Political Parties and Policy Outcomes. Do Parties Block Reforms?

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Abstract

I propose a model of legislative bargaining among endogenous political parties over multiple policy dimensions. I provide a characterization of (i) the partition of the legislature into parties, (ii) the policy reforms that parties propose (if any), and (iii) the policy outcome attained from these proposals. I show that – depending on the position of the status quo – either (1) the presence of parties does not affect the policy outcome and a median voter theorem holds, or (2) a party representing legislators with extreme and opposite political views – i.e., a *coalition of extremes* – can successfully block reforms that would be feasible if parties did not exist. Lastly, I show that the extent to which the existence of parties can increase the set of possible policy reforms is severely limited or null.

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1 Introduction

Political parties play a crucial role in shaping legislative bargaining and policymaking within majoritarian institutions. The structure of the party system and the internal composition of each party jointly determine the policy proposals that are brought to the floor for a vote. Thus, parties can potentially affect policy outcomes.

A successful stream of literature strongly supports the claim that parties *do matter* in shaping policy reforms, even if there is a Condorcet winner – i.e., a policy proposal that defeats all the possible alternatives by simple majority voting – within the choice set of legislators or voters (Austen Smith 1986, Jackson and Moselle 2002, Levy 2004, Morelli 2004). In contrast, there is little theoretical research examining the role played by parties in *preventing* or limiting the extent of policy reforms.

In an assembly, a reform backed by a majority of legislators can be blocked if (i) the *status quo* is preferred to the reform by some agents, and (ii) the institutional setting allows for a situation in which no reform can pass – i.e., a *legislative gridlock*.

Political parties are deemed to play a key role in this process. For instance, by widening the set of available alternatives, they can generate disagreement regarding which reform should be implemented. In turn, disagreement causes the set of feasible reforms to shrink, and in some cases a legislative gridlock may prevail.

In this paper, I study the role of parties in blocking or limiting the magnitude of policy reforms in a legislature. I provide a theoretical framework that endogenizes and admits a general characterization of (i) any stable party structure, (ii) the reforms proposed by each party (if any) in each structure, and (iii) the final policy outcome. I use this characterization to evaluate each party system by assessing its capacity to block reforms that would be approved by a majority of legislators in the absence of parties, or, conversely, to extend the set of feasible reforms.

The goal is to answer the following questions. (1) Can any reform be implemented at all, given the status quo, the party structure, and the distribution of legislators' preferences? If so, what kind of reform will be chosen? (2) What are the features of a party structure that can support or prevent a policy change? (3) In what situations do parties matter in limiting the extent of policy reforms, and how much?

To achieve this goal, I propose a model of legislative bargaining among endogenous political parties. Specifically, I build on the work of Levy (2004), but I depart from this framework in three key directions.[1]

First, I focus on the role of parties in legislative bargaining (as in Eguia 2011a, 2011b, 2012) rather than in elections. Thus, I explicitly account for the role of the status quo, and I allow for it being, at least to some extent, a desirable outcome for some legislators.[2] As a result, such legislators may have an incentive to coalesce into a party aiming to block any possible reform.

Second, I depart from single-peaked preferences and sincere voting. This implies that a preference cycle may occur and a Condorcet winner may fail to exist. As a result, disagreement among legislators may prevail, such that no reform proposal can gain stable support. This is a typical outcome whenever the policy space is multidimensional (Plott 1967, Davis *et al.* 1972, Bernheim and Slavov 2009).

Third, I impose preference restrictions which can be interpreted as a *left vs. right* divide among legislators. Such restrictions are substantially weaker than those sufficient for a median voter theorem to hold in traditional models,[3] such as the (unidimensional) single-crossing condition (Gans and Smart 1996), and those assumed in Levy (2004).[4] Yet, these weaker assumptions are sufficient to identify a pivotal player, which I call the *median legislator*. While the model can deal with a wide range of cases, I focus on legislative bargaining questions in which the median legislator's ideal policy is preferred to the status quo by a majority of legislators, and therefore the assembly always approve a reform if political parties do not exist.

I find that the answers to questions (1), (2) and (3) depend on the position of the status quo in the policy space: specifically, if (i) the status quo is lower or higher[5] than the set of ideal policies[6] of the median legislator, then political parties are not *effective* – i.e., the policy outcome of any stable party system always coincides with

[1] Additional substantial differences are discussed in section 6.2.

[2] Levy (2004) focuses on parties that compete in elections rather than engaging in post-election legislative bargaining. As a result, her analysis does not need to emphasize the role played by the status quo. Thus, she assumes that all citizens strongly dislike the status quo.

[3] Sufficient conditions other than single crossing are discussed in Barberà and Moreno (2011).

[4] See section 6.1 for a comparison between these restrictions and those imposed in this paper.

[5] With respect to the partial order over the policy space.

[6] I.e., a legislator's most preferred policy. There may be more than one such policy.

the one that would prevail if politicians could not compromise.[7] In detail, the set of all possible policy outcomes satisfies a quasi-median voter theorem, and possesses monotone comparative statics properties. Moreover, I show that this case typically results in a *majoritarian* party structure – i.e., a political system in which two or more parties are distributed along a left vs. right dimension, and only one party, clearly identified as left-wing or right-wing, rules.

Conversely, if (ii) the status quo is neither lower nor higher than the ideal policy of the median legislator, then political parties can be effective. In particular, I characterize a *stable legislative gridlock* – i.e., an equilibrium in which no reform proposals can gain stable support in the legislature, and therefore the status quo remains in place. This outcome is the result of a party structure in which there is one active party that includes politicians from both the hard left and hard right, and no moderate ones – i.e., a *coalition of extremes* (Poole and Rosenthal 1997, Hussey 2008).[8] Thus, the presence of political parties can actually block reforms that would be implemented if such institutions did not exist.

Lastly, (iii) I show that even if parties do matter in policymaking, the extent to which they can affect a policy outcome depends on how the median legislator ranks the status quo among the set of alternatives. Namely, the median legislator cannot be made worse off relative to the status quo in any stable party system. Thus, substantial departures from the median legislator's ideal policy can prevail only if the she/he is strongly averse to the status quo.[9]

In detail, I propose a model of legislative bargaining within an assembly in which legislators are organized into parties. A partition of the set of legislators into parties constitutes a party structure. The policy space is a multidimensional partially ordered set. Parties engage in legislative bargaining with the aim of selecting a policy to implement from such a set.

Legislators' preferences over policies satisfy some ordinal properties: namely, legislators can be ordered along a generalized left vs. right preference dimension.

[7]In this first case only, the result is consistent with the view of Krehbiel (1998), who argues against using party-centered theories to explain policy choices.

[8]The occurrence of this non-standard kind of coalition within the Congress has been extensively documented in the Political Science literature. See section 3.3 for details.

[9]Provided that the median legislator's preferences over policies are strictly convex.

This assumption implies that a median legislator can be identified even if neither single-peakedness nor the unidimensional single-crossing condition (Gans and Smart 1996) hold, and no Condorcet winner exists.

The political science literature provides substantial support for this restriction on empirical grounds. For instance, there is convincing evidence that in almost all Western democracies, the perception of political competition is centered on one version or another of the left–right divide (Mair 2007).[10]

I do not model how parties are formed using a non-cooperative approach. Instead, following influential literature (Levy 2004, Anesi and De Donder 2009, 2013), I analyze any party structure that satisfies a stability condition. The concept of stability is that proposed by Ray and Vohra (1997). It applies to the model proposed in this paper as follows. Consider a party structure, and for each party in the structure, consider the corresponding policy proposal (if any). Given legislators' preferences, party members can fully anticipate the equilibrium policy outcome that will prevail. A party structure is stable if no subset of members wishes to quit its party and form a smaller one in order to induce a different equilibrium outcome.

The role played by parties in the models in the literature is one of either sharing the cost of participating in elections (Riviere 1999, Osborne and Tourky 2002, Hansen 2016), or informing voters about the parties' policy positions (Snyder and Ting 2002), or selecting candidates (Bernhardt *et al.* 2009, Krasa and Polborn 2018), or solving the commitment problem of independent candidates (Levy 2004, Anesi and De Donder 2009, 2011, 2013, Becher 2016).

Lastly, some papers model parties as coalitions of legislators aiming to shape the legislative process within an assembly.[11] In these studies, parties are either voting blocs (Eguia 2011a, 2011b, 2012) or cartels that control the agenda of policy proposals considered by the assembly (Cox and McCubbins 2007, Diermeier and Vlaicu 2008).

[10] This is a consistent finding in empirical studies that investigate the opinion of experts and scholars (Huber and Inglehart 1995, Benoit and Laver 2006), of those based on the textual analysis of party programs (Budge et *al.* 2001), as well as those that analyze the voters' self-assessed political position (Inglehart and Klingemann 1976). This regularity appears to hold even if substantial differences in the nature of the left–right dimension are documented across countries (Klingemann 1979) and over time (Lukes 2003).

[11] A review of the early literature is provided in Dhillon (2005). See, also, Eguia (2012).

This paper fits in the last class. However, in contrast with these studies, I model parties as institutions that discipline and constrain the activity of legislators within the assembly. Specifically, parties provide members with a *protocol* – i.e., a set of rules and procedures – that serves two purposes:

(i) It translates the legislators' preferences and relative power within the party into a single policy proposal – i.e., a bundle of reforms that the party pledges to bring to the floor for a vote.

(ii) It enforces commitment. For instance, party membership allows a legislator to credibly commit to making no policy proposal other than the one agreed upon by the party. Moreover, it constrains the set of reforms that legislators can propose. Such commitment is enforced through *party discipline*, which itself relies on institutions and procedures such as whips and voting by list.[12]

Thus, parties can be viewed as institutions that allow legislators to write and enforce "contracts" between themselves and their party leaders and/or constituents before engaging in legislative bargaining. This corresponds to a description of political parties as *bounded oligarchies* (Ceron 2012) – i.e., coalitions of legislators within the assembly that are constrained in their activity by external forces (such as electoral incentives, activists, and lobbying).[13]

In keeping with the literature (Roemer 1999, Levy 2004, 2005), I assume that a party's proposal is *credible* only if it consists of a policy that is in the Pareto set of the party members. Thus, a proposal represents a compromise among internal factions that hold different ideological positions. This assumption is admittedly restrictive, but it is imposed for two reasons. Firstly, it is consistent with recent empirical findings showing that each party's ideological position within a legislature can be largely explained as a weighted average of the positions of its internal factions (Ceron 2012, Romeijn 2018). Secondly, it eases the intuitive understanding of the mechanisms that underpin the main results of the paper. All the results are shown to be robust to less restrictive assumptions regarding the credibility of reform

[12]The empirical research provides convincing evidence of the effectiveness of such procedures in enforcing discipline in the behavior of legislators (Budge *et al.* 1987). Levy (2004) extensively discusses how this literature justifies similar modeling choices.

[13]As such, this analysis admits an interpretation – briefly discussed in section 6.3 – of a model of pre-electoral coalitions with post-election bargaining, similar to Bandypadhyay *et al.* (2011).

proposals.[14]

Departing from the literature, I assume that a party's proposal is *feasible* only if a group choice function exists that translates the preferences of internal factions into policy proposals and that satisfies *neutrality*. This assumption is equivalent to stating that each available policy, with the possible exception of the status quo, is ex-ante equal to any other with respect to the formal procedure that disciplines the choice of the reform proposal.

Lastly, formal models of political parties typically aim to provide a characterization of either (i) the policy outcome of the political process and how this is shaped by the presence of parties (Baron 1993, Riviere 1999, Levy 2004, Anesi and De Donder 2009, 2011, 2013), (ii) the proposals and/or candidates chosen by each party and the extent of their heterogeneity/polarization (Roemer 1999, 2011, Lee and Roemer 2006, Hansen 2016, Krasa and Polborn 2018), or (iii) the stable party structures or political coalitions (Levy 2004, Eguia 2011a, 2011b, 2012, Peeters *et al.* 2016).

In this paper, I simultaneously endogenize and provide a general characterization of all these three outcomes: namely, the partition of the legislature into parties, the policies that parties propose, and the policy outcome attained from these proposals.

The key mechanisms that underpin the results of the paper are illustrated in the following example.

1.1 Motivating Example

Consider a spatial model of legislative bargaining under majority rule. The set of legislators is $\mathcal{N} = \{1, 2, 3, 4, 5, 6, 7\}$. The policy space is $X = [0, 7]^2$, and the *status quo* is x^0. Legislator i's preferences are Euclidean with ideal point x^i.[15] The legislators' ideal points are $\{(1, 1); (2, 3); (2, 3); (2, 6); (2, 6); (4, 6); (7, 7)\}$. Ideal points are totally ordered in X but not aligned, as illustrated in *Fig. 1.1* and *Fig.*

[14]Notice that Roemer 1999, Levy 2004 and 2005 model the role of parties in elections rather than in legislative bargaining. One may object that the kind of credibility issues that prevail in the former strategic environment differ substantially from that in the latter. Thus, if on one hand there is evidence that politicians are constrianed in their legislative activity by the presence of political parties, on the other hand the specific assumption that a party's policy proposal must lie within the Pareto set of its members is strong. Neverthless, this restriction can be substantially weakened with no effect on the predictions of the model. This robustness result is outlined in section 5.

[15]I.e., they can be represented by the objective function $u^i(x) = -\left[x - x^i\right]^T \left[x - x^i\right]$.

2.1. The latter assumption is necessary to rule out the existence of a Condorcet winner (Plott 1967).[16] The ideal point of the median legislator is $x^4 = (2, 6)$.

A partition $\mathbb{P}$ of $\mathcal{N}$ is a *party structure.* Each set $\mathcal{C}^j \in \mathbb{P}$ is a *party.* The political process is divided in two stages. In the first stage, each party $\mathcal{C}^j$ simultaneously make a *proposal* a^j. A proposal can be any policy in the Pareto set of the party members, or $a^j = \emptyset$ if party $\mathcal{C}^j$ is inactive. In the second stage, the legislators use the method of majority rule to select one proposal, called *policy outcome*, among those available. If the majority rule does not deliver a stable outcome, then the policy outcome is the status quo x^0.

Lastly, a party structure is required to be *stable.* Informally, a party structure $\mathbb{P}$ is stable if, given A, no subset of members has a strict incentive to quit his/her party in order to induce a different policy outcome.[17]

Compare two cases. In case (1), the status quo $x^0 = (3, 6)$ is higher than the ideal point of the median legislator x^4 (*Fig. 1*). In case (2), the status quo $x^0 = (3.45, 5.45)$ is neither higher nor lower than x^4 (*Fig. 2*).

What kind of reform can be implemented (if any) in the two cases? What kind of party structure can support a reform? Can any political alliance successfully block any reform and maintain the *status quo*?

First, notice that in both cases (1) and (2) x^4 is preferred to x^0 by a majority of legislators, and there is at least one party structure $\mathbb{P}$ that allows the median legislator to achieve his/her ideal point x^4. For instance, consider the partition $\mathbb{P} = \{\{1, 2, 3, 4, 5\}; \{6, 7\}\}$. The Pareto set of each of the two parties in $\mathbb{P}$ is illustrated in *Fig 1.2.* The proposal must lie within the highlighted area (party $\{1, 2, 3, 4, 5\}$) and on the solid line (party $\{6, 7\}$), respectively. If party $\{1, 2, 3, 4, 5\}$ proposes x^4, then a majority of legislators strictly prefers this proposal to the status quo x^0. Moreover, there is no point on the solid line connecting x^6 and x^7 that can obtain a majority of votes against x^4. Lastly, if the members of party $\{1, 2, 3, 4, 5\}$ try to put forward a proposal other than x^4, then individuals 4 and 5 have a strict incentive to quit

[16] Formally $x^j \leq x^k$ for all $j \leq k$, and for any three distinct ideal points x^k, x^l, x^j, one gets $\left[x^k - x^l\right]^T \left[x^l - x^j\right] \neq 0$. This assumption ensures that the condition of *radial symmetry* in Plott (1967) is violated for any subset of three alternatives.

[17] A formal definition of stability is provided in section 2.

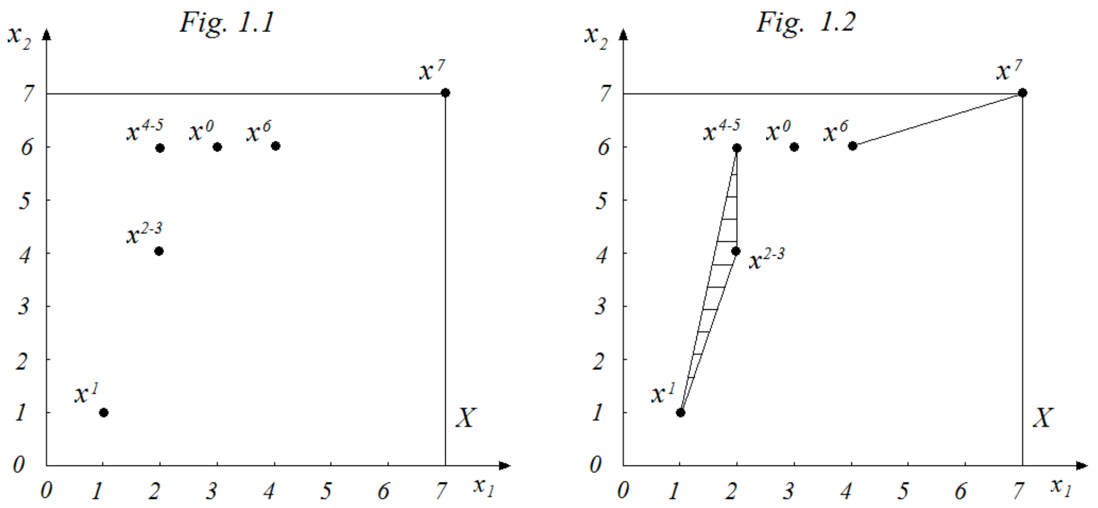

Figure 1. Motivating example, Case (1).

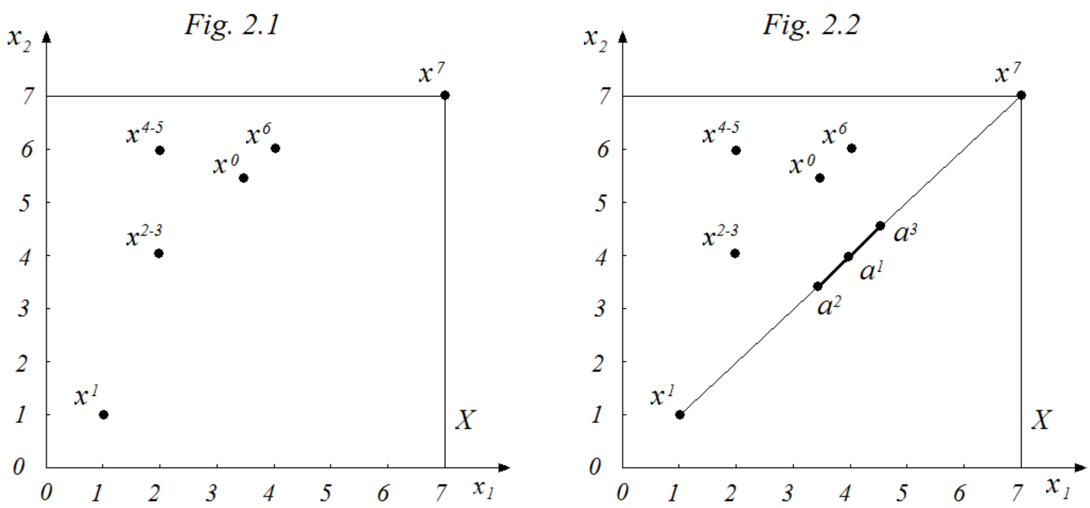

Figure 2. Motivating example, Case (2).

the party, inducing the partition $\mathbb{P}' = \{\{1,2,3\};\{4,5\};\{6,7\}\}$, and securing their most preferred policy outcome x^4.[18] As a result, party structure $\mathbb{P}$ is stable only if $\{1,2,3,4,5\}$ proposes x^4 and the reform is implemented. One can show that if the position of the status quo is as given in case (1), then a reform is approved in *all* stable party structures.[19]

[18] Under party structure $\mathbb{P}'$, neither party $\{1,2,3\}$ nor party $\{6,7\}$ can propose a policy capable of defeating proposal x^4. Thus, party $\{4,5\}$ always wins if it proposes x^4.

[19] In particular, this is true for $\mathbb{P} = \{\{i\}_{i=1}^{7}\}$, in which each party is a singleton. This stable party structure resembles the equilibrium of a citizen–candidate model (Besley and Coate 1997) with no parties. See section 6.2 for details.

On the other hand, in case (2), an outcome such that no reform is implemented is possible. Consider the partition $\mathbb{P}'' = \{\{1,7\};\{2,3\};\{4,5\};\{6\}\}$. The Pareto set of each party in $\mathbb{P}'''$ is illustrated in *Fig. 2.2*. Notice that party $\{1,7\}$'s members exhibit extreme opposite ideological positions – i.e., $\{1,7\}$ is a *coalition of extremes.*

Suppose parties $\{4,5\}$ and $\{6\}$ propose their ideal points x^4 and x^6, respectively, party $\{2,3\}$ remains inactive, and party $\{1,7\}$ proposes a compromise policy such as $a^1 = (3.99, 3.99)$.[20] Firstly, notice that, as in case (1), it is still true that a majority of legislators strictly prefers proposal x^4 to the status quo x^0. Secondly, it is easy to show that there is a *Condorcet cycle* over alternatives $\{x^4, x^6, a^1\}$, because a^1 defeats x^4, x^4 defeats x^6, and x^6 defeats a^1 by simple majority voting. Thus, no stable reform can be chosen by the method of majority rule, and the policy outcome is the status quo x^0.

Moreover, none of the members of each party has a strict incentive to induce his/her party to play a different strategy. Specifically, party $\{4,5\}$'s only available alternative to proposing x^4 is being inactive. In such a case, proposal x^6 becomes a Condorcet winner, making both members of party $\{4,5\}$ worse off. The same argument applies to party $\{6\}$. Party $\{2,3\}$'s only alternative is proposing x^2, but this does not change the final policy outcome.

Lastly, the members of party $\{1,7\}$ can propose policy $\hat{a}^1 \neq a^1$ (including $\hat{a}^1 = \emptyset$) and induce a policy outcome x'. For such an outcome to be stable, x' must be weakly preferred to x^4 by both members of party $\{1,7\}$, or else at least one member has a strict incentive to quit the party, inducing the partition $\mathbb{P}''' = \{\{1\};\{2,3\};\{4,5\};\{6\};\{7\}\}$ and securing x^4 (see case (1)). Recall that $\hat{a}^1$ must lie in the Pareto set of party $\{1,7\}$. Thus, the stability requirement is satisfied only if $\hat{a}^1$ lies on the thick line between $a^2 = (1-\sqrt{13}, 1-\sqrt{13})$ and $a^3 = (1+\sqrt{13}, 1+\sqrt{13})$, or if $\hat{a}^1 = \emptyset$. But in all such cases the policy outcome is still x^0. Thus, both party members are ultimately weakly worse off if their party splits or makes a proposal other than a^1. In turn, this implies that the partition $\mathbb{P}''$ given party proposals $\{a^1;\emptyset;x^4;x^6\}$ is a stable party structure.

This description informally shows how case (2) can result in a legislative gridlock

[20] Any proposal $\tilde{a}^1$ on the thick line between a^1 and a^2 in *Fig. 2.2* delivers the same result.

– i.e., a stable party structure such that no reform is implemented.

Lastly, notice that the members of the coalition of extremes $\{1, 7\}$ in *Fig. 2.2* do not cooperate in hope of having their proposal approved by the assembly. Conversely, the alliance between extremists only serves the purpose of preventing reforms that would otherwise be implemented, i.e. it is purely tactical.
In the next section, I propose a model that generalizes and extends this key insight.

2 The Model

Consider an assembly that consists of an odd number $N \geq 3$ of legislators. The set of legislators is $\mathcal{N} = \{1, 2, ..., i, ..., N\}$. The median element of $\mathcal{N}$ is the *median legislator*, denoted by m. The policy space X is such that $(X, \geq)$ is a complete sublattice of $(\mathbb{R}^d, \geq)$ under the partial order $\geq$. Examples of policy spaces that satisfy these assumptions are $\mathbb{R}^d$ and its subset $[0, 1]^d$. A party structure is a partition $\mathbb{P}$ of the set $\mathcal{N}$. A typical element $\mathcal{C}^j \in \mathbb{P}$ is called a *party*.
Each party $\mathcal{C}^j$ makes a proposal $a^j \in X^j \cup \{\emptyset\}$, where $X^j \subseteq X$, and $a^j = \emptyset$ means that party $\mathcal{C}^j$ proposes no policy reform – i.e., it is *inactive*. An *action profile* $\{a^j\}_{\mathcal{C}^j \in \mathbb{P}}$ is the collection of the proposals made by each party in $\mathbb{P}$. The *set of proposals* $A := \{x \in X \mid a^j = x \textit{ for some } \mathcal{C}^j \in \mathbb{P}\}$ is the corresponding set of all policy proposals that parties have brought to the floor.[21]
Given a party structure $\mathbb{P}$, the legislators play a two-stage game:

1. the members of each party $\mathcal{C}^j$ collectively make a proposal a^j (proposal stage);

2. each legislator votes for one of the available proposals $a^j \in A$ (voting stage).

A voting rule selects the winning proposal (if any exists), which is called the *policy outcome* denoted by w.

2.1 Preferences

Preferences over policy outcomes. Each legislator $i \in \mathcal{N}$ ranks policy outcomes according to a weak preference ordering $\succeq^i$ over X. I denote with $\succ^i$ the corresponding

[21] Thus, A does not include null proposals in $\{a^j\}_{\mathcal{C}^j \in \mathbb{P}}$, i.e. $\emptyset \notin A$, and each element of A is a policy $x \in X$ that has been proposed by *at least* one active party in $\mathbb{P}$.

strict preference relation. Let $I(\succeq^i)$ denote the set of ideal policies of legislator i – i.e., $I(\succeq^i) := \{x | x \succeq^i x' \; \forall x' \in X\}$,[22] with typical element x^i. I assume that a majority of legislators $i \in \mathcal{N}$ has preferences such that $x^m \succeq^i x^0$ for all $x^m \in I(\succeq^i)$, i.e. any ideal policy of the median legislator is a reform that has the support of a majority of the assembly.[23]

Let $\mathbf{\Pi}$ denote the N-fold Cartesian product of the set of individual preferences. An element $(\succeq)$ of $\mathbf{\Pi}$ is called a *preference profile* – i.e., the collection of the preference orderings of all $i \in \mathcal{N}$. Legislators are ordered according to their preference type $\theta^i \in \Theta$, where Θ is a totally ordered set. Two legislators $i, j \in \mathcal{N}$ have the same type $\theta^i \in \Theta$ if and only if they have the same preferences – i.e., if $x \succeq^i x' \longleftrightarrow x \succeq^j x'$ for all $x \in X$.[24]

I impose two ordinal restrictions on legislators' preferences over policies. Following Milgrom and Shannon (1994), I assume *Quasisupermodularity (QSM)* and *Strict Single Crossing Property (SSC)*.[25] Formally, preferences satisfy:

1. *(QSM in x)* if, for all $i \in \mathcal{N}$ and for all $x', x'' \in X$:
   (a) $x'' \succeq^i x' \wedge x'' \rightarrow x' \vee x'' \succeq^i x'$
   (b) $x' \succeq^i x' \vee x'' \rightarrow x' \wedge x'' \succeq^i x''$
2. *(SSC in x, θ)* if, for all $x', x'' \in X$ such that $x' > x''$:[26]
   (a) $x' \succeq^i x'' \rightarrow x' \succeq^j x''$ for all $j \geq i$, and
   (b) $x' \succeq^i x'' \rightarrow x' \succ^j x''$ for all $j > i$ such that $\theta^j > \theta^i$.

where $\wedge, \vee$ are the meet and join operators. These two restrictions imply that legislators are totally ordered with respect to their preference type θ^i, and that

[22] Notice that the completeness of X implies compactness in the order-interval topology (Birkhoff 1967). Hence, $I(\succeq^i) \neq \emptyset$ for all i.

[23] The case in which this assumption is relaxed is discussed in section 5.

[24] I denote each type with the index of the legislator of that type with the lowest index in $\mathcal{N}$ – e.g., if legislators with index 3, 4 and 5 are of the same type, and no other $i \in \mathcal{N}$ is of that type, then I denote their type with θ^3. Notice that this implies that the legislators in $\mathcal{N}$ are ordered such that their type θ^i is weakly increasing in the legislators' index i.

[25] The relationship between the concepts of *QSM* and *SSC* in Milgrom and Shannon (1994).

[26] The strict order $x' > x''$ indicates that $x' \geq x''$ and $x'_k > x''_k$ for at least one policy dimension k.

there is a form of weak complementarity between each pair of policy dimensions.[27] If preferences can be represented by an objective function $V: \ X \times \Theta \times \Phi \to \mathbb{R}$, in which Φ is totally ordered sets of individual taste parameters, then these two conditions are satisfied if V is quasisupermodular in x and satisfies the strict single crossing property in x, θ under the definition in Milgrom and Shannon (1994).

Lastly, notice that whenever X is multidimensional *(SSC* in x, θ*)* is a much weaker assumption than both the (unidimensional) single crossing condition (Gans and Smart 1996) and single-peakedness.[28]

Conditional preferences over policy proposals. Each legislator i member of party $\mathcal{C}^j \in \mathbb{P}$ also has preferences over X^j in relation to which proposal in X^j his/her party should put forward conditional on the proposals of other parties a^{-j}.[29] I define for each legislator $i \in \mathcal{C}^j$ and each pair of proposals $x', x'' \in X^j$ the conditional preference relation $R^{j,i}(a^{-j})$ (with strict preferences denoted by $P^{j,i}(a^{-j})$) over $X^j \times X^j$. For instance, $x' R^{j,i}(a^{-j}) x''$ means that legislator i, member of party $\mathcal{C}^j \in \mathbb{P}$, prefers his/her party to propose x' rather than x'' conditional on other parties proposing a^{-j}. I impose no further restrictions on $R^{j,i}(a^{-j})$. In section 2.4, I restrict the type of $R^{j,i}(a^{-j})$ that can be part of an equilibrium to those that are consistent with purely policy-motivated legislators.

Lastly, let $R^j(a^{-j})$ denote the conditional preference profile of all $i \in \mathcal{C}^j$, and $\mathbf{R}^j$ be the $|\mathcal{C}^j|$-fold Cartesian product of the set of individual preferences over policy proposals in X^j.[30]

2.2 Proposal Stage

Following Levy (2004), I assume that the proposal of each party $\mathcal{C}^j$ must lie in the Pareto set of its members, such that the set of available proposals is $X^j := \{x \in X | \nexists x' \in X \ s.t. \ x' \succeq^i x \ \forall i \in \mathcal{C}^j \ and \ x' \succ^k x \ for \ some \ k \in \mathcal{C}^j\}$.[31] This assumption is

[27] Similar restrictions are common in many fields of economic theory, particularly in the analysis of games with strategic complementarities (Milgrom and Shannon 1994). Notice that, in contrast to Levy (2004) and Morelli (2004), I do not assume *single-peakedness*.

[28] Details of how these two assumptions compare to *(SSC)* are provided in section 6.1.

[29] Notice that a^{-j} lists all the proposals of other parties, including null proposals $\emptyset$.

[30] Both $R^j(a^{-j})$ and $\mathbf{R}^j$ do not depend on the party structure $\mathbb{P}$.

[31] Notice that if a party is a singleton, then X^j reduces to the set of ideal policies of its unique member (as in a citizen–candidate model).

common in models of endogenous parties (e.g., Roemer 1999, Levy 2004), and it is further discussed in Levy (2004).[32] I am going to relax it in section 5.

Departing from Levy (2004), I assume that each party chooses its proposal a^j according to a *protocol* in the form of a surjective conditional group choice function:

$$g^j_{a^{-j}} : \mathbf{R}^j \to X^j \cup \{\emptyset\}$$

that must satisfy the following two conditions.

1. *Monotonicity (M).* Consider any two conditional preference profiles $R^j(a^{-j})$, $\widehat{R}^j(a^{-j}) \in \mathbf{R}^j$ such that for all $i \in \mathcal{C}^j$ and all $x', x'' \in X^j$, $x'' \widehat{R}^{j,i}(a^{-j}) x' \to x'' R^{j,i}(a^{-j}) x'$, and $x'' \widehat{P}^{j,i}(a^{-j}) x' \to x'' P^{j,i}(a^{-j}) x'$. Then $g^j_{a^{-j}}(\hat{R}^j(a^{-j})) = x''$ implies $g^j_{a^{-j}}(R^j(a^{-j})) \neq x'$.[33]

2. *Neutrality (N).* For any conditional preference profile $R^j(a^{-j}) \in \mathbf{R}^j$, let $\rho : X^j \to X^j$ be a permutation of X^j and let $R^j_\rho(a^{-j}) \in \mathbf{R}^j$ be such that for all $i \in \mathcal{C}^j$ and all $x, x' \in X^j$ with $x \neq x'$, $x R^{j,i}_\rho(a^{-j}) x'$ if and only if $\rho(x) R^{j,i}(a^{-j}) \rho(x')$. The function $g^j_{a^{-j}}$ is neutral if $\rho(g^j_{a^{-j}}(R^j(a^{-j}))) = g^j_{a^{-j}}(R^j_\rho(a^{-j}))$.

Let $w(A, \succeq)$ denote the expected policy outcome given policy proposals A and preference profile $\succeq$, and define $A^{-j} = A \setminus \{a^j\}$. I impose a tie-break rule on $g^j_{a^{-j}}$ that satisfies the following conditions.

1. *TB1 (Inaction of irrelevant parties*). If $w(\{\emptyset\} \cup A^{-j}, \succeq) = w(\{x\} \cup A^{-j}, \succeq)$ for some $x \in X^j$, then $g^j_{a^{-j}}(R^j(a^{-j})) \neq x$.

2. *TB2 (Robustness to trivial challenges*). If $\exists x' \in X^j$ such that $w(\{x'\} \cup A, \succeq) = x'$ and $w(\{x'\} \cup A^{-j}, \succeq) \succeq^i w(A, \succeq)$ $\forall i \in \mathcal{C}^j$, then $g^j_{a^{-j}}(R^j(a^{-j})) \neq a^j$.

The conditions *(M)* and *(N)*[34] discipline the choice of the party proposal. In particular, *(M)* ensures that (conditional) Pareto inferior proposals are never chosen.[35]

[32] The author suggests some possible interpretations of this assumption, which are consistent with the setting of this paper.

[33] Monotonicity implies the *weak Pareto principle* (with respect to the conditional preferences) because the function $g^j_{a^{-j}}$ is surjective.

[34] Conditions *(M)* and *(N)* are widely adopted – and often deemed desirable – in social choice literature. They are satisfied by any weighted majority voting rule. See Dasgupta and Maskin (2008).

[35] *(M)* implies *conditional* weak Pareto efficiency, and it is motivated by standard social choice

Monotonicity is admittedly a strong assumption, but is deemed to be credible for this application. Specifically, it is typically satisfied by the choice methods adopted by most political parties in Western countries such as (i) voting within the party congress, (ii) the dictatorship of the leader, and (iii) many forms of bargaining within a small committee (Ceron 2012).

Condition *(N)* implies that the choice protocol treats all alternatives in X^j symmetrically: if the alternatives are relabeled via ρ, then the chosen proposal is relabeled in the same way. Thus, *(N)* implies that the choice over policy proposals is solely driven by the party members' preferences over policy proposals, and by their relative political power within the party.

Condition *TB1* can be rationalized as the effect of a small cost of participating in the legislative process and/or of evaluating an additional alternative, which is not explicitly modeled in the proposed setting. Specifically, it states that a party chooses to be inactive rather than making a proposal that has no influence on the policy outcome. A very similar assumption is imposed in Levy (2005).

Lastly, condition *TB2* states that no party ever chooses a proposal a^j (or no proposal) if there exists an alternative in X^j that is a winning policy over $\{x'\} \cup A$ and that induces a Pareto-superior outcome for all the party members. This assumption prevents instability in cases in which all party members are indifferent between two proposals given other party proposals A^{-j}.

2.3 Voting Stage

In the second stage, legislators must select a proposal in A. The assembly ranks the alternatives in A using the *method of majority rule* whenever this method delivers an outcome, and selects the status quo x^0 otherwise. Each legislator i's voting behavior is driven solely by his/her preferences over policy outcomes $\succeq^i$, and it is not affected by party membership.[36]

Formally, let $MV(\succeq)$ be the complete social preference relation induced by the

considerations. The assumption on the codomain of $g^j_{a^{-j}}$ corresponds to *unconditional* Pareto efficiency, and it is justified by commitment issues.

[36] The results are unaffected if one restricts the attention to equilibria in which no legislator votes against the proposal made by his/her party.

majority rule under a preference profile $\succeq$.[37] Given this social preference relation $MV(\succeq)$ and a subset $A \subseteq X$, I define $K(A, \succeq) := \{x \in A | \, xMV(\succeq)x' \ \ \forall x' \in A\}$ to be the set of *MV-maximal* alternatives in A, which corresponds to the core of the majority voting game over A.

The assembly chooses the outcome from a set of available alternatives $A \subseteq X$ according to a social choice function:

$$W: \ \mathcal{P}(X) \times \Pi \rightarrow X \cup \{x^0\}$$

Thus, the policy outcome w is the outcome of the function W. The social choice function W satisfies the following conditions.

1. *Majority Rule (MR).* If $K(A, \succeq)$ is nonempty, then $W(A, \succeq) = w$ for some $w \in K(A, \succeq)$.

2. *Inertia (I).* If $K(A, \succeq) = \{\emptyset\}$, then $W(A, \succeq) = x^0$ – i.e., the *status quo* is maintained.

3. *Revealed Social Preferences* (*RSP*). For any $A' \subseteq A$, such that $x \in A'$, if $W(A', \succeq) = x'$, $W(A, \succeq) = x$, and $x \in K(A, \succeq)$, then $x' = x$.

Assumptions *(MR)* and *(I)* correspond to the social choice procedure that results from the *method of majority rule.* In such a case, if there is a (weak) Condorcet winner among the set of alternatives, then the core of the voting game is nonempty[38] and the policy outcome lies in the core. Conversely, if no Condorcet winner exists, then no reform proposal can gain a stable support.

Assumption *(RSP)* disciplines the collective choice whenever the core of the voting game is not a singleton. The condition states that if the social choice function W selects $x \in K(A, \succeq)$ over set A, then x is revealed as being collectively preferred to any other proposal in $K(A, \succeq)$. The core of the game is played over a subset $A' \subseteq A$, such that $x \in A'$ is a subset of $K(A, \succeq)$. Thus, x is also revealed as being

[37] Thus, $x'MV(\succeq)x''$ if and only if $\sum_{i=1}^{N} \mathbf{1}[x' \succeq^i x''] \geq N/2$. This definition can be easily extended to the case in which the set of voters is a continuum. Notice that $MV(\succeq)$ is complete, reflexive, but not necessarily transitive.

[38] The relationship between the concepts of (strong) *core* and *Condorcet winner* is described in Ordershook (1986, pp. 347–349).

collectively preferred to any other proposal in $K(A', \succeq)$. In such a case, *(RSP)* states that W must also select x over A'.[39] Notice that if the majority core $K(A, \succeq)$ is not a singleton, then there may be more than one function W that satisfies these conditions. I consider all such functions. Specifically, I denote with $\mathbf{W}(A, \succeq)$ the set of all functions[40] W that satisfy *(MR)*, *(I)*, and *(RSP)*, for a given set of proposals $A \subseteq X$ and preference profile $\succeq \in \Pi$.

I abstract from the specific protocol adopted by the assembly regarding the decision of starting a reform process (e.g. the initial vote of a first amendable bill against the status quo), or to conclude it (e.g. final approval vote). As a result, the status quo x^0 may not be an element of A. For instance, in the case in which a single party exists – i.e. $\mathbb{P} = \{\mathcal{C}^j\}$ with $\mathcal{C}^j = \mathcal{N}$, then $A = \{a^j\}$ and typically $a^j \neq x^0$. This is admittedly an unrealistic description of a typical legislative process. Nevertheless, this assumption simplifies the analysis and it is mostly innocuous in this framework, under the assumption that a majority of legislator prefers the ideal policy of the median legislator to x^0.[41] In particular, in any equilibrium in which the reform preferred by the median legislator is approved, such reform must be the outcome of a proposal that can defeat the status quo in pairwise voting both at the beginning and at the end of the legislative process. Similarly, any outcome in which no proposal gains a stable support within the assembly (a legislative gridlock) can be triggered by an initial proposal consisting in the ideal policy of the median legislator.

2.4 Equilibrium of the Proposal Game

For a given party structure $\mathbb{P}$, let $a(\mathbb{P})$ denote the action profile in an equilibrium of the proposal game and $A(\mathbb{P})$ the corresponding set of proposals. I define an equilibrium of the proposal game as follows.

Definition 1. (*Equilibrium of the Proposal Game*). A collection $\{a^j\}_{\mathcal{C}^j \in \mathbb{P}} \equiv a(\mathbb{P})$ is an equilibrium of the proposal game if there exists $\{R^j(a^{-j}), g^j_{a^{-j}}\}_{\mathcal{C}^j \in \mathbb{P}}$, such that for all $\mathcal{C}^j \in \mathbb{P}$, the following conditions hold: (i) $R^j(a^{-j}) \in \mathbf{R}^j$; (ii) for any $i \in \mathcal{C}^j$, and for

[39] Notice that if the core of the game is a singleton the condition is always trivially satisfied.

[40] Notice that $\mathbf{W}$ is always nonempty under *(QSM)* and *(SSC)*.

[41] The effects of relaxing this assumption by introducing a final approval vote are described and discussed in section 5.

any $x, x' \in X^j$, $xR^{j,i}(a^{-j})x' \iff w(\{x\} \cup A(\mathbb{P}) \setminus \{a^j\}, \succeq) \succeq^i w(\{x'\} \cup A(\mathbb{P}) \setminus \{a^j\}, \succeq)$; (iii) $g^j_{a^{-j}}(R^j(a^{-j})) = a^j$. An equilibrium of the proposal game $a(\mathbb{P})$ is either (i) a *regular equilibrium* if $K(A(\mathbb{P}), \succeq) \neq \{\emptyset\}$, or (ii) a *legislative gridlock* if $K(A(\mathbb{P}), \succeq) = \{\emptyset\}$.

The equilibrium condition on $R^j(a^{-j})$ implies that each legislator i member of party $\mathcal{C}^j$, for the purpose of choosing the party proposal, ranks the policies in X^j solely with respect to the policy outcome they induce conditional on the proposals of other parties. For instance, the policy x' is (conditional) weakly preferred to x'' by legislator i if and only if, given other party proposals a^{-j}, he/she (unconditionally) prefers the policy outcome $w(\{x'\} \cup A(\mathbb{P}) \setminus \{a^j\}, \succeq)$ to $w(\{x''\} \cup A(\mathbb{P}) \setminus \{a^j\}, \succeq)$. Notice that, in contrast to Levy (2004), in this setting an equilibrium of the proposal game may not always exist. This scenario occurs if for any $A \subseteq X$ there is no collection $\{R^j(a^{-j}), g^j_{a^{-j}}\}_{\mathcal{C}^j \in \mathbb{P}}$, such that $g^j_{a^{-j}}(R^j(a^{-j}),) = a^j$ for all $\mathcal{C}^j \in \mathbb{P}$.

In such a case, I assume $A(\mathbb{P}) = \{\emptyset\}$.[42] On the other hand, an equilibrium of the overall game, which I describe in the next section, always exists.

2.5 Stable Party Structures

The concept of stability is borrowed from Ray and Vohra (1997), and it has been implemented in a model of political parties by Levy (2004, 2005). Below, I briefly summarize the stability concept, highlighting the key differences with Levy's (2004) framework.

Players start from some party structure $\mathbb{P}$, and are only allowed to break parties by inducing finer partitions. Let $\mathfrak{R}(\mathbb{P})$ denote all the party structures that are refinements of $\mathbb{P}$. A partition $\mathbb{P}' \in \mathfrak{R}(\mathbb{P})$ is induced from $\mathbb{P}$ if it is generated by breaking a party $\mathcal{C}^j \in \mathbb{P}$ into two. Consider a party $\mathcal{C}^j \in \mathbb{P}$. A party $\mathcal{C}^d \subseteq \mathcal{C}^j$ is a *deviator* if it can induce $\mathbb{P}' \in \mathfrak{R}(\mathbb{P})$ from $\mathbb{P}$. The members of a deviator $\mathcal{C}^d \subseteq \mathcal{C}^j$ take into account future deviations, both by members of their own party $\mathcal{C}^d$ and by members of other parties $\mathcal{C}^s \in \mathbb{P}'$, $s \neq d$.[43] Credible threats are deviations to finer

[42] This assumption is only relevant to evaluate the profitability of a deviation that induces a new proposal game in which no equilibrium exists. See next section.

[43] Because of that, this stability concept satisfies *consistency* and *farsightedness* in the sense of

partitions which are stable – i.e., the definition of stability is recursive.

The only difference in the definition of stability relative to Levy (2004) is that I allow for the outcome of disagreement, which I refer to as a legislative gridlock, to be stable. Formally, consider a sequence of partitions $\{\mathbb{P}_k\}_{k=1}^{K}$ such that for every $k = 2, ..., K$, $\mathbb{P}_k$ is induced by a deviator $\mathcal{C}_k^d \subset \mathcal{C}_{k-1}^j$ for some $\mathcal{C}_{k-1}^j \in \mathbb{P}_{k-1}$. I define stability as follows.

Definition 2. (*Stability*) $\langle \mathbb{P}, a(\mathbb{P}) \rangle$ is *sequentially blocked* by $\langle \mathbb{P}', a(\mathbb{P}') \rangle$ for some $\mathbb{P}' \in \mathfrak{R}(\mathbb{P})$ if there exists a sequence $\{\langle \mathbb{P}_1, a(\mathbb{P}_1) \rangle, \langle \mathbb{P}_2, a(\mathbb{P}_2) \rangle, ..., \langle \mathbb{P}_K, a(\mathbb{P}_K) \rangle\}$, such that:

1. $\langle \mathbb{P}_1, a(\mathbb{P}_1) \rangle = \langle \mathbb{P}, a(\mathbb{P}) \rangle$, $\langle \mathbb{P}_K, a(\mathbb{P}_K) \rangle = \langle \mathbb{P}', a(\mathbb{P}') \rangle$ and for every $k = 2, ..., K$ there is a deviator $\mathcal{C}_k^j$ that induces $\mathbb{P}_k$ from $\mathbb{P}_{k-1}$.

2. $\langle \mathbb{P}', a(\mathbb{P}') \rangle$ is stable with $a(\mathbb{P}') = a(\mathbb{P}_K)$ for some $a(\mathbb{P}_K)$.

3. $\langle \mathbb{P}_k, a(\mathbb{P}_k) \rangle$ is not stable for any $a(\mathbb{P}_k)$ and for $1 < k < K$.

4. $w(A(\mathbb{P}'), \succeq) \succ^i w(A(\mathbb{P}_{k-1}), \succeq)$ for all $k = 2, ..., K$, and $i \in \mathcal{C}_k^j$.

Then, $\langle \mathbb{P}, a(\mathbb{P}) \rangle$ is *stable* if there is no $\langle \mathbb{P}', a(\mathbb{P}') \rangle$ for $\mathbb{P}' \in \mathfrak{R}(\mathbb{P})$ that sequentially blocks $\langle \mathbb{P}, a(\mathbb{P}) \rangle$.

The definition states that a tuple $\langle \mathbb{P}, a \rangle$ is stable if there is no leading deviator $\mathcal{C}^d$ that can induce a sequence of deviations such that at each step of the sequence an equilibrium of the proposal game $\langle \mathbb{P}_k, a(\mathbb{P}_k) \rangle$ (if any exists) is played, and such that it is strictly profitable for each deviator to choose to deviate, with respect to the final outcome of the sequence. Notice that the finest partition of $\mathcal{N}$ – i.e., the one in which each party $\mathcal{C}^j \in \mathbb{P}$ is a singleton – is always stable.

I define an equilibrium for the overall game – which I name a *stable party structure* – as follows.

Definition 3. *(Stable Party Structure).* A Stable Party Structure (*SPS*) is a tuple $\langle \mathbb{P}, a(\mathbb{P}), w \rangle$ such that (i) $a(\mathbb{P})$ is an equilibrium of the proposal game given partition

Ray and Vohra (2015). The main results of the paper go through even if players do not exhibit such sophisticated behavior.

$\mathbb{P}$; (ii) $\langle \mathbb{P}, a(\mathbb{P}) \rangle$ is stable; (iii) w is a policy outcome of the second-stage game – i.e., $W(A(\mathbb{P}), \succeq) = w$. A *SPS* is *a party equilibrium* if $a(\mathbb{P})$ is a regular equilibrium of the proposal game, and it is a *stable legislative gridlock* otherwise.

This definition completes the description of the political process.

3 Results

The main results of the paper are stated in this section. Section 3.1 contains the results regarding the existence and characterization of a stable party structure. Section 3.2 analyzes the comparative statics of the policy outcome. Section 3.3 describes the types of party structure that prevail in any stable party structure.

3.1 Existence and Quasi-Median Voter Theorem

Recall that $I(\succeq^i)$ denotes the set of ideal policies of legislator i with typical element x^i. The supremum and the infimum of $I(\succeq^i)$ are $\bar{x}^i = \sup\{I(\succeq^i)\}$ and $\underline{x}^i = \inf\{I(\succeq^i)\}$,[44] respectively.

Then, I can state the following.

Proposition 1. *(Existence). (i) A stable party structure $\langle \mathbb{P}, a(\mathbb{P}), w \rangle$ always exists for any preference profile $\succeq$; (ii) a party equilibrium such that the policy outcome is an ideal point of the median legislator m – i.e., $\langle \mathbb{P}, a(\mathbb{P}), w \rangle$ with $w \in I(\succeq^m)$ – always exists; (iii) a stable legislative gridlock $\langle \mathbb{P}, a(\mathbb{P}), x^0 \rangle$ exists only if either $x^0 \in I(\succeq^m)$ or if $x^0 \not\geq \bar{x}^m$ and $x^0 \not\leq \underline{x}^m$.*

Proof. See Appendix A.

Proposition 1 (i) and (ii) provide general existence results. Notice that in any party equilibrium only one party is active.[45]

[44] Notice that, because X is a lattice and preferences satisfy *QSM* and *SSC*, then $\bar{x}^i$ and $\underline{x}^i$ exist in X. See Milgrom and Shannon (1994).

[45] This result is a consequence of the assumption *TB1* on the group choice function of non-winning parties. A formal proof of this result is provided in Appendix A.

Proposition 1 (iii) implies that if $x^0 \notin I(\succeq^m)$, then a legislative gridlock is stable only if the status quo is neither higher nor lower than any proposal in the set of ideal policies of the median legislator. This suggests that no stable legislative gridlock ever occurs if the median legislator wants to move each element of the policy vector in the same direction relative to the status quo.

For instance, consider an example in which the policy space is two-dimensional. If the median legislator prefers a shift from the status quo towards a proposal that is more left-wing for both policy dimensions, then a legislative gridlock is not stable. Conversely, if the median legislator supports a shift towards a more left-wing policy for one policy dimension and towards a more right-wing policy for the other, then a legislative gridlock may be part of a stable party structure. This suggests that a legislative gridlock is possible only if the reforms supported by the moderate part of the assembly do not simply consists in a shift towards a more left-wing (or more right-wing) bundle of policies.[46]

Let $UP^m(x) = \{x' \in X |\ x' \succeq^m x\}$ be the upper contour set of $x \in X$ for individual m. Then, I can state the main result of this paper, which is the following.

Corollary 2. *(Quasi-Median Voter Theorem). (i) If either $x^0 \in I(\succeq^m)$, or $x^0 \geq x^m$, or $x^0 \leq x^m$ for all $x^m \in I(\succeq^m)$, then in any stable party structure the policy outcome is an ideal point of the median legislator m. (ii) In any stable party structure, the policy outcome is such that $w \succeq^m x^0$. Thus, (iii) if legislator m's preferences are strictly convex over $\mathbb{R}^d$, then the policy outcome must lie within a circle centered on x^m and with a radius equal to* $\max_{x \in UP^m(x^0)} \|x - x^0\|$.

Proof. See Appendix A.

Corollary 2 provides a sharp characterization of the policy outcome in any possible stable party structure. It states that the policy outcome depends on the position of the status quo in the policy space. Specifically, if the status quo is lower or higher

[46]The political science literature has several examples of episodes in the history of Congress that are deemed to resemble a legislative gridlock (e.g., Alt 1983, Blydenburgh 1971). In most of these examples, the source of the gridlock seems to be the nature of preferences held by members of Congress, rather than the multidimensionality of the policy space.

than the set of ideal policies of the median legislators (or it is part of such a set), then parties are not effective. Moreover, a quasi-median voter theorem holds – i.e., in any stable party structure, the median legislator obtains his/her most preferred policy. It is important to highlight that this result is much stronger than Proposition 1 (ii), which states that an equilibrium in which the quasi-median voter theorem holds always exists. Under the conditions of Corollary 2 (i) the median legislator obtain his/her ideal point in *all* stable party structures. Thus, this result represents a general median voter theorem which requires substantially weaker preference restrictions relative to those that are commonly assumed in the literature.[47]

If the status quo is neither lower nor higher than any ideal policy of the median legislator, then policy outcomes that differ from an ideal policy of the median legislator may prevail. Nevertheless, by part (ii), the median legislator cannot be made worse off relative to the status quo.

Result (ii) states that the equilibrium policy outcome must lie within the upper contour set of x^0 for the median legislator. If preferences are strictly convex (e.g., Euclidean), such a result implies that policies that are "far" from the median legislator's ideal policy can prevail only if the median legislator is sufficiently averse to the status quo.

3.2 Comparative Statics

In this section, I study the comparative statics of the equilibrium policy outcome. The notion of monotonicity is the same as in Milgrom and Shannon (1994). Specifically, I order sets of policy outcomes according to the strong set order.[48]

Let $E(X, \succeq, x^0)$ be the *set of policy outcomes* – i.e., the set of all policies $x \in X$ that are the policy outcome of any stable party structure for some function $W \in \mathbf{W}(A, \succeq)$, given x^0.

The comparative statics exercise consists of a change in the distribution of legislators' preferences. Specifically, I analyze the effect on $E(X, \succeq, x^0)$ of a change in the preference profile consisting in an increase (or decrease) in the median legislator's

[47] See section 6.1 for a comparison between such restrictions and *(SSC)*, *(QSM)*.

[48] Namely, given two sets Y, Z, we say that Y is greater than or equal to Z in the strong set order ($Y \geq_s Z$) if for any $y \in Y$ and $z \in Z$ we have $y \vee z \in Y$ and $y \wedge z \in Z$.

type.

Let $\mathbf{\Pi}_s \subseteq \mathbf{\Pi}$ be the set of all possible preference profiles defined on X that satisfy *QSM* and *SSC*. Define a partial order D_X on $\mathbf{\Pi}_s$ such that $(\underset{t}{\succeq}) D_X (\underset{t'}{\succeq})$ and $t \geq t'$ if and only if, for any $x, x' \in X$ with $x \geq x'$, $x \underset{t}{\succeq}^m x' \rightarrow x \underset{t'}{\succeq}^m x'$ and $x' \underset{t'}{\succeq}^m x \rightarrow x' \underset{t}{\succeq}^m x$. Consider a subset $\mathbf{T} \subseteq \mathbf{\Pi}_s$ such that $(\mathbf{T}, D_X)$ is a totally ordered set, and with typical element $\underset{t}{\succeq}$ – i.e., $\mathbf{T} = \{\underset{1}{\succeq}, \underset{2}{\succeq}, ..., \underset{t}{\succeq}, ..., \underset{T}{\succeq}\}$. Then, I can state the following result:

Proposition 3. *(Monotone Comparative Statics). If $x^0 \in I(\succeq^m)$, or if either $x^0 \geq x^m$ or $x^0 \leq x^m$ for all $x^m \in I(\succeq^m)$, then the set of policy outcomes $E(X, \succeq, x^0)$ of the voting game is (i) a sublattice of X which is (ii) monotonic nondecreasing in t on $\mathbf{T}$.*

Proof. See Appendix A.

Notice that if individual preferences can be represented by a function $V: X \times \Theta \times \Phi \rightarrow \mathbb{R}$ that satisfies *(QSM* in x*)*, *(SSC* in x, θ*)*, and *Single Crossing (SC* in x, φ*)*, as defined in Milgrom and Shannon (1994), then these two results correspond to $E(X, \succeq, x^0)$ being monotonic nondecreasing both in the type of the median legislator θ^m on Θ and on the vector of common taste parameters $\varphi \in \Phi$. Moreover, if V is twice differentiable, then these three properties hold if the sign of the cross derivatives of V are all positive.[49]

The results in this section provide a tool to analyze the effects of a change in the distribution of legislators' types and/or of a common shift in legislators' preferences regarding the policy outcome that prevails in any stable party structure.

Lastly, notice that if the conditions on the status quo in Proposition 3 do not hold, but preferences are strictly convex, then we know from Corollary 2 (iii) that the equilibrium policy must lie within a maximum distance from the median legislator's ideal point. This implies that, even in the latter case, at least one element of $E(X, \succeq, x^0)$ exhibits some monotone comparative statics properties.

[49]See Milgrom and Shannon (1994).

3.3 Stable Party Structures and Legislative Gridlock

A second set of results concerns the characterization of stable party structures in this framework. These results can prove useful in understanding the political content of the analytical results in Propositions 1–3 and Corollary 2.

Definition 4.

(i) A *majoritarian party structure* is a tuple $\langle \mathbb{P}, a(\mathbb{P}) \rangle$ such that each $\mathcal{C}^j \in \mathbb{P}$ (active or not) includes either (a) only individuals of type $\theta^i \leq \theta^m$ or (b) only individuals of type $\theta^i \geq \theta^m$.

(ii) An *ends-against-the-middle party structure* is a tuple $\langle \mathbb{P}, a(\mathbb{P}) \rangle$ such that at least one active party $\mathcal{C}^j \in \mathbb{P}$ that includes both (a) individuals of type $\theta^i < \theta^m$ and (b) individuals of type $\theta^i > \theta^m$, but no individual $i \in \mathcal{C}^j$ of type $\theta^i = \theta^m$.

(iii) A *central party structure* is a tuple $\langle \mathbb{P}, a(\mathbb{P}) \rangle$ such that (1) it is not ends-against-the-middle, and such that (2) at least one active party $\mathcal{C}^j \in \mathbb{P}$ includes both (a) individuals of type $\theta^i < \theta^m$ and (b) individuals of type $\theta^i > \theta^m$, plus at least one individual $i \in \mathcal{C}^j$ of type $\theta^i = \theta^m$.

I call a party that includes both (a) individuals of type $\theta^i < \theta^m$ and (b) individuals of type $\theta^i > \theta^m$, but no individual $i \in \mathcal{C}^j$ of type $\theta^i = \theta^m$ a *coalition of extremes.* Using these definitions, I can state the following results.

Proposition 4. *(Stable Party Structures). Suppose either $x^0 \geq \bar{x}^m$ or $x^0 \leq \underline{x}^m$ is true. (i) If $I(\succeq^m)$ is a singleton, then any majoritarian or central party structure $\langle \mathbb{P}, a(\mathbb{P}) \rangle$, including the grand coalition of all legislators $\langle \{\mathcal{N}\}, a(\{\mathcal{N}\}) \rangle$, always supports a party equilibrium $\langle \mathbb{P}, a(\mathbb{P}), x^m \rangle$. (ii) If $I(\succeq^i) \cap I(\succeq^m) = \emptyset$ for all $i \in \mathcal{N}$ such that $\theta^i \neq \theta^m$, then no ends-against-the-middle party structure $\langle \mathbb{P}, a(\mathbb{P}) \rangle$ supports a party equilibrium.*

Proof. See Appendix B.

A consequence of Proposition 4 is that no proposal $a^j \notin I(\succeq^m)$ made by a coalition of extremes is ever implemented. This proposition also provides an intuitive understanding of the party structure that underpins the median voter result in Corollary 2

– that is, in this framework, a proposal that can defeat x^m and that does not generate instability in the party structure is *credible* – i.e. it is in the Pareto set of a party – only if it is made by a coalition of extremes. But such a proposal is never *feasible* for a coalition of extremes whenever the other parties are inactive, in the sense that no conditional group choice function that satisfies monotonicity and neutrality can be consistent with such a proposal. Thus, there is no party equilibrium that violates the quasi-median voter theorem. This result is coherent with the idea, discussed in Hussey (2008), that parties made up of individuals with opposite political views are less likely to be stable and less capable of putting forward successful reforms.

Proposition 5. *(Legislative Gridlock). If $x^0 \notin I(\succeq^m)$, then (i) in any stable legislative gridlock $\langle \mathbb{P}, a(\mathbb{P}), x^0 \rangle$, the partition $\mathbb{P}$ is either an ends-against-the-middle or a central party structure; (ii) no stable legislative gridlock $\langle \mathbb{P}, a(\mathbb{P}), x^0 \rangle$ exists if $\langle \mathbb{P}, a(\mathbb{P}) \rangle$ is a majoritarian party structure.*

Proof. See Appendix B.

The interpretation of Proposition 5 is that if an active coalition of extremes exists and it is part of a stable party structure, then its role is to block the attempts of other parties to implement a reform.

Propositions 4–5 deliver a rich characterization of the kind of party structures that can be stable in different scenarios. For illustrative purposes, it is useful to analyze a simple example shown in *Fig. 3*. In this example, the policy space is $X = [0, 1]^2$. There are five players – i.e., $\mathcal{N} = \{1, 2, 3, 4, 5\}$. The median legislator is $i = 3$. Each legislator has a unique ideal point represented by a large black dot. Each party is represented as the set of the ideal points of its members. For instance, *Fig. 3.1* represents the partition $\mathbb{P} = \{\{1, 2, 3\}; \{4, 5\}\}$.

The partitions in *Figures 3.1*, *3.2*, and *3.3* can sustain a stable party structure (and a party equilibrium in particular) because they represent either a majoritarian (*3.1* and *3.3*) or central (*3.2*) party structure. Notice that *Fig. 3.3* illustrates a case in which each party is a singleton. In this case, the analysis resembles the one of a

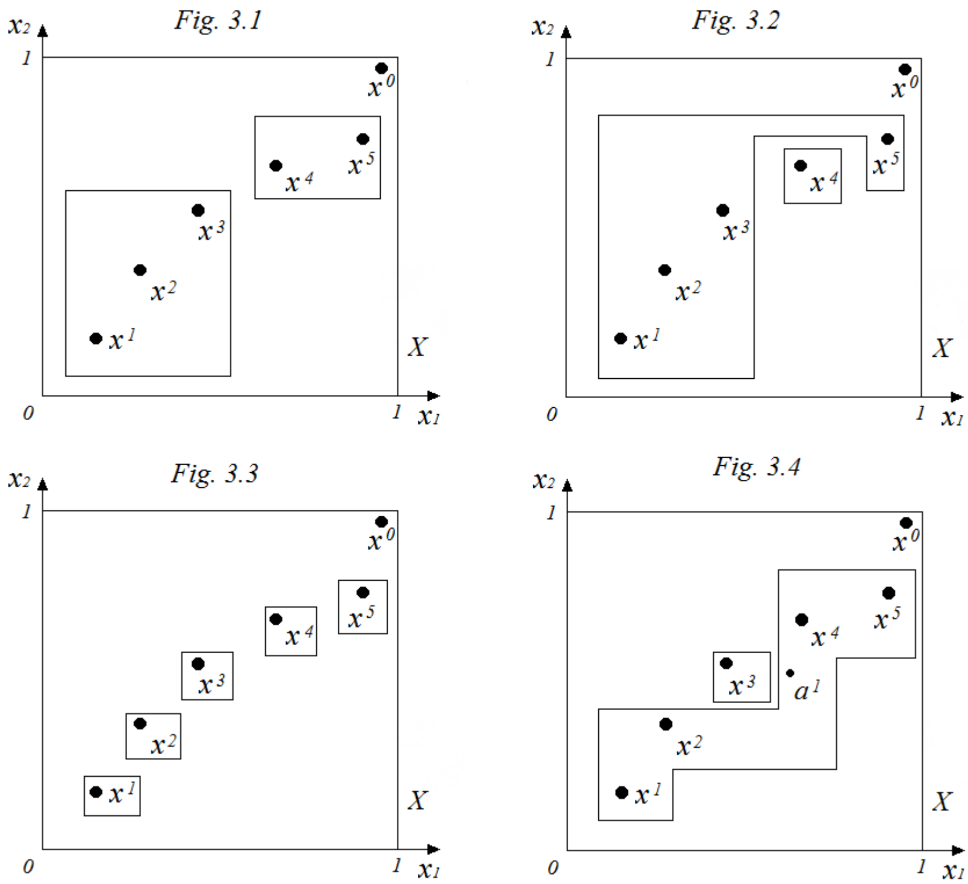

Figure 3. Illustration of four party structures in the policy space $[0,1]^2$.

citizen–candidate model (Osborne and Slivinski 1996, Besley and Coate 1997), in the sense that each legislator can only commit to his/her own ideal policy. Thus, it is equivalent to a model with no political parties. The relationship between the model of electoral competition proposed in this paper and the citizen–candidate model is further discussed in section 6.2.

Lastly, *Fig. 3.4* represents a case in which there is an active coalition of extremes, and therefore the party structure is ends-against-the-middle. The coalition of extremes proposes a^1. Proposition 4 implies that such a structure does not support a party equilibrium.[50]

This example illustrates that in the model, an active coalition of extremes is part of a stable party structure only if a legislative gridlock has occurred. Moreover,

[50] In this case, the party structure does not support a stable legislative gridlock either, because such an outcome requires a minimum of three active parties and because the position of x^0 does not satisfy the conditions in Proposition 1.

it suggests that party structures that can support reforms tend to be in a *left vs. right* form (majoritarian party structures), or to resemble a grand coalition with or without extremist fringes (central party structures). Other configurations – in particular, ends-against-the-middle party structures – are either not stable or imply a legislative gridlock.

The political science literature provides some support for this prediction. Specifically, there is empirical evidence that a coalition of extremes within an elective body, sometimes referred to as COFEX voting behavior in the literature (Hussey 2008), is not a common outcome. For instance, Poole and Rosenthal (1997) analyze the history of voting behavior of the members of Congress and provide evidence that COFEX voting "does not appear very often," and it is documented only "in very unusual circumstances." Their results show little COFEX activity throughout Congress, and further reduction after 1950.

Lastly, notice that in a stable legislative gridlock parties are effective, because the policy outcome x^0 differs from x^m, which prevails with no parties. Moreover, the presence of parties has welfare effects. Namely, in a gridlock the members of an active coalition of extremes are made better off relative to the case in which no parties exist. Conversely, the median legislator is made worse off.

4 Examples

EXAMPLE 2 (Non-convex policy space: Complementary Reforms). Suppose N legislators (N odd) face a discrete choice over a finite two-dimensional set of reform alternatives. The policy space is $X \subseteq \mathbb{R}^2$ with typical element (x_1, x_2) such that $(X, \succeq)$ is a lattice. The status quo is (x_1^0, x_2^0). A legislator i has preferences over X, represented by the preference ordering $\succeq^i$.

The two dimensions of policy reform are (weak) complements, in the sense that a reform that affects one policy dimension is more beneficial (or less harmful) for each legislator whenever the reform also moves the other policy dimension in the same direction. Formally, $(x_1', x_2) \succeq^i (x_1', x_2')$ implies $(x_1, x_2) \succeq^i (x_1, x_2')$ for any $(x_1, x_2) \geq (x_1', x_2')$ and for any $i = 1, 2, ..., N$.

Lastly, legislators' preferences are ordered in intensity such that, for any $(x_1, x_2) \geq (x_1', x_2')$, if $(x_1, x_2) \succeq^k (x_1', x_2')$, then $(x_1, x_2) \succeq^j (x_1', x_2')$ for all $j \geq k$.

What bundle of reforms (if any) will be chosen in the presence of majoritarian institutions? What coalition of legislators supports the reform? Under which conditions is the *status quo* maintained?

Solution. Each individual i is endowed with a different type θ^i in this example. Preferences satisfy *(QSM in* (x_1, x_2)*)* and *(SSC in* $(x_1, x_2), \theta^i$*)*.

All the conditions stated in section 2 are satisfied. As a result, Proposition 1 implies that the policy (x_1^*, x_2^*) chosen in equilibrium is the ideal policy (x_1^m, x_2^m) of median legislator m whenever $(x_1^0, x_2^0) \geq (x_1^m, x_2^m)$ or $(x_1^0, x_2^0) \leq (x_1^m, x_2^m)$. The status quo is maintained only if either $(x_1^0, x_2^0) \ngeq (x_1^m, x_2^m)$ or $(x_1^0, x_2^0) \nleq (x_1^m, x_2^m)$, or $(x_1^0, x_2^0) = (x_1^m, x_2^m)$. Lastly, because $I(\succeq^i) \cap I(\succeq^j) = \emptyset$ for all $i \neq j$, Proposition 4 (ii) implies that a stable party structure that supports a reform (if any exists) can be either majoritarian or central, but cannot be in the ends-against-the-middle form.

EXAMPLE 3 (A reform supported by a large majority of legislators).[51] Consider an assembly consisting of 7 legislators. The policy space is $X = \mathbb{R}^2$. Legislators' preferences are Euclidean. The ideal points of the legislators are: $x^1 = (-4, -5)$, $x^2 = (-4, -4)$, $x^3 = (0, -27/28)$, $x^4 = (0, 0)$, $x^5 = (2, 0)$, $x^6 = (2, 7/2)$, $x^7 = (8, 4)$. The status quo is $x^0 = (1, -1)$. Notice that the ideal point of the median legislator x^4 is strictly preferred to the status quo by a large majority of legislators. Specifically, legislators 2, 3, 4, 6 strictly prefer x^4 to x^0, legislator 1 is indifferent between the two alternatives, and only legislators 5 and 7 strictly prefer x^0 to x^4. Can a policy reform that enjoys such a large support be blocked?

Solution. The answer is yes. Consider the party structure $\mathbb{P} = \{\{1, 7\}; \{2\}; \{3\}; \{4\}; \{5\}; \{6\}\}$ and the vector of policy proposals $\{a^1; \emptyset; \emptyset; x^4; x^5; \emptyset\}$. It is easy to verify that such proposals are optimal for all legislators and parties. Specifically, parties $\{2\}$,$\{3\}$, and $\{6\}$ are inactive because each of them cannot change the policy outcome by proposing the ideal policy of its unique member. Party $\{4\}$ optimally proposes

[51] I would like to thank Saumya Deojain for suggesting this example.

x^4 because its unique member prefers x^0 to the policy outcome that would prevail if the party becomes inactive, which is x^5. Similarly, party $\{5\}$ optimally proposes x^5 because $x^0 \succ^5 a^1$. Lastly, party $\{1, 7\}$ optimally proposes a^1 because both members prefer x^0 to x^4.[52] Moreover, neither 1 nor 7 has a strictly incentive to quit the party, because such a choice would lead to a policy outcome x^4, which both consider weakly worse than x^0. Thus, $\langle \mathbb{P}, \{a^1; x^4; x^5\}, x^0 \rangle$ is a stable legislative gridlock. This implies that the reform proposed by the median legislator does not gain a stable support and the status quo is maintained.

OTHER EXAMPLES (Comparative Statics). The comparative statics results in Proposition 3 have been used in two companion papers – based on a highly simplified version of this setting – to analyze popular Political Economy questions, such as the relationship between income inequality and size of government (Dotti 2019) and the determinants of restrictive immigration policies (Dotti 2016). These and other applications of this framework are briefly described in section 6.3.

5 Robustness

Credibility of Proposals. In models of elections (Roemer 1999, Levy 2004) the assumption that a party's platform must lie within the Pareto set of its members is typically justified on the grounds of commitment issues. Specifically, Levy (2004) argues that, if there exists a feasible alternative that can make all the members of the winning party better off, then after the elections all the party members have a strict incentive to renegotiate their pre-electoral agreement. In turn, this implies that voters should anticipate that such platform is never going to be implemented by the party that proposes it, i.e. the platform is not credible. One may claim that such a restriction is harder to justify in a model of parties within a legislature, in which such commitment issues are less obvious. Nevertheless, it is easy to show that all the results in Propositions 1, 3, 4, 5 and Corollary 2 carry over if a much weaker

[52]Notice that given $a^{-1} = \{\emptyset; \emptyset; x^4; x^5; \emptyset\}$, no proposal in the Pareto set of party $\{1, 7\}$ can make either of its two members strictly better off relative to a^1. Thus, proposal a^1 is consistent with a number of neutral and monotonic conditional group choice functions, including dictatorial ones.

restriction is imposed. Specifically, I propose the following alternative assumption. I define the set of *credible proposals* $\widetilde{X}^j$ for party $\mathcal{C}^j$ as follows.

$$\widetilde{X}^j := \left\{ x \in X \;\middle|\; \nexists x' \in X \; s.t. \; \begin{array}{c} [x' \succeq^i x \; \forall i \in \mathcal{C}^j] \cap \\ \left[x' \succ^k x \; for \; some \; k \in \mathcal{C}^j\right] \cap \\ [x' MV(\succeq) x] \end{array} \right\}$$

This assumption states that a proposal x by party $\mathcal{C}^j$ is not credible if (a) it is strictly Pareto dominated by an alternative x' and (b) it is defeated by x' by majority voting. In other words, policy x it is not credible proposal for party $\mathcal{C}^j$ because if it gains the support of the assembly, then the members of the proposing party have a strict incentive to bring to the floor another proposal that defeats x. Thus, this alternative assumption only excludes proposals that are likely to be immediately amended by the proponents as soon as they obtain the support of the assembly.

Moreover, this notion of credibility is much less demanding than the one that underpins the definition of X^j. In particular, it is easy to show that $\widetilde{X}^j \supseteq X^j$. Details on the proof to this robustness result are provided in the online appendix to this paper.

Party mergers. One aspect that has not yet been considered in this analysis is the possibility of deviations that consist in the merger of two or more parties. A few examples of party mergers are documented in the political science literature examining democratic countries (Mair 1990). One may wonder what happens in this model if one allows for deviations of this sort.

Formally, suppose a deviator can also be a group of parties $\mathbb{C} \subseteq \mathbb{P}$ with typical element $\mathcal{C}^d \in \mathbb{C}$ that induce a new partition $\mathbb{P}'$ such that $\mathcal{C}^m = \cup_{\mathcal{C}^d \in \mathbb{C}}$ and $\mathcal{C}^m \in \mathbb{P}'$. Such deviation is profitable if it sequentially leads to a tuple $\langle \mathbb{P}', a' \rangle$ that is stable and such that $w(A', \succeq) \succ^i w(A, \succeq)$ for all $i \in \mathcal{C}^m$. Lastly, if one allows for mergers, there is the possibility of an infinite sequence of deviations. In such a case, I assume that the deviation is not profitable.[53] Under these assumptions, I can state the following result.

[53]This can be formalized by assuming that the outcome of instability – denoted by x^D – is a condition that legislators strongly dislike – i.e., $x \succ^i x^D$ for all $x \in X \cup \{x^0\}$ and all $i \in \mathcal{N}$.

Proposition 6. *(Mergers of Parties). If either $x^0 \in I(\succeq^m)$, or $x^0 \geq \bar{x}^m$, or $x^0 \leq \underline{x}^m$, then (i) a stable party structure such that the policy outcome is the ideal policy of the median legislator always exists. If (ii) $I(\succeq^m)$ is a singleton, then a two-party majoritarian party structure is stable.*

Proof. See Appendix C.

Proposition 6 states that, even if merger are allowed, an equilibrium that satisfies the quasi-median voter theorem (and therefore possesses monotone comparative properties) always exists. Moreover, if the ideal policy of the median legislator is unique, then any majoritarian two-party structure is stable. This result implies that in a political system in which party aggregation is possible and not costly, legislators tend to sort themselves into each party according to their position in the left vs. right preference dimension. In such a case, a majoritarian two-party structure is always a feasible (and possibly likely) outcome.

Final Approval Vote. Consider an alternative legislative process in which any winning proposal w in the core $K(A, \succeq)$ must be approved in a final vote against x^0 before becoming the policy outcome. All the predictions of this analysis are unaffected as long as the median ideal policy is preferred to the status quo by a majority of individuals ($x^m MV(\succeq) x^0$), as assumed throughout the paper. In particular, this is always the case whenever the ideal point of the median voter is higher or lower than the status quo, as in the motivating example (Case 1) in section 1.2.

If condition ($x^m MV(\succeq) x^0$) does not hold, then the results carry over if one assumes that the group choice function W selects the winning proposal rather than the policy outcome, and conditional preferences $R^{j,i}$ rank proposals according with the outcome of W.[54] All the results carry over under this new set of assumptions, except Proposition 1 part (ii), which becomes *a party equilibrium such that the winning proposal is an ideal point of the median legislator m always exists.* This statement capture the fact that the reform preferred by the median legislator is not approved in the final vote. Thus, the policy outcome is x^0.

[54] Notice that this assumption is not consistent with purely policy-motivated legislators. It can be rationalized, for instance, with ideological considerations that make legislators care about their party proposal, rather than about the final outcome of the legislative process.

Lastly, the main results in Propositions 1, 3, 4, and 5, and Corollary 2, are also robust, at least to some extent, to a number of (i) alternative assumptions on party members' behavior, (ii) refinements of the stability concept, and (iii) specific types of preference perturbations. These robustness results are extensively described in an online appendix.

6 Discussion

In this section, I discuss in further detail some aspects that have been mentioned in the previous sections: (1) how restrictive the assumptions in this paper are relative to those imposed in traditional voting models; (2) the similarities and differences between the model of political parties proposed in this paper and some popular alternatives in the theoretical literature; and (3) the existing and potential applications of this theoretical framework.

6.1 Preference Restrictions

The main result of this paper consists in a quasi-median voter theorem (Corollary 2). How restrictive are the restrictions of *(QSM)* and *(SSC)* relative to those that are sufficient for the traditional median voter theorem to hold with no political parties? In short, they are much less restrictive.

Several different conditions have been shown to be sufficient for a median voter theorem to hold, such as single-peakedness (Black 1958), the single crossing condition (Gans and Smart 1996), and top monotonicity (Barberà and Moreno 2011). All these concepts rely on the existence of a linear order on the set of alternatives that is consistent with a corresponding ordering in voters' preferences.

For instance, consider a total order $>'$ on X and a total order $>''$ on $\mathcal{N}$. The (unidimensional) single crossing condition *(USC)* requires that for all $i, j \in \mathcal{N}$ such that $j >'' i$ and all $x, y \in X$ such that $y >' x$ the voters' preferences satisfy $y \succeq^i x \rightarrow y \succeq^j x$ and $y \succ^i x \rightarrow y \succ^j x$. Similarly, single-peakedness *(SP)* over X – assumed in Levy (2004) – requires that for a given peak $p^i(X) \in X$ of each voter i, $[p^i(X) >' y >' x \text{ or } x >' y >' p^i(X)] \rightarrow y \succ^i x$. The following example illustrates how much more restrictive *(USC)* and *(SP)* are relative to *(SSC)* over multidimensional choice domains.

Consider the lattice $(X, \geq)$ with $X = \{x \vee y, x, y, x \wedge y\}$, $x \not\geq y$, and $x \not\leq y$. Because x and y are not ordered (with respect to $\geq$), no restriction on the voters' preferences between x and y is required for *(SSC)* to hold. Conversely, both *(USC)* and *(SP)* require restrictions on the preferences between x and y.[55] In other words, *(USC)* and *(SP)* restricts preferences between any pair of elements in X. Conversely, *(SSC)* restricts preferences solely between each pair of totally ordered elements of the partially ordered set X.

On the other hand, I impose the additional restriction *(QSM),* which consists in a form of weak complementarity between each two policy dimensions (Milgrom and Shannon 1994). Albeit rather restrictive, *(QSM)* is satisfied in large number of economic applications, such as the four examples illustrated in sections 1.1 and 4 of this paper, and those mentioned in section 6.3.

6.2 Relationship with Alternative Models

Citizen–Candidate Models. The model proposed in this paper is closely related to the class of models of elections with citizen-candidates (Osborne and Slivinski 1996, Besley and Coate 1997). These models can address the problem of multidimensionality by restricting the commitment capability of politicians. The key assumption is that each citizen can run for elections as a candidate, and that each candidate can credibly commit only to his/her ideal point. In Besley and Coate's (1997) model, if there is a strict Condorcet winner in the set of all citizens' ideal points, an equilibrium exists[56] in which a single citizen runs unopposed. Notice that if preferences are strictly convex and satisfy *QSM* and *SSC*, then the condition is satisfied. Thus, the prediction of such an equilibrium of the citizen–candidate model is identical to that of the finest stable party structure in the model presented in this paper.

[55] More generally, single-peakedness is an extremely restrictive assumption over multidimensional choice domains. See Kramer (1973).

[56] The Condorcet winner must be different from the default policy, defined as the policy that is implemented if no candidate runs for election. Moreover, the cost of running for election must be sufficiently small. See Corollary 2 (ii) in Besley and Coate (1997).

However, there are substantial differences between the two frameworks in terms of other possible outcomes. First, even if there is a *Condorcet winner* over the set of ideal points, there may be other equilibria of the citizen–candidate model in which the median ideal policy does not prevail.[57] Second, an outcome in the form of a legislative gridlock can be a stable party structure, but it is never an equilibrium in Besley and Coate's (1997) framework.

Levy's Models of Endogenous Political Parties. The theoretical framework I propose possesses several similarities with that in Levy (2004), which investigates the role of parties in elections. The two models differ in four key aspects. First, I require the parties' internal choice protocol to satisfy some basic properties (monotonicity and neutrality). Second, I assume that individual preferences are ordered – i.e., they satisfy *(SSC),* but they do not need to be single-peaked. Third, I allow for the status quo to be an alternative that could be appealing for some voters. Fourth, I depart from the assumption of sincere voting.

Thus, the model proposed in this paper is more restrictive than Levy's framework in terms of the conditions on the internal party process, but it is more general with respect to individual preferences,[58] and it allows for strategic voting behavior. Moreover, it delivers a sharper characterization of the stable party structures, the proposals made by each party, and the policy outcome of the political process.

In particular, in Levy's model, a coalition of extremes proposing a policy platform that is not an ideal point of the median citizen may be a winning party in some equilibria, while in this paper, there is no party equilibrium featuring an ends-against-the-middle party structure (see Proposition 4).

Such equilibria of Levy's model are sometimes interesting for policy-relevant applications – e.g., the one in Levy (2005). On the other hand, their existence crucially depends on the ability of parties to select their platforms using a choice procedure that violates neutrality (see sections 1 and 2.2). Moreover, the empirical literature deems such ends-against-the-middle structures less likely to propose successful reforms (see section 3.3).

[57] For instance, there may be equilibria in which two (or more) candidates run for elections and each of them wins with positive probability.

[58] See section 6.1.

Lastly, if individual preferences satisfy the conditions of both models (*QSM*, *SSC*, and single-peakedness), then the outcomes of the two models are comparable. In such a case, there is a subset of equilibria in Levy's model that shares the same properties described in Propositions 1–3 and Corollary 2.

6.3 Applications

The comparative statics results in Proposition 3, which may sound unsurprising given the assumptions on preferences, can prove useful in addressing a number of questions in political economy. For instance, they can be applied to revisit some popular questions in the theoretical literature.

Several traditional studies assume a unidimensional policy space in order to exploit the useful properties that deterministic voting models exhibit under such an assumption. Unfortunately, restrictions on the dimensionality of the choice set usually affect the predictions of those studies. Haupt and Peters (1998) show that in some popular application of the unidimensional deterministic voting model, the results are *entirely* driven by the restrictions on the policy space. Typical examples in which this issue may be relevant are the analysis of the determinants of (i) the size of redistributive policies (Meltzer and Richard 1981), (ii) the degree of restrictiveness of immigration policies (Razin and Sadka 1999), and (iii) the relative tax rates on labor and capital income (e.g., Bassetto and Benhabib 2006).[59]

An approach similar to the one proposed in this paper has proved useful in addressing questions of type (i) and (ii) in two companion papers (Dotti 2016, 2019). Moreover, it offers promising perspectives in terms of possible extensions. This claim relies on the preference restrictions that underpin the model, which are borrowed from the literature examining generalized comparative statics (Milgrom and Shannon 1994). This literature has been largely exploited to analyze the equilibrium properties of games characterized by *strategic complementarities*. A typical result of this literature is that in this class of games, even if there may be multiplicity of equilibria, some monotone comparative statics properties are preserved. Thus, this framework

[59]For instance, in the latter example, the authors impose strong restrictions on the policy space in order to deal with the the intrinsic bi-dimensional nature of the problem. This modeling choice plays an important role in shaping the predictions of their model.

represents a promising starting point to tackle a different set of questions.

For instance, a natural extension of this study is the analysis of political processes in which competing institutions – e.g., the parliaments of two different countries – simultaneously decide their policies, and legislators' strategies across different institutions are strategic complements. Examples in which such interdependence may arise include legislative choices over immigration policies and corporate tax policies.

Lastly, the model of political parties proposed in this paper admits an alternative interpretation. Namely, if the players at the voting stage are electors rather than legislators, and parties choose platforms instead of policy proposals, then the structure resembles that of a model of election under proportional representation with post-election legislative bargaining (as in Bandyopadhyay *et al.* 2011). In such a case, voters can perfectly anticipate the composition of the legislature that is going to prevail after the elections. Thus, they can exploit political parties as a tool to induce post-election disagreement in the legislature by inducing parties to commit to policy platforms that result in a legislative gridlock.

7 Concluding remarks

This paper contributes to the literature on endogenous political parties within an assembly in the presence of ordered legislators' preferences. The main result is that the presence of parties may affect policy outcomes by blocking reforms that would be implemented in the absence of such institutions. On the other hand, the extent to which parties can increase the set of possible policy reforms is limited or null.

This core result follows three important findings. First, I show that if the status quo is (weakly) lower or higher than the ideal policy of the median legislator, then a median voter theorem holds and parties are not effective. Second, I show that parties do matter for policy only if the position of the status quo in the policy space is neither higher nor lower than the ideal policy of the median legislator. The distance between these two objects determines how much parties can matter. Specifically, the closer the median legislator's preferred outcome is to the status quo, the less parties can affect policies. Third, I provide a general characterization of the party structures that can prevail in a stable party structure.

The main limitation of this analysis is that its range of application is restricted to a class of problems in which legislators' preferences possess some rather restrictive ordinal properties. While this represents a large (and allegedly interesting) class of questions, much research is needed in order to understand the role of political parties in shaping policies when legislators' preferences differ along multiple (and possibly orthogonal) dimensions.

The model is characterized by a sufficient degree of flexibility, and its predictions, summarized by the quasi-median voter result, are easy to interpret. These desirable features make this framework potentially suitable to analyze several questions in political economy. Many possible extensions of this framework offer promising perspectives, so there is large scope for future research.

Appendices

A Proof of Main Result

The proof to Proposition 1 and Corollary 2 proceeds as follows. First, I prove Lemmas 7, 8, 9, 10a, 10b, 11, 12, 13a, 13b, 14, 15. Then, I use these results to prove Proposition 1 and Corollary 2.

Lemma 7. *If $\succeq$ satisfies QSM and SSC, then the Pareto Set X^j of a party $\mathcal{C}^j \subseteq \mathcal{N}$ is such that either (i) $X^j = I(\succeq^i)$ for some $k \in \mathcal{C}^j$, or (ii) $y \in X^j$ only if $y \geq \sup\left\{I(\succeq^l)\right\}$ and $y \leq \inf\left\{I(\succeq^h)\right\}$ where $l = \min\{\mathcal{C}^j\}$ and $h = \max\{\mathcal{C}^j\}$.*

Proof. Case (i). First of all, notice that - from the definition of the Pareto set - a policy $x \in X^j$ for party $\mathcal{C}^j$ if and only if $\nexists x' \in X$ such that $x' \succeq^i x \ \forall i \in \mathcal{C}^j$ and $x' \succ^i x$ for at least one $k \in \mathcal{C}^j$. Case (i). Suppose $\theta^i = \hat{\theta}$ for some $\theta^k \in \Theta$ and for all $i \in \mathcal{C}^j$. Then $X^j = I(\succeq^k)$ trivially holds. Case (ii). Suppose there is at least two players $i, k \in \mathcal{C}^j$ such that $\theta^i \neq \theta^k$. Denote with $\bar{x}^l = \sup\left\{I(\succeq^l)\right\}$ and $\underline{x}^h = \inf\left\{I(\succeq^h)\right\}$. Notice that by Milgrom and Shannon (1994), $I(\succeq^i)$ is a complete sublattice of X for all $i \in \mathcal{C}^j$, which implies in turn that $\bar{x}^l \in I(\succeq^l)$ and $\underline{x}^h \in I(\succeq^h)$. Suppose $y \ngeq \overline{x}^l$ but $y \in X^j$. Because X is a lattice, $y \wedge \overline{x}^l, y \vee \overline{x}^l \in X$. Thus, because of the optimality of $\overline{x}^k$, it must be true that $\overline{x}^l \succeq^l y \wedge \overline{x}^l$. *QSM* implies $y \vee \overline{x}^l \succeq^l y$. Notice that $y \ngeq \overline{x}^l$ implies $y \vee \overline{x}^l \neq y$. Hence *SSC* implies $y \vee \overline{x}^l \succ^i y$ for all $i \in \mathcal{C}^j$ such that $\theta^i > \theta^l$. Set $x' = y \vee \overline{x}^l$. Given that $\theta^i > \theta^l$ is true for at least one $i \in \mathcal{C}^j$, one gets that $\exists x' \in X$ such that $x' \succeq^i y \ \forall i \in \mathcal{C}^j$ and $x' \succ^k y$ for at least one $k \in \mathcal{C}^j$, which from the definition of Pareto set implies $y \notin X^j$, which leads to a contradiction. Similarly one can show that $y \in X^j$ only if $y \leq \underline{x}^h$. Q.E.D.

Corollary 7b. *If $x^l \in I(\succeq^l)$ and $\theta^l < \theta^k$ (or $x^h \in I(\succeq^h)$ and $\theta^h > \theta^k$) for all $k \in \mathcal{C}^k$, then $x^l \notin X^k$ unless $x^l = \bar{x}^l$ ($x^h \notin X^k$ unless $x^h = \bar{x}^h$).*

Proof. Suppose the opposite is true, $x^l \neq \bar{x}^l$ and $x^l \in X^k$. The definition of $\bar{x}^l$ implies $\bar{x}^l \geq x^l$. Optimality implies $\bar{x}^l \succeq^l x^l$. The *SSC* implies $\bar{x}^l \succ^k x^l$ for all k such that $\theta^l < \theta^k$. Thus, $x^l \notin X^k$. Similarly, one can show that the same result holds for x^h. Q.E.D.

Lemma 8. *A conditional group choice function $g^j_{a^{-j}}$ that satisfies (i) Neutrality, (ii) Monotonicity,* and *(iii) TB1, TB2 always exists.*

Proof. Consider a group choice function that is *lexicographic dictatorial* (1), Consider a total order $\geq^s$ on set $\mathcal{C}^j$, such that $\mathcal{C}^j(\geq^s) = (\mathcal{C}^j, \geq^s)$ is totally ordered. Define the sequence $\{O^1(R^j(a^{-j})), O^2(R^j(a^{-j})), ..., O^{N^j+1}(R^j(a^{-j}))\}$ in which $O^1(R^j(a^{-j})) = X^j$ and in which each element $O^s(R^j(a^{-j}))$ for $s \geq 2$ is defined recursively as $O^{s+1}(R^j(a^{-j})) := \{x \in O^s(R^j(a^{-j}))| \; xR^{j,s}(a^{-j})x' \; \forall x' \in O^s(R^j(a^{-j}))\}$. Then for $x \in X^j$, $g^j_{a^{-j}}(R^j(a^{-j})) = x$ only if $x \in O^{\bar{s}}(R^j(a^{-j}))$ where $\bar{s}$ is the highest s such that $O^s(R^j(a^{-j})) \neq \{\emptyset\}$. I check that the group choice function satisfies (i), (ii) and (iii). For (i), consider a permutation $\rho : \; X^j \to X^j$ and a preference profile $R^j_\rho(a^{-j})$ such that $xR^{j,i}_\rho(a^{-j})x'$ if and only if $\rho(x)R^{j,i}(a^{-j})\rho(x')$. Recall $g^j_{a^{-j}}$ is neutral if for any ρ and any $R^j(a^{-j})$ one gets $\rho(g^j_{a^{-j}}(R^j_\rho(a^{-j}))) = g^j_{a^{-j}}(R^j(a^{-j}))$. Suppose it does not. Then there is R^j, a^{-j} such that $\rho(g^j_{a^{-j}}(R^j_\rho(a^{-j}))) \neq a^j$ (1). Because $g^j_{a^{-j}}$ is dictatorial, $g^j_{a^{-j}}(R^j(a^{-j})) = a^j$ with $a^jR^{j,k}(a^{-j})x'$ for all $x' \in X$. Define ρ such that $x = \rho(\hat{x})$ for all $x \in X^j$ and such that $\hat{x} \in X^j$. Notice that $\hat{x}'R^{j,i}_\rho(a^{-j})\hat{x}$ if and only if $x'R^{j,i}(a^{-j})x$. This implies $g^j_{a^{-j}}(R^j_\rho(a^{-j})) = \hat{a}^j$, where $\hat{a}^j$ solves $a^j = \rho(\hat{a}^j)$, which implies $\rho^{-1}(a^j) = \hat{a}^j$. Thus, $\rho(g^j_{a^{-j}}(R^j_\rho(a^{-j}))) = \rho(\rho^{-1}(a^j)) = a^j$, which represents a contradiction of (1). Thus, $g^j_{a^{-j}}$ satisfies *N*. For part (ii), consider two policies $x, x' \in X^j$, a party $\mathcal{C}^j \subseteq \mathcal{N}$ and two (not necessarily disjoint) sub-parties $\mathcal{C}^k \subseteq \mathcal{C}^j$ and $\mathcal{C}^l \subseteq \mathcal{C}^j$. Also consider two conditional preference profiles $R^j(a^{-j}), \hat{R}^j(a^{-j}) \in \mathbf{R}^j$ such that $x'R^{j,i}(a^{-j})x \to x'\widehat{R}^{j,i}(a^{-j})x$, $x'P^{j,i}(a^{-j})x \to x'\widehat{P}^{j,i}(a^{-j})x$, $x\widehat{R}^{j,i}(a^{-j})x' \to xR^{j,i}(a^{-j})x'$ and $x\widehat{P}^{j,i}(a^{-j})x' \to xP^{j,i}(a^{-j})x'$ for all $i \in \mathcal{C}^j$. Then if $g^j_{a^{-j}}(\hat{R}^j(a^{-j})) = x$ it must be true that $g^j_{a^{-j}}(R^j(a^{-j})) \neq x'$. Suppose $g^j_{a^{-j}}(R^j(a^{-j})) = x'$. This implies either (a) $x'R^{j,s}(a^{-j})x$ for $s = 1, 2, ..., k-1$ and $x'P^{j,k}(a^{-j})x$, or (b) $x'R^{i,s}(a^{-j})x$ for all $i \in \mathcal{C}^j$. In case (a), using the definition of monotonicity, $x'\hat{R}^{j,s}(a^{-j})x$ for $s = 1, 2, ..., k-1$ and $x'P^{j,k}(a^{-j})x$, which implies $g^j_{a^{-j}}(R^j(a^{-j})) \neq x$. In case (b), using the definition of monotonicity, $x'\hat{R}^{j,s}(a^{-j})x$ for $s = 1, 2, ..., \tilde{k}-1$ for some $\widetilde{k} \in \mathcal{C}^j(\geq^s)$ and $x'P^{j,\tilde{k}}(a^{-j})x$, which implies $g^j_{a^{-j}}(R^j(a^{-j})) \neq x$. Thus, in both cases there is a contradiction. Regarding (iii), one can impose condition *TB1* without causing a violation of (M) and (N), because conditions *TB1* affects group choices only between a policies $x \in X$ and the choice of being inactive ($a^j = \emptyset$), in which the latter does not enter the definition of M and N. Lastly, *TB2* does not causes neither a violation of (N) because it affects the choice only in cases in which all the members of the party agree on the ranking of two options, nor of (M). Thus, the group choice function $g^j_{a^{-j}}$ satisfies (i), (ii), and (iii). Q.E.D.

Lemma 9. *In any regular equilibrium of the proposal game (if any exists) only one party proposes a policy $a^j \neq \emptyset$.*

Proof. Suppose that the tuple $\langle \mathbb{P}, a(\mathbb{P}) \rangle$ is a regular equilibrium of the proposal game in which more than one party propose a policy. i.e. $\#A(\mathbb{P}) > 1$. In a regular equilibrium $W(A, \succeq) = a^j$ for some $j \in \{1, 2, ..., J_{\mathbb{P}}\}$ and $a^j \in K(A, \succeq)$. This means $a^j MV(\succeq) a^k$ for any $a^k \in A$. This implies that $a^j \in K(A \backslash \{a^k\}, \succeq)$ for $a^k \neq a^j$. The assumption *RSP* in this case implies $W(A \backslash \{a^k\}, \succeq) = a^j$. Assumption *TB1* states that if $w(\{\emptyset\} \cup A \backslash \{a^k\}, \succeq) = w(\{a^k\} \cup A \backslash \{a^k\}, \succeq)$, then $g^k_{a^{-k}}(R^k(a^{-k})) \neq a^k$. This implies that $\langle \mathbb{P}, a(\mathbb{P}) \rangle$ is not an equilibrium of the proposal game, which leads to a contradiction. Therefore, in a regular equilibrium of the proposal game it must be true that $A(\mathbb{P})$ is a singleton. Q.E.D.

Lemma 10. *There is no regular equilibrium of the proposal game in which $\mathcal{C}^j$ is such that $a^j \notin I(\succeq^m)$ and $a^j \not\succeq \bar{x}^m$, $a^j \not\preceq \underline{x}^m$, and $a^j \neq \emptyset$.*

Proof. In a regular equilibrium of the proposal game the winning policy must be feasible. Suppose this is the case, then there is $\mathcal{C}^j \in \mathbb{P}$ such that $g^j_{a^{-j}}(R^j(a^{-j})) = a^j$, with $a^j \not\succeq \bar{x}^m$, $a^j \not\preceq \underline{x}^m$, and $a^j \neq \emptyset$. Lemma 7 implies that this can be the case only if $\exists l, h \in \mathcal{C}^j$ such that $\theta^l < \theta^m$ and $\theta^h > \theta^m$. First, consider $a^j \vee \underline{x}^m$. Optimality implies $\underline{x}^m \succeq^m a^j \vee \underline{x}^m$. *QSM* implies $a^j \wedge \underline{x}^m \succeq^i a^m$. Condition $a^j \not\preceq \underline{x}^m$ imply $a^j \wedge \underline{x}^m \neq a^j$. Thus, *SSC* implies $a^j \wedge \underline{x}^m \succ^i a^j$ $\forall i \in \mathcal{N}$ *s.t.* $\theta^i < \theta^m$. Thus, either $a^j \wedge \underline{x}^m \in X^j$ (in which case set $\hat{x} = a^j \wedge \underline{x}^m$), or there exists $x' \in X^j$ such that $x' \succeq^i a^j \wedge \underline{x}^m$ $\forall i \in \mathcal{C}^j$ (in which case set $\hat{x} = x'$) and therefore $\exists \hat{x} \in X^j$ such that $\hat{x} \succeq^m a^j$ and $\hat{x} \succ^i a^j$ $\forall i \in \mathcal{C}^j$ *s.t.* $\theta^i < \theta^m$. Denote with $\theta^h \geq \theta^m$ the highest type such that $\hat{x} \succeq^h a^j$ and notice that all types such that $\theta^i < \theta^h$ are such that $\hat{x} \succ^i a^j$ because of *SSC*. By definition of θ^h, one gets $a^j \succ^i \hat{x}$ $\forall i \in \mathcal{C}^j$ *s.t.* $\theta^i > \theta^h$. Moreover, because in a party equilibrium only one policy $a^j \neq \emptyset$ is proposed, then it must be true that $W(\{\hat{x}\} \cup A \backslash$

$\{a^j\}; \succeq) = \hat{x}$, and therefore the following results are true: $\hat{x}R^{j,h}(a^{-j})a^j$; $\hat{x}P^{j,i}(a^{-j})a^j$ $\forall i \in \mathcal{C}^j$ $s.t.$ $\theta^i < \theta^h$, and $a^j P^{j,i}(a^{-j})\hat{x}$ $\forall i \in \mathcal{C}^j$ $s.t.$ $\theta^i > \theta$. Second, consider $a^j \wedge \bar{x}^m$. Following the same procedure of the previous point, one can show that $\exists \check{x} \in X^j$ and $l \leq m$ such that $\check{x}R^{j,l}(a^{-j})a^j$, $\check{x}P^{j,i}(a^{-j})a^j$ $\forall i \in \mathcal{C}^j$ $s.t.$ $\theta^i > \theta^l$, and $a^j P^{j,i}(a^{-j})\check{x}$ $\forall i \in \mathcal{C}^j$ $s.t.$ $\theta^i < \theta^l$. Recall $g^j_{a^{-j}}(R^j(a^{-j})) = a^m$. By assumption, $g^j_{a^{-j}}$ must satisfy neutrality. Namely, consider a permutation $\rho: \ X^j \to X^j$ such that $\rho(\check{x}) = a^j$ and $\rho(a^j) = \hat{x}$ and a preference profile $R^j_\rho(a^{-j})$ such that $a^j R^{j,i}_\rho(a^{-j})\check{x}$ if and only if $\rho(a^j)R^{j,i}(a^{-j})\rho(\check{x})$, that is, $a^j R^{j,i}_\rho(a^{-j})\check{x}$ if and only if $\hat{x}R^{j,h}(a^{-j})a^j$, which is true for $\forall i \in \mathcal{C}^j$ $s.t.$ $\theta^i \leq \theta^h$. Therefore, $a^j R^{j,i}_\rho(a^{-j})\check{x}$ $\forall i \in \mathcal{C}^j$ $s.t.$ $\theta^i \leq \theta^h$, and $\check{x}R^{j,i}_\rho(a^{-j})a^j$ $\forall i \in \mathcal{C}^j$ $s.t.$ $\theta^i > \theta^h$ and $a^j \neg R^{j,i}_\rho(a^{-j})\check{x}$ $\forall i \in \mathcal{C}^j$ $s.t.$ $\theta^i > \theta^h$. Because $g^j_{a^{-j}}$ satisfies neutrality, then it must be true that $\rho(g^j_{a^{-j}}(R^j_\rho(a^{-j}))) = g^j_{a^{-j}}(R^j(a^{-j})) = a^j$, which under $\rho(\check{x}) = a^j$ implies $g^j_{a^{-j}}(R^j_\rho(a^{-j})) = \check{x}$. Lastly, , we know that $\check{x}R^{j,l}(a^{-j})a^j$, and $\check{x}P^{j,i}(a^{-j})a^j$ $\forall i \in \mathcal{C}^j$ $s.t.$ $\theta^i > \theta^l$. Notice that $l \leq h$, thus for any $i \in \mathcal{C}^j$ one gets $\check{x}R^{j,i}_\rho(a^{-j})a^j \to \check{x}R^{j,i}(a^{-j})a^j$, $[\check{x}R^{j,i}_\rho(a^{-j})a^j] \wedge [a^j \neg R^{j,i}_\rho(a^{-j})\check{x}] \to \check{x}P^{j,i}(a^{-j})a^m$, $a^j R^{j,i}(a^{-j})\check{x} \to a^j R^{j,i}_\rho(a^{-j})\check{x}$ and $a^j P^{m,i}(a^{-j})\check{x} \to [a^j R^{j,i}_\rho(a^{-j})\check{x}] \wedge [\check{x}\neg R^{j,i}_\rho(a^{-j})a^j]$. By monotonicity, this implies that if $g^j_{a^{-j}}(R^j_\rho(a^{-j})) = \check{x}$, then $g^j_{a^{-j}}(R^j(a^{-j})) \neq a^j$ (with $a^j \neq \emptyset$). Notice that a group choice function with such characteristics always exist by Lemma 8. Thus, the proposal a^j is not feasible for party $\mathcal{C}^j$. This leads to a contradiction. Q.E.D.

Lemma 10b. *If there is $\mathcal{C}^m \in \mathbb{P}$ such that $\theta^i = \theta^m$ for all $i \in \mathcal{C}^m$, then there is no regular equilibrium of the proposal game in which $w(A(\mathbb{P}), \succeq) \notin I(\succeq^m)$ and either $w(A(\mathbb{P}), \succeq) \geq \bar{x}^m$, or $w(A(\mathbb{P}), \succeq) \leq \underline{x}^m$.*

Proof. Suppose $w(A(\mathbb{P}), \succeq) \geq \bar{x}^m$. In a regular equilibrium of the proposal game the set of proposals $A(\mathbb{P})$ must be feasible. Moreover, Lemma 9 implies $A(\mathbb{P}) = \{a^j\}$ for some $\mathcal{C}^j \in \mathbb{P}$ and $a^j \neq \emptyset$. This implies $w(A(\mathbb{P}), \succeq) = a^j$. Thus, for $A = \{a^j\}$ it must be true that $g^j_{a^{-j}}(R^j(a^{-j})) = a^j$ and $g^k_{a^{-k}}(R^k(a^{-k})) = \emptyset$ for all $\mathcal{C}^k \in \mathbb{P}$ with $\mathcal{C}^k \neq \mathcal{C}^j$. Case 1: $\mathcal{C}^m = \mathcal{C}^j$. In this case, $g^j_{a^{-j}}(R^j(a^{-j})) = x^m$ for some $x^m \in I(\succeq^m)$, thus $g^j_{a^{-j}}(R^j(a^{-j})) \neq a^j$, which leads to a contradiction. Case 2: $\mathcal{C}^m \neq \mathcal{C}^j$. In this case for $A = \{a^j\}$ one finds $g^m_{a^{-m}}(R^m(a^{-m})) = x^m$ for some $x^m \in I(\succeq^m)$, thus $g^m_{a^{-m}}(R^m(a^{-m})) \neq \emptyset$. This because optimality and $a^j \notin I(\succeq^m)$

imply $x^m \succ^m a^j$. The SSC implies $x^m \succ^i a^j$ for all $i \in \mathcal{N}$ such that $\theta^i \leq \theta^m$. This implies $W(\{a^j, x^m\}, \succeq) = x^m$, and therefore $g^m_{a^{-m}}(R^m(a^{-m})) \neq \emptyset$, which leads to a contradiction. Similarly one can show that $w(A(\mathbb{P}), \succeq) \leq \underline{x}^m$ implies a contradiction. Q.E.D.

Lemma 11. *In any stable legislative gridlock* $\langle \mathbb{P}, a(\mathbb{P}), x^0 \rangle$ with $x^0 \notin I(\succeq^m)$, *the partition* $\mathbb{P}$ *is not a majoritarian party structure.*

Proof. Consider a stable legislative gridlock $\langle \mathbb{P}, a(\mathbb{P}), x^0 \rangle$ with $x^0 \notin I(\succeq^m)$, and suppose $\mathbb{P}$ is a majoritarian party structure. Case 1. One active party is such that $a^j = \tilde{x}^m$ where $\tilde{x}^m$ defeats any other $\tilde{x}^m \in I(\succeq^m)$ under W (there is one such policy under the assumption of RSP). In order to have a legislative gridlock, there must be an active party $\mathcal{C}^j \in \mathbb{P}$ such that $g^j_{a^{-j}}(R^j(a^{-j})) = a^j$ with $a^j MV(\succeq)\tilde{x}^m$ (else $\tilde{x}^m$ is in $K(A(\mathbb{P}), \succeq)$, which means that $A(\mathbb{P})$ is not a legislative gridlock). Secondly, the assumptions on W imply that $a^j \notin I(\succeq^m)$. Suppose $a^j \geq \tilde{x}^m$ (or $a^j \leq \tilde{x}^m$). Then optimality and $a^j \notin I(\succeq^m)$ imply $\tilde{x}^m \succ^m a^j$. The SSC implies $\tilde{x}^m \succ^i a^j$ for all $i \in \mathcal{N}$ such that $\theta^i \leq \theta^m$ ($\theta^i \geq \theta^m$). Thus, $a^j \neg MV(\succeq)\tilde{x}^m$. Suppose $a^j \ngeq \tilde{x}^m$ and $a^j \nleq \tilde{x}^m$. Corollary 7b and $a^j \notin I(\succeq^m)$ imply that any party such that all members are either $\theta^i \geq \theta^m$ (or only $\theta^i \leq \theta^m$) can only propose $a^j \geq \bar{x}^m$ (or $a^j \leq \underline{x}^m$), thus this cannot be true. Case 2. No active party is such that $a^j = \tilde{x}^m$ where $\tilde{x}^m$ defeats any other $x^m \in I(\succeq^m)$ under W . Consider some $\mathcal{C}^j \in \mathbb{P}$ that contains at least one i such that $\theta^i = \theta^m$ (there must be at least one of such parties) and construct a deviator $\mathcal{C}^k := \{i \in \mathcal{C}^j |\ \theta^i = \theta^m\}$ that induces a new majoritarian party structure $\mathbb{P}'$. Then $g^k_{a^{-k}}(R^k(a^{-k})) = \tilde{x}^m$ because of *TB2* and $A(\mathbb{P}') = \{\tilde{x}^m\}$. The deviation is blocked if either (1) $g^k_{a^{-k}}(R^k(a^{-k})) = \tilde{x}^m$ or $g^j_{a^{-j}}(R^j(a^{-j})) = \emptyset$ for all $\mathcal{C}^j \in (\mathbb{P}' \setminus \mathcal{C}^k)$ are unfeasible, or (2) $A(\mathbb{P}') = \{\tilde{x}^m\}$ is not an equilibrium of the proposal game, or (3) $\mathbb{P}'$ is not stable. Regarding (1), $g^k_{a^{-k}}(R^k(a^{-k})) = \tilde{x}^m$ is trivially feasible under $A(\mathbb{P}') = \{\tilde{x}^m\}$ given *TB1*, *TB*2, and *RSP*. Corollary 7b implies that any any party such that all members are either $\theta^i \geq \theta^m$ (or only $\theta^i \leq \theta^m$) can only propose $a^j \geq \bar{x}^m$ (or $a^j \leq \underline{x}^m$). If $a^j \notin I(\succeq^m)$, then optimality implies $\tilde{x}^m \succ^m a^j$. The SSC implies $\tilde{x}^m \succ^i a^j$ for all $i \in \mathcal{N}$ such that $\theta^i \leq \theta^m$ ($\theta^i \geq \theta^m$). Thus, $\tilde{x}^m MV(\succeq)a^j$ and $a^j \neg MV(\succeq)\tilde{x}^m$. This means that it must be true that either $a^j = \emptyset$ or $a^j \in I(\succeq^m$

). But in the latter case the assumptions of *RSP* implies $w(\{\tilde{x}^m, a^j\}, \succeq) = \tilde{x}^m$, thus *TB1* implies $g^j_{a^{-j}}(R^j(a^{-j})) = \emptyset$. Regarding (2), trivially $A(\mathbb{P}') = \{\tilde{x}^m\}$ is an equilibrium of the proposal game because $\tilde{x}^m \in K(\{\tilde{x}^m\}, \succeq)$. Regarding (3), notice that for any further immediate deviation induces a new majoritarian party structure $\mathbb{P}_2$, and that $\mathbb{P}_2$ is such that $\mathcal{C}^k \in \mathbb{P}_2$, because none of its members have a strict incentive to deviate. Following points (1) and (2), in this majoritarian party structure there is an equilibrium of the proposal game such that $g^k_{a^{-k}}(R^k(a^{-k})) = \tilde{x}^m$, and $g^j_{a^{-j}}(R^j(a^{-j})) = \emptyset$ for all $\mathcal{C}^j \in (\mathbb{P}_2 \setminus \mathcal{C}^k)$ and therefore $A = \{\tilde{x}^m\}$. Moreover, all the possible equilibria of the proposal game it must be true that $w(A(\mathbb{P}_2), \succeq) = \tilde{x}^m$, because if $w(A(\mathbb{P}_2), \succeq) \neq \tilde{x}^m$, then *TB2* and *RSP* imply $g^k_{a^{-k}}(R^k(a^{-k})) = \tilde{x}^m$. This means that the final outcome $\mathbb{P}''$ of any sequential deviation must also be such that $w(A(\mathbb{P}''), \succeq) = \tilde{x}^m$. Thus, any further deviation cannot be strictly profitable, and therefore $\mathbb{P}'$ is stable. Case 1 and 2 implies that no legislative gridlock can prevail under a majoritarian party structure. This leads to a contradiction. Q.E.D.

Lemma 12. *If $x^0 \leq \underline{x}^m$ or $x^0 \geq \bar{x}^m$ and $x^0 \notin I(\succeq^m)$, then - for any given W - any majoritarian party structure either (i) admits a party equilibrium in the form $\langle \mathbb{P}, a(\mathbb{P}), x^m \rangle$ for some $x^m \in I(\succeq^m)$, or (ii) every sequential deviation leads to a party equilibrium in the form $\langle \mathbb{P}', a(\mathbb{P}'), x^m \rangle$, in which $\mathbb{P}'$ is a refinement of $\mathbb{P}$.*

Proof. Suppose $\mathbb{P}$ is a majoritarian party structure and $\langle \mathbb{P}, a(\mathbb{P}), x^m \rangle$ is not a party equilibrium for any $x^m \in I(\succeq^m)$. Then either (1) $g^m_{a^{-m}}(R^m(a^{-m})) = x^m$ for some $\mathcal{C}^m \in \mathbb{P}$ and/or $g^j_{a^{-j}}(R^j(a^{-j})) = \emptyset$ for each $\mathcal{C}^j \in \mathbb{P}$ such that $\mathcal{C}^j \neq \mathcal{C}^m$ is unfeasible; or (2) $\langle \mathbb{P}, a(\mathbb{P}) \rangle$ is not an equilibrium of the proposal game; or (3) $\langle \mathbb{P}, a(\mathbb{P}), x^m \rangle$ is not stable. Regarding (1) $g^m_{a^{-m}}(R^m(a^{-m})) = x^m$ is feasible for at least one $x^m \in I(\succeq^m)$. If (a) there is at least one $\mathcal{C}^l \in \mathbb{P}$ such that $\theta^i = \theta^m$ for all $i \in \mathcal{C}^l$, then choose $\mathcal{C}^l = \mathcal{C}^m$ and set $x^m = \tilde{x}^m$ where $\tilde{x}^m$ is such that $W(A, \succeq) = \tilde{x}^m$ for all $A \subseteq I(\succeq^m)$ such that $\tilde{x}^m \in A$ (it exists by assumption of *RSP*, and selects $\tilde{x}^m$ because of *TB2*). In this case, notice that Lemma 7 and Corollary 7b imply that all parties $\mathcal{C}^j \in \mathbb{P}$ such that $\theta^i \neq \theta^m$ for some $i \in \mathcal{C}^j$ can only offer proposals $a^j \geq \bar{x}^m$ or $a^j \leq \underline{x}^m$. Moreover, none of these proposals can defeat $\tilde{x}^m$. Thus, under $A = \{\tilde{x}^m\}$ *TB1* implies $g^j_{a^{-j}}(R^j(a^{-j})) = \emptyset$ for each $\mathcal{C}^j \in \mathbb{P}$ such that $\mathcal{C}^j \neq \mathcal{C}^m$. Thus this is feasible too. If (a) does not apply, then choose $\mathcal{C}^m$ such that either (b) $\theta^i = \theta^m$ for at least one $i \in \mathcal{C}^m$ and $\theta^i > \theta^m$ for at least one $h \in \mathcal{C}^m$, or (c) $\theta^i = \theta^m$ for at least one $i \in \mathcal{C}^m$ and $\theta^i < \theta^m$ for at least one $l \in \mathcal{C}^m$ (notice that at least one party of type (a) or (b) must exists in $\mathbb{P}$). Consider case (a). Set $g^m_{a^{-m}}(R^m(a^{-m})) = \bar{x}^m$ for $A^{-m} = \{\emptyset\}$. This is feasible for a dictatorial and Pareto efficient $g^m_{a^{-m}}$ (with a dictator such that $\theta^i = \theta^m$). Notice that any other party is either of type (b) or type (c). Lemma 7 and Corollary 7b imply that these parties can offer only proposals such that $a^j \geq \bar{x}^m$ or $a^j \leq \underline{x}^m$. Notice that none of these proposals can defeat $\bar{x}^m$ - except possibly $\bar{x}^m$ and $\underline{x}^m$. *TB1* implies that no party $\mathcal{C}^j \neq \mathcal{C}^m$ can play $g^j_{a^{-j}}(R^j(a^{-j})) = \bar{x}^m$ given $A = \{\bar{x}^m\}$. Thus, the only possibility is $g^d_{a^{-d}}(R^d(a^{-d})) = \underline{x}^m$ for some $\mathcal{C}^d \in \mathbb{P}$ such that $\mathcal{C}^d \neq \mathcal{C}^m$. This can be true only if $W(\{\bar{x}^m, \underline{x}^m\}, \succeq) = \underline{x}^m$. If that is the case, set $a^d = \underline{x}^m$ and $a^j = \emptyset$ for all $\mathcal{C}^j \in \mathbb{P}$ such that $\mathcal{C}^j \neq \mathcal{C}^d$. It is easy to show that $g^d_{a^{-d}}(R^d(a^{-d})) = \underline{x}^m$ and $g^j_{a^{-j}}((a^{-j})) = \emptyset$ are feasible. Similarly, one can show that the same is true if $\mathcal{C}^m$ is of type (b). Thus there is a contradiction to (1). Regarding (2), notice that $x^m \in K(\{x^m\}, \succeq)$ for any $x^m \in I(\succeq^m)$. Thus,$\langle \mathbb{P}, \{x^m\} \rangle$ is an equilibrium of the proposal game, which is a contradiction to (2). Lastly, regarding (3), consider any sequence of deviations from equilibria of the stage game of types (a), (b) or (c) in part (i). If $\mathbb{P}$ is of type (a), then the resulting partition is of type (a); If $\mathbb{P}$ is of type (b) or (c), then the resulting partition is of type (a) or (b) or (c). In all cases, the resulting stable party structure cannot be neither a stable gridlock, because of Lemma 11, nor a disequilibrium $A(\mathbb{P}) = \{\emptyset\}$, because an equilibrium of the proposal game exists. Thus it is must be a party equilibrium. Moreover, parts (1) and (2) imply that it must have form$\langle \mathbb{P}', a(\mathbb{P}'), x^m \rangle$ for some $x^m \in I(\succeq^m)$. Thus, if a sequential deviation occurs, it must lead to a party equilibrium $\langle \mathbb{P}', a(\mathbb{P}'), x^m \rangle$ in which $\mathbb{P}'$ is a refinement of $\mathbb{P}$. Q.E.D.

Lemma 13a. *If $x^0 \leq \underline{x}^m$ or $x^0 \geq \bar{x}^m$and $x^0 \notin I(\succeq^m)$, then there is no stable party structure in which $w(A(\mathbb{P}), \succeq)$ is such that $w(A(\mathbb{P}), \succeq) \notin I(\succeq^m)$ and $w(A(\mathbb{P}), \succeq) \geq \bar{x}^m$, $w(A(\mathbb{P}), \succeq) \leq \underline{x}^m$.*

Proof. Suppose there is a stable party structure $\langle \mathbb{P}, a(\mathbb{P}), w, \succeq) \rangle$ such that $w = w(A(\mathbb{P}), \succeq)$ *with* $w \notin I(\succeq^m)$ and $w \geq \bar{x}^m$. Consider a sequence of deviations

$\{\mathbb{P}_1, \mathbb{P}_2, ..., \mathbb{P}_K\}$ where each $\mathbb{P}_k$ is induced by a deviator $\mathcal{C}_k^d \subseteq \mathcal{C}_{k-1}^j$ for some $\mathcal{C}_{k-1}^j \in \mathbb{P}_{k-1}$. Lemma 10 implies that if $\langle \mathbb{P}_k, a(\mathbb{P}_k) \rangle$ is a regular equilibrium of the proposal game, it must be such that $A(\mathbb{P}_k) = \{a_k^j\}$ with $a_k^j = w_k(A(\mathbb{P}_k), \succeq) = w_k$ such that either $w_k \in I(\succeq^m)$, or $w_k \geq \bar{x}^m$ or $w_k \leq \underline{x}^m$. If there is a legislative gridlock, or no equilibrium of the proposal game, then the outcome is x^0 which is also by assumption $x^0 \leq \underline{x}^m$ or $x^0 \geq \bar{x}^m$. Thus, it is always the case that either $w_k \in I(\succeq^m)$, or $w_k \geq \bar{x}^m$ or $w_k \leq \underline{x}^m$. Suppose the players expect a final outcome $\mathbb{P}_K = \mathbb{P}'$ such that $A(\mathbb{P}') = \{\hat{x}^m\}$ for some $\hat{x}^m \in I(\succeq^m)$. Say (1) $w_k \neq \hat{x}^m$, then a deviator is either (a) $\mathcal{C}_k^d := \{i \in \mathcal{C}_{k-1}^j | \ \theta^i > \theta^m\}$ if $w_k \leq \underline{x}^m$, or (b) $\mathcal{C}_k^e := \{i \in \mathcal{C}_{k-1}^j | \ \theta^i < \theta^m\}$ if $w_k \geq \bar{x}^m$ whenever $w_k \neq \hat{x}^m$ and any of such deviation is possible. If $w_k = \hat{x}^m$ but $\langle \mathbb{P}_k, a(\mathbb{P}_k) \rangle$ is not stable, then (c) $\mathcal{C}_k^f$ is any deviator that can induce $\mathbb{P}_{k+1}$ and such that this deviation is strictly profitable for all $i \in \mathcal{C}_k^f$ if the final outcome is $\hat{x}^m$ (there must be one, else $\mathbb{P}_k$ would be stable). Notice that the deviation (a) is profitable for all deviators as long as they expect it to induce $\hat{x}^m$ because optimality implies $\hat{x}^m \succeq^m w_k$, and the *SSC* together with $w_k \neq \hat{x}^m$ and $w_k \leq \underline{x}^m$ imply $\hat{x}^m \succ^i w_k$ for all i such that $\theta^i < \theta^m$. Similarly, deviations of type (b) are always profitable. The sequence continues until either it reaches a party equilibrium with $A(\mathbb{P}_k) = \{\hat{x}^m\}$, or until no deviations of type (a), (b) or (c) are possible. In the former case, the deviation is profitable and stable, thus $\langle \mathbb{P}', a(\mathbb{P}') \rangle$ sequentially blocks $\langle \mathbb{P}, a(\mathbb{P}) \rangle$. Consider the latter case. Because all the possible deviations with $\mathcal{C}_k^d := \{i \in \mathcal{C}_{k-1}^j | \ \theta^i > \theta^m\}$ and $\mathcal{C}_k^e := \{i \in \mathcal{C}_{k-1}^j | \ \theta^i < \theta^m\}$ have been carried out, then all the parties that includes both individuals of type $\theta^i < \theta^m$ and individuals of type $\theta^i > \theta^m$ have been broken. Thus, the resulting party structure is a majoritarian party structure. Lemma 12 implies that in the cases in which this proposition applies any majoritarian party structure either (a) supports a party equilibrium in the form $\langle \mathbb{P}', a(\mathbb{P}'), x^m \rangle$ for some $x^m \in I(\succeq^m)$, or it induces a sequential deviation that leads to a party equilibrium in the form $\langle \mathbb{P}', a(\mathbb{P}'), x^m \rangle$ for some $x^m \in I(\succeq^m)$. Set $\hat{x}^m = x^m$ such that the expectations of all the deviators are correct. This means that the sequence of deviations leads to a tuple $\langle \mathbb{P}_K, a(\mathbb{P}_K) \rangle = \langle \mathbb{P}', a(\mathbb{P}') \rangle$ such that $\langle \mathbb{P}', a(\mathbb{P}'), w(A(\mathbb{P}')) \rangle$ is stable, and that each step of the sequence corresponds to a deviator that induce a strictly profitable deviation. This implies that $\langle \mathbb{P}', a(\mathbb{P}') \rangle$

sequentially blocks $\langle \mathbb{P}, a(\mathbb{P})\rangle$, which implies that $\langle \mathbb{P}, a(\mathbb{P}), w(A(\mathbb{P}), \succeq)\rangle$ is not a stable party structure. This leads to a contradiction. Similarly, one can show that a contradiction occurs if $w(A(\mathbb{P}), \succeq) \leq \underline{x}^m$. Q.E.D.

Lemma 13b. *If $x^0 \notin I(\succeq^m)$, then a stable legislative gridlock occurs only if $x^0 \ngeq \bar{x}^m$ and $x^0 \nleq \underline{x}^m$ for some $x^m \in I(\succeq^m)$.*

Proof. Follows directly from Lemma 13a. Q.E.D.

Lemma 14. *If $x^0 \leq \underline{x}^m$ or $x^0 \geq \overline{x}^m$ and $x^0 \notin I(\succeq^m)$, then in a party equilibrium for any party $\mathcal{C}^j \in \mathbb{P}$ (could be a singleton) one gets $a^j \in I(\succeq^m) \cup \{\emptyset\}$.*

Proof. Suppose $a^j \notin I(\succeq^m)$ and $a^j \neq \emptyset$. In a party equilibrium this implies $A(\mathbb{P}) = \{a^j\}$ because of Lemma 9, and therefore $W(A(\mathbb{P}), \succeq) = a^j$. The condition $a^j \notin I(\succeq^m$) cannot be true if $\theta^i = \theta^m$ for all members of $\mathcal{C}^j$, because in such case $X^j = I(\succeq^m)$. Thus, there must be at least one $i \in \mathcal{C}^j$ such that either $\theta^i > \theta^m$ or $\theta^i < \theta^m$. There are two possible cases.
(i) Say $a^j \geq \bar{x}^m$ or $a^j \leq \underline{x}^m$. Lemma 13a implies that $\langle \mathbb{P}, a(\mathbb{P}), a^j\rangle$ is not a party equilibrium. This leads to a contradiction.
(ii) If $a^j \ngeq \bar{x}^j$ and $a^j \nleq \underline{x}^m$. Lemma 10 implies that there is no regular equilibrium of the proposal game in this case, thus $\langle \mathbb{P}, a(\mathbb{P}), a^j\rangle$ is not a party equilibrium. This leads to a contradiction. Q.E.D.

Lemma 15. *In any stable party structure $\langle \mathbb{P}, a(\mathbb{P}), w(A(\mathbb{P}), \succeq)\rangle$ (i) the winning policy is such that $w(A(\mathbb{P}), \succeq) \succeq^m x^0$. Thus, if (ii) preferences are strictly convex over $\mathbb{R}^d$ and $X \subseteq \mathbb{R}^d$, then the winning policy must lie within a circle centered in x^m and with radius equal to* $\max_{x \in UP^m(x^0)} \|x - x^m\|$.

Proof. Part (i). Suppose $w(A(\mathbb{P}), \succeq) \prec^m x^0$. This can occurs only if there exists in a party equilibrium in the form $\langle \mathbb{P}, a(\mathbb{P}), a^j\rangle$, because in a stable legislative gridlock - if any exists - one must get $w(A(\mathbb{P}), \succeq) = x^0$, which trivially implies $w(A(\mathbb{P}), \succeq) \nprec^m x^0$. Lemma 13b implies that a legislative gridlock can occur only if $x^0 \ngeq \bar{x}^m$ and $x^0 \nleq \underline{x}^m$. There are two possible cases.

(1) Say $a^j \geq \bar{x}^m$ or $a^j \leq \underline{x}^m$. The fact that $w(A(\mathbb{P}), \succeq) \prec^m x^0$ implies $w(A(\mathbb{P}), \succeq) \notin I(\succeq^m)$. Lemma 10b states that If there is $\mathcal{C}^m \in \mathbb{P}$ such that $\theta^i = \theta^m$ for all $i \in \mathcal{C}^m$, then there is no regular equilibrium of the proposal game in which $w(A(\mathbb{P}), \succeq) \notin I(\succeq^m)$ and $w(A(\mathbb{P}), \succeq) \geq \bar{x}^m$ (or $w(A(\mathbb{P}), \succeq) \leq \underline{x}^m$). Thus, a party with such characteristics must not exist in $\mathbb{P}$ in this case. Consider a sequence of deviations $\{\mathbb{P}_1, \mathbb{P}_2, ..., \mathbb{P}_K\}$ such that the initial deviator is a party $\mathcal{C}_2^k := \{i \in \mathcal{C}^m | \; \theta^i = \theta^m\}$ (notice that one party of this kind must exists) and that lead to any stable party structure with a sequence of strictly profitable steps (if any additional profitable steps exists). Because the final partition $\mathbb{P}_K = \mathbb{P}'$ must contain a party $\mathcal{C}^k \in \mathbb{P}'$ such that $\theta^i = \theta^m$ for all $i \in \mathcal{C}^k$, then if $\langle \mathbb{P}, a(\mathbb{P}), w(A(\mathbb{P}), \succeq) \rangle$ is a regular equilibrium of the proposal game, Lemma 10 and 10b imply that it must be such that $w(A(\mathbb{P}), \succeq) \in I(\succeq^m)$. In this case, the deviation is strictly profitable and stable. The only other possibilities are that (b) the deviation leads to a stable legislative gridlock $\langle \mathbb{P}', a(\mathbb{P}'), x^0 \rangle$, or (c) the deviation leads to a disequilibrium $\langle \mathbb{P}', a, x^0 \rangle$. In both these cases case, given that $w(A(\mathbb{P}), \succeq) \prec^m x^0$, the deviation to $\mathbb{P}'$ represents a stable and strictly profitable deviation for all the deviators at each step. Thus, $\langle \mathbb{P}, a(\mathbb{P}), w(A(\mathbb{P}), \succeq) \rangle$ is not stable, which implies that it is not a stable party structure. This represents a contradiction.

(2) Say $a^j \ngeq \bar{x}^m, a^j \nleq \underline{x}^m$. Lemma 10 implies that $\langle \mathbb{P}, a(\mathbb{P}), w(A(\mathbb{P}), \succeq) \rangle$ is not a stable party structure. This leads to a contradiction. Thus it must be true that $w(A(\mathbb{P}), \succeq) \succeq^m x^0$.

Part (ii). If preferences are strictly convex over $\mathbb{R}^d$ and $X \subseteq \mathbb{R}^d$, then $I(\succeq^m)$ is a singleton. Moreover, there is an indifference curve in $\mathbb{R}^d$ for each player i with $\theta^i = \theta^m$ passing through x^0, which defines a convex upper contour set in $\mathbb{R}^d$ $UP^m_{\mathbb{R}^d}(x^0) := \{x \in \mathbb{R}^d | \; x \succeq^m x^0\}$. Because $X \subseteq \mathbb{R}^d$ it must be the case that the upper contour set in X is such that $UP^m(x^0) \subseteq UP^m_{\mathbb{R}^d}(x^0)$. Thus, if we select $\tilde{x} = \arg\max_{x \in UP^m(x^0)} \|x - x^m\|$, then it must be true that, for any $x \in UP^m(x^0)$, one gets $\|x - x^m\| \leq \|\tilde{x} - x^m\|$. Because of part (i), we know that at any stable party structure $w(A(\mathbb{P}), \succeq) \succeq^m x^0$, which means that $w(A(\mathbb{P}), \succeq) \in UP^m(x^0)$. This is implies that $w(A(\mathbb{P}), \succeq)$ must lie within a circle centered in x^m and with radius equal to $\max_{x \in UP^m(x^0)} \|x - x^m\|$. Q.E.D.

Proposition 1. *(Existence). (i) A stable party structure* $\langle \mathbb{P}, a(\mathbb{P}), w \rangle$ *always exists for any preference profile* $\succeq$*; (ii) a party equilibrium such that the policy outcome is an ideal point of the median legislator* m *– i.e.,* $\langle \mathbb{P}, a(\mathbb{P}), w \rangle$ *with* $w \in I(\succeq^m)$*, always exists; (iii) a stable legislative gridlock* $\langle \mathbb{P}, a(\mathbb{P}), x^0 \rangle$ *exists only if either* $x^0 \in I(\succeq^m)$*, or if* $x^0 \ngeq \bar{x}^m$ *and* $x^0 \nleq \underline{x}^m$*.*

Proof. Part (i) Consider the finest partition $\widetilde{\mathbb{P}}$, i.e. a partition in which each party contains only one legislator. It is easy to show that $\left\langle \widetilde{\mathbb{P}}, a(\widetilde{\mathbb{P}}), x^m \right\rangle$ is an equilibrium with agreement under any x^0. Denote with $\mathcal{C}^m$ the only active party, which has only one member of type $\theta^i = \theta^m$. Consider some $x^m \in I(\succeq^m)$ such that $g^m_{a^{-m}}(R^m(a^{-m})) = a^m = x^m$ in which $g^m_{a^{-m}}$ satisfies the required conditions. This means that x^m is a feasible proposal for party $\mathcal{C}^m$. Because of *TB2*, g^m chooses the $x^m \in I(\succeq^m)$ such that $w(\{x^m, \tilde{x}^m\}, \succeq) = x^m$ for all $\tilde{x}^m \in I(\succeq^m)$. Notice that all the other parties $\mathcal{C}^k$ such that the unique member i has type $\theta^i \neq \theta^m$ are such that any feasible proposal is either $a^k \geq \bar{x}^m$ (or $a^k \leq \underline{x}^m$). If $a^k \neq \bar{x}^m$ (or $a^k \neq \underline{x}^m$), optimality implies $x^m \succ^m a^k$ and *SSC* implies $x^m \succ^i a^k$ $\forall i \in \mathcal{N}$ such that $\theta^i \geq \theta^m$ (or $\theta^i \leq \theta^m$). This means $x^m MV(\succeq) a^k$ and $a^k \neg MV(\succeq) x^m$. If $a^k = \bar{x}^m$ (or $a^k = \underline{x}^m$), *TB2* implies that no a^k can change the winning policy, because $w(\{x^m, \bar{x}^m\}, \succeq) = x^m$ ($w(\{x^m, \underline{x}^m\}, \succeq) = x^m$), thus each $\mathcal{C}^k$ is stable by playing $a^k = \emptyset$, which is the only credible proposal for such parties. Moreover, any other party $\mathcal{C}^q$ such that the unique member i has type $\theta^i = \theta^m$ (if any) is trivially stable at $a^q = \emptyset$, because each of its member is already obtaining her ideal policy. Lastly, because $\widetilde{\mathbb{P}}$ is the finest possible partition, deviation to proper subparties $\mathcal{C}^j \subset \mathcal{C}^k$ are obviously impossible. Thus, partition $\widetilde{\mathbb{P}}$ is stable, each $a^j \in A$ is a credible, and $W(\{x^m\}, \succeq) = x^m$. Therefore $\left\langle \widetilde{\mathbb{P}}, a(\widetilde{\mathbb{P}}), x^m \right\rangle$ is a *SPS*, which proves (i). Part (ii). The equilibrium in part (i) is such that $w(A, \succeq) = x^m$ with $x^m \in K(A, \succeq)$, i.e. it is a party equilibrium, which proves (ii). Part (iii). Follows directly from Lemma 13b and Lemma 11.

Corollary 2. *(Quasi-Median Voter Theorem). (i) If either* $x^0 \in I(\succeq^m)$*, or* $x^0 \geq x^m$*, or* $x^0 \leq x^m$ *for all* $x^m \in I(\succeq^m)$*, then in any stable party structure the policy outcome is an ideal point of the median legislator* m*. (ii) In any stable party structure, the policy outcome is such that* $w \succeq^m x^0$*. Thus, (iii) if legislator* m*'s preferences are* *strictly convex over* $\mathbb{R}^d$*, then the policy outcome must lie within a circle centered in* x^m *and with radius equal to* $\max_{x\in UP^m(x^0)} \|x - x^0\|$.

Proof. Part (i). If $x^0 \in I(\succeq^m)$ then Lemma 15 trivially implies $w(A(\mathbb{P}), \succeq) \in I(\succeq^m)$. If $x^0 \notin I(\succeq^m)$, because either $x^0 \geq x^m$ or $x^0 \leq x^m$ for all $x^m \in I(\succeq^m)$, then Lemma 14 implies that any party equilibrium has form $\langle \mathbb{P}, a(\mathbb{P}), x^m \rangle$ for some $x^m \in I(\succeq^m)$. Moreover, Lemma 13b implies that there is no stable legislative gridlock. This means that in any stable party structure has the form $\langle \mathbb{P}, a(\mathbb{P}), x^m \rangle$ for some $x^m \in I(\succeq^m)$. Part (ii) - (iii). See Lemma 15. Q.E.D.

Corollary 2b. *If* $x^0 \in I(\succeq^m)$*, or if either* $x^0 \geq x^m$ *or* $x^0 \leq x^m$ *for all* $x^m \in I(\succeq^m)$*, then the set of policy outcomes is* $E(X, \succeq, x^0) = I(\succeq^m)$.

Proof. Consider a party equilibrium $\left\langle \widetilde{\mathbb{P}}, a(\widetilde{\mathbb{P}}), x \right\rangle$ with each party $\mathcal{C}^j \in \widetilde{\mathbb{P}}$ is a singleton. Also, consider a social choice function W such that $W(\{x^m, \tilde{x}^m\}, \succeq) = x^m$ for all $\tilde{x}^m \in I(\succeq^m)$ (it must exists by *RSP*). Proposition 1 implies that in a party equilibrium $w(A(\mathbb{P}), \succeq) \in I(\succeq^m)$. Suppose $\exists x^m \in I(\succeq^m)$ such that $x^m \notin E(X, \succeq, x^0)$. This implies that for $A^{-m} = \{\emptyset\}$ either (i) a $g^m_{a^{-m}}$ such that $g^m_{a^{-m}}(R^m(a^{-m})) = x$ does not exist, or that (ii) $\exists x' \in X^k$ for some party $\mathcal{C}^k$ such that $W(\{x^m, x'\}, \succeq) = x'$. Regarding (i), it is never true because a group choice function $g^m_{a^{-m}}$ that selects $x^m \in I(\succeq^m)$ exists and does not violate any of the required conditions. Regarding (ii), notice that no subgroup of any party $\mathcal{C}^j$ exists because any party $\mathcal{C}^j \in \widetilde{\mathbb{P}}$ is a singleton. Thus, if it exists, it must be $\mathcal{C}^k \in \widetilde{\mathbb{P}}$ for some $k \neq m$. If this is the case, denote with θ^k the type of the unique member of $\mathcal{C}^k$. Recall that in this case $x^k \in I(\succeq^k)$. Notice that Lemma 7 implies that an active party can propose only (i) $a^k \geq \inf\left\{I(\succeq^k)\right\}$ if $\theta^k > \theta^m$, or (ii) $a^l \leq \sup\left\{I(\succeq^l)\right\}$ if $\theta^l < \theta^m$, or (iii) $a^j \in I(\succeq^m)$ if $\theta^j = \theta^m$. In case (i), first of all notice that $x^m \leq x^k$ by Corollary 7b. Also notice that Corollary 7b implies $x^m \notin I(\succeq^k)$ unless $x^m = \bar{x}^m$. Thus, either (a) $x^k = \bar{x}^m$ or (b) $x^k \neq \bar{x}^m$ and $x^m \leq \bar{x}^m \leq x^k$. In case (a) the choice of W such that $W(\{x^m, \bar{x}^m\}; \succeq) = x^m$ rules out this possibility. Namely, the policy outcome w is unaffected, thus the only possibility is $g^k_{a^{-k}}(R^k(a^{-k})) = \emptyset$ for all $\mathcal{C}^k \neq \mathcal{C}^m$ and $g^m_{a^{-m}}(R^m(a^{-m})) = x^m$ which are consistent with $A = \{x^m\}$.

In case (b) $x^k \neq \bar{x}^m$ and $x^m \leq \bar{x}^m \leq x^k$. This implies $x^k \notin I(\succeq^m)$. Thus, optimality implies $x^m \succ^m x^k$, and that *SSC* implies $x^m \succ^i x^k \; \forall i \in \mathcal{N} : \theta^i > \theta^m$. This means $x^m MV(\succeq) x^k$ and $x^m \neg MV(\succeq) x^k$. This implies $W(\{x^m, x^k\}, \succeq) = x^m$, which implies that the only possibility is $A = \{x^m\}$ and $g^k_{a^{-k}}\{R^k(a^{-k})) = \emptyset$ for all $\mathcal{C}^k \neq \mathcal{C}^m$ and $g^m_{a^{-m}}\{R^m(a^{-m})) = x^m$. Lastly, in case (iii), trivially *TB1* implies for $A = \{x^m\}$ it must be true that $g^j_{a^{-j}}\{R^j(a^{-j})) = \emptyset$. This leads to a contradiction. Apply the same procedure to each $\hat{x}^m \in I(\succeq^m)$, choosing a social choice function $W(\{\hat{x}^m, \tilde{x}^m\}, \succeq) = \hat{x}^m$ for all $\tilde{x}^m \in I(\succeq^m)$. Thus, the party equilibrium in each case is a $\hat{x}^m \in I(\succeq^m)$, for all $\hat{x}^m \in I(\succeq^m)$. Lastly, Corollary 2 (i) implies that, under the stated assumptions, in any stable party structure $\langle \mathbb{P}, A(\mathbb{P}), w \rangle$ one gets $w \in I(\succeq^m)$. Therefore, $E(X, \succeq, x^0) = I(\succeq^m)$. Q.E.D.

Proposition 3. *(Monotone Comparative Statics). If $x^0 \in I(\succeq^m)$, or if either $x^0 \geq x^m$ or $x^0 \leq x^m$ for all $x^m \in I(\succeq^m)$, then the set of policy outcomes $E(X, \succeq, x^0)$ of the voting game is (i) a sublattice of X which is (ii) monotonic nondecreasing in $\succeq$ on* $\mathbf{T}$.

Proof. Corollary 2b implies $E(X, \succeq, x^0) = I(\succeq^m)$. Notice that the definition of $\mathbf{T}$ implies that all $\underset{t}{\succeq} \in \mathbf{T}$ satisfy *QSM* and *SSC*. Part (i). Consider any two policies $x, x' \in I(\succeq^m)$ (if is a singleton, the only possibility is $x = x'$). Because X is a lattice, $x \wedge x', x \vee x' \in X$. Optimality implies $x \succeq^m x \wedge x'$. *QSM* implies $x \vee x' \succeq^m x'$. Hence $x \vee x' \in I(\succeq^m)$. Similarly, one can show that $x \wedge x' \in I(\succeq^m)$. Thus, $I(\succeq^m)$ is a sublattice of X. Part (ii). Consider a preference profile $\hat{\succeq} \in \mathbf{T}$ such that $\hat{\succeq} D_X \succeq$. Notice that in $\mathbf{T}$, $(\hat{\succeq}) D_X (\succeq)$ if and only if, for any $x, x' \in X$ with $x \geq x'$, one gets $x \succeq^m x' \rightarrow x \hat{\succeq}^m x'$ and $x' \hat{\succeq}^m x \rightarrow x' \succeq^m x$. This is equivalent to say that the median legislator m under the new preference profile has type $\hat{\theta}^m$ that is (weakly) higher than θ^m, i.e. $\hat{\theta}^m \geq \theta^m$. Consider any couple of policies $x', \hat{x} \in X$ such that $x' \in I(\succeq^m)$ and $\hat{x} \in I(\hat{\succeq}^m)$. Optimality implies $x' \succeq^m \hat{x} \wedge x'$. *QSM* implies $\hat{x} \vee x' \succeq^m \hat{x}$. The condition $\hat{\succeq} D_X \succeq$ implies $\hat{x} \vee x' \succeq^m \hat{x}$. Thus, $\hat{x} \vee x' \in$ Hence $\hat{x} \vee x' \in I(\hat{\succeq}^m)$. Similarly, one can show that $\hat{x} \wedge x' \in I(\succeq^m)$. Because this is true for all possible policies $x', \hat{x} \in X$ such that $x' \in I(\succeq^m)$ and $\hat{x} \in I(\hat{\succeq}^m)$, it implies that $I(\hat{\succeq}^m) \geq_s I(\succeq^m)$, where $\geq_s$ is the operator that denotes the strong set order.

Therefore $E(X, \hat{\succeq}, x^0) \geq_s E(X, \succeq, x^0)$. Lastly, because this result is true for any two preference profiles $\hat{\succeq}, \succeq \in \mathbf{T}$, then $E(X, \succeq, x^0)$ is monotonic nondecreasing in $\succeq$ on $\mathbf{T}$. Q.E.D.

B Stable Party Structures

Proposition 4. *(Stable Party Structures). Suppose either $x^0 \geq \bar{x}^m$ or $x^0 \leq \underline{x}^m$ is true. (i) If $I(\succeq^m)$ is a singleton, then any majoritarian or central party structure $\langle \mathbb{P}, a(\mathbb{P})\rangle$, including the grand coalition of all legislators $\langle \{\mathcal{N}\}, a(\{\mathcal{N}\})\rangle$, always supports a party equilibrium $\langle \mathbb{P}, a(\mathbb{P}), x^m\rangle$. (ii) If $I(\succeq^i) \cap I(\succeq^m) = \emptyset$ for all $i \in \mathcal{N}$ such that $\theta^i \neq \theta^m$, then no ends-against-the-middle party structure $\langle \mathbb{P}, a(\mathbb{P})\rangle$ supports a party equilibrium.*

Proof. Part (i). Suppose $\langle \mathbb{P}, a(\mathbb{P})\rangle$ is a majoritarian party structure, $\langle \mathbb{P}, a(\mathbb{P}), x^m\rangle$ with $A(\mathbb{P}) = \{x^m\}$ is not a party equilibrium and $I(\succeq^m) = \{x^m\}$. Then, either (1) a^j is not feasible for any $\mathcal{C}^j \in \mathbb{P}$, or (2) $a(\mathbb{P})$ is not an equilibrium of the proposal game, or (3) $\langle \mathbb{P}, a(\mathbb{P})\rangle$ is not stable. Regarding (1), consider a party $\mathcal{C}^m \in \mathbb{P}$ that includes at least one individual i with type $\theta^i = \theta^m$ and such that $\theta^i \geq \theta^m$ ($\theta^i \leq \theta^m$) for all $i \in \mathcal{C}^m$ (there exists at least one). Notice that for such party there exists at least one policy $x^m \in I(\succeq^m)$ such that $x^m \in X^m$. Also notice that a dictatorial and Pareto efficient group choice function that always selects the best alternative of individual m would meet all the properties listed in section 2.3 (see proof to Lemma 8). Thus, for $A = \{x^m\}$ there exists at least one group choice function $g^m_{a^{-m}}$ that satisfies the required property and such that $g^m_{a^{-m}}(R^m(a^{-m})) = x^m$. This ensures that a^j is feasible for party $\mathcal{C}^m$. Similarly, I need $g^j_{a^{-j}}(R^j(a^{-j})) = \emptyset$ for all $\mathcal{C}^j \neq \mathcal{C}^m$. Notice that by *TB1* any other party $\mathcal{C}^j \neq \mathcal{C}^m$ such that $\theta^i = \theta^m$ for all $i \in \mathcal{C}^j$ (if any exists) has never a strict incentive propose an action $a^j \neq \emptyset$ for all its members. Thus, suppose, there is $\mathcal{C}^j \in \mathbb{P}$ such that either $\theta^i \geq \theta^m$, (or $\theta^i \leq \theta^m$) for all $i \in \mathcal{C}^j$ such that $g^j_{a^{-j}}(R^j(a^{-j})) = a^j \neq \emptyset$. This is true only if $w(\{x^m\} \cup \{a^j\}, \succeq) \succ^i w(\{x^m\}, \succeq)$ for at least one $i \in \mathcal{C}^j$. Lemma 7 and Corollary 2b imply that $x \in X^j$ only if $x \geq x^m$ ($x \leq x^m$). Moreover, *TB1* implies that $a^j \neq x^m$. Thus, if $a^j \neq \emptyset$, it must be true that (a) $a^j \neq x^m$, and (b) $a^j \geq x^m$

or $a^j \leq x^m$. Recall optimality and $a^j \neq x^m$ imply $x^m \succ^m a^j$. The *SSC* implies $x^m \succ^i a^j$ for all $i \in \mathcal{N}$ such that $\theta^i \leq \theta^m$ ($\theta^i \geq \theta^m$). Thus, $a^j \neg MV(\succeq) x^m$ and therefore $w(\{x^m\} \cup \{a^j\}; \succeq) = x^m$, which implies $w(\{x^m\} \cup \{a^j\}, \succeq) \not\succ^i w(\{x^m\}, \succeq)$ for all $i \in \mathcal{C}^j$, which leads to a contradiction. Regarding (2), notice that in the cases in the previous paragraph, $A(\mathbb{P}) = \{x^m\}$ is an equilibrium of the proposal game, because trivially $w(\{x^m\}, \succeq\} = x^m$ and $x^m \in K(\{x^m\}, \succeq)$. Regarding (3), consider a sequential deviation with an initial deviation $\mathbb{P}_2$ induced by a deviator $\mathcal{C}_2^k$. Notice that Corollary 2 implies that any stable party structure that prevails in the final partition $\mathbb{P}_K = \mathbb{P}'$ must be in the form $\langle \mathbb{P}', a(\mathbb{P}'), \hat{x}^m \rangle$ with $A(\mathbb{P}') = \{\hat{x}^m\}$ for some $\hat{x}^m \in I(\succeq^m)$. Because $I(\succeq^m)$ is a singleton, then the deviation is trivially not strictly profitable. Thus, the only equilibrium of the proposal game must have form $A(\mathbb{P}_2) = \{x^m\}$. Notice that x^m cannot be defeated by any credible proposal of other parties. Thus, there is no gridlock unless $x^0 = x^m$. Because the same is true for any further step $\mathbb{P}_k$, the final outcome must be a stable party structure in the form $\langle \mathbb{P}', a(\mathbb{P}'), x^m \rangle$. Thus, the deviation is not profitable, which implies that $\langle \mathbb{P}, a(\mathbb{P}), x^m \rangle$ with $A(\mathbb{P}) = \{x^m\}$ is stable. This leads to a contradiction to (3). Thus, $\langle \mathbb{P}, a(\mathbb{P}), x^m \rangle$ is a party equilibrium and this leads to a contradiction.

Now suppose that $\langle \mathbb{P}, a(\mathbb{P}) \rangle$ is a central party structure, $\langle \mathbb{P}, a(\mathbb{P}), x^m \rangle$ is not a party equilibrium and $\{x^m\} = I(\succeq^m)$, Following exactly the same steps as in the previous paragraph, one can show that this leads to a contradiction. Notice that the partition $\mathbb{P} = \{\mathcal{N}\}$ (grand coalition) trivially satisfies the conditions of part (i). Thus, $\langle \{\mathcal{N}\}, a(\{\mathcal{N}\}) \rangle$ supports a party equilibrium if $I(\succeq^m)$ is a singleton.

Part (ii). Suppose $\langle \mathbb{P}, a(\mathbb{P}) \rangle$ is an Ends-Against-the-Middle party structure and it does supports a party equilibrium . Then $\langle \mathbb{P}, a(\mathbb{P}), x \rangle$ is a party equilibrium for some $x \in X$ and some coalition of extremes $\mathcal{C}^j \in \mathbb{P}$. This means that a^j is feasible for some $\mathcal{C}^j \in \mathbb{P}$, where $\mathcal{C}^j$ is a coalition of extremes. Therefore, it must be true that for $A^{-j} = \{\emptyset\}$one gets $g^j_{a^{-j}}(R^j(a^{-j})) = a^j$. Assume $I(\succeq^i) \cap I(\succeq^m) = \emptyset$ for all $i \in \mathcal{C}^j$. Lemma 9 implies that in any a party equilibrium only one proposal $a^j \neq \emptyset$ is made, then it must be true that $A(\mathbb{P}) = \{a^j\}$. Moreover, Lemma 14 implies $a^j = x^m$ for some $x^m \in I(\succeq^m)$. Therefore, it must be true that for $A^{-j} = \{\emptyset\}$ one gets $g^j_{a^{-j}}(R^j(a^{-j})) = x^m$.

Consider the lowest type in $\mathcal{C}^j \in \mathbb{P}$ such that $\theta^h > \theta^m$, and recall $I(\succeq^h) \cap I(\succeq^m) = \emptyset$. Thus, $x^h \neq x^m$ for any $x^h \in I(\succeq^h)$. *QSM* and the *SSC* imply $x^h \geq x^m$. Optimality implies $x^h \succ^h x^m$, and the *SSC* implies $x^h \succ^i x^m$ for all $\forall i \in \mathcal{C}^j$ that $\theta^i \geq \theta^h$. Notice that all types such that $\theta^i < \theta^h$ are such that $x^m \succ x^h$ because of *SSC*. By definition of θ^h, one gets $a^j \succ^i x^h \; \forall i \in \mathcal{C}^j : \theta^i < \theta^m$. Because in a party equilibrium only one proposal $a^j \neq \emptyset$ is made, then it must be true that $A(\mathbb{P}) = \{a^j\}$ and $A^{-j} = \emptyset$, the following results must be true: $x^h P^{j,i}(a^{-j}) a^j \; \forall i \in \mathcal{C}^j : \theta^i > \theta^m$, and $a^j P^{j,i}(a^{-j}) x^h \; \forall i \in \mathcal{C}^j : \theta^i < \theta^m$. Following the same procedure of the previous point, consider the highest type in $\mathcal{C}^j \in \mathbb{P}$ such that $\theta^l < \theta^m$. One can show that for some $x^l \in I(\succeq^l)$ it must be true that $x^l P^{j,i}(a^{-j}) a^j \; \forall i \in \mathcal{C}^j : \theta^i < \theta^m$, and $a^j P^{j,i}(a^{-j}) x^l \; \forall i \in \mathcal{C}^j : \theta^i > \theta^m$. Recall $g^j_{a^{-j}}(R^j(a^{-j})) = a^j$. By assumption, $g^j_{a^{-j}}$ must satisfy neutrality. Namely, consider any permutation $\rho : X^j \to X^j$ such that $\rho(x^h) = a^j$ and $\rho(a^j) = x^l$ and a preference profile $R^j_\rho(a^{-j})$ such that $a^j P^{j,i}_\rho(a^{-j}) x^h$ if and only if $\rho(a^j) P^{j,i}(a^{-j}) \rho(x^h)$, that is, $a^j P^{j,i}_\rho(a^{-j}) x^h$ if and only if $x^l P^{j,h}(a^{-j}) a^j$, which is true for $\forall i \in \mathcal{C}^j : \theta^i < \theta^m$. Therefore, $a^j P^{j,i}_\rho(a^{-j}) x^h$ $\forall i \in \mathcal{C}^j : \theta^i < \theta^m$, and $x^h R^{j,i}_\rho(a^{-j}) a^j \forall i \in \mathcal{C}^j : \theta^i > \theta^m$ and $a^j \neg R^{j,i}_\rho(a^{-j}) x^h$ $\forall i \in \mathcal{C}^j : \theta^i > \theta^m$. Thus, the null permutation $R^j_\rho(a^{-j}) = R^j(a^{-j})$ satisfies such conditions. Because $g^j_{a^{-j}}$ satisfies neutrality, then it must be true that for any of such permutations one gets $\rho(g^j_{a^{-j}}(R^j_\rho(a^{-j}))) = g^j_{a^{-j}}(R^j(a^{-j})) = a^j$, which under $\rho(x^h) = a^j$ implies $g^j_{a^{-j}}(R^j_\rho(a^{-j})) = x^h$. But we know that under $R^j_\rho(a^{-j}) = R^j(a^{-j})$, it must be trivially true that $g^j_{a^{-j}}(R^j_\rho(a^{-j})) = a^j$. Notice that a group choice function with such characteristics always exist by Lemma 8. Thus, it must be such that $g^j_{a^{-j}}(R^j(a^{-j})) \neq a^j$, i.e. the proposal a^j is not feasible for party $\mathcal{C}^j$. This leads to a contradiction. Q.E.D.

Proposition 5. *(Legislative Gridlock). If $x^0 \notin I(\succeq^m)$, then (i) in any stable legislative gridlock $\langle \mathbb{P}, a(\mathbb{P}), x^0 \rangle$, the partition $\mathbb{P}$ is either an ends-against-the-middle or a central party structure; (ii) no stable legislative gridlock $\langle \mathbb{P}, a(\mathbb{P}), x^0 \rangle$ exists if $\langle \mathbb{P}, a(\mathbb{P}) \rangle$ is a majoritarian party structure.*

Proof. Part (i) follows directly from Lemma 13b and Lemma 11a. Part (ii) is straightforward from Lemma 13b. Q.E.D.

C Other Proofs

Proposition 6. *(Mergers of Parties). If either $x^0 \in I(\succeq^m)$, or $x^0 \geq \bar{x}^m$, or $x^0 \leq \underline{x}^m$, then (i) a stable party structure such that the policy outcome is the ideal policy of the median legislator always exists. If (ii) $I(\succeq^m)$ is a singleton, then a two-party majoritarian party structure is stable.*

Proof. Part (i). Suppose a stable party structure such that $w(A(\mathbb{P}), \succeq) \in I(\succeq^m)$ does not exists. Consider a majoritarian multi-party structure $\langle \mathbb{P}, a(\mathbb{P}) \rangle$ with $A(\mathbb{P}) = \{\tilde{x}^m\}$ such that there exists $\mathcal{C}^m \in \mathbb{P}$ with $\theta^i = \theta^m$ for all $i \in \mathcal{C}^m$. Suppose it is not a stable party structure. Given that it is feasible and it is an equilibrium of the proposal game, it must be not stable. Consider the final outcome of a sequential deviation. Because by assumption a permanently not-stable sequence of deviations is not profitable, the outcome must be a stable party structure $\langle \mathbb{P}', a(\mathbb{P}') \rangle$. If $w = \tilde{x}^m$, then the deviation is trivially not profitable. Thus, suppose $\langle \mathbb{P}', a(\mathbb{P}') \rangle$ is a legislative gridlock with $x^0 \geq \tilde{x}^m$, or a party equilibrium with $w(A(\mathbb{P}'), \succeq) \geq \tilde{x}^m$, then the final outcome is either x^0, or $\hat{w} \geq \tilde{x}^m$. The initial deviation cannot include any player in $\mathcal{C}^m$, nor any party $\mathcal{C}^j$ such that $\theta^i \leq \theta^m$ for all $i \in \mathcal{N}$, because it is never strictly profitable for such players to deviate. Thus initial deviators must be only individuals such that $\theta^i > \theta^m$, because *SSC* implies $\tilde{x}^m \succeq^i x^0$, and $\tilde{x}^m \succeq^i \hat{w}$ for all i such that $\theta^i \leq \theta^m$. Because the deviation (division or merger) only involves players such that $\theta^i > \theta^m$, this implies that $\mathbb{P}_2$ is another majoritarian partition that includes $\mathcal{C}^m$. But in this case, the only equilibrium of the proposal game is $\langle \mathbb{P}_2, a(\mathbb{P}_2) \rangle$ with $A(\mathbb{P}_2) = \{\tilde{x}^m\}$. Therefore, the same is true for the next step of the sequential deviation. Namely, $\mathbb{P}_3$ can only be finer partition of $\mathbb{P}_2$ contains $\mathcal{C}^m$, which implies an equilibrium of the proposal game $\langle \mathbb{P}_3, a(\mathbb{P}_3) \rangle$ with $A(\mathbb{P}_3) = \{\tilde{x}^m\}$. Sequentially, this process can only lead to an equilibrium of the proposal game $\langle \mathbb{P}', a(\mathbb{P}') \rangle$ with $A(\mathbb{P}') = \{\tilde{x}^m\}$. Lastly, because no merger ever occurs, the process must stop at the finest partition. which means the deviation cannot lead to a legislative gridlock with $x^0 \geq \tilde{x}^m$, nor to a party equilibrium with $w(A(\mathbb{P}'), \succeq) \geq \tilde{x}^m$. Similarly one can show that $\langle \mathbb{P}', a(\mathbb{P}') \rangle$ cannot be neither a legislative gridlock with $x^0 \leq \tilde{x}^m$, nor a party equilibrium with $w(A(\mathbb{P}'), \succeq) \leq \tilde{x}^m$. The other possibility is that the deviation leads to a party equilibrium with $w \not\succeq \tilde{x}^m$, $w \not\preceq \tilde{x}^m$. But if $w \not\succeq \tilde{x}^m$ then $w \not\succeq \bar{x}^m$, and if $w \not\preceq \tilde{x}^m$ then $w \not\preceq \underline{x}^m$. This case cannot occur if $w \notin I(\succeq^m)$ because of Lemma 10. Thus, if $w \notin I(\succeq^m)$, then no sequential deviation can ever be profitable and stable, which means that $\langle \mathbb{P}, a(\mathbb{P}) \rangle$ with $A(\mathbb{P}) = \{\tilde{x}^m\}$ is stable in such case. The only possibility left is $w \in I(\succeq^m)$ and $w \not\succeq \tilde{x}^m$, $w \not\preceq \tilde{x}^m$. But if $\langle \mathbb{P}', a(\mathbb{P}') \rangle$ with $A(\mathbb{P}') = \{w\}$ is stable, and $w \in I(\succeq^m)$, then there exist a stable party structure $\langle \mathbb{P}', a(\mathbb{P}'), w \rangle$ in which an ideal policy of the median legislator wins. This leads to a contradiction. Part (ii). Suppose $I(\succeq^m)$ is a singleton, and a majoritarian two-party structure $\left\langle \hat{\mathbb{P}}, a(\hat{\mathbb{P}}) \right\rangle$ with $A(\hat{\mathbb{P}}) = \{x^m\}$ is not stable. then there exists a sequential deviation that leads to a new stable party structure $\left\langle \hat{\mathbb{P}}', a(\hat{\mathbb{P}}') \right\rangle$, such that $\hat{w}' \neq x^m$. Because $I(\succeq^m)$ is a singleton, this implies $\hat{w}' \notin I(\succeq^m)$. Now consider a deviator $\mathcal{C}^k \subseteq \mathcal{C}^j$ for some $\mathcal{C}^j \in \hat{\mathbb{P}}'$, such that $\theta^i = \theta^m$ for all $i \in \mathcal{C}^m$ (there exists at least one party $\mathcal{C}^j$ that allows for this deviation). Call the new partition $\mathbb{P}$. Such partition satisfies the conditions of point (i), thus it supports a party equilibrium $\langle \mathbb{P}, a(\mathbb{P}) \rangle$ with $A(\mathbb{P}) = \{x^m\}$. Notice that this deviation is feasible, profitable for all $i \in \mathcal{C}^k$, and stable because of point (i). This implies that the party structure $\left\langle \hat{\mathbb{P}}', a(\hat{\mathbb{P}}') \right\rangle$ is not stable, which implies in turn that $\left\langle \hat{\mathbb{P}}, a(\hat{\mathbb{P}}) \right\rangle$ with $A(\hat{\mathbb{P}}) = \{x^m\}$ is stable. This leads to a contradiction. Q.E.D.

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