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A characterization of lexicographic preferences

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Abstract

This paper characterizes lexicographic preferences over alternatives that are identified by a finite number of attributes. We say two alternatives are ‘totally different’ if they are different with respect to every attribute. Our characterization is based on two key concepts: a weaker notion of continuity called ‘mild continuity’ (strict preference order between any two totally different alternatives is preserved around their small neighborhoods) and an ‘unhappy set’ (any alternative outside the set is preferred to all alternatives inside).

Keywords: lexicographic preference; mild continuity; unhappy set; inclusion of marginally improved alternatives; preference nonreversibility

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1 Introduction

Consider an individual choice problem over alternatives that have multiple attributes. For instance, suppose an individual faces a choice over different cellphones where each cellphone has three attributes: data capacity, battery life and quality of camera. For an individual having a lexicographic preference over such choices, there is an order of importance of different attributes. The individual first looks at the attribute most important to him, then looks at the second most important attribute and so on. In the example of cellphones, suppose data capacity is the most important attribute for an individual, followed by battery life and quality of camera. Then for any two cellphones having different data capacity, this individual prefers the one that has a higher data capacity; for any two phones with the same data capacity, the individual prefers the one that has a higher battery life and finally for any two phones that are same with respect to both data capacity and battery life, prefers the one that has a better quality camera. Lexicographic preferences are particularly useful to understand choice problems where alternatives have a large number of attributes.

This paper presents a characterization of lexicographic preferences in an individual choice framework. The early literature can be traced back to the seminal paper of Fishburn (1975), whose axiomatization of lexicographic preferences is closely connected with Arrow's (1951) impossibility theorem. For a lexicographic preference over a finite-dimensional product set, the most dominant factor is dictatorial in the sense that for two points that differ with respect to that factor, the preference order is solely determined by that specific factor; if two points are same with respect to the most dominant factor, the second most dominant factor is dictatorial and so on. In the words of Fishburn, there is a "hierarchy of dictators" for lexicographic preferences. Fishburn's proof begins by establishing the existence of a smallest decisive subset of the set of factors to show that this subset must contain a single element. A recent alternative proof by Mitra and Sen (2014) reconfirms the Arrow-Fishburn interconnections by determining an extremely pivotal factor along the same lines Geanakoplos (2005) identifies an extremely pivotal voter to prove Arrow's theorem.

More recently, Petri and Voorneveld (2016) propose another characterization of lexicographic preferences. They point out the unsatisfactory restriction of domain in Fishburn's approach that is needed to apply Arrow's proof technique. Specifically, one of the axioms of Fishburn requires that for every coordinate, only two equivalence classes are important to identify the entire order, but to use Arrow's technique, each coordinate must have at least three equivalence classes. The characterization of Petri and Voorneveld is based on robustness of preference ordering between two alternatives for changes in a few rather than a large number of coordinates. However, it requires Fishburn's *noncompensation* axiom, which "prohibits tradeoffs between factors" and thus already has a lexicographic flavor.

In this paper we consider an individual choice problem over alternatives that are identified by a finite number n of attributes, where the domain of each attribute is the set of all non-negative real numbers, so that the set of all alternatives is \mathbb{R}_+^n . In this framework, we provide a

different characterization of lexicographic preferences. Unlike Fishburn (1975) and Petri and Voorneveld (2016), our characterization does not require any noncompensation condition.¹ Our key distinction from the Arrow-Fishburn way of reasoning is that continuity arguments play a prominent role in our analysis. This approach can be of interest, since lexicographic preference is perhaps the most well known textbook example of a discontinuous preference.

Our characterization is based on two central concepts: a weaker version of continuity called mild continuity and the notion of unhappy sets, which is an extension of lower contour sets. To define mild continuity, call two alternatives totally different if they are different with respect to *every* attribute. A preference relation is mildly continuous if strict preference order between any two totally different alternatives is preserved around their small neighborhoods. A set of alternatives is called an unhappy set if any alternative outside the set is preferred to all alternatives inside.²

One of our axioms imposes a certain structure on the unhappy sets of the preference relation. Consider two totally different alternatives that belong to an unhappy set. Suppose the second alternative is superior with respect to a subset S of attributes while the first is superior with respect to the complementary subset. We say an unhappy set satisfies inclusion of marginally improved alternatives (IMIA) if it includes a third alternative that does better than the first but worse than the second with respect to the attributes in S but has the same levels as the first alternative for the remaining attributes. First we show that when there are only two attributes (that is, the choice problem is on \mathbb{R}_+^2), any rational preference relation is lexicographic if and only if it is strong monotone, mildly continuous and any unhappy set satisfies IMIA (Theorem 1). The result on \mathbb{R}_+^2 is of independent interest. In addition, we build on this result for the analysis of cases with more than two attributes.

Extending to the general case of more than two attributes, the starting point is to compare alternatives for which all but two attributes have level zero.³ The zero level can be interpreted as the minimum or basic level of an attribute. Fix any subset of two attributes and consider all alternatives for which all attributes outside that subset have zero level. The original preference relation gives an induced preference between alternatives for which all but two fixed attributes have level zero (such an induced preference is defined on \mathbb{R}_+^2). One of our axioms requires that any closed unhappy set for any such induced preference satisfies IMIA.

Our last axiom requires preference nonreversibility under additional attributes (NRAA). As before consider two alternatives for which all but two attributes have zero level. Suppose one of these alternatives is preferred to the other. Now add positive levels of one or more attributes to each of these alternatives keeping the levels of additional attributes same across the two. NRAA holds if the preference order between such new pairs of alternatives stays the

¹Consider any alternatives x, y, w, z . The noncompensation condition says: suppose for every attribute i , $x_i > y_i$ if and only if $w_i > z_i$ and $y_i > x_i$ if and only if $z_i > w_i$; then the individual prefers x to y if and only if the individual prefers w to z .

²Unhappy sets were introduced in Mitra and Sen (2017).

³This approach is closely related to choice rules based on “elimination by aspects” proposed by Tversky (1972).

same as before. Consider the example of cellphones given before in which any cellphone has three attributes: data capacity, battery life and quality of camera. Consider two cellphones, each of which has zero level (that is, the minimum level) of battery life and suppose the first cellphone is preferred to the second. NRAA says that raising battery life by the same amount to each of them without changing their levels of data capacity or quality of camera does not alter the preference order.⁴

Our main result (Theorem 2) shows that in the general case of more than two attributes, a rational preference relation is lexicographic if and only if (a) any induced preference between alternatives for which all but two same attributes have zero levels satisfies (i) strong monotonicity, (ii) mild continuity, (iii) IMIA for any closed unhappy set and (b) preference nonreversibility holds. The guiding principle of our characterization is facing choices involving multiple attributes, the individual decision maker proceeds by looking at two attributes at a time. This can arguably be a good behavioral rule in many problems in which looking at a large number attributes together might seem too complex and demanding. Accordingly we look for structures on the induced preferences defined on \mathbb{R}_+^2 and finally preference nonreversibility is used to gradually go from two to many attributes.

The paper is organized as follows. We present the analytical framework in Section 2. The axioms of lexicographic preference and the main result are presented in Section 3. Section 4 gives examples to illustrate the robustness of the axioms. Most proofs are presented in the Appendix (Section 5).

2 The analytical framework

Consider an individual who has a preference relation \succsim on a set of alternatives X . Each alternative is characterized by n attributes. Let $N = \{1, \dots, n\}$ be the set of attributes. The domain of any attribute $i \in N$ is \mathbb{R}_+ . Therefore an alternative is given by a vector $x = (x_1, \dots, x_n) \in \mathbb{R}_+^n$ and the set of all alternatives is $X = \mathbb{R}_+^n$.

For any non empty $S \subseteq N$, denote $X_S = \mathbb{R}_+^{|S|}$. For any $x \in X$ and $S \subseteq N$, we write $x = (x^S, x^{N \setminus S})$ where $x^S \in X_S$ and $x^{N \setminus S} \in X_{N \setminus S}$ (note that $x = x^N$). If $x_i = 0$ for all $i \in S$, we write $x^S = 0^S$.

In the special case when S is the singleton set $\{i\}$, it will be convenient to use the simpler notation $x^S = x_i$, $x^{N \setminus S} = x_{-i}$, $X_S = X_i$ and $X_{N \setminus S} = X_{-i}$. We write $x = (x_i, x_{-i})$ where $x_i \in X_i$ and $x_{-i} \in X_{-i}$.

The distance between $x, y \in X$, denoted by $d(x, y)$, is given by the Euclidean metric. For $x^S, y^S \in X_S$, $d(x^S, y^S)$ is the same metric d restricted to X_S . A neighborhood of x^S is a set $B_\varepsilon(x^S)$ consisting of all $y^S \in X_S$ such that $d(x^S, y^S) < \varepsilon$ for some $\varepsilon > 0$.

The individual's preference on X is defined using the binary relation \succsim where " $x \succsim y$ "

⁴Although NRAA has some resemblance with the independence axiom of Fishburn, they are not same. We refer to Example 2, which gives a preference that satisfies NRAA, but for which Fishburn's independence axiom does not hold.

stands for “the individual prefers x to y ”. The strict preference is defined as $x \succ y \Leftrightarrow [x \succsim y]$ and $[\text{not } y \succsim x]$. The indifference relation is defined as $x \sim y \Leftrightarrow [x \succsim y]$ and $[y \succsim x]$.

For any $x \in X$, the lower contour set under \succsim is $L(x) = \{y \in X \mid x \succsim y\}$. The strict lower contour set is $\underline{L}(x) = \{y \in X \mid x \succ y\}$ and the indifference set is $I(x) = \{y \in X \mid x \sim y\}$.

A preference relation \succsim on X is *complete* if for any $x, y \in X$, either $x \succsim y$ or $y \succsim x$. It is *transitive* if for any $x, y, z \in X$, whenever $x \succsim y$ and $y \succsim z$, we have $x \succsim z$.

A preference relation \succsim on X is *rational* if it is complete and transitive.

Let $x, y \in X$. If $x_i > y_i$ for all $i \in N$, we write $x > y$. If $x_i \geq y_i$ for all $i \in N$, we write $x \geq y$. Two alternatives x, y are *non-comparable*, denoted by $x \bowtie y$, if there are $i, j \in N$ such that $y_i > x_i$ and $x_j > y_j$.

A preference relation \succsim on X is *monotone* if for any $x, y \in X$ with $x > y$, we have $x \succ y$. It is *strong monotone* if for any $x, y \in X$ with $x \geq y$ and $x \neq y$, we have $x \succ y$.

Definition 1 A preferences \succsim on X is *lexicographic* if there is a linear order $i(1) <_0 i(2) <_0 \dots <_0 i(n)$ on N such that for any $x, y \in X$, $x \succ y$ if and only if either $[x_{i(1)} > y_{i(1)}]$ or $[x_{i(1)} = y_{i(1)}, x_{i(2)} > y_{i(2)}]$ or \dots or $[x_{i(1)} = y_{i(1)}, \dots, x_{i(n-1)} = y_{i(n-1)}, x_{i(n)} > y_{i(n)}]$.

For such a preference, $i(1)$ will be called the most important attribute, $i(2)$ the next most important attribute and so on.

2.1 Some useful concepts

To characterize lexicographic preferences, it will be useful to introduce two concepts: (i) mild continuity of a preference relation, which is a weaker version of continuity and (ii) unhappy sets, which are related to lower contour sets. The notion of induced preferences will be also useful.

2.1.1 Mild continuity

Definition 2 For $x, y \in X$, we say x and y are *totally different*, denoted by $x \hat{\neq} y$, if $x_i \neq y_i$ for all $i \in N$.

Note that $x \hat{\neq} y$ if and only if $y \hat{\neq} x$. For $x^S, y^S \in X_S$, we define $x^S \hat{\neq} y^S$ similarly. A preference relation is mildly continuous if strict preference order between any two totally different points is preserved around their small neighborhoods.

Definition 3 A preference relation \succsim is *mildly continuous* on X if for any $x, y \in X$ with $x \hat{\neq} y$ and $x \succ y$, there exists $\varepsilon > 0$ such that if $\tilde{x} \in B_\varepsilon(x)$ and $\tilde{y} \in B_\varepsilon(y)$, then $\tilde{x} \succ \tilde{y}$.

We recall that a preference relation \succsim is continuous on X if for any $x, y \in X$ with $x \succ y$, there exists $\varepsilon > 0$ such that if $\tilde{x} \in B_\varepsilon(x)$ and $\tilde{y} \in B_\varepsilon(y)$, then $\tilde{x} \succ \tilde{y}$. Thus for a continuous preference strict preference order between any two points, totally different or otherwise, is preserved around their small neighborhoods.

2.1.2 Unhappy sets

Definition 4 A set $A \subseteq X$ is an *unhappy set* for a preference relation \succsim on X if for any $y \in X \setminus A$, $y \succ x$ for every $x \in A$.

Observe that lower contour and strict lower contour sets are unhappy sets. So are the sets X and \emptyset .

Let A be a subset of a metric space X . A point $x \in X$ is a boundary point of A if every neighborhood of x contains at least one point in A and at least one point in $X \setminus A$. The set of all boundary points of A is called the boundary of A and denoted by ∂A . To characterize unhappy sets we recall the following result. For the proof, see, e.g., Mendelson (1990: Theorem 4.23, Chapter 3 and Definition 2.1 of Chapter 4).

Result 1 *If A is a non empty proper subset of a connected space, then (i) $\partial A \neq \emptyset$ and (ii) the set A cannot be both open and closed.*

Since \mathbb{R}_+^n is a connected set, we can use this result for subsets of $X = \mathbb{R}_+^n$.

Proposition 1 *Consider a rational preference relation \succsim on $X = \mathbb{R}_+^n$.*

- (i) *If A, B are unhappy sets, then either $A \subseteq B$ or $B \subseteq A$.*
- (ii) *Let A be a non empty proper subset of X . If A is an unhappy set and \succsim is mildly continuous then for any $x \in \partial A$, the following hold for any $y \neq x$.*
 - (a) *If $y \in \partial A$, then $x \sim y$.*
 - (b) *If $y \in A$, then $x \succsim y$.*
 - (c) *If $x \succ y$, then $y \in A$.*

Proof See Section 5.2 of the Appendix. ■

For a continuous preference relation, the last part of the proposition also holds for y that are not totally different from x . In that case unhappy sets can be more precisely characterized.

Corollary 1 *Consider a rational and continuous preference relation \succsim on $X = \mathbb{R}_+^n$. Let A be an unhappy set which is a non empty proper subset of X . Then the following hold for any $x \in \partial A$.*

- (i) *$x \sim y$ for any $y \in \partial A$.*
- (ii) *$\underline{L}(x) \subseteq A \subseteq L(x)$.*
- (iii) *The set A must be either closed or open, but not both. If A is closed, then $A = L(x)$ and if A is open, then $A = \underline{L}(x)$.*

Proof See Section 5.2 of the Appendix. ■

Thus for a continuous preference a set is an unhappy set if and only if it is a lower contour or a strict lower contour set. A lexicographic preference is not continuous, although it is mildly continuous. Any lower contour set of a lexicographic preference is neither open nor closed.

2.1.3 Induced preferences

Fix a subset of attributes and consider all points for which attributes in that subset have zero level. The original preference relation gives a preference order between alternatives for which a fixed subset of attributes have level zero. This is formalized by the notion of an induced preference, defined as follows.

Definition 5 Let S be a non empty subset of N . For a preference relation \succsim on X , the *induced preference* \succsim_S on $X_S = \mathbb{R}_+^{|S|}$ is defined as follows: for $y^S, z^S \in X_S$, $y^S \succsim_S z^S$ if and only if $(y^S, 0^{N \setminus S}) \succsim (z^S, 0^{N \setminus S})$.

Thus \succsim_S is a preference relation over all alternatives for which the attributes in the set $N \setminus S$ have zero level. Note that the induced preference \succsim_N coincides with \succsim . We can define unhappy sets for induced preferences. We say a set $A \subseteq X_S$ is an unhappy set for the induced preference \succsim_S if for any $b \in X_S \setminus A$, $b \succ_S a$ for every $a \in A$.

For $x \in X$ and a preference relation \succsim on X , denote by $\bar{L}(x)$ the closure of the lower contour set $L(x)$, that is, $\bar{L}(x) := L(x) \cup \partial L(x)$. The next proposition shows mild continuity of induced preferences ensures that the closure of a lower contour set is an unhappy set.

Proposition 2 *Consider a rational and strong monotone preference relation \succsim on $X = \mathbb{R}_+^n$. Suppose for any $S \subseteq N$, the induced preference \succsim_S is mildly continuous on X_S . Then for every $x \in X$, the set $\bar{L}(x)$ is an unhappy set for \succsim .*

Proof See Section 5.2 of the Appendix. ■

A strong monotone preference relation on \mathbb{R}_+ is continuous, and so it is mildly continuous. Suppose \succsim is strong monotone and let $S \subseteq N$ be a singleton set. Then the induced preference \succsim_S , which is defined on \mathbb{R}_+ , is strong monotone and therefore mildly continuous. When $X = \mathbb{R}_+^2$ in Proposition 2, then any non empty proper subset of X is a singleton set and mild continuity already holds for the corresponding induced preference. This gives the following result.

Corollary 2 *For a rational, strong monotone and mildly continuous preference relation on \mathbb{R}_+^2 , the closure of any lower contour set is an unhappy set.*

Note that for a lexicographic preference, any induced preference is also lexicographic.

Observation 1 If \succsim is lexicographic with linear order $1 <_0 2 <_0 \dots <_0 n$ and $S = \{i_1, \dots, i_s\} \subseteq N$ where $i_1 < \dots < i_s$, then \succsim_S is lexicographic with linear order $i_1 <_0 i_2 <_0 \dots <_0 i_s$.

The converse of Observation 1 does not hold. We give an example where an induced preference is lexicographic, but the original preference is not.

Example 1 Let $N = \{1, 2, 3\}$. Consider a preference relation \succsim on $X = \mathbb{R}_+^3$ that is reflexive ($x \sim x$ for all $x \in X$) and for which following hold for any $x_i, y_i \in \mathbb{R}_+$: (a) $(x_1, x_2, x_3) \succ (y_1, y_2, y_3)$ if $x_1 > y_1$, (b) $(0, x_2, x_3) \succ (0, y_2, y_3)$ if $x_2 > y_2$, (c) $(0, x_2, x_3) \succ (0, x_2, y_3)$ if $x_3 > y_3$, (d) $(x_1, x_2, x_3) \succ (x_1, y_2, y_3)$ if $x_1 > 0$ and $x_2 + x_3 > y_2 + y_3$ and (e) $(x_1, x_2, x_3) \sim (x_1, y_2, y_3)$ if $x_1 > 0$ and $x_2 + x_3 = y_2 + y_3$. For any $S \subseteq N$ with $|S| = 2$, the induced preference \succsim_S is lexicographic, but \succsim is not lexicographic.

3 Characterization of lexicographic preferences

We begin with a property which together with strong monotonicity and mild continuity can characterize a lexicographic preference in the case where there are only two attributes (that is, when $X = \mathbb{R}_+^2$). This property will also be the basis of an axiom for the characterization of a lexicographic preference with more than two attributes.

3.1 Inclusion of marginally improved alternatives

Consider $x, y \in X$ such that $x \not\hat{=} y$ and let $S \subseteq N$. We say (i) y is S -superior to x if $y^S > x^S$ and (ii) y is *exhaustively* S -superior to x if $y^S > x^S$ and $x^{N \setminus S} > y^{N \setminus S}$. If $y > x$, then clearly y is (exhaustively) N -superior to x .

Definition 6 For a preference relation \succsim on X , an unhappy set A satisfies *inclusion of marginally improved alternatives* (IMIA) if the following hold for any $x \not\hat{=} y$ and any $S \subseteq N$: if $x, y \in A$ with $y^S > x^S$ and $x^{N \setminus S} > y^{N \setminus S}$, then $\exists \tilde{x}^S \in X_S$ with $y^S > \tilde{x}^S > x^S$ such that $(\tilde{x}^S, x^{N \setminus S}) \in A$.

For an unhappy set A satisfying IMIA, if $x, y \in A$ and y is S -superior to x , then any new alternative that does better than x but worse than y with respect to the attributes in S but is the same as x for the remaining attributes does not necessarily take us outside of the unhappy set A .

Observation 2 Let A be an unhappy set for a monotone preference relation \succsim on X . If A satisfies IMIA, then the following hold for $x, y \in A$:

- (i) If $y^S > x^S$ and $x^{N \setminus S} > y^{N \setminus S}$, then for any $T \subseteq S$, $\exists \tilde{x}^T \in X_T$ with $y^T > \tilde{x}^T > x^T$ such that $(\tilde{x}^T, x^{N \setminus T}) \in A$.
- (ii) If $y_i > x_i$ and $x_{-i} \not\hat{=} y_{-i}$, then $\exists \tilde{x}_i$ with $y_i > \tilde{x}_i > x_i$ such that $(\tilde{x}_i, x_{-i}) \in A$.

Proof (i) By IMIA, $\exists \tilde{x}^S \in X_S$ with $y^S > \tilde{x}^S > x^S$ such that $(\tilde{x}^S, x^{N \setminus S}) \in A$. As the preference is monotone, for any $T \subseteq S$, we have $(\tilde{x}^S, x^{N \setminus S}) \succsim (\tilde{x}^T, x^{N \setminus T})$. As A is an unhappy set, the result follows. \blacksquare

(ii) Since $y_i > x_i$ and $x_{-i} \not\hat{=} y_{-i}$, there is a non empty set S (in particular, $i \in S$) such that $y^S > x^S$ and $x^{N \setminus S} > y^{N \setminus S}$ (the set $N \setminus S$ can be possibly empty). Then the result follows by taking $T = \{i\}$ in part (i). \blacksquare

Remark 1 When $S = N$ in Definition 6, we have $y > x$. So for a monotone preference, the requirement of IMIA always holds when $S = N$. IMIA imposes additional structure on a monotone preferences only when $S \subset N$.

Remark 2 If x is an interior point of A then there is a neighborhood $B_\varepsilon(x) \subset A$ and we can find a point $\tilde{x} = (\tilde{x}^S, x^{N \setminus S}) \in B_\varepsilon(x)$ such that $y^S > \tilde{x}^S > x^S$. Thus IMIA always holds for open unhappy sets such as strict lower contour sets. IMIA imposes additional structure on the preference only for unhappy sets that are not open. For example, for continuous preferences such as Cobb-Douglas, perfect substitutes or perfect complements IMIA does not hold for lower contour sets. For a lexicographic preference IMIA does not hold for lower contour sets either (which are neither open nor closed). However, any closed unhappy set of a lexicographic preference satisfies this property.

Lemma 1 *Any closed unhappy set for a lexicographic preference satisfies IMIA.*

Proof See Section 5.2 of the Appendix. \blacksquare

When a closed unhappy set satisfies IMIA, it can be bounded with respect to at most one attribute. This gives a structure to such sets, which is stated in the next lemma. See Section 6 for more details of this structure. For a preference relation \succsim , let \mathcal{A}_{\succsim} be a family of subsets of X defined as follows

$$\mathcal{A}_{\succsim} = \{A \mid A \subset X; \exists y \in A \text{ with } y > 0^N; A \text{ is a closed unhappy set that satisfies IMIA}\} \quad (1)$$

Lemma 2 *Consider a rational, strong monotone and mildly continuous preference relation \succsim on X . Let $A \in \mathcal{A}_{\succsim}$. Then \exists a positive number α^A and an attribute $i^* \in N$ such that $A = \{x \in X \mid 0 \leq x_{i^*} \leq \alpha^A\}$. Moreover the attribute i^* is the same for all sets in \mathcal{A}_{\succsim} .*

Proof See Section 5.2 of the Appendix. \blacksquare

3.2 Lexicographic preference with two attributes

Applying Lemma 2 for the case $X = \mathbb{R}_+^2$ and using Corollary 2, we can characterize lexicographic preferences with two attributes. This result is of independent interest. Moreover, it will be also useful for characterizing lexicographic preferences with more than two attributes.

Theorem 1 *A rational preference relation \succsim on \mathbb{R}_+^2 is lexicographic if and only if it is strong monotone, mildly continuous and any closed unhappy set of \succsim satisfies IMIA.*

Proof See Section 5.2 of the Appendix. ■

3.3 Lexicographic preference with more than two attributes

Now we are in a position to characterize lexicographic preferences that have more than two attributes. A key aspect of our approach is all axioms are on induced preferences \succsim_S for $|S| = 2$ (that is, on preferences defined on \mathbb{R}_+^2). As we mentioned before, the guiding principle here is facing a choice problem involving multiple attributes, the individual decision maker proceeds by looking at two attributes at a time.

3.3.1 Strong monotonicity, IMIA

Axiom 1 For any $S \subseteq N$ with $|S| = 2$, the induced preference \succsim_S is strong monotone.

Axiom 2 Consider any $S \subseteq N$ with $|S| = 2$. Any closed subset of $X_S = \mathbb{R}_+^2$ that is an unhappy set for the preference \succsim_S , satisfies IMIA.

Remark Let A be an unhappy set for a monotone preference relation on \mathbb{R}_+^2 . Then A satisfies IMIA if for any $x, y \in A$ with $y_1 > x_1$ and $x_2 > y_2$, there is \tilde{x}_1 with $y_1 > \tilde{x}_1 > x_1$ such that $(\tilde{x}_1, x_2) \in A$.

3.3.2 Mild continuity of induced preference

Consider two points both of which have zero levels for all but two attributes, say 1 and 2. Also suppose these two points are different with respect to both attributes 1, 2 and one of them is strictly preferred to other. The next axiom requires that the initial preference order is preserved when we move to new pairs of points that are found by small changes in attributes 1 or 2, still keeping all other attributes at zero level. This is simply the requirement that the induced preference \succsim_S is mildly continuous for any S that contains only two attributes.

Axiom 3 For any $S \subseteq N$ with $|S| = 2$, the induced preference \succsim_S is mildly continuous on $X_S = \mathbb{R}_+^2$.

3.3.3 Nonreversibility under additional attributes

As before consider two points for which all but two attributes have zero level. Suppose one of these points is strictly preferred to the other. Now suppose we add one or more attributes to each of these points keeping the levels of additional attributes same across the two. Preference nonreversibility holds if the preference order between such new pairs of points stays the same as before.

Definition 7 A preference relation \succsim on X satisfies *nonreversibility under additional attributes* (NRAA) if for any $S \subseteq N$ with $|S| = 2$, the following hold: if $(x^S, 0^{N/S}) \succ (y^S, 0^{N/S})$, then $(x^S, z^{N/S}) \succ (y^S, z^{N/S})$ for any $z^{N/S} \in X_{N \setminus S}$.

Axiom 4 The preference relation \succsim satisfies NRAA.

Remark Fishburn's independence axiom (see Axiom 2, p.416 of Fishburn, 1975) requires for any for any $S \subseteq N$ with $|S| = 1$ and any $z^{N/S}, \tilde{z}^{N/S} \in X_{N \setminus S}$: $(x^S, z^{N/S}) \succ (y^S, z^{N/S})$ if and only if $(x^S, \tilde{z}^{N/S}) \succ (y^S, \tilde{z}^{N/S})$. We give an example where NRAA holds, but Fishburn's independence axiom does not.

Example 2 Let $N = \{1, 2, 3\}$. Consider a preference relation \succsim on $X = \mathbb{R}_+^3$ that has utility function $u(x_1, x_2, x_3) = \min\{x_1, x_2\} + \min\{x_1, x_2\}x_3$. Observe that if $S = \{1, 3\}$ or $\{2, 3\}$, then $u(x^S, 0^{N/S}) = u(y^S, 0^{N/S}) = 0$, so we do not have $(x^S, 0^{N/S}) \succ (y^S, 0^{N/S})$. For $S = \{1, 2\}$, if $(x^S, 0^{N/S}) \succ (y^S, 0^{N/S})$, then $\min\{x_1, x_2\} > \min\{y_1, y_2\}$, which implies that $u(x_1, x_2, z_3) > u(y_1, y_2, z_3)$ for all $z_3 \geq 0$. This shows that NRAA holds for this preference.

To see Fishburn's independence axiom does not hold for this preference, let $S = \{1\}$, so that $N/S = \{2, 3\}$. Take $x^S = x_1 = 4$, $y^S = y_1 = 1$, $z^{N/S} = (z_2, z_3) = (2, 2)$ and $\tilde{z}^{N/S} = (\tilde{z}_2, \tilde{z}_3) = (1/2, 3)$. Then $u(x_1, z_2, z_3) = 6 > u(y_1, z_2, z_3) = 3$ so $(x^S, z^{N/S}) \succ (y^S, z^{N/S})$. But $u(x_1, \tilde{z}_2, \tilde{z}_3) = u(y_1, \tilde{z}_2, \tilde{z}_3) = 2$ so $(x^S, \tilde{z}^{N/S}) \sim (y^S, \tilde{z}^{N/S})$.

3.4 The main result

Now we are in a position to state the main result.

Theorem 2 Consider any rational preference relation \succsim on $X = \mathbb{R}_+^n$ where $n \geq 3$. The following statements are equivalent.

(L1) The preference relation \succsim satisfy Axiom 1, Axiom 2, Axiom 3 and Axiom 4.

(L2) The preference relation \succsim is lexicographic.

Proof

Proof of (L2) \Rightarrow (L1)

Consider a lexicographic preference relation \succsim . Since it is strong monotone, any induced preference \succsim_S is also strong monotone for any $S \subseteq N$ with $|S| = 2$, so Axiom 1 holds. Since for \succsim , the induced preference \succsim_S is mildly continuous for any $S \subseteq N$, Axiom 3 holds. Since for any $S \subseteq N$, whenever $(x^S, 0^{N \setminus S}) \succ (y^S, 0^{N \setminus S})$, we have $(x^S, z^{N \setminus S}) \succ (y^S, z^{N \setminus S})$ for any $z^{N \setminus S} \in X_{N \setminus S}$, Axiom 4 also holds.

Finally recall from Observation 1 that for a lexicographic preference \succsim , the induced preference \succsim_S is also lexicographic for any $S \subseteq N$ (and in particular when $|S| = 2$). Applying Lemma 1 for \succsim_S it follows that any closed unhappy set for \succsim_S satisfies IMIA. Hence Axiom 2 also holds. This shows that (L2) \Rightarrow (L1). \blacksquare

Proof of (L1) \Rightarrow (L2)

We begin with a lemma, which will be useful for the proof.

Notation Let $i, j \in N$ and $S = \{i, j\}$. We write “ $i \succ^* j$ ” to mean “ \succsim_S is a lexicographic preference with linear order $i <_0 j$ ”.

Lemma 3 Consider a rational preference relation \succsim that satisfy Axiom 1, Axiom 2, Axiom 3 and Axiom 4.

- (i) Consider any $S \subseteq N$ with $|S| = 2$. Then $\exists i, j \in S$ such that
 - (a) $i \succ^* j$ so that $(x^S, 0^{N \setminus S}) \succ (y^S, 0^{N \setminus S})$ if and only if either $x_i > y_i$ or $(x_i = y_i$ and $x_j > y_j)$.
 - (b) For any $z^{N \setminus S} \in X_{N \setminus S} : (x^S, z^{N \setminus S}) \succ (y^S, z^{N \setminus S})$ if and only if either $x_i > y_i$ or $(x_i = y_i$ and $x_j > y_j)$.
- (ii) Let $i, j, k \in N$. If $i \succ^* j$ and $j \succ^* k$, then $i \succ^* k$.
- (iii) Based on \succ^* , the attributes of N can be ordered. That is, we can write $N = \{i_1, \dots, i_n\}$ such that $i_1 \succ^* \dots \succ^* i_n$.

Proof See Section 5.2 of the Appendix. ■

Proof of (L1) \Rightarrow (L2) Consider a rational preference relation \succsim that satisfies Axiom 1, Axiom 2, Axiom 3, and Axiom 4. By Lemma 3 we can write $N = \{i_1, \dots, i_n\}$ such that $i_1 \succ^* \dots \succ^* i_n$. Without loss of generality let $i_1 = 1, \dots, i_n = n$. Fix any $m \in \{1, \dots, n-1\}$. Let $S = \{i \in N | i < m\}$ and $T = \{i \in N | i > m\}$. Let $x, y \in X$ such that $x^S = y^S$ and $x_m > y_m$. We can find numbers x_m^0, \dots, x_m^{n-m} such that $x_m = x_m^0 > x_m^1 > \dots > x_m^{n-m} = y_m$. Construct $z(k) \in X$ recursively as follows: $z(0) = x$ and for $k = 1, \dots, n-m$, $z(k)$ is such that

$$z(k)_m = x_m^k, z(k)_{m+k} = y_{m+k}, z(k)^S = z(0)^S = x^S \text{ and } z(k)^{T \setminus \{m+k\}} = z(k-1)^{T \setminus \{m+k\}}$$

That is,

$$z(1)_i = x_i \text{ for } i < m, z(1)_m = x_m^1, z(1)_{m+1} = y_{m+1}, z(1)_i = x_i \text{ for } i > m+1$$

$$z(2)_i = x_i \text{ for } i < m, z(2)_m = x_m^2, z(2)_i = y_i \text{ for } i = m+1, m+2, z(2)_i = x_i \text{ for } i > m+2$$

and in general

$$z(k)_i = x_i \text{ for } i < m, z(k)_m = x_m^k, z(k)_i = y_i \text{ for } i = m+1, \dots, m+k, z(k)_i = x_i \text{ for } i > m+k$$

Observe that $z(k)$ and $z(k-1)$ have same levels for all but two attributes (m and $m+k$). Since $m \succ^* m+k$ and $z(k-1)_m = x_m^{k-1} > z(k)_m = x_m^k$, by Lemma 3(i)(b) we have $z(k-1) \succ z(k)$. Noting that $z(n-m) = y$, we conclude that $x = z(0) \succ \dots \succ z(n-m) = y$. Hence $x \succ y$.

Thus for any x, y with $x_i = y_i$ for all $i < m$ and $x_m > y_m$, we have $x \succ y$. Applying this result for $m = 1, \dots, n-1$ it follows that \succsim is a lexicographic preference with linear order $1 <_0 \dots <_0 n$. \blacksquare

4 Robustness of the axioms

In conclusion we give examples of rational preferences to show that if one of the four axioms does not hold, then we do not get a lexicographic preference.

Example 3 (All axioms except Axiom 1 hold) Let $N = \{1, 2, 3\}$. Consider a rational and continuous preference relation \succsim on $X = \mathbb{R}_+^3$ that has utility function $u(x_1, x_2, x_3) = x_1$. Note that the induced preference \succsim_S is not strong monotone for $S = \{1, 2\}$, so Axiom 1 is violated. We show the remaining three axioms hold.

Note that for any $S \subseteq N$ with $|S| = 2$, \succsim_S is continuous, so Axiom 3 holds. As \succsim_S is continuous, any closed unhappy set for \succsim_S is a lower contour set. To see Axiom 2 and Axiom 4 hold, first let $S = \{1, 2\}$. Since $x^S = (x_1, x_2) \succ_S y^S = (y_1, y_2)$ if and only if $x_1 > y_1$, we have $L^*(x^S) = \{y^S \in \mathbb{R}_+^2 \mid 0 \leq y_1 \leq x_1\}$. If $y, z \in L^*(x^S)$ such that $z_1 < y_1$, then we can find $z_1 < \tilde{z}_1 < y_1 \leq x_1$, so $(\tilde{z}_1, z_2) \in L^*(x^S)$. If $y, z \in L^*(x^S)$ such that $z_2 < y_2$, then for any $z_2 < \tilde{z}_2 < y_2$, $(z_1, \tilde{z}_2) \in L^*(x^S)$. This shows IMIA holds for any closed unhappy set for \succsim_S . Similar reasoning applies when $S = \{1, 3\}$.

Note that if $(x_1, x_2, 0) \succ (y_1, y_2, 0)$, then $x_1 > y_1$. So for any x_3 we have $u(x_1, x_2, x_3) = x_1 > u(y_1, y_2, x_3) = y_1$. This shows the requirement for NRAA is satisfied when $S = \{1, 2\}$. Similar reasoning applies when $S = \{1, 3\}$.

Finally let $S = \{2, 3\}$. Then $u(0, x_2, x_3) = 0$. Hence for any $x^S, y^S \in X_S$ we have $x^S \sim_S y^S$ so that $L^*(x^S) = \mathbb{R}_+^2$. This shows IMIA trivially holds for any closed unhappy set for \succsim_S . Since for any $x^S = (x_2, x_3), y^S = (y_2, y_3)$, we have $(0, x_2, x_3) \sim_S (0, y_2, y_3)$, the requirement for NRAA is satisfied vacuously when $S = \{2, 3\}$. This shows both Axiom 2 and Axiom 4 hold.

Example 4 (All axioms except Axiom 2 hold) Let $N = \{1, 2, 3\}$. Consider a rational, continuous preference relation \succsim on $X = \mathbb{R}_+^3$ with utility function $u(x) = x_1 + x_2 + x_3$ (perfect substitutes). Since \succsim is strong monotone, so is \succsim_S for any non empty $S \subseteq N$, so Axiom 1 holds. Since it is continuous, Axiom 3 also holds. Since the utility function is additively separable, NRAA (Axiom 4) holds.

However, Axiom 2 does not hold. For instance, let $S = \{1, 2\}$. Consider the induced preference \succsim_S , which has utility function $u_S(x^S) = x_1 + x_2$. It is continuous, so by Corollary 2 any closed unhappy set of \succsim_S is a lower contour set. Consider the lower contour set $L^*(x^S)$ for $x^S = (1, 3)$. Let $y^S = (3, 1)$. Then $x^S, y^S \in L^*(x^S)$. Also note that $x^S \not\hat{\succ} y^S$ with $y_1 > x_1$ and $x_2 > y_2$. But there is no \tilde{x}_1 with $y_1 > \tilde{x}_1 > x_1$ such that $(\tilde{x}_1, x_2) \in L^*(x^S)$. This is because for any $\tilde{x}_1 > x_1$, we have $\tilde{x}_1 + x_2 > x_1 + x_2 = u_S(x^S)$, so such a point will be outside $L^*(x^S)$.

Example 5 (All axioms except Axiom 3 hold) Let $N = \{1, 2, 3\}$. Consider the lexi-max preference \succsim introduced by Bossert, Pattanaik and Xu (1994). For $x = (x_1, x_2, x_3) \in \mathbb{R}_+^3$, denote by x_i^* the i -th highest order statistics of x so that $x_1^* \geq x_2^* \geq x_3^*$. For the lexi-max preference, for any $x_i, y_i \in \mathbb{R}_+$ we have: (a) $x \succ y$ if either $[x_1^* > y_1^*]$, or $[x_1^* = y_1^*, x_2^* > y_2^*]$, or $[x_1^* = y_1^*, x_2^* = y_2^*, x_3^* > y_3^*]$ and (b) $x \sim y$ if $[x_1^* = y_1^*, x_2^* = y_2^*, x_3^* = y_3^*]$.

Note that \succsim is rational. It does not satisfy Axiom 3, but satisfies the remaining three axioms. To see Axiom 3 is violated, let $S = \{1, 2\}$ and consider the induced preference \succsim_S . Let $x = (3, 2, 0)$, $y = (1, 3, 0)$, so that $x^S = (3, 2)$, $y^S = (1, 3)$. Note that $x^S \not\hat{=} y^S$. Since $x \succ y$, we have $x^S \succ_S y^S$. Any neighborhood of y^S contains a point $z^S = (1, z_2)$ such that $z_2 > 3$, so that $z_1^* = z_2 > 3 = x_1^*$ and we have $z^S \succ_S x^S$. This shows \succsim_S is not mildly continuous, so Axiom 3 does not hold.

To see that the remaining three axioms hold, note that since \succsim is strong monotone, so is \succsim_S for any non empty $S \subseteq N$, so Axiom 1 holds. To see Axiom 2 and Axiom 4 hold, consider any $S \subseteq N$ with $|S| = 2$. Without loss of generality let $S = \{1, 2\}$. For $x^S = (x_1, x_2)$, $y^S = (y_1, y_2)$, recall that the induced preference \succsim_S is defined as

$$x^S \succsim_S y^S \text{ if and only if } (x_1, x_2, 0) \succsim (y_1, y_2, 0)$$

So the induced preference \succsim_S is a lexi-max preference on $X_S = \mathbb{R}_+^2$. That is, (a) $x^S \succ_S y^S$ if either $[x_1^* > y_1^*]$, or $[x_1^* = y_1^*, x_2^* > y_2^*]$ and (b) $x^S \sim_S y^S$ if $[x_1^* = y_1^*, x_2^* = y_2^*]$. Let A be any closed unhappy set for \succsim_S . In what follows we show A satisfies IMIA.

Let $x^S, y^S \in A$ be such that $y_1 > x_1$ and $x_2 > y_2$. If $x_1 \geq x_2$, then we have $y_1^* = y_1 > x_1^* = x_1$, so that $y^S \succ_S x^S$. Consider any \tilde{x}_1 with $y_1 > \tilde{x}_1 > x_1$ and let $\tilde{x}^S = (\tilde{x}_1, x_2)$. Then $y^S \succ_S \tilde{x}^S$, so we have $\tilde{x}^S \in A$.

Finally let $x^S, y^S \in A$ be such that $y_1 > x_1$, $x_2 > y_2$ and $x_2 > x_1$. Consider any \tilde{x}_1 with $\min\{x_2, y_1\} > \tilde{x}_1 > x_1$ and let $\tilde{x}^S = (\tilde{x}_1, x_2)$. Then any neighborhood of \tilde{x}^S contains a point $z^S = (z_1, z_2)$ such that $z_1 = \tilde{x}_1 < z_2 < x_2$. Then $z_1^* = z_2 < x_1^* = x_2$ and we have $x^S \succ_S z^S$, so $z^S \in A$. This shows \tilde{x}^S is a limit point of A . Since A is closed, we must have $\tilde{x}^S \in A$. This shows A satisfies IMIA, so Axiom 2 holds.

To verify Axiom 4 (NRAA) holds, let $(x_1, x_2, 0) \succ (y_1, y_2, 0)$. Then either $[x_1^* > y_1^*]$ or $[x_1^* = y_1^*, x_2^* > y_2^*]$. Take any $x_3 \in \mathbb{R}_+$ and let $a, b \in \mathbb{R}_+^3$ be such that

$$a_1 = x_1, a_2 = x_2, a_3 = x_3 \text{ and } b_1 = y_1, b_2 = y_2, b_3 = x_3$$

We have to show $a \succ b$. First let $x_3 \geq x_1^* \geq y_1^*$. Then $a_1^* = b_1^* = x_3$, $a_2^* = x_1^*$, $b_2^* = y_1^*$, $a_3^* = x_2^*$ and $b_3^* = y_2^*$. Since either $[a_2^* > b_2^*]$ or $[a_2^* = b_2^*, a_3^* > b_3^*]$, we conclude that $a \succ b$.

Next suppose $x_3 < x_1^*$. If $x_1^* > y_1^*$, then $a_1^* = x_1^* > b_1^* = \max\{y_1^*, x_3\}$, so we have $a \succ b$.

Finally suppose $x_3 < x_1^*$ and let $x_1^* = y_1^*$, $x_2^* > y_2^*$. Then $a_1^* = b_1^* = x_1^*$. If $x_2^* \leq x_3$, then $a_2^* = b_2^* = x_3$ and $a_3^* = x_2^* > b_3^* = y_2^*$, so we have $a \succ b$. If $x_2^* > x_3$, then $a_2^* = x_2^*$ and $b_2^* = \max\{y_2^*, x_3\} < a_2^*$, so we have $a \succ b$. This shows that Axiom 4 (NRAA) also holds.

The preference of Example 1 (All axioms except Axiom 4 hold) This preference is rational. For any $S \subseteq N$ with $|S| = 2$, the induced preference \succsim_S is lexicographic, so it is strong monotone and mildly continuous, implying Axiom 1 and Axiom 3 holds. By Lemma 1, any closed unhappy set of \succsim_S satisfies IMIA, so Axiom 2 also holds. Note from (c) and (d) that $(0, 6, 4) \succ (0, 3, 8)$ but $(1, 3, 8) \succ (1, 6, 4)$ so it violates Axiom 4 (NRAA).

The following two examples show without rationality we do not get lexicographic preference even with all four axioms. For these two examples, it will be useful to recall $x, y \in X$, we say x, y are non-comparable, denoted by $x \bowtie y$, if $\exists i, j$ such that $y_i > x_i$ and $x_j > y_j$.

Example 6 Let $N = \{1, 2, 3\}$. Consider a preference relation \succsim on $X = \mathbb{R}_+^3$ that is reflexive, strong monotone and $x \sim y$ whenever $x \bowtie y$. Note that \succsim is complete but not transitive. For example, let $x = (3, 2, 0)$, $y = (2, 2, 0)$ and $z = (4, 1, 0)$. By strong monotonicity, $x \succ y$. Since $y \bowtie z$, we have $y \sim z$. For transitivity to hold we must have $x \succ z$. However, since $x \bowtie z$, we have $x \sim z$, so transitivity does not hold.

For any $S \subseteq N$ with $|S| = 2$, the induced preference \succsim_S is strong monotone, so Axiom 1 holds. To see the remaining axioms also hold, consider any $S \subseteq N$ with $|S| = 2$. Without loss of generality let $S = \{1, 2\}$. Let A be an unhappy set for \succsim_S and let $x^S, y^S \in A$ such that $y_1 > x_1$, $x_2 > y_2$. Then $(x_1, x_2, 0) \bowtie (y_1, y_2, 0)$, so we have $x^S \sim_S y^S$. For any \tilde{x}_1 with $y_1 > \tilde{x}_1 > x_1$, let $\tilde{x}^S = (\tilde{x}_1, x_2)$. Since $(\tilde{x}_1, x_2, 0) \bowtie (y_1, y_2, 0)$, we have $\tilde{x}^S \sim_S y^S$. As A is an unhappy set, we must have $\tilde{x}^S \in A$. This shows any unhappy set for \succsim_S satisfies IMIA, so Axiom 2 holds.

To see Axiom 3 holds, let $x^S \not\sim_S y^S$ and $x^S \succ_S y^S$. Then we must have $x^S > y^S$. Since there are neighborhoods $B_\varepsilon(x^S), B_\varepsilon(y^S)$ such that $\tilde{x}^S > \tilde{y}^S$ for all $\tilde{x}^S \in B_\varepsilon(x^S)$ and $\tilde{y}^S \in B_\varepsilon(y^S)$, by strong monotonicity of \succsim_S it follows that \succsim_S is mildly continuous, so Axiom 3 holds.

Finally to see Axiom 4 (NRAA) holds, let $(x_1, x_2, 0) \succ (y_1, y_2, 0)$. Then we must have $x^S = (x_1, x_2) \geq y^S = (y_1, y_2)$ and $x^S \neq y^S$. So for any x_3 , $(x_1, x_2, x_3) \geq (y_1, y_2, x_3)$ and $(x_1, x_2, x_3) \neq (y_1, y_2, x_3)$. This shows Axiom 4 also holds.

Example 7 Let $N = \{1, 2, 3\}$. Consider a preference relation \succsim on $X = \mathbb{R}_+^3$ that is reflexive, strong monotone and whenever $x \bowtie y$, we have neither $x \succsim y$ nor $y \succsim x$. This preference relation is transitive but not complete.

For any $S \subseteq N$ with $|S| = 2$, the preference \succsim_S is strong monotone, so Axiom 1 holds. By the same reasoning as in Example 6, Axiom 3 and Axiom 4 (NRAA) also hold. To see Axiom 2 holds, consider any $S \subseteq N$ with $|S| = 2$. Without loss of generality let $S = \{1, 2\}$. Let A be an unhappy set for \succsim_S and let $x^S, y^S \in A$ such that $y_1 > x_1$, $x_2 > y_2$. For any \tilde{x}_1 with $y_1 > \tilde{x}_1 > x_1$, let $\tilde{x}^S = (\tilde{x}_1, x_2)$. Since $(\tilde{x}_1, x_2, 0) \bowtie (y_1, y_2, 0)$, we have neither $\tilde{x}^S \succ_S y^S$ nor $y^S \succ_S \tilde{x}^S$. So in particular we do not have $\tilde{x}^S \succ_S y^S$. Since A is an unhappy set, we must have $\tilde{x}^S \in A$. This shows Axiom 2 holds.

5 Appendix

5.1 Boundedness of unhappy sets under IMIA

When an unhappy set is inclusive of marginally improved alternatives, there are some useful implications with respect to boundedness of that set. Let $A \subseteq X$ and $x = (x_i, x_{-i}) \in A$. Denote $A(x_{-i}) := \{y_i \in X_i \mid (y_i, x_{-i}) \in A\}$. We say $A(x_{-i})$ is bounded above if there is a number k such that $y_i \leq k$ for all $y_i \in A(x_{-i})$.

Proposition 3 *Consider a rational and strong monotone preference relation \succsim on X . Let A be a proper subset of X . Suppose A is a closed unhappy set which satisfies IMIA. Let $x \in A$ be such that $x > 0^N$.*

- (i) *Let $i \in N$. If $A(x_{-i})$ is bounded above, then*
 - (a) $\exists \alpha(x_{-i})$ (the least upper bound of $A(x_{-i})$) such that $A(x_{-i}) = [0, \alpha(x_{-i})]$.
 - (b) *For any $z \in A$, $A(z_{-i})$ is also bounded above with the same least upper bound. That is, $\alpha(x_{-i}) = \alpha(z_{-i}) = \alpha_i > 0$ and $A(x_{-i}) = A(z_{-i}) = [0, \alpha_i]$. Moreover the points $(\alpha_i, x_{-i}), (\alpha_i, z_{-i})$, are both boundary points of A .*
- (ii) *Let $i \in N$. If $A(x_{-i})$ is not bounded above, then for any $z \in A$, $A(z_{-i})$ is also not bounded above and $A(x_{-i}) = A(z_{-i}) = \mathbb{R}_+$.*
- (iii) *Suppose \succsim is also mildly continuous. Then there can be at most one $i \in N$ where the following hold: $\exists z \in A$ with $z > 0^N$ such that $A(z_{-i})$ is bounded above.*

Proof (i)(a) Since $x = (x_i, x_{-i}) \in A$, we have $x_i \in A(x_{-i})$, so $A(x_{-i})$ is a non empty subset of \mathbb{R} . As $A(x_{-i})$ is bounded above, by the least-upper-bound property of \mathbb{R} (see, e.g., Theorem 1.19, Rudin, 1976) $A(x_{-i})$ has a least upper bound $\alpha(x_{-i}) \geq x_i > 0$. As A is an unhappy set, strong monotonicity of the preference implies whenever $y_i \in A(x_{-i})$, any $y'_i < y_i$ also belongs to $A(x_{-i})$. Hence $(y_i, x_{-i}) \in A$ for $0 \leq y_i < \alpha(x_{-i})$ and $(y_i, x_{-i}) \notin A$ for $y_i > \alpha(x_{-i})$. Finally note that any neighborhood of $(\alpha(x_{-i}), x_{-i})$ contains a point (y_i, x_{-i}) with $y_i < \alpha(x_{-i})$. So $(\alpha(x_{-i}), x_{-i})$ is a limit point of A . Since A is a closed set, $(\alpha(x_{-i}), x_{-i}) \in A$. This proves that $A(x_{-i}) = [0, \alpha(x_{-i})]$.

(i)(b) Since $z \in A$, we have $z_i \in A(z_{-i})$, so $A(z_{-i})$ is non empty. First suppose $z_{-i} \not\hat{=} x_{-i}$. Since A satisfies IMIA and $(\alpha(x_{-i}), x_{-i}) \in A$, if $(y_i, z_{-i}) \in A$ with $y_i > \alpha(x_{-i})$, then by Observation 2(ii), $\exists \tilde{x}_i > \alpha(x_{-i})$ such that $(\tilde{x}_i, x_{-i}) \in A$, implying $\tilde{x}_i \in A(x_{-i})$, contradicting (a). So $A(z_{-i})$ must be bounded above by $\alpha(x_{-i})$ and it has a least upper bound $\alpha(z_{-i}) \leq \alpha(x_{-i})$.

If $\alpha(z_{-i}) = 0$, it is the only point of $A(z_{-i})$ and $(\alpha(z_{-i}), z_{-i}) \in A$. If $\alpha(z_{-i}) > 0$, using strong monotonicity of the preference and closedness of the unhappy set A as in (a) we again have $(\alpha(z_{-i}), z_{-i}) \in A$. Since $(\alpha(x_i), x_{-i}) \in A$, if $\alpha(z_{-i}) < \alpha(x_{-i})$, then by Observation 2(ii), $\exists \tilde{z}_i > \alpha(z_{-i})$ such that $(\tilde{z}_i, z_{-i}) \in A$. This implies $\tilde{z}_i \in A(z_{-i})$, a contradiction. So

we must have $\alpha(z_{-i}) = \alpha(x_{-i}) > 0$. Denoting their common value by α_i , it follows that $A(x_{-i}) = A(z_{-i}) = [0, \alpha_i]$ for any $z_{-i} \neq x_{-i}$.

Now consider the case where z_{-i}, x_{-i} are not totally different. Since $x > 0^N$, we can construct \tilde{x} such that for all $j \in N$: $0 < \tilde{x}_j < x_j$ and $\tilde{x}_j \neq z_j$. Since $x \in A$ and $x > \tilde{x}$, by strong monotonicity, $\tilde{x} \in A$. Since $\tilde{x}_{-i} \neq x_{-i}$, applying the result of the last paragraph we have $A(\tilde{x}_{-i}) = [0, \alpha_i]$. Finally observing that $\tilde{x} > 0^N$ and $z_{-i} \neq \tilde{x}_{-i}$, reapplying the result of the last paragraph we have $A(z_{-i}) = [0, \alpha_i]$.

As any neighborhood of (α_i, x_{-i}) contains two points $y = (y_i, x_{-i}), y' = (y'_i, x_{-i})$ such that $y'_i < \alpha_i < y_i$, we have $y \in A$ and $y' \in X \setminus A$. This shows $(\alpha_i, x_{-i}) \in \partial A$. The same holds for (α_i, z_{-i}) .

(ii) As $A(x_{-i})$ is not bounded above, for any $k > 0$, there is $y_i > k$ such that $y_i \in A(x_{-i})$. As A is an unhappy set, strong monotonicity of the preference implies any $y'_i < y_i$ also belongs to $A(x_{-i})$. This shows $A(x_{-i}) = \mathbb{R}_+$.

For $z \in A$, if the set $A(z_{-i})$ is bounded above, it has a least upper bound $\alpha(z_{-i}) \geq z_i$ and by the same reasoning as before, $(\alpha(z_{-i}), z_{-i}) \in A$. Consider any $k > \alpha(z_{-i})$. Since $A(x_{-i}) = \mathbb{R}_+$, $(k, x_{-i}) \in A$. As $x > 0^N$, we can construct \tilde{x} such that $\tilde{x}_i = k$ and for all $j \in N \setminus \{i\}$: $0 < \tilde{x}_j < x_j$ and $\tilde{x}_j \neq z_j$. Since $(k, x_{-i}) \geq \tilde{x}$, by strong monotonicity $\tilde{x} \in A$. Since $(\alpha(z_{-i}), z_{-i}) \in A$, $\tilde{x}_{-i} \neq z_{-i}$ and $\tilde{x}_i = k > \alpha(z_{-i})$, by Observation 2(ii) $\exists \tilde{z}_i > \alpha(z_{-i})$ such that $(\tilde{z}_i, z_{-i}) \in A$, a contradiction. This shows $A(z_{-i})$ must be also not bounded above and $A(z_{-i}) = \mathbb{R}_+$.

(iii) We have to show that there can be at most one $i \in N$ where the following hold:

$$\exists z \in A \text{ with } z > 0^N \text{ such that } A(z_{-i}) \text{ is bounded above} \quad (2)$$

If (2) does not hold, then there are $i, k \in N$ and $y, z \in A$ with $y > 0^N, z > 0^N$ such that $A(y_{-i})$ and $A(z_{-k})$ are both bounded above. By part (i), there are positive numbers α_i, α_k such that $A(y_{-i}) = [0, \alpha_i]$ and $A(z_{-k}) = [0, \alpha_k]$.

Since $y > 0^N$ and $z > 0^N$, we can construct \tilde{x}, \tilde{y} such that $0 < \tilde{y}_j < \tilde{x}_j < \min\{y_j, z_j\}$ for all $j \in N$ and $\tilde{y}_k < \tilde{x}_k < \alpha_k$. Then $y > \tilde{x} > \tilde{y}$. As $y \in A$, by strong monotonicity, $\tilde{x}, \tilde{y} \in A$. Then by part (1)(i), $A(\tilde{x}_{-i}) = A(\tilde{y}_{-i}) = [0, \alpha_i]$. Let $\bar{x} = (\alpha_i, \tilde{x}_{-i}), \bar{y} = (\alpha_i, \tilde{y}_{-i})$. Then $\bar{x}, \bar{y} \in \partial A$.

Since $z > 0^N$, we can construct \tilde{z} such that $0 < \tilde{z}_i < \min\{z_i, \alpha_i\}$ and $\tilde{z}_j = z_j$ for all $j \in N \setminus \{i\}$. Then $z \geq \tilde{z}$. As $z \in A$, by strong monotonicity, $\tilde{z} \in A$. Then by part (1)(i), $A(\tilde{z}_{-k}) = [0, \alpha_k]$. Then $\bar{z} = (\alpha_k, \tilde{z}_{-k}) \in \partial A$.

Note that (a) $\bar{y}_j < \bar{x}_j < \bar{z}_j$ for $j \in N \setminus \{i, k\}$, (b) $\bar{y}_i = \bar{x}_i = \alpha_i > \bar{z}_i$ and (c) $\bar{y}_k < \bar{x}_k < \alpha_k = \bar{z}_k$. This shows $\bar{x} \not\sim \bar{z}$ and $\bar{y} \not\sim \bar{z}$. Since $\bar{x}, \bar{y}, \bar{z}$ are all boundary points of the unhappy set A , by Proposition 1(ii), we conclude: $\bar{x} \sim \bar{z}$ and $\bar{y} \sim \bar{z}$. Then transitivity implies $\bar{x} \sim \bar{y}$. However, since $\bar{x}_i = \bar{y}_i$ and $\bar{x}_{-i} > \bar{y}_{-i}$, by strong monotonicity of the preference relation have $\bar{x} \succ \bar{y}$, which is a contradiction. This proves that there is at most one $i \in N$ where (2) holds.

5.2 Proofs

Proof of Proposition 1 (i) Suppose there are two unhappy sets A, B and $x \in A, y \in B$ such that $x \notin B, y \notin A$. By definition of unhappy sets we must have $x \succ y$ and $y \succ x$, a contradiction.

For (ii), by the Result 1 we know there exists $x \in \partial A$.

(ii)(a) If the assertion is not true, then by completeness one of x, y is strictly preferred to the other. Without loss of generality, let $x \succ y$. Then mild continuity implies $\exists \varepsilon > 0$ such that all points in $B_\varepsilon(x)$ is strictly preferred to all points in $B_\varepsilon(y)$. Since $x, y \in \partial A$, $\exists \tilde{x} \in B_\varepsilon(x), \tilde{y} \in B_\varepsilon(y)$ such that $\tilde{x} \in A, \tilde{y} \notin A$ and we have $\tilde{x} \succ \tilde{y}$. A contradiction since A is an unhappy set.

(b) If the assertion is not true, then $y \succ x$ and mild continuity implies $\exists \varepsilon > 0$ such that y is strictly preferred to all points in $B_\varepsilon(x)$. Since $x \in \partial A$, $\exists \tilde{x} \in B_\varepsilon(x)$ such that $\tilde{x} \notin A$ and we have $y \succ \tilde{x}$, a contradiction since $y \in A$ and A is an unhappy set.

(c) As $x \succ y$, mild continuity implies $\exists \varepsilon > 0$ such that all points in $B_\varepsilon(x)$ is strictly preferred to y . Since $x \in \partial A$, $\exists \tilde{x} \in B_\varepsilon(x)$ such that $\tilde{x} \in A$ and we have $\tilde{x} \succ y$. Since A is an unhappy set, we must have $y \in A$. ■

Proof of Corollary 1 (i)-(ii) As the preference relation is continuous, strict preference orders are preserved around small neighborhoods of *any two* points rather than only totally different points. Thus for any $x \in \partial A$, conclusions of Proposition 1(iv) hold for *any* $y \in X$.

Part (i) of the corollary follows by applying Proposition 1(iv)(a) for any $y \in X$. Part (ii) follows by applying Proposition 1(iv)(b)-(c) for any $y \in X$.

(iii) By Result 1, the set A cannot be both open and closed, so we can have either (a) A is neither open nor closed, or (b) A is either open or closed, but not both. First we rule out (a). To see this, suppose there is an unhappy set A that is neither open nor closed. Since A is not open, $\exists x \in A$ such that every neighborhood of x contains a point outside A , so we have $x \in \partial A$. Since A is not closed, $\exists y \notin A$ which is a limit point of A , that is, every neighborhood of y contains a point in A , so we have $y \in \partial A$. Since $x, y \in \partial A$, by (i) we have $x \sim y$. However, since $x \in A, y \notin A$ and A is an unhappy set, we must have $y \succ x$, a contradiction. This rules out (a). So the set A is either open or closed, but not both.

Suppose A is open. Let $x \in \partial A$. We must have $x \notin A$ since every neighborhood of x contains a point outside A . As A is an unhappy set, if $y \sim x$, we must have $y \notin A$. This shows $A \cap I(x) = \emptyset$. Since $A \subseteq L(x) = \underline{L}(x) \cup I(x)$ (by (ii)), we must have $A \subseteq \underline{L}(x)$ and again by (ii) we have $A = \underline{L}(x)$.

Suppose A is closed. Let $x \in \partial A$. As A is closed, we have $x \in A$. Since A is an unhappy set, if $y \sim x$, we must have $y \in A$. This shows $I(x) \subseteq A$. Since $\underline{L}(x) \subseteq A$, (by (ii)), we conclude $L(x) = \underline{L}(x) \cup I(x) \subseteq A$. Again by (ii) it follows that $A = L(x)$. ■

Proof of Proposition 2 Note by definition of lower contour set that any point in $X \setminus \bar{L}(x)$ is strictly preferred to any point in $L(x)$. To prove $\bar{L}(x)$ is an unhappy set, it remains to show that if $a \in X \setminus \bar{L}(x)$ and $b \in \partial L(x)$, then $a \succ b$.

Since \succsim is rational (complete and transitive), if the result does not hold, there are $a \in X \setminus \bar{L}(x)$, $b \in \partial L(x)$ such that $b \succsim a$. Denote

$$E = \{i \in N | a_i = b_i = 0\} \text{ and } S = N \setminus E$$

so that $S \cup E = N$. Since \succsim is monotonic preference relation, $0^N \in L(x)$, so $a \neq 0^N$. Thus $S \neq \emptyset$ and there is $i \in S$ with $a_i > 0$.

Since $a \in X \setminus L(x)$ and $a \notin \partial L(x)$, there is a neighborhood $B_\varepsilon(a) \subseteq X \setminus L(x)$. We can construct $\tilde{a} \in B_\varepsilon(a)$ (so that $\tilde{a} \in X \setminus L(x)$) such that (i) if $i \in E$, then $\tilde{a}_i = a_i = 0$; (ii) if $i \in S$ and $a_i = 0$, then $\tilde{a}_i = 0$; and (iii) if $i \in S$ and $a_i > 0$, then $0 < \tilde{a}_i < a_i$ and $\tilde{a}_i \neq b_i$. Note that $\tilde{a}^S \not\hat{=} b^S$. By strong monotonicity of \succsim , $a \succ \tilde{a}$ and transitivity implies $b \succ \tilde{a}$.

If $E = \emptyset$, then $N = S$. So $\tilde{a} = \tilde{a}^S$, $b = b^S$ and we have $\tilde{a} \hat{=} b$. Since $L(x)$ is an unhappy set, $b \in \partial L(x)$ and $b \succ \tilde{a}$, by Proposition 1(ii)(c) we must have $\tilde{a} \in L(x)$ which is a contradiction since $\tilde{a} \in X \setminus L(x)$.

If $E \neq \emptyset$, then $\tilde{a} = (\tilde{a}^S, 0^E)$ and $b = (b^S, 0^E)$. Since $b \in \partial L(x)$, every neighborhood $B_\varepsilon(b)$ contains a point $\bar{b} \in L(x)$. From such \bar{b} , construct c as $c = (\bar{b}^S, 0^E)$. By monotonicity of \succsim , we have $\bar{b} \succ c$. Hence $c \in L(x)$. Moreover $d(b^S, \bar{b}^S) = d(b, c) \leq d(b, \bar{b}) < \varepsilon$, so $\bar{b}^S \in B_\varepsilon(b^S)$. This shows for every neighborhood $B_\varepsilon(b^S)$ of b^S :

$$\exists \bar{b}^S \in B_\varepsilon(b^S) \text{ such that } (\bar{b}^S, 0^E) \in L(x) \quad (3)$$

Consider the induced preference \succsim_S on X_S . Note that for $y^S, z^S \in X_S$:

$$y^S \succsim_S z^S \Leftrightarrow (y^S, 0^E) \succsim (z^S, 0^E) \quad (4)$$

It is given \succsim_S is mildly continuous on X_S . Since $b = (b^S, 0^E) \succ \tilde{a} = (\tilde{a}^S, 0^E)$, by (4) we have $b^S \succ_S \tilde{a}^S$. Since $b^S \not\hat{=} \tilde{a}^S$, by mild continuity of \succsim_S , there is a neighborhood $B_\varepsilon(b^S)$ such that for every $\tilde{b}^S \in B_\varepsilon(b^S)$ we have $\tilde{b}^S \succ_S \tilde{a}^S$, so by (4)

$$(\tilde{b}^S, 0^E) \succ (\tilde{a}^S, 0^E) = \tilde{a} \text{ for all } \tilde{b}^S \in B_\varepsilon(b^S) \quad (5)$$

By (3) and (5), it follows that $\tilde{a} \in L(x)$, which is a contradiction since $\tilde{a} \in X \setminus L(x)$. ■

Proof of Lemma 1 Let the set of all attributes be $N = \{1, \dots, n\}$. Let \succsim be a lexicographic preference on $X = \mathbb{R}_+^n$. Suppose 1 is the most important attribute of \succsim .

Let $A \subseteq X$ be a closed unhappy set for \succsim . Let $x, y \in A$ be such that $x \not\hat{=} y$ with $y^S > x^S$ and $x^{N \setminus S} > y^{N \setminus S}$. Then we have either $x \succ y$ or $y \succ x$.

If $x \succ y$, then we must have $1 \in N \setminus S$ and $x_1 > y_1 \geq 0$. Consider any $\tilde{x}^S > x^S$ and let $\tilde{x} = (\tilde{x}^S, x^{N \setminus S})$. So in particular, $\tilde{x}_1 = x_1$. Note that any neighborhood of \tilde{x} contains a point

z such that $z_1 < \tilde{x}_1 = x_1$. So we have $x \succ z$. As A is an unhappy set, we have $z \in A$. Thus any neighborhood of \tilde{x} contains a point $z \neq \tilde{x}$ such that $z \in A$. So \tilde{x} is a limit point of A . As A is closed, $\tilde{x} \in A$. This shows $\tilde{x} = (\tilde{x}^S, x^{N \setminus S}) \in A$ for any $\tilde{x}^S > x^S$.

If $y \succ x$, then $1 \in S$ and $y_1 > x_1 \geq 0$. Consider any \tilde{x}^S such that $y^S > \tilde{x}^S > x^S$, so in particular $y_1 > \tilde{x}_1 > x_1$. Let $\tilde{x} = (\tilde{x}^S, x^{N \setminus S})$. Then $y \succ \tilde{x}$. As A is an unhappy set, we have $\tilde{x} \in A$. This shows $\tilde{x} = (\tilde{x}^S, x^{N \setminus S}) \in A$ for any $y^S > \tilde{x}^S > x^S$. This proves that any closed unhappy set A for \succsim satisfies IMIA. \blacksquare

Proof of Lemma 2 Let $A \in \mathcal{A}_{\succsim}$. By Proposition 3 there can be at most one $i \in N$ where (2) holds. Suppose (2) fails to hold for all $i \in N$. We know $\exists x \in A$ with $x > 0^N$. Consider any point $z = (z_1, \dots, z_n) \in X$. Since $A(x_{-1})$ is unbounded, $z^1 = (z_1, x_2, \dots, x_n) \in A$. Since $A(z_{-2}^1)$ is unbounded, $z^2 = (z_1, z_2, x_3, \dots, x_n) \in A$. After n steps we can show $z^n = (z_1, \dots, z_n) \in A$. Using this reasoning for any $z \in X$, we have $X = A$, which cannot happen since A is a proper subset of X . This shows there is exactly one $i \in N$ for which (2) holds. Denote this i by i^* .

Then by Proposition 3(i) it follows that $\exists \alpha^A > 0$ such that for any $z \in A$: (a) $A(z_{-i^*}) = [0, \alpha^A]$ and (b) $A(z_{-j}) = \mathbb{R}_+$ for any $j \in N \setminus \{i^*\}$. Note from (a) that if $y \in X$ has $y_{i^*} > \alpha^A$, then $y \in X \setminus A$. Since $x \in A$, by (a), $(\alpha^A, x_{-i^*}) \in A$. Then by (b) it follows that $(\alpha^A, y_{-i^*}) \in A$ for any $y_{-i^*} \in X_{-i^*}$. Since A is an unhappy set, strong monotonicity of the preference implies that $(z_{i^*}, y_{-i^*}) \in A$ for any $z_{i^*} \in [0, \alpha^A]$. This proves that $A = \{y \in X \mid 0 \leq y_{i^*} \leq \alpha^A\}$.

To complete the proof consider another set $B \in \mathcal{A}_{\succsim}$. Then there is $k \in N$ and $\alpha^B > 0$ such that $B = \{x \in X \mid 0 \leq x_k \leq \alpha^B\}$. It only remains to show that $k = i^*$. Recall by Proposition 1 that since A, B are both unhappy sets, either $A \subseteq B$ or $B \subseteq A$. Without loss of generality let $A \subseteq B$. If $k \neq i^*$, then for any $z \in A$, $A(z_{-k}) = \mathbb{R}_+$, so there is a $z \in A$ such that $z_k > \alpha^B$, so that $z \notin B$, a contradiction. So we must have $k = i^*$. \blacksquare

Proof of Theorem 1 A lexicographic preference on $X = \mathbb{R}_+^2$ is strong monotone and mildly continuous and by Lemma 1, any closed unhappy set of such a preference satisfies IMIA.

To prove the converse, let \succsim be a rational, strong monotone and mildly continuous preference relation on $X = \mathbb{R}_+^2$ for which any closed unhappy set satisfies IMIA. Consider any $x = (x_1, x_2) \in \mathbb{R}_+^2$ such that x_1, x_2 are both positive. Let $L(x)$ be the lower contour set of x and $\bar{L}(x) = L(x) \cup \partial L(x)$ be its closure. By Corollary 2, $\bar{L}(x)$ is a closed unhappy set, so it belongs to the family \mathcal{A}_{\succsim} given in (1). By Lemma 2 we conclude there is $\kappa(x) > 0$ and a unique $i^* \in \{1, 2\}$ such that $\bar{L}(x) = \{y \in \mathbb{R}_+^2 \mid 0 \leq y_{i^*} \leq \kappa(x)\}$. Without loss of generality, let $i^* = 1$. Then for any x where x_1, x_2 are both positive we have $\bar{L}(x) = \{y \in \mathbb{R}_+^2 \mid 0 \leq y_1 \leq \kappa(x)\}$.

In what follows we show $\kappa(x) = x_1$. Note that if $\kappa(x) < x_1$, then x will be outside $\bar{L}(x)$, so we must have $\kappa(x) \geq x_1$. If $\kappa(x) > x_1$, we can construct $\tilde{x} \in \mathbb{R}_+^2$ such that $\tilde{x}_1 = \kappa(x) > x_1$ and $\tilde{x}_2 > x_2$. Observe that $\tilde{x} \in \bar{L}(x)$ and any neighborhood of \tilde{x} contains a point y such that $y_1 > \kappa(x)$, so that $y \notin \bar{L}(x)$. This shows $\tilde{x} \in \partial \bar{L}(x)$.

Next observe that $\tilde{x} > x$ so by monotonicity $\tilde{x} \succ x$. Since \succsim is mildly continuous and $\tilde{x} \not\hat{\succ} x$, there exists a neighborhood $B_\varepsilon(\tilde{x})$ such that all points there is strictly preferred to x .

So we have $\tilde{x} \notin \partial L(x)$. But we know $\tilde{x} \in \partial \bar{L}(x)$. This is a contradiction since for any set A , the boundary of its closure is a subset of ∂A (see, e.g., Chapter 3 of Mendelson, 1990). This shows we must have $\kappa(x) = x_1$. So for any x where x_1, x_2 are both positive:

$$\bar{L}(x) = \{y \in \mathbb{R}_+^2 \mid 0 \leq y_1 \leq x_1\} \quad (6)$$

To show that \succsim is lexicographic, consider any $y, z \in \mathbb{R}_+^2$ such that $y \neq z$. If $z_1 > y_1$, then $\exists x$ with x_1, x_2 both positive such that $z_1 > x_1 > y_1$. Then by (6) it follows that $y \in \bar{L}(x)$ and $z \notin \bar{L}(x)$. Since $\bar{L}(x)$ is an unhappy set, we must have $z \succ y$. This shows whenever $z_1 > y_1$, we must have $z \succ y$. Finally let $y \neq z$ such that $z_1 = y_1$. Then by strong monotonicity, $z \succ y$ if $z_2 > y_2$ and $y \succ z$ if $y_2 > z_2$. This shows that \succsim is a lexicographic preference with linear order $1 <_0 2$. \blacksquare

Proof of Lemma 3 (i)(a) Take any $S \subseteq N$ with $|S| = 2$. Consider the induced preference \succsim_S defined on $X_S = \mathbb{R}_+^2$. Note that \succsim_S is rational (since \succsim is rational), strong monotone (by Axiom 1) and mildly continuous (by Axiom 3). By Axiom 2, any closed unhappy set for \succsim_S satisfies IMIA. Then by Proposition 1 we conclude that \succsim_S is a lexicographic preference on $X_S = \mathbb{R}_+^2$. Since $|S| = 2$, $\exists i, j \in S$ such that $i \succ^* j$, that is, the linear order of the lexicographic preference \succsim_S is $i <_0 j$.

(i)(b) Take any $x^S, y^S \in X_S = \mathbb{R}_+^2$ where $x^S \neq y^S$. Then either (a) $x_i > y_i$ or (b) $y_i > x_i$ or (c) ($x_i = y_i$ and $x_j > y_j$) or (d) ($x_i = y_i$ and $y_j > x_j$).

Consider any $z^{N \setminus S} \in X_{N \setminus S}$. Since $i \succ^* j$, if (a) or (c) hold, then $(x^S, 0^{N \setminus S}) \succ (y^S, 0^{N \setminus S})$ and by Axiom 4 we have $(x^S, z^{N \setminus S}) \succ (y^S, z^{N \setminus S})$. If (b) or (d) hold, then $(y^S, 0^{N \setminus S}) \succ (x^S, 0^{N \setminus S})$ and by Axiom 4 we have $(y^S, z^{N \setminus S}) \succ (x^S, z^{N \setminus S})$.

(ii) Suppose $i \succ^* j$ and $j \succ^* k$. Let $S = \{i, k\}$. First observe that if x^S, y^S are such that $x_i = y_i$ and $x_k > y_k$, then by strong monotonicity of \succsim_S , we have $(x^S, 0^{N \setminus S}) \succ (y^S, 0^{N \setminus S})$.

So let x^S, y^S are such that $x_i > y_i$. Consider non negative numbers x_k, y_k, \tilde{x}_j such that $\tilde{x}_j > 0$. Denote $T = \{i, j, k\}$ and let

$$x^T = (x_i, 0, x_k), \tilde{x}^T = (y_i, \tilde{x}_j, x_k), y^T = (y_i, 0, y_k) \text{ and}$$

$$x = (x^T, 0^{N \setminus T}), \tilde{x} = (\tilde{x}^T, 0^{N \setminus T}), y = (y^T, 0^{N \setminus T}).$$

Note that $x_i > \tilde{x}_i = y_i$ and $x_\ell = \tilde{x}_\ell$ for any $\ell \neq i, j$. Since $i \succ^* j$, by (i)(b), we have $x \succ \tilde{x}$. Next observe that $\tilde{x}_j > y_j = 0$ and $\tilde{x}_\ell = y_\ell$ for any $\ell \neq j, k$. Since $j \succ^* k$, by (i)(b) we have $\tilde{x} \succ y$. By transitivity of \succsim , we have $x \succ y$. Noting that $x = (x^S, 0^{N \setminus S})$ and $y = (y^S, 0^{N \setminus S})$ we conclude that if $x_i > y_i$, then $(x^S, 0^{N \setminus S}) \succ (y^S, 0^{N \setminus S})$. This shows that $i \succ^* k$.

(iii) Note from (i)-(ii) for any two different $i, j \in N$, either $i \succ^* j$ or $j \succ^* i$. Moreover for $i, j, k \in N$, if $i \succ^* j$ and $j \succ^* k$, then $i \succ^* k$. We prove the result by induction on n . The result clearly holds for $n = 2$. For $m \geq 3$, assume the result holds for all $n \leq m - 1$ and let $M = \{1, \dots, m\}$. Let $S = \{i \in M \mid i \succ^* m\}$ and $T = \{i \in M \mid m \succ^* i\}$. Then S, T are disjoint

and $S \cup T = M$. Let $|S| = s$, $|T| = t$. Then $0 \leq s, t \leq m - 1$. So by induction hypothesis, we have $S = \{i_1, \dots, i_s\}$ such that $i_1 \succ^* \dots \succ^* i_s$ and $T = \{j_1, \dots, j_t\}$ such that $j_1 \succ^* \dots \succ^* j_t$. Since $i_s \succ^* m$ and $m \succ^* j_1$, we have $i_1 \succ^* \dots \succ^* i_s \succ^* m \succ^* j_1 \dots \succ^* j_t$. ■

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