Rationalist Explanations for Two-Front War

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Abstract

By extending the extant costly-lottery models of war to three-party bargaining scenarios, we offer rationalist explanations for two-front war, where a state at the center is fought by two enemies at opposing peripheries. We found that even though private information exists only in one front, war can break out in both fronts. Because the war outcome in one front can affect the outcome in the other through the shift of military balance, the central state may preemptively initiate war in one front to establish its preponderance in the other (e.g., World War I), or a peripheral state may preventively join the war waging in the other front to leverage its power (e.g., Napoleonic Wars). These findings echo Waltz's neorealism concern that a multi-polar system may not be so stable as the bipolar system that bargaining models of dyadic war commonly presume. (JEL: C78; D74; F51)

Keywords: costly-lottery models, rationalist explanations for war, three-party bargaining, two-front war.

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1 Introduction

Presumably due to difficulties with modeling multilateral interactions across states (Jackson and Morelli 2011), formal theorists in international relations have developed few models of multilateral war (for an exception, Krainin and Wiseman 2016), while they have devoted much more efforts to modeling dyadic war (Fearon 1995; Powell 2004; Slantchev 2003a, 2003b; Smith 1998; Smith and Stam 2004; Wagner 2000). To model war fought by multiple parties, further simplification need to be undertaken.

One approach to such simplification is to focus on a particular form of war such as war fought by one against N parties, as often found in rebellions and revolutions (Alt, Calvert, and Humes 1988; Bueno de Mesquita 2010; Fearon 2011; Ginkel and Smith 1999; Nakao 2015, 2018; Roemer 1985; Weingast 1995), or war intervened by a third party (Altfeld and Bueno de Mesquita 1979; Gartner and Siverson 1996; Powell 2017; Smith 1996). In this article, we explore a particular form of war—*two-front war*, where a state at the center is fought by two enemies at opposing peripheries.[1] Possibly due to its geographic nature, two-front wars were repeatedly experienced in Europe, which is stretched along with major powers in a row—from Spain, France, and Germany to Russia. If two peripheral states are strongly committed to a prewar alliance, they could be treated as a single actor, and two-front war reduced to dyadic war (e.g., Arab states in the Palestine War), but such an alliance is not necessarily formed.[2] We thus develop a theory of two-front war, which is built upon costly-lottery models with ultimatum games. Major two-front wars in history are listed in Table 1.[3]

In modeling two-front war, we illuminate two causal mechanisms of triadic war, which cannot be captured by the extant models of dyadic war. In a mechanism, a peripheral state plays a critical role in spreading war from one theater to the other. In the other mechanism, the central state initiates war against either peripheral state, followed by the outbreak of war in the other theater. While the former mechanism resembles the Napoleonic Wars, where Russia challenged the French hegemony by "bandwagoning" on the uprising in the Iberian Peninsula, the latter might better

[1] At the tactical level, simultaneous attacks on the enemy's flanks from the two opposing sides are called the Hammer and Anvil.

[2] A static model of alliance formation among three players with complete information has been developed by Krainin (2014).

[3] Among the wars in Table 1, those won by the central belligerents are the Palestine War, the Six-Day War, the War of the League of Cognac with the Siege of Vienna.

Belligerent's Position	West	Center	East
First Congo War 1996-1997	Angola	Zaire	Uganda, Rwanda, & Burundi
Yom Kippur War 1973	Egypt	Israel	Syria
Six-Day War 1967	Egypt	Israel	Syria & Jordan
Palestine War, 1947-1949	Egypt	Israel	Lebanon, Syria, & Transjordan,
World Wars II 1939-1945	France, Britain, & U.S.	Germany & Austria	Russia
World Wars I 1914-1918	France, Britain, & U.S.	Germany & Austria	Russia
Napoleonic Wars 1807-1814	Portugal & Spain	France	Sweden, Russia, & Prussia
War of the League of Cognac with Siege of Vienna 1526-1530	France	Holy Roman Empire	Ottoman Empire

Table 1: Major two-front wars in history.

capture World War I, where Germany declared war against France in hope to shatter it before Russia was ready to fight. These mechanisms are missing in the extant costly-lottery models in that unlike them, our models allow the military balance to shift endogenously, as a result of the interplay among the three states.

The rest of the paper proceeds as follows. Section 2 offers a benchmark model of bargaining and fighting among three states. Sections 3 and 4 present costly-lottery models of two-front war, which are compared in Section 5. Section 6 summarizes our theoretical findings. All the proofs appear in Appendix.

Front	West	East
Belligerents	(α, β)	(α, γ)
Probabilities of winning	(p_α, p_β)	(q_α, q_γ)
Value of issue	U	V
Costs of fighting	(c_α, c_β)	(d_α, d_γ)

Table 2: Parameters for the two fronts.

2 Benchmark Model of Two-Front Bargaining

To illuminate the causes of two-front war—a war between a party at the center and two others at opposing peripheries, we develop bargaining models of war, which comprises two ultimatum games. We begin with a benchmark model, where peace is the unique equilibrium, and subsequently seek conditions for the outbreak of two-front war.

In the benchmark model, there are three states $\{\alpha, \beta, \gamma\}$, among which α is located at the center, β at the western end, and γ in the eastern end. In the west, α and β are in conflict over resources with value $U > 0$. In the east, α and γ also have a dispute about resources valued $V > 0$. Since β and γ are far from contiguous, a war between them is geographically impossible.

The game proceeds as follows: At the beginning, α chooses its demands $\theta_\beta \in [0, U]$ in the west and $\theta_\gamma \in [0, V]$ in the east. In response, β and γ simultaneously decide to accept α's proposal or to fight α. If β accepts, α gains $U - \theta_\beta$, and β gains θ_β. If γ accepts, α gains $V - \theta_\gamma$, and γ gains θ_γ. If β fights, α (β) wins the entire U with probability $p_\alpha > 0$ ($p_\beta > 0$) such that $p_\alpha + p_\beta = 1$. If γ fights, α (γ) wins V with probability $q_\alpha > 0$ ($q_\gamma > 0$) such that $q_\alpha + q_\gamma = 1$.

Definition 1 *Western war refers to the fight between α and β. Eastern war refers to the fight between α and γ. Two-front war refers to the combination of both the western and eastern wars simultaneously prosecuted by α.*

Given α and β's costs of fighting $c_\alpha > 0$ and $c_\beta > 0$, their *ex ante* payoffs from fighting the western war are $p_\alpha U - c_\alpha > 0$ and $p_\beta U - c_\beta > 0$, respectively. Given α and γ's costs of fighting $d_\alpha > 0$ and $d_\gamma > 0$, their *ex ante* payoffs from fighting the eastern war are $q_\alpha V - d_\alpha > 0$ and $q_\gamma V - d_\gamma > 0$, respectively. In the game as a whole, α's payoff equals the sum of the payoffs it gains from the two fronts.[4] The key parameters are summarized in Table 2.

[4] For instance, if β accepts θ_β in the west and γ fights in the east, α's *ex ante* payoff will be $U - \theta_\beta + q_\alpha V - d_\alpha$.

If the bargaining outcomes in the two fronts do not influence each other as presumed above, war never emerges in equilibrium:

Lemma 1 *In the unique subgame perfect Nash equilibrium of the benchmark model, war never breaks out in either front; i.e.,* α *offers*

$$\begin{aligned} \theta_\beta &= p_\beta U - c_\beta \\ \theta_\gamma &= q_\gamma V - d_\gamma, \end{aligned}$$

both of which are accepted by β *and* γ*, respectively.*

As in the dyadic bargaining situation (Fearon 1995), the outbreak of two-front war is a puzzle even in the triadic bargaining situation—given a war is costly, there always exists a peaceful settlement that is Pareto superior to war.

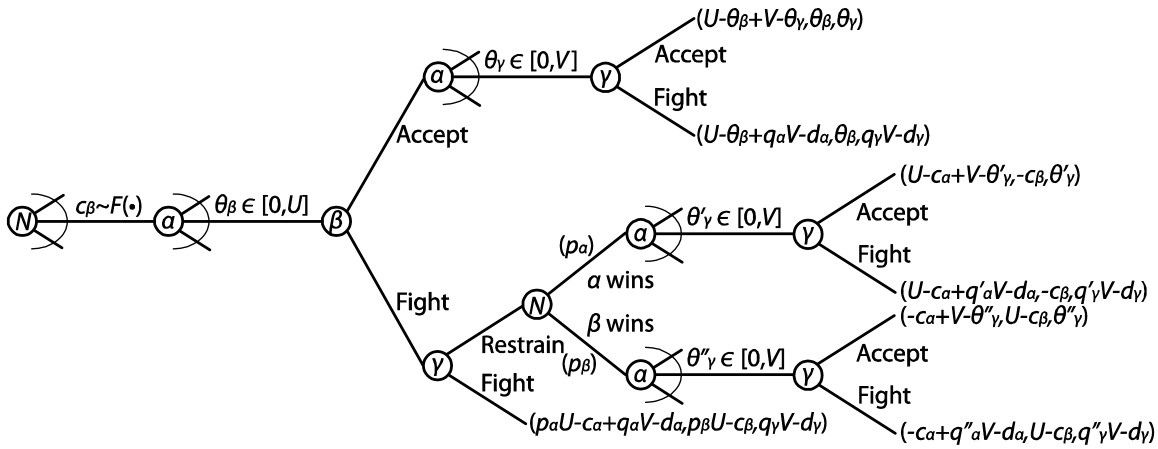

Figure 1: Two-front war with preventive fight.

3 Model I: Reactive, Preventive Fight

By extending the benchmark model, we next seek the conditions with which two-front war can break out. Two-front war can trivially arise from private information (or commitment problems) in both the fronts. However, we demonstrate that two-front war can break out despite private information only in one front.

The next model, labeled Model I, differs from the benchmark model in a threefold manner: (i) β has private information on its own cost c_β; (ii) a time lag exists in bargaining and fighting between the two fronts; (iii) the war outcome in the west can affect the military balance in the east. This endogenous shift of the military balance is missing in extant bargaining models of dyadic war and forms our model's novelty. The extensive form of Model I appears in Figure 1.

As to (i), when placing the offer θ_β, α does not know the true value of c_β, but it still knows the cumulative distribution $F(c_\beta)$ and density $f(c_\beta)$ with non-decreasing hazard rate $f(c_\beta)/(1-F(c_\beta))$ (Fudenberg and Tirole 1991: 267). As to (ii), an ultimatum game in the west is played earlier than in the east. Thus the western war can begin before bargaining in the east takes place. However, γ has the chance to fight α before the western war ends. As to (iii), if α wins in the west, it could reallocate all its forces to the east, so that the probability that α wins in the east increases from q_α to $q'_\alpha > 0$. Conversely, if β wins in the west, α would lose a part of its military resources prepared for the east, so that the probability that α wins decreases to $q''_\alpha > 0$ such that $q''_\alpha < q_\alpha < q'_\alpha$ with $q'_\alpha + q'_\gamma = 1$, and $q''_\alpha + q''_\gamma = 1$, where

$q'_\gamma > 0$ ($q''_\gamma > 0$) denotes the probability that γ wins in the east if α wins (loses) in the west.[5]

In this game, the asymmetry of information between α and β can cause the war in the west, which may, in turn, induce γ to *preventively* fight α in the east. If the war outcome in the west is likely to produce a disadvantageous military imbalance in the east, γ—tacitly allied with β—would fight α in the east before the western war ends.

Proposition 1 *In the unique subgame perfect Nash equilibrium of Model I, two-front war can break out if*

$$f(0) < \frac{1}{c_\alpha} \tag{1}$$

$$p_\alpha q'_\gamma + p_\beta q''_\gamma < q_\gamma; \tag{2}$$

i.e., β fights α in the west with a positive probability if Inequality (1) holds; conditional on β's fight in the west, γ fights α in the east if Inequality (2) holds.

The strategy profile of the equilibrium appears in Lemma 2 of Appendix. In the west, β fights with probability $F\left(p_\beta U - \theta_\beta^\dagger\right)$, where $\theta_\beta^\dagger$ is α's equilibrium offer to β. Inequality (1) guarantees that this probability is positive. As with other costly-lottery models with private information (Fearon 1995), in choosing θ_β, α weighs the balance between the terms of peaceful agreement and the risk of war.

In the east, γ chooses to "fight" instead of "refrain" immediately after β's fight if Inequality (2) holds—it fights, because pincer attacks jointly with β would give γ a better prospect of war if α has difficulties in maintaining two battlefronts simultaneously. In other words, γ jumps on β's fight in the west, since a delay in fight would leave γ in isolation. On the other hand, if Inequality (2) is violated, the alliance with β would not be so helpful for γ's prosecution of war, and thus γ would refrain from fighting before it bargains with α. Inequality (2) holds with a large q_γ, large

[5] The shift of military balance in the east can be formally explained as follows: With Tullock's (1980) contest success function, the probability of α's winning can be shown as: $p_\alpha = \frac{(m_\alpha)^P}{(m_\alpha)^P + (m_\beta)^P}$ in the west and $q_\alpha = \frac{(n_\alpha)^Q}{(n_\alpha)^Q + (n_\gamma)^Q}$ in the east, where $m_i > 0$ ($n_i > 0$) is i's strength in the west (east) with $i \in \{\alpha, \beta, \gamma\}$, $P \geq 1$, and $Q \geq 1$. If α wins in the west, α can deploy all its forces in the east, so that the probability of α's winning in the east increases: $q'_\alpha = \frac{(m_\alpha + n_\alpha)^Q}{(m_\alpha + n_\alpha)^Q + (n_\gamma)^Q}$. If α loses, it loses a part of its resources, so that $q''_\alpha = \frac{(n''_\alpha)^Q}{(n''_\alpha)^Q + (n_\gamma)^Q}$ with $n''_\alpha \leq n_\alpha$.

p_α, and small q'_γ, implying that a preventive war in the east is likely if β is a great help for γ (with a large q_γ), or if β's defeat in the west (with a large p_α) significantly disadvantages γ in the east (with a small q'_γ).

In particular, if a uniform distribution is presumed (i.e., $c_\beta \sim U[0, \bar{c}_\beta]$), the result is much simplified:

Corollary 1 *Given* $F(c_\beta) = \frac{c_\beta}{\bar{c}_\beta}$ *with* $\bar{c}_\beta \in (c_\alpha, c_\alpha + 2p_\beta U)$*, two-front war breaks out with probability* $\frac{1}{2} - \frac{c_\alpha}{2\bar{c}_\beta}$ *if Inequality (2) holds.*

The restriction that $\bar{c}_\beta \in (c_\alpha, c_\alpha + 2p_\beta U)$ satisfies Inequality (1), guaranteeing that α's equilibrium offer to β is interior, or $\theta^\dagger_\beta \in (0, p_\beta U)$. With this distribution, α's offer to β is $\theta^\dagger_\beta = p_\beta U - \frac{\bar{c}_\beta - c_\alpha}{2}$, which is positive by $\bar{c}_\beta < c_\alpha + 2p_\beta U$. The western war breaks out with probability $F\left(p_\beta U - \theta^\dagger_\beta\right) = \frac{1}{2} - \frac{c_\alpha}{2\bar{c}_\beta}$, which is positive by $\bar{c}_\beta > c_\alpha$. By Inequality (2), the eastern war also breaks out, conditional on the western war's outbreak.

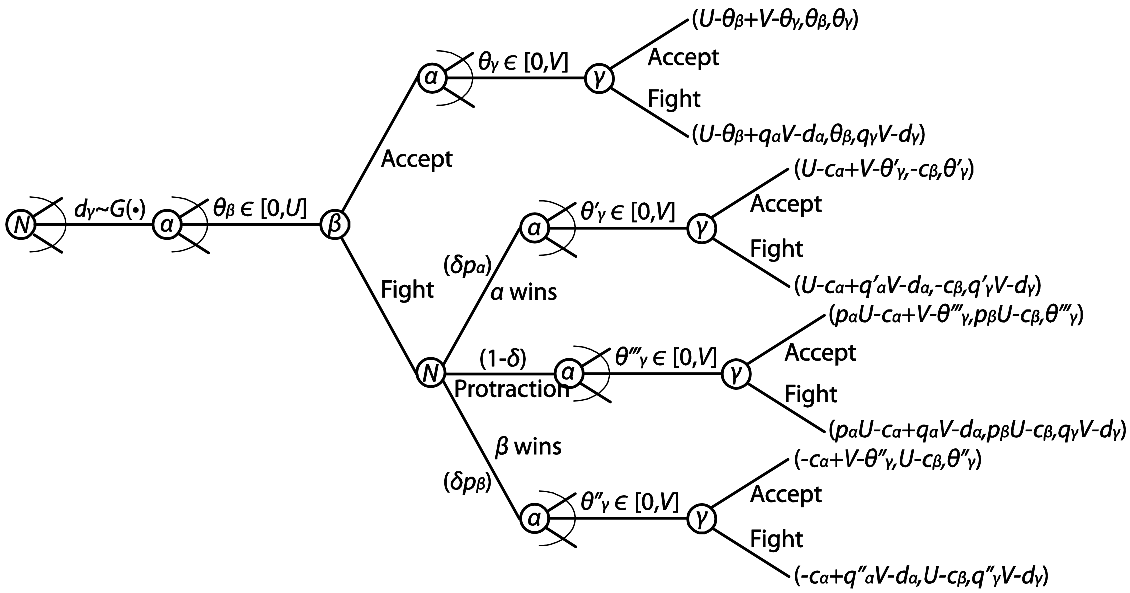

Figure 2: Two-front war with preemptive fight.

4 Model II: Proactive, Preemptive Fight

The last model, labeled Model II, delineates the other cause and pattern of two-front war. It differs from Model I in a threefold manner: (i') instead of β, γ has private information on its cost d_γ, so that α only knows that d_γ follows the cumulative distribution $G(d_\gamma)$ and density $g(d_\gamma)$ with non-decreasing hazard rate $g(d_\gamma)/(1-G(d_\gamma))$; (ii') unlike Model I, γ cannot fight α immediately after β's fight due to geographic constraints, time for γ's mobilization, or other obstacles; (iii') the western war decisively ends before bargaining in the east begins with probability $\delta > 0$ and is indecisively protracted with probability $1-\delta$. If the western war is protracted, it will end only after γ decides whether to fight or not.

To elaborate on (iii'), if β fights in the west, α (β) immediately wins with probability δp_α (δp_β) before bargaining in the east begins, Furthermore, as the war outcome in the west can affect the military balance in the east (for the same reasons as in Model I), if β fights and immediately wins (loses) in the west, the probability that α wins in the east changes to $q'_\alpha > 0$ ($q''_\alpha > 0$) such that $q''_\alpha < q_\alpha < q'_\alpha$. If the western war is protracted, the military balance in the east is unaffected, so that the probability that α wins remains q_α. The extensive form of Model II is shown in Figure 2.

In this game, α may induce β's fight in the west to improve its bargaining position in the east, whereas the eastern war can be caused by the asymmetry of information between α and γ.

Proposition 2 *In any subgame perfect Nash equilibria of Model II, two-front war can break out if*

$$\frac{c_\alpha + c_\beta}{\delta} < \Psi \tag{3}$$

$$g(0) < \frac{1}{d_\alpha}, \tag{4}$$

where

$$\Psi \equiv \sum_{q \in \left\{q_\gamma, q'_\gamma, q''_\gamma\right\}} p(q) \left(\begin{array}{c} G\left(qV - \theta_\gamma(q)\right)\left((1-q)V - d_\alpha\right) \\ +\left(1 - G\left(qV - \theta_\gamma(q)\right)\right)\left(\left(V - \theta_\gamma(q)\right)\right) \end{array} \right) \tag{5}$$

with

$$p(q) = \begin{cases} -1 & \text{if } q = q_\gamma \\ p_\alpha & \text{if } q = q'_\gamma \\ p_\beta & \text{if } q = q''_\gamma \end{cases}$$

$$\theta_\gamma(q) = \begin{cases} 0 & \text{if } \frac{g(qV)}{1-G(qV)} \leq \frac{1}{qV+d_\alpha} \\ \widehat{\theta}_\gamma & \text{otherwise} \end{cases}$$

such that

$$\frac{g\left(qV - \widehat{\theta}_\gamma\right)}{1 - G\left(qV - \widehat{\theta}_\gamma\right)} = \frac{1}{qV + d_\alpha - \widehat{\theta}_\gamma};$$

i.e., β fights α in the west if Inequality (3) holds; γ fights α in the east with a positive probability if Inequality (4) holds.

The strategy profile of the equilibria appears in Lemma 3 of Appendix. There exist multiple equilibria in Model II due to the flexibility of α's best-response offer to β ($\theta^\ddagger_\beta$), which can be any θ_β less than β's reservation payoff, or $\theta^\ddagger_\beta < p_\beta U - c_\beta$. All other best-response actions are uniquely determined.

In the west, α places an offer that is unacceptable to β if Inequality (3) holds. By placing an unacceptable offer, α induces β to fight. Although β is the player who chooses to "fight" in the game, it is actually α who in effect triggers the war—α *preemptively* initiates the war, because by defeating β, it can invest more military resources in the eastern front, so that that α can draw more compromise from γ. In other words, since it is costly for α to maintain its standing forces in the west, it would annihilate the threat in the west to deploy more forces in the east (Coe 2012). The standing forces in the west are costly not in the budgetary sense, but they entail the loss of opportunity to garner more favorable outcomes in the east. On the other hand, if Inequality (3) is violated, α would place the acceptable offer to β, or $\theta_\beta^\ddagger = p_\beta U - c_\beta$, so that the western war would be avoided. Inequality (3) is likely to hold, or the western war is plausible if the costs of fighting are small for α and β (with small c_α and c_β), and if α's decisive victory in the west (with a large δp_α) generates its military advantage in the east (with a large q'_α).

In the east, conditional on the western war's outbreak and protraction, γ fights with probability $G\left(q_\gamma V - \theta_\gamma\left(q_\gamma\right)\right)$, which is positive by Inequality (4). Without knowing γ's cost of fighting d_γ, α would take the risk of war to a reasonable extent for the sake of favorable terms upon peace.

Although the interpretation of Inequality (3) is difficult, it can be simplified by assuming a uniform distribution of d_γ (i.e., $d_\gamma \sim V[0, \overline{d}_\gamma]$):

Corollary 2 *Given* $G\left(d_\gamma\right) = \frac{d_\gamma}{\overline{d}_\gamma}$ *with* $\overline{d}_\gamma \in \left(d_\alpha, d_\alpha + 2q'_\gamma V\right)$*, two-front war breaks out with probability* $(1-\delta)\left(\frac{1}{2} - \frac{d_\alpha}{2\overline{d}_\gamma}\right)$ *if*

$$q_\alpha + \frac{c_\alpha + c_\beta}{\delta V} < p_\alpha q'_\alpha + p_\beta q''_\alpha. \tag{6}$$

With this distribution, Inequality (3)—the condition for β's "fight" in the west—can be reduced to Inequality (6). By the restriction $\overline{d}_\gamma \in \left(d_\alpha, d_\alpha + 2q'_\gamma V\right)$, the eastern war can break out, or Inequality (4) holds. In the east, α's offer to γ is $\theta_\gamma(q) = qV - \frac{\overline{d}_\gamma - d_\alpha}{2}$, which depends on $q \in \left\{q_\gamma, q'_\gamma, q''_\gamma\right\}$ but is always positive by $\overline{d}_\gamma < d_\alpha + 2q'_\gamma V$. Regardless of the subgames with $q \in \left\{q_\gamma, q'_\gamma, q''_\gamma\right\}$, the probability of the eastern war is $G\left(qV - \theta_\gamma^\ddagger\right) = \frac{1}{2} - \frac{d_\alpha}{2\overline{d}_\gamma}$, which is also positive by $\overline{d}_\gamma > d_\alpha$.

5 Comparison

The assumption of the uniform distributions enables the comparison of the conditions for two-front war between Models I and II. In either model, it is presumed that the central state (α) is the proposer of offers, while the peripheral states (β and γ) the receivers. As shown below, this proposer-receiver relationship in the bargaining protocol affects the condition for two-front war's outbreak. Because the models are built upon ultimatum games, the proposer possesses the full bargaining power, whereas the receivers have no such power. Consequently, the proposer can grab the entire surplus (e.g., $c_\alpha + c_\beta$ in the west) upon peace by settling with its most preferred outcome in the bargaining range; in contrast, the receivers cannot gain any surplus. Those say, the distribution of bargaining power, determined by the bargaining protocol, generates different incentives to fight between the proposer and the receivers—in provoking war, the proposer must abandon the surplus from peace that the receivers would not entertain regardless of their decisions. Therefore, the condition on the proposer's incentive to fight should be more restrictive than on the receivers'.

In Model I with the uniform distribution, the expansion of war from the west to the east hinges on γ's incentive to fight. That is, γ decides to join the eastern war if Inequality (2) holds, or equivalently if

$$q_\alpha < p_\alpha q'_\alpha + p_\beta q''_\alpha.$$

Because γ is the receiver in the east, no surplus can affect its decision, and thus only the shift of the military balance matters for the expansion. That means, γ decides to fight without delay if its expected payoff from fighting immediately exceeds the reservation payoff from fighting in the future. If the western war is likely to result in γ's disadvantage, γ would preventively fight α in the east before the western war ends, leading to wars waged simultaneously in both the fronts.

In Model II with the uniform distribution, on the other hand, the outbreak of two-front war hinges on α's incentive to fight in the west, or α decides to fight β if

$$q_\alpha + \frac{c_\alpha + c_\beta}{\delta V} < p_\alpha q'_\alpha + p_\beta q''_\alpha,$$

as shown in Inequality (6). As the proposer, α must incorporate the loss of surplus in its decision calculus. Thus, unlike Inequality (2) of Model I, this condition contains an additional term $\frac{c_\alpha + c_\beta}{\delta V}$, which makes α more hesitant to fight, because it must abandon the surplus when it fights. The condition also shows that α is more likely to fight in the west if if the eastern war is more likely to end decisively (with a large δ) and also if the issue at stake is more valuable in the east (with a large V).

The comparison between Models I and II reveals that whether wars are waged in both the fronts depends on which party (proposer or receivers) plays the pivotal role among the three states. Although the two models depict different channels to wars, the key mechanism is in common—two-front war is a result of the interplay among three states in light of the shifting military balance. While the peripheral states (γ in Model I) strive to prevent the rise of hegemony, the central state (α in Model II) aspires to establish its preponderance. Unlike extant costly-lottery models, our model elucidates the shift as a product of bargaining and fighting in another area.

Model	Model I	Model II
Private information	Cost c_β in the west	Cost d_γ in the east
Pivotal state	Receiver γ in periphery	Proposer α at center
Bargaining power of the pivotal state	No power held by receiver γ	Full power held by proposer α
Timing of the pivotal decision	Reactive (after the western war)	Proactive (before the eastern war)
Motive as to the military balance	Prevention of isolation	Preemption for preponderance
Exemplary war	Napoleonic Wars	World War I

Table 3: Comparison of the implications between the two models.

6 Conclusion

By extending the extant costly-lottery models of dyadic war (Fearon 1995), we have developed a theory of two-front war, where a state at the center is fought by two enemies at opposing peripheries. Since bargaining and fighting in one front can affect the military balance in the other, war can spread from one front to the other. By analyzing two models of combined ultimatum games, we have uncovered two channels through which war can break out and expand.

In one of the two models, labeled Model I, war is originally caused by private information in the west. Because this war could change the military balance in the east, the peripheral state in the east would join the war before it ends, leading to the expansion of war waged in both the fronts. The eastern state's "bandwagoning" is regarded as both reactive and preventive—it is reactive in that the state fights after the western war begins—and also preventive in that the state fights before the military disadvantage materializes upon itself.

This pattern can be found in the Napoleonic Wars. During the closing phases of the Wars, Portugal refused Napoleon's Continental Blockage in 1807, leading to the Peninsular War in the West from 1808. As France was troubled over the Spanish resistance in the Peninsular, Russia provoked its challenge to France in 1810 (Haldi 2003). Russia and other following states in the East presumably leveraged their military power by exploiting the Iberian resistance (Ellis 2003).

In the equilibrium of the other model, or Model II, war is initiated by the central state toward the west. The western state is targeted due to the time lag for mobilization between the two peripheral states—the eastern state needs a longer time to deploy its forces on its border than the western state. That means, the central state intends to defeat the western enemy shortly and decisively before the eastern state is ready to fight. The central state's decision to initiate war is both proactive and preemptive—it is proactive in that the decision is made before fighting begins in the east—and also preemptive in that it gives no room for negotiation in the west. In other words, the central state seeks its preponderance, or military superiority, in the east by forestalling its enemies. By disallowing a defacto coalition by the peripheral states, the central state could avoid the simultaneous fights in both the fronts. However, the failure to swiftly defeat the western state would drag the central state into the devastating scenario of two-front war.

World War I resembles this pattern. Long before the War's onset, Germany adopted the Schlieffen Plan in 1905, which was based on the presumption that due to geographic, technological, and other constraints, Russia needed at least six weeks to overrun the eastern approaches of Berlin. Within the six weeks—according to the Plan—Germany could shatter the French forces by introducing the vast majority of its army, and after the French defeat, Germany would swiftly relocate its entire army to counter the Russian forces in the East. In other words, the Plan was to decouple the combat between the two fronts. However, the War did not proceed, as Germany planned. Possible causes of the German failure in the West were the Belgian tenacious scorched-earth resistance, the stretched supply lines to the German troops, and the loss of quantitative military advantage (Creveld 2004; Keegan 1998; Winter 1989). Moreover, Russia enabled its army to take a quicker offensive by shortcutting its mobilization timeline (Cashman and Robinson 2007: 38), putting Germany into the position of simultaneously maintaining both the fronts. The Plan was dismissed after the First Battle of Marne, where Germany halted and withdrew its forces.

Although the two models portray different channels, they share the common factor—the timing of fight. The timing can matter, because it affects the relative strength between the central and peripheral states. While the peripheral states pursue the simultaneous confrontation against the central state, the central state attempts to disallow such coordination by dealing with them sequentially or separately. In this sense, the states might disagree not only about the division of benefits, but also the timings of fights across the fronts. War could be waged in two fronts simultaneously if the central state fails to keep its adversaries in isolation. The complexity caused by geography, the timing of fight, and the shift of balance among more states may raise the risk of war, echoing Waltz's (1979) neorealism concern that peace is more difficult in a multipolar world than in a bipolar world that dyadic models commonly presume.

Moreover, our models indicate that the likelihood of two-front war hinges on the distribution of bargaining power across states. A state with more bargaining power is less prone to fight, because it must abandon greater surplus once bargaining fails. Therefore, if the central state possesses more bargaining power than the peripheral counterparts, as presumed in our models, two-front war of the Napoleonic-Wars type should be more likely, common, or frequent than those of the WWI type, posing an empirical question. The comparison between Models I and II is summarized in Table 3.

Finally, we close the discussion by suggesting several agendas for future research. The models we have developed in this article are presumably the simplest possible formal descriptions of two-front war. While focusing on the timing of fight and the shift of military balance, the models assumed away other important elements that may affect the form of war such as the forth and other states, arms races, geography, and duration. Richer implications could be garnered by incorporating some of these elements. Thus one of the possible extensions would be to include more states which may seek an alliance, bargain, and fight multilaterally (Krainin and Wiseman 2016), although such an extension would be theoretically difficult (Jackson and Morelli 2011). Two-front war can be categorized as a particular form of multilateral war. Arms races can also be an important element that is missing in our models. Especially, the central state must engage not only in production, but also in the allocation and reallocation of its forces between two fronts, as war evolves. Another extension would be to lay out the geographic distances across states in a more explicit manner, as found in random-walk models (Slantchev 2003b; Smith 1998; Smith and Stam 2003, 2004). In addition, the costly-lottery models presented in the article are an illustration of war more parsimonious than the costly-process models (Reiter 2003). It would thus be meaningful to delineate the entire process of war from its onset toward the termination, as was done by some theorists of dyadic war (Powell 2004; Slantchev 2003a; Wagner 2000). Modelling of multilateral war should have a spacious room for further research.

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APPENDIX

The following claim applies to all three models.

Claim 1 *(a) With complete information on c_β and d_γ, α offers*

$$\begin{aligned} \theta_\beta(p) &= pU - c_\beta &(7)\\ \theta_\gamma(q) &= qV - d_\gamma, &(8) \end{aligned}$$

where $p \in \{p_\beta, p'_\beta, p''_\beta\}$ and $q \in \{q_\gamma, q'_\gamma, q''_\gamma\}$ are the probabilities that β wins and that γ wins, respectively; (b) with incomplete information, α offers

$$\theta_\beta(p) = \begin{cases} 0 & \text{if } \frac{f(pU)}{1-F(pU)} \leq \frac{1}{pU+c_\alpha} \\ pU & \text{if } f(0) \geq \frac{1}{c_\alpha} \\ \widehat{\theta}_\beta & \text{otherwise} \end{cases} \tag{9}$$

$$\theta_\gamma(q) = \begin{cases} 0 & \text{if } \frac{g(qV)}{1-G(qV)} \leq \frac{1}{qV+d_\alpha} \\ qV & \text{if } g(0) \geq \frac{1}{d_\alpha} \\ \widehat{\theta}_\gamma & \text{otherwise,} \end{cases} \tag{10}$$

where $\widehat{\theta}_\beta \in (0, pU)$ and $\widehat{\theta}_\gamma \in (0, qV)$ hold that

$$\frac{f\left(pU - \widehat{\theta}_\beta\right)}{1 - F\left(pU - \widehat{\theta}_\beta\right)} = \frac{1}{pU + c_\alpha - \widehat{\theta}_\beta} \tag{11}$$

$$\frac{g\left(qV - \widehat{\theta}_\gamma\right)}{1 - G\left(qV - \widehat{\theta}_\gamma\right)} = \frac{1}{qV + d_\alpha - \widehat{\theta}_\gamma}; \tag{12}$$

(c) whether information is complete or incomplete, β and γ respond to θ_β and θ_γ by

$$\sigma_\beta(p) = \begin{cases} \text{Accept} & \text{if } \theta_\beta \geq pU - c_\beta \\ \text{Fight} & \text{if } \theta_\beta < pU - c_\beta \end{cases} \tag{13}$$

$$\sigma_\gamma(q) = \begin{cases} \text{Accept} & \text{if } \theta_\gamma \geq qV - d_\gamma \\ \text{Fight} & \text{if } \theta_\gamma < qV - d_\gamma. \end{cases} \tag{14}$$

Proof of Claim 1. (a,c) With complete information, β accepts θ_β if it is larger than or equal to the expected payoff from fighting, or β chooses $\sigma_\beta(p)$ in Equation (13). Expecting this $\sigma_\beta(p)$, α chooses the smallest θ_β that is acceptable to β, or $\theta_\beta(p)$ in Equation (7). Similarly, the best responses in the east are $\theta_\gamma(q)$ and $\sigma_\gamma(q)$ in Equations (8, 14).

(b,c) With incomplete information, given $\sigma_\beta(p)$ in Equation (13), α determines $\theta_\beta \in [0, p_\beta U]$ to maximize its expected payoff:

$$\max_{\theta_\beta} F(pU - \theta_\beta)(pU - c_\alpha) + (1 - F(pU - \theta_\beta))(U - \theta_\beta),$$

where $F(pU - \theta_\beta)$ is the probability that β fights. The derivative of the objective is:

$$-1 - f(pU - \theta_\beta)(-pU - c_\alpha + \theta_\beta) + F(pU - \theta_\beta),$$

which is positive if

$$\frac{f(pU - \theta_\beta)}{1 - F(pU - \theta_\beta)} > \frac{1}{pU + c_\alpha - \theta_\beta}.$$

Since the left-hand side is non-increasing in θ_β and the right-hand side is strictly increasing for $\theta_\beta < pU$, $\theta_\beta(p) = 0$ if $\frac{f(pU)}{1-F(pU)} \leq \frac{1}{pU+c_\alpha}$; $\theta_\beta(p) = pU$ if $f(0) \geq \frac{1}{c_\alpha}$; otherwise, $\theta_\beta(p) = \widehat{\theta}_\beta(p)$. Equivalently, $\theta_\beta(p)$ is as in Equation (9). Similarly, given $\sigma_\gamma(q)$ in Equation (14), $\theta_\gamma(q)$ can be derived as in Equation (10). ■

Let an asterisk (*) denote the best-response actions in the baseline model.

Proof of Lemma 1. The proof is immediate from Claim 1-(a,c); α's strategy is $\theta^*_\beta = \theta_\beta(p_\beta)$ in Equation (7) and $\theta^*_\gamma = \theta_\gamma(q_\gamma)$ in Equation (8); β's strategy $\sigma^*_\beta = \sigma_\beta(p_\beta)$ in Equation (13); and γ's $\sigma^*_\gamma = \sigma_\gamma(q_\gamma)$ in Equation (14). ■

Let a dagger (†) denote the best-response actions in Model I.

Lemma 2 *In the unique subgame perfect Nash equilibrium of Model I, α's strategy is $\theta_\beta^\dagger = \theta_\beta(p_\beta)$ in Equation (9) and $\left(\theta_\gamma^\dagger, \theta_\gamma'^\dagger, \theta_\gamma''^\dagger\right) = \left(\theta_\gamma(q_\gamma), \theta_\gamma\left(q_\gamma'\right), \theta_\gamma\left(q_\gamma''\right)\right)$ in Equation (8); β's strategy $\sigma_\beta^\dagger = \sigma_\beta(p_\beta)$ in Equation (13); and γ's $\left(\sigma_\gamma^\dagger, \sigma_\gamma'^\dagger, \sigma_\gamma''^\dagger\right) = \left(\sigma_\gamma(q_\gamma), \sigma_\gamma\left(q_\gamma'\right), \sigma_\gamma\left(q_\gamma''\right)\right)$ in Equation (14) and*

$$\phi_\gamma^\dagger = \begin{cases} \textit{Refrain} & \textit{if } q_\gamma \leq p_\alpha q_\gamma' + p_\beta q_\gamma'' \\ \textit{Fight} & \textit{if } q_\gamma > p_\alpha q_\gamma' + p_\beta q_\gamma'', \end{cases} \tag{15}$$

where $\phi_\gamma^\dagger$ is γ's action immediately after β fights.

Proof of Lemma 2. The model is solved by backward induction. In the subgame where β accepts α's offer θ_β, α and γ's best responses are $\left(\theta_\gamma^\dagger, \sigma_\gamma^\dagger\right)$. Similarly, in the two subgames where β fights and γ restrains, α and γ's best responses are: $\left(\theta_\gamma'^\dagger, \sigma_\gamma'^\dagger\right)$ if α wins; and $\left(\theta_\gamma''^\dagger, \sigma_\gamma''^\dagger\right)$ if β wins.

Expecting $\theta_\gamma'^\dagger$ and $\theta_\gamma''^\dagger$, γ decides whether to fight before the western war ends:

$$\phi_\gamma^\dagger = \begin{cases} \text{Refrain} & \text{if } q_\gamma V - d_\gamma \leq p_\alpha\left(q_\gamma' V - d_\gamma\right) + p_\beta\left(q_\gamma'' V - d_\gamma\right) \\ \text{Fight} & \text{if } q_\gamma V - d_\gamma > p_\alpha\left(q_\gamma' V - d_\gamma\right) + p_\beta\left(q_\gamma'' V - d_\gamma\right), \end{cases}$$

or Equation (15).

Given $\sigma_\beta^\dagger$, α maximizes its expected payoff by choosing $\theta_\beta^\dagger$. ■

Proof of Proposition 1. The proof is immediate from Lemma 2. Given $\theta_\beta^\dagger$, the probability of the western war is $F\left(p_\beta U - \theta_\beta^\dagger\right)$, which is positive if $f(0) < \frac{1}{c_\alpha}$. The condition for γ's fight appears in Equation (15). ■

Proof of Corollary 1. By Equation (11),

$$
\begin{aligned}
\theta_\beta^\dagger &= p_\beta U + c_\alpha - \frac{1 - F\left(p_\beta U - \theta_\beta^\dagger\right)}{f\left(p_\beta U - \theta_\beta^\dagger\right)} \\
&= p_\beta U + c_\alpha - \frac{1 - \frac{p_\beta U - \theta_\beta^\dagger}{\overline{c}_\beta}}{\frac{1}{\overline{c}_\beta}} \\
&= p_\beta U - \frac{\overline{c}_\beta - c_\alpha}{2},
\end{aligned}
$$

which is interior, or $\theta_\beta^\dagger \in (0, p_\beta U)$ by $\overline{c}_\beta \in (c_\alpha, c_\alpha + 2p_\beta U)$. Given $\theta_\beta^\dagger$, the probability that β fights in the west is:

$$
\begin{aligned}
F\left(p_\beta U - \theta_\beta^\dagger\right) &= \frac{p_\beta U - \theta_\beta^\dagger}{\overline{c}_\beta} \\
&= \frac{p_\beta U - \left(p_\beta U - \frac{\overline{c}_\beta - c_\alpha}{2}\right)}{\overline{c}_\beta} \\
&= \frac{1}{2} - \frac{c_\alpha}{2\overline{c}_\beta},
\end{aligned}
$$

which is positive by $\overline{c}_\beta > c_\alpha$. ■

Let a double dagger (‡) denote the best-response actions in Model II.

Lemma 3 *In the subgame perfect Nash equilibrium of Model II, α's strategy is*

$$\theta_{\beta}^{\ddagger}=\begin{cases} p_{\beta}U-c_{\beta} & \text{if } \Psi\leq\frac{c_{\alpha}+c_{\beta}}{\delta} \\ \text{any } \theta_{\beta}<p_{\beta}U-c_{\beta} & \text{if } \Psi>\frac{c_{\alpha}+c_{\beta}}{\delta}, \end{cases} \tag{16}$$

where Ψ is defined by Equation (5) of Proposition 2, and $\left(\theta_{\gamma}^{\ddagger},\theta_{\gamma}^{\prime\ddagger},\theta_{\gamma}^{\prime\prime\ddagger},\theta_{\gamma}^{\prime\prime\prime\ddagger}\right)=(\theta_{\gamma}(q_{\gamma}),$ $\theta_{\gamma}\left(q_{\gamma}^{\prime}\right),\theta_{\gamma}\left(q_{\gamma}^{\prime\prime}\right),\theta_{\gamma}(q_{\gamma}))$ in Equation (10); β's strategy $\sigma_{\beta}^{\ddagger}=\sigma_{\beta}(p_{\beta})$ in Equation (13); and γ's $\left(\sigma_{\gamma}^{\ddagger},\sigma_{\gamma}^{\prime\ddagger},\sigma_{\gamma}^{\prime\prime\ddagger},\sigma_{\gamma}^{\prime\prime\prime\ddagger}\right)=\left(\sigma_{\gamma}(q_{\gamma}),\sigma_{\gamma}\left(q_{\gamma}^{\prime}\right),\sigma_{\gamma}\left(q_{\gamma}^{\prime\prime}\right),\ \sigma_{\gamma}(q_{\gamma})\right)$ in Equation (14).

Proof of Lemma 3. The equilibrium is derived by backward induction. In the subgame where β accepts, γ's best response $\sigma_{\gamma}^{\ddagger}$. Given $\sigma_{\gamma}^{\ddagger}$, α chooses $\theta_{\gamma}^{\ddagger}$ to maximize its expected payoff. In the subgame where β fights with protraction, the best responses are the same: $\left(\theta_{\gamma}^{\prime\prime\prime\ddagger},\sigma_{\gamma}^{\prime\prime\prime\ddagger}\right)=\left(\theta_{\gamma}^{\ddagger},\sigma_{\gamma}^{\ddagger}\right)$.

Similarly, in the subgame where β fights and α wins, α and γ's best responses are $\left(\theta_{\gamma}^{\prime\ddagger},\sigma_{\gamma}^{\prime\ddagger}\right)$. Also, in the subgame where β fights and wins, α and γ's best responses are $\left(\theta_{\gamma}^{\prime\prime\ddagger},\sigma_{\gamma}^{\prime\prime\ddagger}\right)$.

Given $\sigma_{\beta}^{\ddagger}$, α's expected payoff from placing the minimum acceptable offer ($\theta_{\beta}=p_{\beta}U-c_{\beta}$) is:

$$U-(p_{\beta}U-c_{\beta})+G\left(q_{\gamma}V-\theta_{\gamma}^{\ddagger}\right)(q_{\alpha}V-d_{\alpha})+\left(1-G\left(q_{\gamma}V-\theta_{\gamma}^{\ddagger}\right)\right)\left(V-\theta_{\gamma}^{\ddagger}\right),$$

while α's expected payoff from placing an unacceptable offer (any $\theta_{\beta}<p_{\beta}U-c_{\beta}$) is:

$$\begin{aligned}&p_{\alpha}U-c_{\alpha}+\delta p_{\alpha}\left(G\left(q_{\gamma}^{\prime}V-\theta_{\gamma}^{\prime\ddagger}\right)(q_{\alpha}^{\prime}V-d_{\alpha})+\left(1-G\left(q_{\gamma}^{\prime}V-\theta_{\gamma}^{\prime\ddagger}\right)\right)\left(\left(V-\theta_{\gamma}^{\prime\ddagger}\right)\right)\right)\\&+\delta p_{\beta}\left(G\left(q_{\gamma}^{\prime\prime}V-\theta_{\gamma}^{\prime\prime\ddagger}\right)(q_{\alpha}^{\prime\prime}V-d_{\alpha})+\left(1-G\left(q_{\gamma}^{\prime\prime}V-\theta_{\gamma}^{\prime\prime\ddagger}\right)\right)\left(\left(V-\theta_{\gamma}^{\prime\prime\ddagger}\right)\right)\right)\\&+(1-\delta)\left(G\left(q_{\gamma}V-\theta_{\gamma}^{\prime\prime\prime\ddagger}\right)(q_{\alpha}V-d_{\alpha})+\left(1-G\left(q_{\gamma}V-\theta_{\gamma}^{\prime\prime\prime\ddagger}\right)\right)\left(\left(V-\theta_{\gamma}^{\prime\prime\prime\ddagger}\right)\right)\right).\end{aligned}$$

By comparing these two payoffs, α chooses to fight by placing an unacceptable offer if the latter payoff is larger, so that $\theta_{\beta}^{\ddagger}$ is choosen. ■

Proof of Proposition 2. The proof is immediate from Lemma 3. The condition for β's fight appears in Equation (16). Conditional on $\theta_{\gamma}^{\prime\prime\prime\ddagger}$ in the subgame of protraction, the probability of the eastern war is $G\left(q_{\gamma}V-\theta_{\gamma}^{\prime\prime\prime\ddagger}\right)$, which is positive if $g(0)<\frac{1}{d_{\alpha}}$. ■

Proof of Corollary 2. By Equation (12), α's offers to γ are $\left(\theta_\gamma^\ddagger, \theta_\gamma'^\ddagger, \theta_\gamma''^\ddagger, \theta_\gamma'''^\ddagger\right) = \left(\theta_\gamma^\ddagger\left(q_\gamma\right), \theta_\gamma^\ddagger\left(q_\gamma'\right), \theta_\gamma^\ddagger\left(q_\gamma''\right), \theta_\gamma^\ddagger\left(q_\gamma\right)\right)$, where

$$\theta_\gamma^\ddagger(q) = qV - \frac{\overline{d}_\gamma - d_\alpha}{2}.$$

They are all interior by $\overline{d}_\gamma \in \left(d_\alpha, d_\alpha + 2q_\gamma' V\right)$ with $q_\gamma' < q_\gamma < q_\gamma''$. Regardless of $q \in \left\{q_\gamma, q_\gamma', q_\gamma''\right\}$, the probability that γ fights is:

$$\begin{aligned} G\left(qV - \theta_\gamma^\ddagger(q)\right) &= \frac{qV - \theta_\gamma^\ddagger(q)}{\overline{d}_\gamma} \\ &= \frac{qV - \left(qV - \frac{\overline{d}_\gamma - d_\alpha}{2}\right)}{\overline{d}_\gamma} \\ &= \frac{1}{2} - \frac{d_\alpha}{2\overline{d}_\gamma}. \end{aligned} \tag{17}$$

which is positive by $\overline{d}_\gamma > d_\alpha$.

By $\Psi > \frac{c_\alpha + c_\beta}{\delta}$ in Equation (16) with Equation (17), the condition for the western war is summarized as Inequality (5). ■