $0 \geq e^i \cdot \zeta(e^i) = \zeta_i(e^i)$. We label $R(e^i) = i$.

We now verify that this labeling satisfies the Sperner condition. Let π be in a face $[[e^{i_1}, \ldots, e^{i_m}]]$ with $m < n$. We have in this case $\pi_j = 0, \forall j \notin \{i_1, \ldots, i_m\}$. We have

$$0 \geq \sum_{j=1}^{N} \pi_j \zeta_j(\pi) = \sum_{j \in \{i_1, \ldots, i_m\}} \pi_j \zeta_j(\pi).$$

Hence, there must be $j \in \{i_1, \ldots, i_m\}$ with $\zeta_j(\pi) \leq 0$. We define $R(x) = j$. We have proved that our labeling satisfies the Sperner condition.

From Sperner lemma, for any K, there exists a completely labeled subsimplex $[[\bar{\pi}^{K,1}, \bar{\pi}^{K,2}, \ldots, \bar{\pi}^{K,n}]]$ such that $R(\bar{\pi}^{K,j}) = j$, i.e., $\zeta_j(\bar{\pi}^{K,j}) \leq 0$, $\forall j = 1, \ldots, N$.

When K tends to $+\infty$, we can suppose that the sequence of subsimplices $\left\{[[\bar{\pi}^{K,1}, \bar{\pi}^{K,2}, \ldots, \bar{\pi}^{K,N}]]\right\}_K$ converges. Since $Mesh(T^K) < 1/K$ tends to zero, the vertices $\{\bar{\pi}^{K,j}\}$ converge to the same point $\pi^* \in \Delta$. Recall that, for any K, any j, $\zeta_j(\bar{\pi}^{K,j}) \leq 0$, by the continuity of ζ, we have

$$\zeta_j(\pi^*) \leq 0, \forall j.$$

From Remark 4, Walras Law holds. Hence, $\sum_j \pi_j^* \zeta_j(\pi^*) = 0$ and we have actually $\pi_j^* \zeta_j(\pi^*) = 0, \forall j$.

Finally, we claim that $\Pi^{j,\alpha}(p^*, q^*) = \max\{p^* \cdot F^j(y^j) - q^* \cdot y^j : y^j \in Y\}$. Indeed, if there exists $y \in Y$ such that $p^* \cdot F^j(y) - q^* \cdot y > \Pi^{j,\alpha}(p^*, q^*) \geq 0$, then $q^* \cdot y - p^* F^j(y) < 0 < \alpha$ and that is a contradiction. We have proved that there exists an equilibrium in the intermediate economy. □

The following proposition allows us to move from an equilibrium in the intermediate economy to an equilibrium in the initial economy.

Proposition 2. *$((x^{i*})_{i=1,\ldots,I}, (y^{j*})_{j=1,\ldots,J}, p^*, q^*)$ is an equilibrium for the initial economy.*

Proof. First observe that if there exists $y \in \mathbb{R}^K_+$ such that

$$p^* \cdot F^j(y) - q^* \cdot y > p^* \cdot F^j(y^*) - q^* \cdot y^* = \Pi^{j,\alpha}(p^*, q^*) \geq 0$$

then $q^* \cdot y - p^* F^j(y) < 0 < \alpha$ and that is a contradiction. By consequence, we get that

$$p^* \cdot F^j(y^*) - q^* \cdot y^* = \Pi^j(p^*, q^*) = \max\{p^* \cdot F^j(y^j) - q^* \cdot y^j : y^j \in \mathbb{R}^K_+\}.$$

Now fix some i and take $x \in \mathbb{R}^L_+$ satisfying $u^i(x) > u^i(x^{i*})$. We have to prove that $p^* \cdot x > p^* \cdot \omega^i + \sum_j \theta^{ij} \Pi^j(p^*, q^*) + q^* \cdot y^i_0$. Of course, this is the case if $x \in X$. We now consider the case where $x \notin X$. Since x^{i*} is in the interior of X, there exists $\lambda \in (0, 1)$ such that $\lambda x + (1-\lambda)x^{i*} \in X$. We have $u^i(\lambda x + (1-\lambda)x^{i*}) \geq \lambda u^i(x) + (1-\lambda)u^i(x^{i*}) > u^i(x^{i*})$. Hence, we have

$$p^* \cdot (\lambda x + (1-\lambda)x^{i*}) > p^* \cdot \omega^i + \sum_j \theta^{ij} \Pi^j(p^*, q^*) + q^* \cdot y^i_0 = p^* \cdot x^{i*}$$

$$\Leftrightarrow \lambda p^* \cdot x > \lambda p^* \cdot x^{i*} \Leftrightarrow p^* \cdot x > p^* \cdot x^{i*} = p^* \cdot \omega^i + \sum_j \theta^{ij} \Pi^j(p^*, q^*) + q^* \cdot y^i_0.$$

□

3.3.2 Equilibrium existence in an economy with financial assets

In this section, we use Sperner lemma to prove the existence of an equilibrium in an economy with nominal assets. We briefly present here some essential notions. For a full exposition, see Florenzano (1999).

Consider a pure exchange economy with two periods ($t = 0$ and $t = 1$), L consumption goods, J financial assets, and I agents. There is no uncertainty in period 0 while there are S possible states of nature in period 1. In period 0, each agent $i \leq I$ consumes and purchases assets. The consumption prices are denoted by $p_0 \in \mathbb{R}^L_+$ in the first period, $p_s \in \mathbb{R}^L_+$ in the state s of period 1. Let $\pi \equiv (p_0, p_1, \ldots, p_S)$. Each consumer has endowments of consumption good $\omega^i_0 \in \mathbb{R}^L_+$ in period 0 and $\omega^i_s \in \mathbb{R}^L_+$ in state s of period 1. Any agent i has a utility function $U^i(x^i_0, x^i_1, \ldots x^i_S)$ where x^i_s is her consumption at state s. There is a matrix of returns depending on π of financial assets which is the same for any agent. Typically, if agent $i \leq I$ purchases z^i quantity of assets in period 0, in period 1, at state s, she/he will obtain an income (positive or negative) $\sum_{j=1}^{J} R_{s,j}(\pi)z^j$. The returns $R(\pi)$ can be represented by a matrix

$$R = \begin{bmatrix} R_{1,1}(\pi) & R_{1,2}(\pi) & \dots & R_{1,J}(\pi) \\ R_{2,1}(\pi) & R_{2,2}(\pi) & \dots & R_{2,J}(\pi) \\ & \cdots\cdots & & \\ & \cdots\cdots & & \\ & \cdots\cdots & & \\ R_{S,1}(\pi) & R_{S,2}(\pi) & \dots & R_{S,J}(\pi) \end{bmatrix}$$

We denote by $R_s(\pi) = (R_{s,1}(\pi), R_{s,2}(\pi), \dots, R_{s,J}(\pi))$, the s^{th} row of $R(\pi)$. Typically, the constraints faced by agent i are

$$\begin{aligned} &p_0 \cdot (x_0^i - \omega_0^i) + q \cdot z^i \leq 0, \\ &p_s \cdot (x_s^i - \omega_s^i) \leq R_s(\pi) \cdot z^i \ \forall s = 1, \dots, S. \end{aligned}$$

We make use of the following set of standard assumptions.

Assumption 2. *(i) For any $i = 1, \dots, I$, the consumption set is $\mathbb{R}_+^L$ the assets set $Z^i = \mathbb{R}^J$.*
(ii) For any $i = 1, \dots, I$, $\omega_0^i \in \mathbb{R}_{++}^L$, $\omega_s^i \in \mathbb{R}_+^L$ for any state s in period 1.
(iii) $R_{s,j}(\pi) > 0$, for any s, any j, any π.
(iv) rank $R(\pi) = J$, for any π and the map $\pi \to R(\pi)$ is continuous.
(v) For any $i = 1, \dots, I$, U^i is strictly increasing, continuous, and strictly concave.

We now introduce the definitions of complete and incomplete asset markets, feasible allocations, and the notion of equilibrium in an economy with financial assets.

Definition 8. *The assets market is called complete if $S = J$ and incomplete if $S > J$.*

Definition 9. *An equilibrium of this economy is a list $\left(x^{i*}, z^{i*})_{i=1}^I, x^{I+1*}, (p^*, q^*)\right)$ where $(x^{i*}, z^{i*})_{i=1}^I \in (X^i)^I \times (Z^i)^J$, $(p^*, q^*) \in \mathbb{R}_{++}^L \times \mathbb{R}_{++}^J$ such that*

(i) For any i, (x^{i*}, z^{i*}) solve the problem

$$\max U^i(x_0^i, x_1^i, \dots, x_S^i)$$

$$\text{subject to: } p_0^* \cdot (x_0^i - \omega_0^i) + q^* \cdot z^i \leq 0 \tag{8a}$$

$$p_s^* \cdot (x_s^i - \omega_s^i) \leq R_s(\pi) \cdot z^i, \ s = 1, \dots, S \tag{8b}$$

(ii) $\sum_{i=1}^I (x_s^{*i} - \omega_s^i) = 0$ for any $s = 0, 1, \dots, S$ and $\sum_{i=1}^I z^{*i} = 0$.

Definition 10. *The allocations $((x^i, z^i)_i) \in (X^i)^I \times (Z^I)^I$ are feasible if (i) $\sum_{i=1}^I (x^i - \omega^i) \leq 0$ and (ii) $\sum_{i=1}^I z^i = 0$. Accordingly, take $\alpha > 0$ and define the sets $F^c = \{(x^i)_i \in (X^i)^I : \sum_{i=1}^I (x^i - \omega^i) \leq \alpha\}$ and $F^f = \{(z^i)_i \in (Z^i)^I : \sum_i z_j^i = 0, \forall j\}$. Moreover, denote the projection of F^c on X^i by $\widehat{X}^i$.*

The following lemma will be useful in proving the existence of equilibrium.

Lemma 7. *Let* $(z^i) \in \mathbb{R}^{J\times I}$ *satisfy that: for all* i*, there exists* $(x^i) \in F^c$ *such that*

$$\forall s = 1, \dots, S,\ p_s \cdot (x_s^i - \omega_s^i) = R_s(\pi) \cdot z^i$$

where $\|p_s\| \leq 1, \forall s$*. Then there exists* $\beta > 0$ *such that* $\|z^i\| \leq \beta, \forall i$.[17]

Let B^c be a ball of $\mathbb{R}^L$, centered at the origin, which contains any $\hat{X}^i$ in its interior. Let us consider an intermediate economy in which the consumption set is $\widetilde{X}^i = B^c$ for any i.

Definition 11. *An equilibrium of this intermediate economy is a list* $\big((x^{i*}, z^{i*})_{i=1}^I, (p^*, q^*)\big)$ *where* $(x^{i*}, z^{i*})_{i=1}^I \in (\widetilde{X}^i)^I \times (\widetilde{Z}^i)^J,\ (p^*, q^*) \in \mathbb{R}^L_{++} \times \mathbb{R}^J_{++}$ *such that*

(i) For any i*,* x^{i*} *solve the problem*

$$\max U^i(x_0^i, x_1^i, \dots, x_S^i) \tag{9a}$$
$$\textit{subject to: } \exists z^i \in \mathbb{R}^J,\ p_0^* \cdot (x_0^i - \omega_0^i) + q^* \cdot z^i \leq 0, \tag{9b}$$
$$p_s^* \cdot (x_s^i - \omega_s^i) \leq R_s(\pi) \cdot z^i,\ s = 1, \dots, S \tag{9c}$$
$$x^i \in \widetilde{X}^i\ \forall s = 0, 1, \dots, S. \tag{9d}$$

(ii) $\sum_{i=1}^I (x_s^{*i} - \omega_s^i) = 0$ *for any* $s = 0, 1, \dots, S$ *and* $\sum_{i=1}^I z^{*i} = 0$.

We aim to provide a new proof (by using Sperner lemma) of the following result which corresponds to Theorem 1 in Cass (2006) or Theorem 7.1 in Florenzano (1999).

Proposition 3. *Under above assumptions, there exists an equilibrium* $\big((x^{i*}, z^{i*})_{i=1}^I, (p^*, q^*)\big)$ *with* $q^* = \sum_{s=1}^S R_s(\pi)$.

Proof. Observe that, by using the same argument in the proof of Proposition 2 in Section 3.3.1, we can prove that an equilibrium of the intermediate economy is indeed an equilibrium for the initial economy. As such, it remains to prove the existence of equilibrium in the intermediate economy. To do so, we proceed in two steps. First, we use Sperner lemma to prove that there exists actually a *Cass equilibrium*. Second, we show that this equilibrium constitutes an equilibrium of the intermediate economy.

Following Cass (2006) and Florenzano (1999), we define Cass equilibrium.

Definition 12. *Cass equilibrium is a list* $\big((\bar{x}^i)_{i=1}^I, (\bar{z}^i)_{i=2}^I, (\bar{p}, \bar{q})\big)$ *such that* $(\bar{x}^i)_{i=1}^I, (\bar{z}^i)_{i=2}^I \in (B^c)^I \times (B^f)^{I-1}$*,* $(\bar{p}, \bar{q})) \in \mathbb{R}^L_{++} \times \mathbb{R}^J_{++}$*, and* $\bar{\pi} = (\bar{p}, \bar{q})$ *where*

[17] Indeed, assume that there exists a sequence $(z^i(n))_n$ with $\|z^i(n)\| \to +\infty$ when $n \to +\infty$. We have, for any n, $\forall s = 1, \dots, S,\ p_s(n) \cdot (x_s^i(n) - \omega_s^i) = R_s(\pi(n)) \cdot z^i(n)$. We can assume that $\pi(n) \to \pi \in \Delta$. We obtain that, $\forall s = 1, \dots, S$, $\frac{p_s(n)\cdot(x_s^i(n) - \omega_s^i)}{\|z^i(n)\|} = R_s(\pi(n)) \cdot \frac{z^i(n)}{\|z^i(n)\|}$. We can suppose $\frac{z^i(n)}{\|z^i(n)\|} \to \zeta \neq 0$. Let $n \to +\infty$. We get $0 = R_s(\pi) \cdot \zeta$. Since rank $R(\pi) = J$, we have $\zeta = 0$: a contradiction.

(i) $\bar{x}^1$ *solves the consumer* 1 *problem under the constraint* $x^1 \in B^c,\ \ \bar{\pi}\cdot(x^1-\omega^1) \leq 0$.

(ii) For $i = 2, \ldots, I$, $\bar{x}^i$ *solves the consumer* i*'s problem*

$$\max U^i(x_0^i, x_1^i, \ldots, x_S^i) \text{ subject to: } \exists z^i \in \mathbb{R}^J,\ \bar{p}_0 \cdot (x_0^i - \omega_0^i) + \bar{q} \cdot z^i \leq 0,$$
$$\bar{p}_s \cdot (x_s^i - \omega_s^i) \leq R_s(\pi) \cdot z^i \ \forall s \geq 1$$
$$x^i \in B^c \ \forall i.$$

(iii) $\bar{q} = \sum_s R_s(\pi)$ *and* $\sum_{i=1}^I (\bar{x}^i - \omega^i) = 0$.

Lemma 8. *There exists a Cass equilibrium.*

Proof. Let $\pi = (p_0, p_1, \ldots, p_S) \in \Delta$ where Δ denotes the unit-simplex of $\mathbb{R}^{L(S+1)}$. Assume that ϵ satisfies $0 < \epsilon < \frac{\alpha}{(I-1)}$.

Agent 1 solves the following problem

$$\max U^1(x^1) \text{ subject to } x^1 \in \widetilde{X}^1,\ \pi \cdot (x^1 - \omega^1) \leq 0.$$

Any agent i ($i \geq 2$) solves the following problem

$$\max U^i(x^i) \text{ subject to: } x^i \in \widetilde{X}^i,\ z^i \in \widetilde{Z}^i,$$
$$\exists z^i \in \mathbb{R}^J,\ p_0 \cdot (x_0^i - \omega_0^i) + (\sum_s R_s(\pi)) \cdot z^i \leq \epsilon,$$
$$p_s \cdot (x_s^i - \omega_s^i) \leq R_s(\pi) \cdot z^i \ \forall s \geq 1.$$

The budget set of agent 1 has a nonempty interior since $\pi \in \Delta$. To prove the budget sets of the agents $i \geq 2$ have nonempty interiors, we observe that $x_s^i = \omega_s^i$, $s = 0, 1, \ldots, S$ and $z^i > 0$ such that $\sum_s R_s(\pi) z^i < \epsilon$ are in the interior of these budget sets. Therefore, the optimal value $(x^{*1}, x_\epsilon^{*2} \ldots, x_\epsilon^{*I})$, $(z_\epsilon^{*2}, \ldots, z_\epsilon^{*I})$ are continuous mappings with respect to π . For any π, we have

$$\pi \cdot \sum_{i=1}^I (x^{*i}(\pi) - \omega^i) \leq (I-1)\epsilon.$$

Define the excess demand mapping ξ by

$$\xi(\pi) = \sum_{i=1}^I (x^{*i}(\pi) - \omega^i).$$

It is obvious that $\forall \pi \in \Delta, \pi \cdot \xi(\pi) \leq (I-1)\epsilon$.

Denote $N = (S+1)L$. Let $K > 0$ be an integer and consider a simplicial subdivision T^K of the unit-simplex Δ of $\mathbb{R}^N$ such that $Mesh(T^K) < 1/K$. We define the following labeling r. For any $\pi \in \Delta, r(\pi) = t$ if $\xi_t(\pi) \leq (I-1)\epsilon$. Such a labeling is well defined. Moreover, it satisfies Sperner condition. Indeed, we see that:

- For $t \in \{1, \dots, N\}$. If $\pi = e^t$ (recall that e^t is a unit-vector of $\mathbb{R}^N$), then $(I-1)\epsilon \geq e^t \cdot \xi(e^t) = \xi_t(e^t)$. We label $r(e^t) = t$.

- If $\pi \in [[e^{i_1}, \dots, e^{i_m}]]$ with $m < N$, then $(I-1)\epsilon \geq \pi \cdot \xi(\pi) = \sum_{q \in \{i_1, \dots, i_m\}} \pi_q \xi_j(\pi)$. There must exists $q \in \{i_1, \dots, i_m\}$ with $\xi_q(\pi) \leq (I-1)\epsilon$. We label $r(\pi) = q$ with some $q \in \{i_1, \dots, i_m\}$.

So, the labeling r satisfies Sperner condition. Hence, there exists a completely labeled subsimplex $[[\bar{\pi}^1(K), \dots, \bar{\pi}^N(K)]]$, i.e., $\xi_t(\bar{\pi}^t(K)) \leq (I-1)\epsilon \ \forall t = 1, \dots, N$. Observe that

$$\forall t = 1, \dots, N, \ \sum_{i=1}^{I} \Big(x^{*i}(\bar{\pi}^t(K)) - \omega^i\Big) \leq (I-1)\epsilon < \alpha. \tag{10}$$

Let $K \to +\infty$. Then, for any $t \in \{1, \dots, N\}$, $\bar{\pi}^t(K) \to \pi^*(\epsilon) \in \Delta$. We have $\xi_q(\pi^*(\epsilon)) \leq (I-1)\epsilon < \alpha$, for all q. It follows from (10) that

$$\sum_{i=1}^{I} \Big(x^{*i}(\pi^*(\epsilon)) - \omega^i\Big) \leq (I-1)\epsilon < \alpha. \tag{11}$$

Write $\pi^*(\epsilon) = (p_0^*(\epsilon), p_1^*(\epsilon), \dots, p_S^*(\epsilon))$. Because of (11) and the fact that utility functions are strictly increasing, we obtain

$$\pi^*(\epsilon) \cdot (x^{*1}(\pi^*(\epsilon)) - \omega^1) = 0 \tag{12}$$

that implies $\pi^*(\epsilon) \gg 0$. Hence, for any $i \geq 2$,

$$p_0^*(\epsilon) \cdot (x_0^{*i}(\pi^*(\epsilon)) - \omega_s^i) + (\sum_s R_s(\pi^*(\epsilon)) z^{*i}(\pi^*(\epsilon)) = \epsilon,$$
$$p_s^*(x^{*i}(\pi^*(\epsilon)) - \omega_s^i) = R_s(\pi^*(\epsilon)) \cdot z^{*i}(\pi^*(\epsilon)), s = 1, \dots, S.$$

From Lemma 7, we have $\|z^{*i}(\pi^*(\epsilon))\| \leq \beta$.

Let $\epsilon \to 0$, we have that

- $\pi^*(\epsilon) \to \bar{\pi}$,

- $x^{*1}(\pi^*(\epsilon)) \to \bar{x}^1 = x^{*1}(\bar{\pi}) \Rightarrow \bar{\pi} \gg 0$,

- $\bar{\pi} \gg 0 \Rightarrow \forall i \geq 2, x^{*i}(\pi^*(\epsilon)) \to \bar{x}^i = x^{*i}(\bar{\pi}),\ z^{*i}(\pi^*(\epsilon)) \to \bar{z}^i = z^{*i}(\bar{\pi})$, i.e., for $i \geq 2$, $(\bar{x}^i, \bar{z}^i)$ solves the problem of agent i for given prices $\bar{\pi}$.

Note from (11) that $\sum_{i=1}^{I} (\bar{x}^i - \omega^i) \leq 0$ and from (12) that $\bar{\pi} \cdot (\sum_{i=1}^{I} (\bar{x}^i - \omega^i) = 0 \Rightarrow \bar{\pi}_p \sum_i (\bar{x}_p^i - \omega_p^i) = 0, p = 1, \dots, N$.

Since $\bar{\pi} \gg 0$, we deduce that $\sum_{i=1}^{I} (\bar{x}_p^i - \omega_p^i) = 0, \forall p = 1, \dots, N$, or equivalently $\sum_{i=1}^{I} (\bar{x}^i - \omega^i) = 0$. We have proved the existence of a *Cass equilibrium*. $\square$

We move from Cass equilibrium to an equilibrium in the intermediate economy.

Lemma 9. *There exists an equilibrium in the intermediate economy with $\bar{q} = \sum_s R_s(\pi)$.*

Proof. Since $\sum_{i=1}^I (\bar{x}_s^i - \omega_s^i) = 0 \ \forall s \geq 1$, we get that

$$\forall s \geq 1 \ , 0 = \bar{p}_s \cdot \sum_{i=1}^{I} (\bar{x}_s^i - \omega_s^i) = \bar{p}_s \cdot (\bar{x}_s^1 - \omega_s^i) + \bar{p}_s \cdot \sum_{i=2}^{I} (\bar{x}_s^i - \omega_s^i).$$

Denote $\bar{z}^1 = -\sum_{i \geq 2} \bar{z}^i$. We have $\bar{p}_s \cdot \sum_{i=2}^I (\bar{x}_s^i - \omega_s^i) = R_s(\bar{\pi}) \cdot \bar{z}^1$ which implies that

$$\sum_{s \geq 1} \bar{p}_s \cdot (\bar{x}_s^1 - \omega_s^i) = \Big(\sum_s R_s(\bar{\pi}) \Big) \cdot \bar{z}^1 = \bar{q} \cdot \bar{z}^1.$$

By combining this with the fact that $\bar{p}_0 \cdot (\bar{x}_0^1 - \omega_0^1) + \sum_{s \geq 1} \bar{p}_s \cdot (\bar{x}_s^1 - \omega_s^1) = 0$, we get that $\bar{p}_0 \cdot (\bar{x}_0^1 - \omega_0^1) + \bar{q} \cdot \bar{z}^1 = 0$.

It is easy to prove that $\bar{x}^1$ solves the problem (9a-9d).

□

□

Remark 5 (equilibrium price versus no-arbitrage price)**.** *Our above proof of the existence of competitive equilibrium leads to a conclusion that: an equilibrium exists if and only if there exists a no-arbitrage assets price. Indeed, any no-arbitrage price is the strictly positive convex combination of financial returns. Accordingly, take a no-arbitrage price. Using the Cass trick we obtain an equilibrium. Conversely, for any financial equilibrium, under the assumption that the utility functions are strictly increasing, the first order conditions show that an equilibrium asset price is a no-arbitrage price.*

Remark 6. *When we use the utility functions and production functions, we can skip the use of Kakutani Theorem. This theorem is required when the utility functions or the production functions are not strictly concave, or instead of utility functions and production functions we have preference orders for the consumers and production sets. In these cases, the demands of the consumers or of the firms are not necessarily single valued. They are upper semicontinuous correspondences with convex compact values. However, if the utility functions and the production are only concave, we can approximate them by a family of strictly concave utility functions and production functions as follows*

$$\textit{For } \varepsilon > 0, \ \textit{define } u_\varepsilon^i(x) = u^i(x) + \varepsilon v(x) \ , F_\varepsilon^j(k) = F^j(k) + \varepsilon G(k)$$

where $\varepsilon > 0$, v and G are strictly concave.
For any $\varepsilon > 0$ we get an equilibrium $\big((x^{i*}(\varepsilon))_{i=1,\dots,I}, (y^{j*}(\varepsilon))_{j=1,\dots,J}, p^*(\varepsilon), q^*(\varepsilon)\big)$. Let ε go to zero. It is easy to prove that the limit of this list constitutes an equilibrium for the initial economy.

4 Conclusion

In this paper, we have established that Sperner lemma can be used as a powerful tool for studying the existence of a general equilibrium. This allow us to skip the use of either Brouwer and Kakutani fixed point theorems or Gale-Nikaido-Debreu lemma. In doing so, we first pointed out these theorems and lemma can be proved using solely Sperner lemma. For a demonstration of possible applications of this new approach in general equilibrium models, we have separately studied competitive economies with either production or financial assets respectively. We have successfully provided novel proofs for the existence of competitive equilibrium using Sperner lemma for each case. Since Brouwer theorem was proved in 1910, Sperner lemma in 1928, from our paper, may we say that Kakutani theorem might be proved before 1941 (Kakutani, 1941), and the existence of general equilibrium before the years of 1950? In any case, one cannot change History.

We hope that our paper provides an important first step towards strengthening mathematical background for the analysis of general equilibrium models.

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