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17 May 2020

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Abstract. This paper explores optimal location-dependent but time-invariant peak-load charges on a road and a train in a continuous closed city with bottleneck road congestion and rail overcrowding. In our model, rail and car commuters both choose their departure times, considering their schedule delay costs and dynamically changing transportation costs, and their residential locations. Our theoretical results show that when the bottleneck is located at the fringe of the CBD area (Situation 1), the optimal uniform toll and fares are determined by the difference in price distortions between the train and cars. When the bottleneck on the road is located some distance from the CBD (Situation 2), the optimal uniform toll and fares are represented by price distortions of the cars and train, respectively. Our quantitative results show that, in Situation 1, our toll and fares can achieve 25% of the first-best welfare gains, whereas, in Situation 2, our toll and fares can achieve approximately 30% of the first-best welfare gains.

Key words: Bottleneck road congestion, Congestion toll, Railway fare, Rail overcrowding
JEL Classification: H21; H23; R48

Acknowledgments. This research was supported by the Ministry of Education, Culture, Sports, Science and Technology, which is gratefully acknowledged (Grant-in-Aid for Scientific Research B 17H02517). In addition, we would like to thank the late Xiangchun Lu for his valuable supports such as discussing this topic with us and developing the first simulation model.

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1. Introduction

We explore location-dependent but time-invariant peak-load charges on cars and trains in a continuous city, considering the interplay between cars and trains. We consider a situation with dynamic change in road congestion and train overcrowding, in which commuters living all over the city choose their departure times as well as transport mode, and their residential locations.

Severe road congestion is one of the most important problems in urban areas. The first-best policy is to impose congestion tolls that vary at each point of driving time such that every motorist always faces the marginal social cost. However, it is difficult to implement the first-best congestion tolls because of the potentially enormous implementation and operation costs, as indicated by Rouwendel and Verhoef (2006). Accordingly, second-best policies such as cordon pricing are implemented in cities such as London and Singapore because they are easier to introduce than the first-best.

Severe overcrowding on commuter trains is also an important problem in urban areas. Possible solutions are an increase in the number of trains, and construction of new railway lines, etc. But these are not always possible on every line because they take a lot of construction time and cost. Correspondingly, we focus on congestion fares, which are easier to implement than construction of railways. In fact, these have been introduced in Washington and London subways for the purpose of relieving overcrowding.

We focus on the congestion problems of the two modes and seek congestion charge levels for road and rail in a continuous closed monocentric city. Congestion charges for cars and trains change the population density, the modal choice at each location and the size of the city. Accordingly, we need to consider a continuous city.

Since the two modes have different congestion mechanisms, many papers have explored a two-transport-mode system. For example, Tabuchi (1993), Arnott and Yan (2000) and Romeo et al. (2002) explore a railway system in a model with congested roads. However, most related papers do not consider urban space, setting only one set of origin and destination points. Haring et al. (1976), Anas and Moses (1979) and Sasaki (1989) suppose an urban space with two transport modes but do not study transportation charge policies. Although Arnott and Yan (2000) and Quentin and Renaud (2014) consider congestion in a public transportation system, population density is given exogenously.

The common way of modeling congestion in urban areas is static flow congestion, where the timing of travel is not a choice variable, and where traffic flows and speeds at each location are constant over time. The bottleneck model, developed by Vickrey (1969), can also express congestion externality. The bottleneck model is a dynamic model of traffic congestion, in which the choice of departure times is endogenous, and where dynamic patterns of travel delays are key features. However, most papers exploring bottleneck congestion do not consider urban spaces in the city.

A few recent papers have taken account of dynamic congestion in urban space. For instance, Fosgerau and de Palma (2012) and Takayama and Kuwahara (2016) consider a continuous city with a central bottleneck, exploring welfare improvements of heterogeneous residents. These papers mainly focus on the effects of bottleneck congestion and an optimal time-varying congestion toll on the spatial structure of cities.

The first novel point of the current paper is that it explores simultaneous optimization of time-invariant charges at the bottleneck of the road and railway fares at each location in the continuous city with endogenously determined residential locations. These charges cannot exclude congestion externalities completely because the dynamic distribution of departure times is not the first-best. Indeed, in the cities imposing congestion tolls, they impose time-invariant tolls, or small variations at best during peak hours. This practical imposition is useful for calculation of optimal tolls and fares as well as ease of understanding for users. But it generates deadweight losses due to inefficient dynamic distribution of transport demand. Our examination can identify how the deadweight losses can be minimized, focusing on the interplay between the two modes[1].

The second novel point is that we model railway commuters who compare the schedule delay cost and train overcrowding cost in a continuous city. This model theoretically demonstrates a novel property about location- and departure-time-dependent crowding of a train (See Lemma 1). This property is verified by the data of a railway in Tokyo. With these railway commuters and the car commuters who compare the schedule delay and the transportation costs, we express the interplay between the two modes.

The third novel point is that we consider the end point of the railway. This is realistic, but previous theoretical papers have not actively considered this. As a result, we can show the optimal policies for the residents living beyond the end point of the railway. In addition, we exogenously set different road bottleneck locations.

Our results show how price distortions of a bottleneck and railway fares are related in a city with a bottleneck in the road and multiple overcrowded trains on a line, depending on the geographical location of the bottleneck in the road. Furthermore, we perform numerical simulations to obtain changes in the welfare level.

The remainder of this paper is as follows. Section 2 develops an urban model with car and train commuting depending on the location of the road bottleneck. Section 3 explores optimal peak-load time-invariant car tolls and railway fares. Numerical simulations are shown in Section 4. Finally, Section 5 concludes the paper.

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[1] Many other second-best situations have been explored (e.g. Tikoudis, et al. (2018) and Kono and Kawaguchi (2017) considering distortions in the housing market; Parry and Bento (2001), Tikoudis et al. (2015), and Kono et al. (2020) considering distortions in the labor market.) The current paper focuses on two-mode problems, so ignores the distortions in markets other than transportation.

2. The model

2.1. The City

We construct a congested monocentric rectangular city. The width of the residential area is normalized to one, and the area extends from $\underline{x}$ at the CBD edge to $\overline{x}$ at the urban growth boundary (UGB). The city has two transport modes with road congestion and rail overcrowding. When car commuters pass through a single bottleneck located at x_b, they incur dynamic traffic congestion.

The city is composed of four essentially-heterogenous zones: zone 0 (CBD area), which extends from 0 to $\underline{x}$, zone 1, which extends from $\underline{x}$ to the location of the bottleneck x_b, zone 2, which extends from x_b to the end point of the railway x_r, and zone 3, which extends from x_r to the urban boundary $\overline{x}$ (See Fig. 1).

The commuters living in zones 1 and 2 choose road or rail, and the commuters living in zone 3 use only cars because they live beyond the end point of the railway. We assume that the desired arrival time is the same. Car commuters living in zones 2 and 3 have to incur time cost if congestion arises at the bottleneck. Assuming a so-called point queue, bottleneck congestion does not have a physical distance. Due to a physical constraint of the bottleneck's capacity, car commuters necessarily arrive earlier[2] than the desired arrival time. In such a case, they have to incur the schedule cost. As a result of trade-off between time cost and schedule cost, they incur the same total private cost.

We assume that there is an effective bottleneck in the road, and that there is no road congestion at other locations. Rail commuters incur discomfort cost due to train overcrowding in addition to travel time cost and fare. Due to the physical capacity of the railway car, not all the rail commuters can arrive at the desired arrival time but the

[2] For simplicity, in this research, we do not consider commuters who arrive later than the desired arrival time.

majority of them have to arrive earlier. In such a case, they have to incur the schedule delay cost. As a result of trade-off between the discomfort cost and the schedule delay cost, all the commuters incur the same private cost.

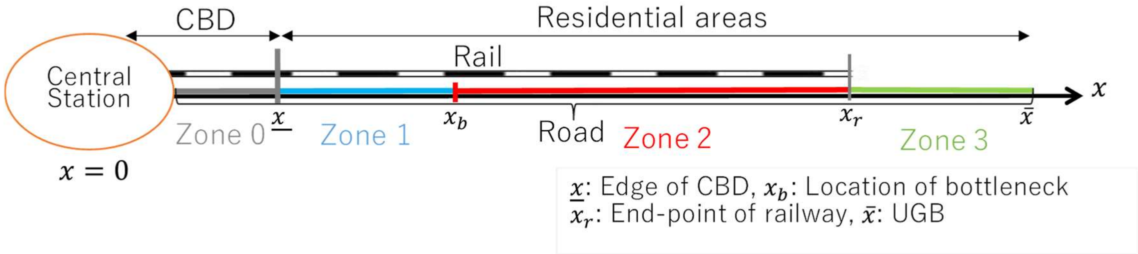

Fig. 1 The city

We consider a closed city. The land is publicly owned and the land is rented by developers to build dwelling units. The total supply per unit area of land at location x is 1 for simplicity. The road operator introduces toll τ, which is levied uniformly throughout peak hours at the bottleneck. The railway operator optimizes location-dependent railway fare $e(x)$, but impose it uniformly throughout peak hours.

2.2. Behavior of commuters

We assume that N identical households reside in the city. For simplicity, we suppose the number of households in the city is equal to the number of commuters. A household has a utility function

$$v = \upsilon(k + \textstyle\sum_{j=1}^{J} \zeta_j \varepsilon_j^{car}, q), \tag{1}$$

where $v = \upsilon(k + \sum_{j=1}^{J} \zeta_j \varepsilon_j^{car}, q)$ is a function of the numeraire composite goods k, the change in utility from rail use to road use ε_j^{car}, and housing square footage q. k includes all goods except for floor space. ε_j^{car} depends on day j, but the distribution of ε_j^{car} follows a certain probability function. ζ_j is a dummy variable which equals 1 if a person uses his car and equals 0 if he uses a train. J is a certain period and j is one day of J.

The simple sum of k and $\sum_{j=1}^{J} \zeta_j \varepsilon_j^{car}$ in Eq. (1) might appear specific. This requires an explanation. First, normally we can set a utility function as

$u(k, \sum_{j=1}^{J} \zeta_j \varepsilon_j^{car}, q)$. In this situation, if $\partial u / \partial k = \partial u \big/ \partial(\sum_{j=1}^{J} \zeta_j \varepsilon_j^{car})$, we can represent the utility function as Eq. (1). This condition implies that k and $\sum_{j=1}^{J} \zeta_j \varepsilon_j^{car}$ are perfectly substitutable. In other words, ε_j^{car} is measured in terms of k. Since the probability distribution of ε_j^{car} can be any function, Eq. (1) represents a large variety of preferences over composite goods, transportation, and housing.

An example of the probability function $\chi(\varepsilon^{car})$ is shown in Fig. 1, where $\Phi(\kappa(x))$ is a probability of using a car, given as $\Phi(\kappa(x)) = \int_{\kappa(x)}^{\infty} \chi(\varepsilon^{car}) d\varepsilon^{car}$. $\kappa(x)$ is a value of ε_j^{car} when the difference between car and train commuting costs equals ε_j^{car} (See Fig. 2), which is mathematically shown as

$$\kappa(x) = t^{car}(x) - t^{rail}(x), \tag{2}$$

where $t^{car}(x)$ is the commuting cost by car and $t^{rail}(x)$ is the commuting cost by rail, which is composed of travel time cost and railway fare.

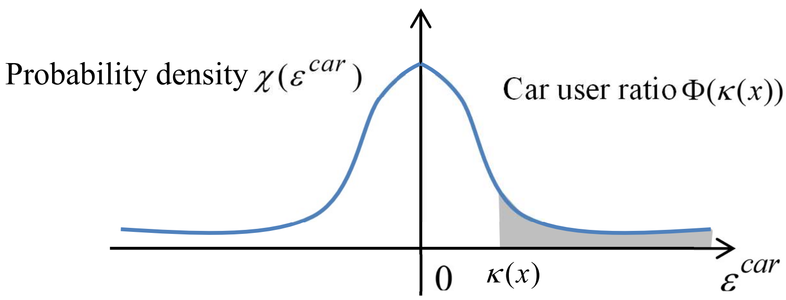

Fig. 2 Train and car user ratios

The revenues of land rent R, congestion toll T, and rail fare E are equally distributed among all households. In addition, all households equally incur the railway maintenance cost Z, and construction cost X. That is, $[R+T+E-Z-X]/N \equiv G/N$ is distributed to each household, where G is non-labor income.

Each household earns income y per period. A household rents floor space from developers at price r per unit space per period. The income constraint is expressed as

$$k+rq=y-\left\{\sum_{j=1}^{J}\zeta_j t^{car}(x)+\sum_{j=1}^{J}\left[1-\zeta_j\right]t^{rail}(x)\right\}+\frac{G}{N}. \tag{3}$$

We assume that J is a sufficiently long term. Because ε_j^{car} has a certain probability function, the values of $\sum_{j=1}^{J}\zeta_j\varepsilon_j^{car}$, which is the number of car commutes, and $\sum_{j=1}^{J}[1-\zeta_j]$, which is the number of rail commutes, will converge to certain values, which are hereafter defined by $J^{car}(x)$ and $J^{rail}(x)$. In addition, the certain values will be common over the households living at location x.

Floor rent r equals the maximum floor rent bid by a household as a result of competition among residents. Mathematically, such behavior is expressed by

$$\max_{q(x),k,J^{car},J^{rail}} \quad r(x)=\frac{y-[J^{car}(x)t^{car}(x)+J^{rail}(x)t^{rail}(x)]+\frac{G}{N}-k}{q(x)} \quad \text{s.t. Eq. (1).} \tag{4}$$

Solving Eq. (4) and using utility level $v(x)$ at location x yields

$$r=r\left(t^{car}(x),\kappa(x),G,v(x)\right) \text{ and } q=q\left(t^{car}(x),\kappa(x),G,v(x)\right). \tag{5}$$

2.3. Population at location x and car commuting cost

The commuters living in zones 1 and 2 choose road or rail and the commuters living in zone 3 use only road. Variables ($n(x),n^{car}(x),t^{car}(x),r(x),e(x),\kappa(x)$) are separated into zone 0 ($0\le x\le \underline{x}$), zone 1 ($\underline{x}\le x\le x_b$), zone 2 ($x_b\le x\le x_r$), and zone 3 ($x_r\le x\le \overline{x}$), with subscripts 0-3 added to denote zone number. $n(x)$ denotes the number of households residing beyond x, and represents the number of commuters passing location x. The number of households residing beyond x, $n_i(x)$, $i\in\{1,2,3\}$ are defined as

$$n_1(x)=\int_x^{x_b}\frac{1}{q_1(s)}ds+\int_{x_b}^{x_r}\frac{1}{q_2(s)}ds+\int_{x_r}^{\overline{x}}\frac{1}{q_3(s)}ds,\ x\in[\underline{x},x_b], \tag{6}$$

$$n_2(x)=\int_x^{x_r}\frac{1}{q_2(s)}ds+\int_{x_r}^{\overline{x}}\frac{1}{q_3(s)}ds,\ x\in[x_b,x_r], \tag{7}$$

$$n_3(x)=\int_x^{\overline{x}}\frac{1}{q_3(s)}ds,\ x\in[x_r,\overline{x}]. \tag{8}$$

Since $\underline{x}$ is the inner edge of the residential area, $n_1(\underline{x})$ represents the total number of commuters (i.e., $n_1(\underline{x}) = N$). $n^{car}(x)$ and $n^{rail}(x)$ are the number of commuters who use road and rail, respectively, at location x.

Moving on to the commuting cost, we will first explain the commuting cost of a car. We assume that the desired arrival times of all commuters are the same, so all car commuters in zones 2 and 3 incur the same private cost at the bottleneck. Using $n_2^{car}(x_b)$, which represents the number of car commuters passing location x_b (i.e., the total number of car commuters in zones 2 and 3), this is deterministically obtained as $\delta_c n_2^{car}(x_b)/s_c$, where δ_c is a parameter and s_c is the bottleneck capacity, as shown in Arnott et al. (1993). All car commuters incur free flow cost, which linearly increases with respect to the distance. Using parameter b, the generalized cost per distance is expressed as $\dot{t}^{car}(x) = b$, where a dot expresses a differentiation with respect to distance. In addition, car commuters incur congestion tolls τ, which the road operator may introduce at the bottleneck. The cost of a car commuter is shown as

$$t_1^{car}(x) = bx, \tag{9}$$

$$t_h^{car}(x) = \frac{\delta_c n_2^{car}(x_b)}{s_c} + bx + \tau, \ h \in \{2,3\}. \tag{10}$$

2.4. Railway commuting cost

Next, we set the commuting cost of rail. Van den Berg and Verhoef (2012) construct a dynamic model regarding rail congestion, assuming an origin-destination pair. We extend this to include many origin points. The number of trains to run is M, and all the trains start at different times and all the trains run between the CBD and the end point of the railway. Rail commuters choose train m $(1 \leq m \leq M)$. Train 1 reaches the CBD at the commuters' desired arrival time t_1. As the train number increases, the train arrives earlier, train M arriving at the earliest time t_M. We assume that no rail commuter reaches the CBD later than the desired arrival time. Fig. 3 shows train m, with train numbers on the vertical axis, and the distance from the CBD on the horizontal axis.

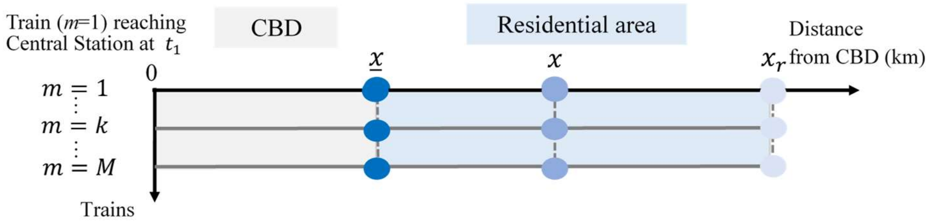

Fig. 3 Train service

Rail commuters at location x incur rail fare $e_h(x)$, train overcrowding cost, schedule delay cost, and travel time cost $a\,x$, and a is a parameter, where $h \in \{1,2\}$ is zone number. Train overcrowding cost of train m for a resident at location x is expressed as $\int_{\underline{x}}^{x} \rho\left(n_m^{rail}(x)\right) dx$, where ρ is a monotonically increasing function and $n_m^{rail}(x)$ $(1 \leq m \leq M)$ is the number of rail commuters who use train m. $n_m^{rail}(x)$ decreases with respect to train number. Schedule delay cost of train m is expressed as $\sigma(t_m - t_1)$, where σ is a parameter, t_m $(1 \leq m \leq M)$ is arrival time when a commuter takes train m. σ represents per unit cost of early arrival.

When a commuter takes train M, they do not incur the train overcrowding cost, but incurs the highest schedule delay cost. On the other hand, when a commuter takes train 1, they do not incur the schedule delay cost but incurs the highest train overcrowding cost. The commuting cost of railway at location x is expressed as

$$c_m^{rail}(x) = ax + e(x) + \int_0^x \rho\left(n_m^{rail}(s)\right) ds + \sigma\left(t_m - t_1\right) \quad (m = 1,2,3,\cdots,M). \tag{11}$$

At equilibrium, any rail commuter at location x is unable to find a train which reduces his total cost. In other words, C_m^{rail} should be constant for all m.

$$C^{rail}(x) = c_1^{rail}(x) = \cdots = c_m^{rail}(x) = \cdots = c_{M(x)}^{rail}(x) < c_{M(x)+1}^{rail}(x), \tag{12}$$

where $C^{rail}(x)$ is the commuter cost of rail at location x in equilibrium. The constraint of the number of rail commuters is expressed as

$$\sum_{i=1}^{M(x)} n_i^{rail}(x) = n^{rail}(x)\,. \tag{13}$$

We explore the equilibrium of rail commuters in the residential areas. First, we consider the equilibrium for rail commuters living at $\underline{x}$. Commuting costs of the lth train $c_l^{rail}(\underline{x})$ and that of the kth train $c_k^{rail}(\underline{x})$ are equal for $k \le l$, where k and l are arbitrary numbers. Therefore,

$$\underbrace{\int_0^{\underline{x}} \rho\left(n_l^{rail}(s)\right) ds - \int_0^{\underline{x}} \rho\left(n_k^{rail}(s)\right) ds}_{\text{(Train overcrowding cost)}} = \underbrace{\sigma\left(t_k - t_l\right)}_{\text{(Schedule delay cost)}} \quad \text{for } k \le l, \tag{14}$$

where the left-hand side means the difference in train overcrowding cost in the lth train and that in the kth train. The right-hand side means the difference in schedule delay cost in the lth train and that in the kth train. This equality implies that overcrowding cost of each train is balanced with the schedule delay cost.

At equilibrium, every commuter living at the city boundary $\underline{x}$ has no incentive to change their train. That implies that the congestion cost in the trains running from $\underline{x}$ to 0 should be balanced with the opportunity cost of early arrival. This discussion is for commuters residing at $\underline{x}$. However, every commuter residing beyond $\underline{x}$ also has the same condition that the congestion cost in the trains running from $\underline{x}$ to 0 should be balanced with the opportunity cost of early arrival because they face the same situations for this interval $(0, \underline{x})$. This point is important for the next discussion.

We next consider the equilibrium for rail commuters living between $\underline{x}$ and x_r. The commuting cost from $\underline{x}$ to 0 has already been determined based on the equilibrium of residents living at $\underline{x}$, i.e., Eq. (14). So, it is sufficient to check the commuting cost from the residence to $\underline{x}$. Commuting cost of the lth train service $c_l^{rail}(\underline{x})$ and that in the kth train service $c_k^{rail}(\underline{x})$ are equal ($k \le l$), shown as

$$\int_{\underline{x}}^{x} \rho\left(n_l^{rail}(s)\right) ds = \int_{\underline{x}}^{x} \rho\left(n_k^{rail}(s)\right) ds \quad \text{for } x \in (\underline{x}, x_r) \quad \text{for any } k \text{ and } l. \tag{15}$$

Eq. (15) means all rail commuters at location x incur the same level of train overcrowding cost between $\underline{x}$ and x regardless of train number. In addition, the number of rail commuters at location x is equal between $\underline{x}$ and x regardless of train number. From Eq. (13), the number of rail commuters at location x ($\underline{x} \leq x \leq x_r$) is proportional to the number of total rail commuters $n^{rail}(x)$. These results are summarized in Lemma 1.

Lemma 1 (Overcrowding level on each train at each location). (1) The overcrowding cost of the trains running from $\underline{x}$ to 0 should be balanced with the opportunity cost of the early arrival of the trains. **(2)** Rail commuters at location x incur the same level of train overcrowding cost between $\underline{x}$ and x regardless of which train they use.

Van den Berg and Verhoef (2012) already show Lemma 1 (1). But since they set one origin-destination pair, Lemma 1 (2) is not obtained[3]. We extend their model to many pairs with multiple origins and a single destination. Due to the constraints of data availability, it is difficult to completely check these properties shown in Lemma 1 (1) and (2) in real situations. But, using the data of two routes in Tokyo, we are able to verify that Lemma (1) and (2) hold in real situations. The analysis is included in Appendix B.

Using Lemma 1 (2), train overcrowding cost between $\underline{x}$ and x is expressed as

$$\int_{\underline{x}}^{x} \rho\left(n_m^{rail}(s)\right) ds = \int_{\underline{x}}^{x} f\left(n^{rail}(s)\right) ds, \tag{16}$$

where f is a monotonically increasing positive function. This implies that, in equilibrium, train overcrowding cost for rail commuters from location x to $\underline{x}$ is determined by a function of the number of total rail commuters at location x. With Lemma 1 (1), following Van den Berg and Verhoef (2012), if we specify the crowding

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[3] One might feel that this result is not realistic. In real situations, there are some different conditions such as multiple destination points and heterogenous desired times. In the real world, Lemma 1 would not hold due to these conditions. However, it is important to focus on the mechanisms. So, we set the current homogenous conditions.

cost function $\rho\left(n_m^{rail}(\underline{x})\right) = \rho \cdot \left(n_m^{rail}(\underline{x})\right)^\theta$, where θ is a positive parameter, the commuting cost of rail is expressed as[4]

$$t_h^{rail}(x) = ax + e_h(x) + \underline{x}f(n_1(\underline{x}) - n_1^{car}(\underline{x})) + \sum_{i=1}^{h} \int_{\underline{x}}^{x} f(n_i(s) - n_i^{car}(s))ds, \ h \in \{1,2\}, \quad (17)$$

$$t_1^{rail}(\underline{x}) = a\underline{x} + e_1(\underline{x}) + \underline{x}f(n_1(\underline{x}) - n_1^{car}(\underline{x})). \quad (18)$$

2.5. Landowners, road operators and railway operators

The revenue of land rent is expressed as

$$\begin{aligned} R &= \int_0^{\underline{x}} r_0(x)dx + \int_{\underline{x}}^{x_b} r_1(t_1^{car}(x), \kappa_1(x), G, v(x))dx \\ &+ \int_{x_b}^{x_r} r_2(t_2^{car}(x), \kappa_2(x), G, v(x))dx + \int_{x_r}^{\overline{x}} r_3(t_3^{car}(x), G, v(x))dx \end{aligned}. \quad (19)$$

Road operators impose congestion toll τ at the bottleneck. Since all car commuters pass the bottleneck, the total revenue of congestion toll is expressed as

$$T = Jn_2^{car}(x_b)\tau. \quad (20)$$

Railway operators construct a railway in the city and maintain it. The cost related to rail is divided into operation cost and construction cost. The operation cost varies in accordance with total transport distance, which is the sum of personal commuting distance of all individuals. It is expressed as

$$Z = Jz\left[\underline{x}\left\{n_1(\underline{x}) - n_1^{car}(\underline{x})\right\} + \int_{\underline{x}}^{x_b}\left\{n_1(x) - n_1^{car}(x)\right\}dx + \int_{x_b}^{x_r}\left\{n_2(x) - n_2^{car}(x)\right\}dx\right], \quad (21)$$

where Z is marginal cost per passenger distance. The construction cost varies with the distance. Since the railway extends from $\underline{x}$ to x_r, the construction cost is expressed as

$$X = Ix_r, \quad (22)$$

where I is construction cost of the railway per distance.

The total revenue of rail fare $e(x)$ is expressed as

[4] The specification of the crowding cost function is necessary for the derivation. The derivation process is shown on the authors' website.

$$E = -J\int_{\underline{x}}^{x_b} e_1(x)\left\{\dot{n}_1(x) - \dot{n}_1^{car}(x)\right\}dx - J\int_{x_b}^{x_r} e_2(x)\left\{\dot{n}_2(x) - \dot{n}_2^{car}(x)\right\}dx \,. \quad (23)$$

2.6. Market equilibrium conditions

Since households can choose any residential location, household utility is equal to a common level u at any locations. The level of u is endogenously determined as

$$v(x) = u \quad \forall x \in [1, \overline{x}] \,. \quad (24)$$

The population constraint at each location, which is obtained by differentiating Eqs. (6) - (8) with respect to x, should hold, as

$$q(t_h^{car}(x), \kappa_h(x), G, v(x))\left(-\dot{n}_h(x)\right) \le 1, h \in \{1,2\}, q(t_3^{car}(x), G, v(x))\left(-\dot{n}_3(x)\right) \le 1 \,. \quad (25)$$

Using Eqs. (6)-(8), (25) and the modal ratio $\Phi(\kappa(x))$, we obtain the following relations.

$$n_1^{car}(x) = -\int_x^{x_b} \dot{n}_1(s)\Phi(\kappa_1(s))ds - \int_{x_b}^{x_r} \dot{n}_2(s)\Phi(\kappa_2(s))ds - \int_{x_r}^{\overline{x}} \dot{n}_3(s)ds \,, \quad (26)$$

$$n_2^{car}(x) = -\int_x^{x_r} \dot{n}_2(s)\Phi(\kappa_2(s))ds - \int_{x_r}^{\overline{x}} \dot{n}_3(s)ds \quad (27)$$

$$n_3^{car}(x) = -\int_x^{\overline{x}} \dot{n}_3(s)ds \quad (28)$$

2.7. Social welfare function

The optimality of car and rail charges is defined as maximizing social welfare. The social welfare W is the total utility of households. The optimal car toll and rail fare are given by

$$[\tau, e_1(x), e_2(x)] = \arg \max_{\tau, e_1(x), e_2(x)} W = Nu \quad (29)$$

s.t. Eqs. (1)-(28).

Because we assume a closed city, the total population N is exogenous.

3. Theoretical examination

3.1. Optimal car toll and rail fare when bottleneck is located at the edge of CBD

As a result of examination, we obtain Proposition 1.

Proposition 1 (Optimal toll and fare with the bottleneck at the CBD edge)

When the bottleneck is at the edge of the CBD, the optimal car toll and rail fares *satisfy*

$$\tau - e_2(x) = \frac{\delta_c n_2^{car}(\underline{x})}{s_c} - \begin{bmatrix} zx + \underline{x}\left(n_2(\underline{x}) - n_2^{car}(\underline{x})\right) f'(n_2(\underline{x}) - n_2^{car}(\underline{x})) \\ + \int_{\underline{x}}^{x} \left(n_2(s) - n_2^{car}(s)\right) f'(n_2(s) - n_2^{car}(s)) ds \end{bmatrix}. \tag{30}$$

for $x \in (\underline{x}, x_r)$

See Appendix A for the detailed derivation.

Proposition 1 shows the condition for optimal car toll and rail fares. The difference between car toll and rail fare at x equals the difference between the cost of passing through the bottleneck and the sum of railway marginal operation cost and congestion cost at x.

3.2. Optimal car toll and rail fares when bottleneck is located in residential area

We obtain Proposition 2 when the bottleneck is located in the residential area.

Proposition 2 (Optimal fare on the CBD side of the bottleneck with the bottleneck in the residential area)

When the bottleneck is located in the residential area, the optimal rail fare inside the location of the bottleneck satisfies

$$e_1(x) = \begin{bmatrix} zx + \underline{x}\left(n_1(\underline{x}) - n_1^{car}(\underline{x})\right) f'(n_1(\underline{x}) - n_1^{car}(\underline{x})) \\ + \int_{\underline{x}}^{x} \left\{\left(n_1(s) - n_1^{car}(s)\right) - \left(n_1(x_b) - n_1^{car}(x_b)\right)\right\} f'(n_1(s) - n_1^{car}(s)) ds \end{bmatrix}. \tag{31}$$

for $x \in (\underline{x}, x_b)$

See Appendix A for the detailed derivation.

Proposition 2 implies that the optimal rail fare at x on the CBD side of the bottleneck equals the sum of marginal operation cost and train overcrowding cost from x to the CBD. This is equal to the first-best charge in terms of formulation. This holds because car congestion is simultaneously controlled by the optimal uniform toll (a second-best toll).

Proposition 3 (Optimal fare outside the bottleneck with the bottleneck in the residential area)

When the bottleneck is in the residential area, the optimal rail fares beyond the bottleneck satisfy

$$e_2(x)=\left[\begin{array}{l} zx+\underline{x}\left(n_1(\underline{x})-n_1^{car}(\underline{x})\right)f'(n_1(\underline{x})-n_1^{car}(\underline{x})) \\ +\int_{\underline{x}}^{x_b}\left\{\left(n_1(s)-n_1^{car}(s)\right)-\left(n_1(x_b)-n_1^{car}(x_b)\right)\right\}f'(n_1(s)-n_1^{car}(s))ds \\ +\left(n_1(x_b)-n_1^{car}(x_b)\right)f'(n_1(x_b)-n_1^{car}(x_b))+\int_{x_b}^{x}\left(n_2(s)-n_2^{car}(s)\right)f'(n_2(s)-n_2^{car}(s))ds \end{array}\right]. \tag{32}$$

for $x\in(x_b,x_r)$

Proposition 3 implies that the optimal rail fare at x beyond the bottleneck equals the sum of marginal operation cost and train overcrowding cost from x to the CBD. Similar to Proposition 2, optimal rail fares are determined only by rail-related costs, not by car-related cost even though the optimal uniform toll and fares are second-best. See Chapter 7 of Kono and Joshi (2019) for the optimal conditions of multiple second-best policies.

Proposition 4 (The optimal car toll, as in Arnott et al. (1993))

When the bottleneck is in the residential area, the optimal car toll satisfies

$$\tau=\frac{\delta_c n_2^{car}(x_b)}{s_c}. \tag{33}$$

Proposition 4 implies that, when the bottleneck is in the residential area, the optimal car toll at the location of the bottleneck is the same as shown in Arnott et al. (1993), who consider elastic car demand for an origin-destination pair without a railway, even if we consider a continuous and closed city[5].

[5] The current paper assumes that the end of the railway is given. But, if that is optimized, when optimal car toll and rail fares are imposed, the optimal end point of the railway line satisfies $r_2(x_r)-r_3(x_r)=I$.

4. Numerical Situations

4.1. The Setup

Similar to the setup adopted by Kono et al.(2012), we divide the residential areas into narrow, discrete rings with an equal width c. We call the rings 'bands'. The length of each band is denoted as ε_i, where i represents band number. The distance of each band from the CBD is denoted as $x_i = \sum_{j=1}^{i} \varepsilon_j$. A station is at the center of each band. Commuting cost for a resident in a band is set as the commuting cost from the center point of each band to the CBD. We specify the utility function as $v(k + \sum_{j=1}^{J} \zeta_j \varepsilon_j^{car}, q) = k + \sum_{j=1}^{J} \zeta_j \varepsilon_j^{car} + \alpha \ln q + 31000$, where α is a positive parameter.

From the viewpoint of calculation burden, we consider only six bands, which corresponds to zones in Figure 1, as follows.

Table 1. Zones in Figure 1 and the corresponding bands

Zones in Figure 1	Zone 1	Zone 2				Zone 3
Corresponding zones	Band 1	Band 2	Band 3	Band 4	Band 5	Band 6

Next, the total cost of cars t_i^{car} is expressed as

$$t_i^{car} = \frac{\delta_c n^{car}(x_b)}{s_c} + b\sum_{j=1}^{i} \varepsilon_j + \tau\,, \tag{34}$$

which is essentially equal to Eq. (10). The only difference in the notation is $b\sum_{j=1}^{i} \varepsilon_j$, which is the generalized travel cost including time and distance cost. Parameter b expresses the generalized travel cost incurred while driving 1 km. Using average car speed v^{car}, unit-distance travel expense φ and value of time w, b is calculated as $b = w/v^{car} + \varphi$.

The total commuting cost of railway t_i^{rail} is expressed as

$$t_i^{rail} = ax_i + e_i + x_i f(n_i^{rail}) + Q\,, \tag{35}$$

where Q is access cost. The rest of the notation is similar to the notation in Eq. (18). Parameter a expresses the travel time cost per distance. Using average train speed v^{rail} and value of time w, a is calculated as $a = w / v^{rail}$. $x_i f(n_i^{rail})$ is train overcrowding cost and $f(n_i^{rail})$ is a train overcrowding cost function.

The function of the train overcrowding is specified as follows. Δt is the service interval of trains during peak hours. c_{train} is the capacity of a train. However, regardless of the actual train time schedule, to smoothly evaluate the welfare effect of toll and fare policies, we regard trains to run every minute. Accordingly, the capacity of a train $\bar{c}_{train}$ can be calculated as $c_{train} / \Delta t$. So, we suppose that passengers incur train overcrowding cost when there are more than $\bar{c}$ persons in a train. The congestion cost is set as congestion parameter f multiplied by the number of commuters, as expressed by[6]

$$f(n_i^{rail}) = f\, n_i^{rail} \quad \left(\bar{c}_{train} \le n_i^{rail}\right), \tag{36}$$

$$f(n_i^{rail}) = 0 \quad \left(0 \le n_i^{rail} \le \bar{c}_{train}\right). \tag{37}$$

The frequency of trains is exogenously given. From Eq. (14), the commuting cost of rail in the *k*th train for residents in Band 1 is equal to that in the *k-1*th train. So,

$$\underbrace{2 \cdot f \cdot \left(n_k^{rail} - n_{k-1}^{rail}\right)}_{\text{(Train overcrowding cost)}} = \underbrace{\delta_c / 60}_{\text{(Schedule delay cost)}}, \tag{38}$$

where the left-hand side means the difference in train overcrowding cost and the right-hand side means the difference in scheduling delay cost. In Eq. (38), $\left(n_k^{rail} - n_{k-1}^{rail}\right)$ means the difference in the number of rail commuters between the trains. Accordingly, the total number of rail commuters S_m can be expressed as

[6] Congestion cost in a railway car normally increases non-linearly. However, it is difficult to obtain the numerical solutions if such a function is used. So, we use the linear function as an approximation.

$$S_m = \sum_{k=1}^{m} n_k^{rail} = \frac{m-1}{2} \cdot \left\{ 2 \cdot n_f + \frac{\delta_c}{120 \cdot f} \cdot (m-1) \right\}, \tag{39}$$

where n_f is the number of rail commuters on the first train to arrive at the CBD during peak hours. m means the total number of commuter trains during rush hours. In equilibrium, the total volume of rail commuters is affected by the land rent. Therefore, the number of commuter trains during peak hours is determined by Eq. (39).

The total population beyond i, n_i is given by $n_i = c\sum_{j=i}^{i^*} \varepsilon_i D_i$, and the total number of car commuters and train commuters beyond i, n_i^{car} and n_i^{rail} are given by $n_i^{car} = c\sum_{j=i}^{i^*} \varepsilon_i D_i \Phi_i$ and $n_i^{rail} = c\sum_{j=i}^{i^*} \varepsilon_i D_i (1-\Phi_i)$, where i^* denotes the outermost rings, such that $n_{i^*} = 0$, and D_i is the population density. Here, we assume that the probability function $\chi(\varepsilon^{car})$ is uniform. Then, Φ_i is given by

$$\Phi_i = \frac{b_1 - \kappa_i}{b_1 - a_1}, \tag{40}$$

where a_1 and b_1 denote the smallest ε^{car} and the largest ε^{car} of the probability density function. κ_i is the difference in commuting cost between car and train in Band i, which is expressed by

$$\kappa_i = t_i^{car} - t_i^{rail}. \tag{41}$$

Finally, the social welfare is expressed as

$$W = c\sum_{j=1}^{i^*} \varepsilon_j \left[D_j u + r_j \right] + J\tau n^{car}(x_b) + J\sum_{j=1}^{i^*} \left(e_j + \left(1-2x_j\right) z_o \right)\left(n_j^{rail} - n_{j+1}^{rail} \right) - Ix_r \tag{42}$$

where y is income and N is the total number of households. The iterative process begins $i=1$ with $n_1 = N$ and is implemented conditionally on the value of u, which should satisfy the equilibrium condition.

We calibrate parameters, using some real data for the city of Sendai, Japan. The total number of households N is set at 20,800, which is equal to the total number of commuters of the households living in the area within 1.5 km on either side of the Sendai Subway Namboku Line. The CBD edge $\underline{x}$ is set at 1km. The UGB $\overline{x}$ is set at 13 km. The end point of the railway is located at 10 km. The length of each band ε_i is 2 km. The width c is set as 0.6 km.[7] The income per household per year is set at \$42,628.555, which is generalized income. Housing parameter α in the utility function is set at 8,000, implying 20 percent of the income of \$40,000. The value of time w equals \$24.06/hour. The number of trips to the CBD is set at 230 per person per year. Other parameters are explained in Appendix C.

We analyze two patterns of the bottleneck point located at the edge of the CBD and between Band 1 and Band 2 to gauge the efficacy of toll and fare policies obtained as Propositions 1, 2, 3, and 4. We call the two patterns Situations1 and 2, defined as

Situation 1: the bottleneck point is located in the edge of the CBD.

Situation 2: the bottleneck point is located between Band 1 and Band 2.

4.2. Numerical result

We show the results in Table 2 and Table 3. We can analyze how much road congestion and train overcrowding influence welfare gains. For each situation, we calculated three equilibria: (a) laissez-faire, (b) first-best toll and fares (i.e., optimal dynamic toll and fares), (c) optimal uniform toll and fares (second-best). Under laissez-faire, toll τ is 0 and each rail fare is set at marginal cost. Under the first-best, optimal dynamic toll is levied at the location of the bottleneck. Optimal dynamic tolls and fine (the first-best) rail fares eliminate congestion externalities perfectly.

The capacity of a train $\overline{c}_{train}$ is set as follows. Under laissez-faire, because trains are operated every four minutes during the peak hours in Sendai, $\overline{c}_{train}$ is set as 144

[7] In the area within 1.5 km on either side of the railway, the ratio of commuters to households is approximately 0.4. In Sendai, the inhabitable area is half of the total area. So, 3 (km) × 0.4 × 0.5 = 0.6 (km).

(persons/minute). However, under the first-best and the uniform toll and fares, we set $\overline{c}_{train}$ as 288 (persons/minute) because we can increase the frequency of trains on a line such as the Tokyo Metro Ginza Line, where trains run every two minutes during rush hour.

Table 2. Numerical results: social welfare

	W (10^7\$)	Welfare gain (%)	U (\$)	T (10^7\$)	E (10^6\$)	Z (10^6\$)	Road congestion (10^7\$)	Train overcrowding (10^7\$)
Situation 1								
(a)	3.066	—	1474.1	0	3.950	3.950	5.462	1.423
(b)	6.849	100.0	3292.8	2.635	8.694	4.086	0	0
(c)	3.999	24.6	1922.4	2.945	4.017	4.478	4.734	1.071
Situation 2								
(a)	5.311	—	2553.5	0	3.648	3.648	3.299	1.287
(b)	7.881	100.0	3789.0	1.599	7.922	3.782	0	0
(c)	6.052	28.8	2909.5	2.677	1.240	3.841	2.672	0.8570

Note: (a) laissez-faire; (b) first-best; (c) second-best, T: congestion toll revenue, E: railway fare revenue, Z: total cost of railway Z.

Table 3. Numerical results: toll and fares

	τ (\$/day)	e_1 (\$/day)	e_2 (\$/day)	e_3 (\$/day)	e_4 (\$/day)	e_5 (\$/day)
Situation 1						
(a)	0	44	88	131	175	219
(c)	1025	44	88	131	175	219
Situation 2						
(a)	0	44	88	131	175	219
(c)	1210	565	609	653	697	741

Note: (a) laissez-faire; (c) second-best

We will now discuss welfare gains. In Situation 1, the welfare gains under the optimal uniform toll and fares are \$9.33 million, which is equal to 24.6 percent of the first-best welfare gains. The welfare gains per household are \$448. In Situation 2, the welfare gains are \$7.41 million, which is equal to 28.8 percent of the first-best welfare gains.

Factors which affect welfare gains are 1) the number of car commuters, 2) the number of rail commuters, and 3) population in each band. First, we discuss the numbers of car commuters and rail commuters. Fig. 4 and Fig. 5 indicate the number of car commuters and that of rail commuters, respectively, in Situation1 and Situation 2. In Situation 1, the number of car commuters in each band decreases and that of rail commuters increases by our (second-best) toll system because car congestion is more severe than train overcrowding in our numerical model. In this way, optimal uniform charges improve social welfare. However, in Situation 2, the number of car commuters increases greatly on the CBD side of the bottleneck (i.e. in Band 1) although the number of rail commuters in Band 1 increases mildly. The number of car commuters in each band is equal to the product of population and the car share in each band. We will discuss population and the modal share in each band.

Population in each band is shown in Fig. 6. In Situation 1, population in Band 6 decreases and population in other bands slightly increases with the optimal uniform toll and fares. In other words, this indicates that households in Band 6 move to Bands 1, 2, 3, 4, and 5 because rail is not available in Band 6 and total commuting costs incurred by households in Band 6 in a year are more than those incurred by households in other bands. In addition, most commuters do not change residential location with the optimal uniform toll and fare system. In Situation 2, population in Band 6 decreases as in Situation 1. However, population in Band 1 increases and populations in Bands 2, 3, 4, and 5 decrease because one year’s commuting costs incurred by households in Band 1 are different from those incurred by households in other bands.

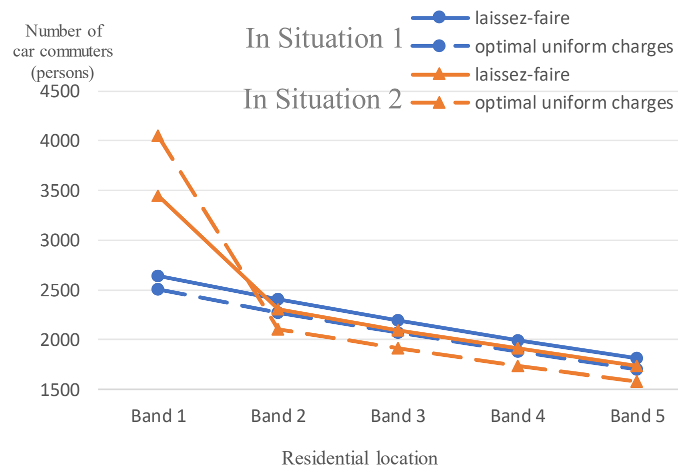

Fig. 4 The number of car commuters

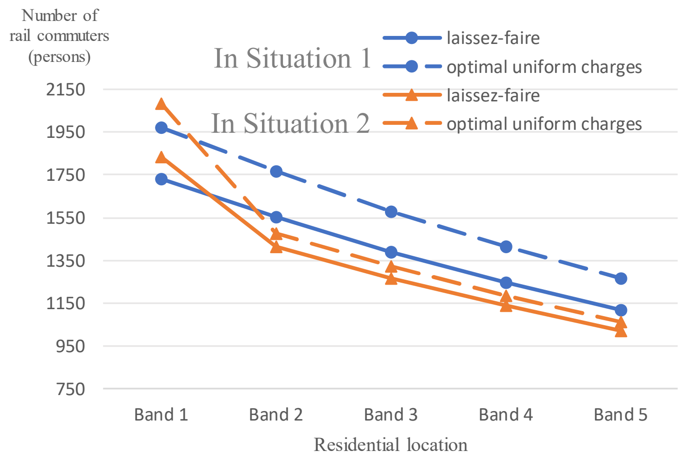

Fig. 5 The number of rail commuters

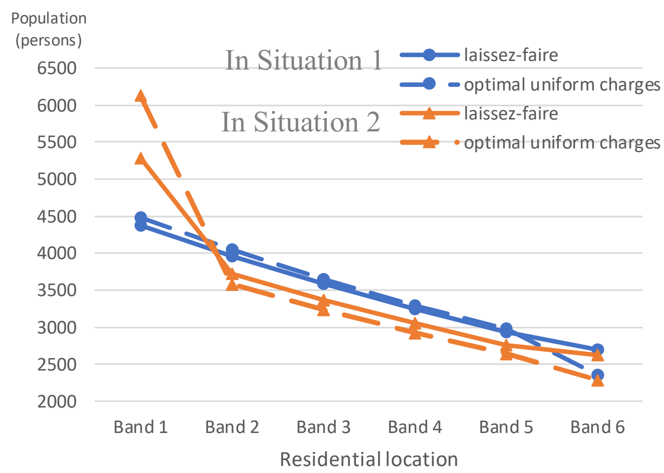

Fig. 6 Population distribution

Next, we will discuss commuting costs in each band in Situation 2. Fig. 7 shows increases in commuting costs per person in each band after imposing our toll and fare system. All the commuters using rail are influenced equally by train overcrowding externalities because our numerical setup generates rail overcrowding only in Band 1. Households using cars outside the location of the bottleneck are influenced equally by bottleneck congestion externalities. However, households using cars inside the location of the bottleneck are not influenced by them at all because they do not pass through the bottleneck. So, households outside the location of the bottleneck have more commuting costs in a year than households inside the location of the bottleneck and move to the CBD side of the bottleneck in order to escape the car congestion externalities. Population in Band 1 increases and populations in other bands decrease as in Fig. 6.

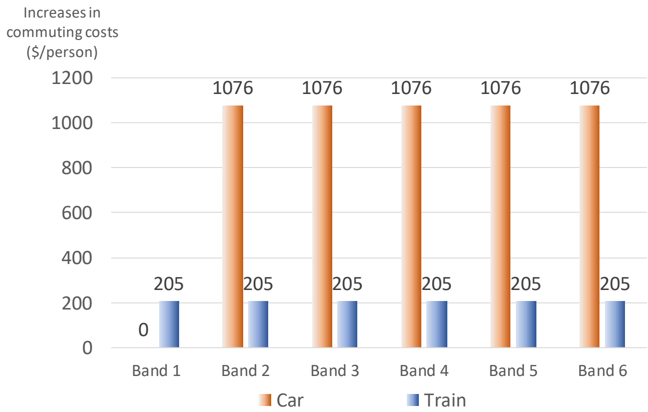

Fig. 7 Increases in commuting costs in Situation 2

Fig. 7 can explain how the modal shares change by using the relative costs of the two modes. The car share on the CBD side of the bottleneck increases and the car shares beyond the bottleneck decrease. On the CBD side of the bottleneck, the increase in train commuting cost is higher than that of car, which is equal to 0. So, households there try to avoid train overcrowding and the car share increases. Outside the location of the bottleneck, the increase in rail commuting cost is less than that of car. So, households there try to avoid car congestion and use rail, and the car shares decrease. Welfare gains are influenced by the changes in residential locations and modal shares.

Finally, in order to analyze how much road congestion affects welfare gains[8], we changed bottleneck capacity δ_c from 11500 to 10350 and 12650. Table 5 presents the results in the case of each capacity when the bottleneck is located at the edge of the CBD. The welfare gains at low capacity ($\delta_c = 10350$) are the highest among the three cases, \$10.46 million and the gains per household are \$503. The welfare gains at $\delta_c = 11500$ and $\delta_c = 12650$ are \$9.32 million and \$8.45 million, respectively, so the gains per household are \$448 and \$406 per year, respectively. This is because road congestion decreases and the total number of rail commuters increases as bottleneck capacity δ_c increases. Fig. 8 shows the increases in population when the optimal uniform toll and fares are implemented. As the bottleneck capacity increases, the increases change greatly, because total commuting costs in Band 6 increase more than those in other bands and households try to move to other bands in order to decrease their total commuting costs.

Table 4. Modal shares in each Band in Situation 2

	Band 1	Band 2	Band 3	Band 4	Band 5
(a): Modal share (%)	65.3	62.0	62.4	62.7	63.1
(c): Modal share (%)	66.0	58.7	59.1	59.5	59.8

Note: (a) laissez-faire; (c) uniform toll and fares

Table 5. Sensitivity analysis with regard to congestion degree in Situation 1

	W	Welfare gain	T	e_1	e_2	e_3	e_4	e_5	n_{rail}	Road congestion
	(10^7\$)	(%)	(\$)	(\$)	(\$)	(\$)	(\$)	(\$)	(persons)	(10^7\$/yr)
Situation 1										
$\delta_c = 10350$										
(a)	2.499	–	0	0.44	0.88	1.31	1.75	2.19	7150	5.977
(b)	6.574	100.0	–	–	–	–	–	–	7396	0
(c)	3.545	25.69	11.61	0.44	0.88	1.31	1.75	2.19	8195	5.097

[8] In this sensitivity analysis, we consider how much the relative levels of train overcrowding and car congestion affect welfare gains. When congestion cost parameter f increases, rail crowding costs and road congestion decreases. This is equal to the case of a decrease of the bottleneck capacity.

$\delta_c = 11500$ (=original parameter)										
(a)	3.066	—	0	0.44	0.88	1.31	1.75	2.19	7045	5.462
(b)	6.849	100.0	—	—	—	—	—	—	7289	0
(c)	3.999	24.65	10.25	0.44	0.88	1.31	1.75	2.19	7995	4.734
$\delta_c = 12650$										
(a)	3.543	—	0	0.44	0.88	1.31	1.75	2.19	6959	5.028
(b)	7.079	100.0	—	—	—	—	—	—	7201	0
(c)	4.388	23.90	9.10	0.44	0.88	1.31	1.75	2.19	7828	4.407
Situation 2										
$\delta_c = 11500$ (=original parameter)										
(a)	5.311	—	0	0.44	0.88	1.31	1.75	2.19	6669	1.287
(b)	7.881	100.0	—	—	—	—	—	—	6907	0
(c)	6.052	28.81	12.10	5.65	6.09	6.53	6.97	7.41	7134	0.857

Note: (a) laissez-faire; (b) first-best; (c) uniform toll and fares

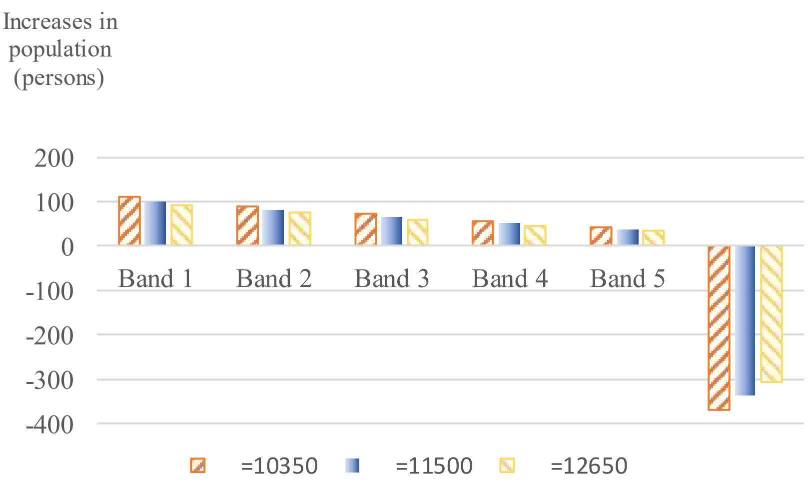

Fig. 8 Population increases

5. Conclusion

This paper explores optimal uniform rail fares and car toll in a continuous and closed city with bottleneck congestion. When the bottleneck is located some distance from the CBD, the optimal fares and toll are represented by price distortion of the railway and cars, respectively. When the bottleneck is located at the edge of the CBD, the optimal fares and toll are determined by the difference in price distortions of the railway and cars.

When the bottleneck is at the edge of the CBD, the optimal uniform toll and fares achieve approximately 25% of the first-best. This is because road congestion is excluded to a certain degree by the optimal uniform toll. In addition, road congestion is mitigated because some car commuters change modes from car to rail. In addition, households in Band 6, which has no railway stations, move to the other bands because they incur higher total commuting costs than households in other bands.

When the bottleneck is located in the residential area, welfare gains of the optimal uniform toll and fares achieve approximately 30 % of the first-best. Households residing beyond the bottleneck contribute to road congestion, while households residing on the CBD side of the bottleneck have no effect on road congestion. Therefore, households beyond the bottleneck move to the area between the CBD and the bottleneck.

Future research on this topic will extend to a consideration of commuter heterogeneity. The existence of user heterogeneity requires the analysis of departure-time decisions and residential location choices.

Appendix A. Lagrangian function of the model

A.1 First order conditions

We obtain the optimal solution of Eq. (29) with a Lagrangian function. Appendix A derives the first order conditions in the case in which the bottleneck is in the residential area. The Lagrangian is very long because multiple heterogenous areas are linked with boundary conditions. This method is adopted in Kono and Joshi (2018) and Kono and Kawaguchi (2017). Some explanation of the mathematics (e.g., the relationship with the Hamiltonian) is shown in Kono and Joshi (2019). The Lagrangian function is given as

$$
\begin{aligned}
L = {} & \frac{Nu}{\eta} + \int_{\underline{x}}^{x_b} \lambda_1(x)\left[\dot{t}_1^{car}(x) - b\right]dx + \int_{x_b}^{x_r} \lambda_2(x)\left[\dot{t}_2^{car}(x) - b\right]dx + \int_{x_r}^{\overline{x}} \lambda_3(x)\left[\dot{t}_3^{car}(x) - b\right]dx \\
& + \int_{\underline{x}}^{x_b} \varphi_1(x)\left[\dot{t}_1^{car}(x) - a - \dot{e}_1(x) - f'(n_1(x) - n_1^{car}(x))\left(\dot{n}_1(x) - \dot{n}_1^{car}(x)\right) - \dot{\kappa}_1(x)\right]dx \\
& + \int_{x_b}^{x_r} \varphi_2(x)\left[\dot{t}_2^{car}(x) - a - \dot{e}_2(x) - f'(n_2(x) - n_2^{car}(x))\left(\dot{n}_2(x) - \dot{n}_2^{car}(x)\right) - \dot{\kappa}_2(x)\right]dx \\
& + \int_{\underline{x}}^{x_b} \mu_1(x)\left[q(t_1^{car}(x), \kappa_1(x), G, u)\dot{n}_1(x) + 1\right]dx + \int_{x_b}^{x_r} \mu_2(x)\left[q(t_2^{car}(x), \kappa_2(x), G, u)\dot{n}_2(x) + 1\right]dx \\
& + \int_{x_r}^{\overline{x}} \mu_3(x)\left[q(t_3^{car}(x), G, u)\dot{n}_3(x) + 1\right]dx + \int_{\underline{x}}^{x_b} \theta_1(x)\left[\dot{n}_1^{car}(x) - \dot{n}_1(x)\Phi(\kappa_1(x))\right]dx \\
& + \int_{x_b}^{x_r} \theta_2(x)\left[\dot{n}_2^{car}(x) - \dot{n}_2(x)\Phi(\kappa_2(x))\right]dx + \int_{x_r}^{\overline{x}} \theta_3(x)\left[\dot{n}_3(x) - \dot{n}_3^{car}(x)\right]dx \\
& - G + \int_0^{\underline{x}} r_0(x)dx + \int_{\underline{x}}^{x_b} r_1(t_1^{car}(x), \kappa_1(x), G, u)dx + \int_{x_b}^{x_r} r_2(t_2^{car}(x), \kappa_2(x), G, u)dx \\
& + \int_{x_r}^{\overline{x}} r_3(t_3^{car}(x), G, u)dx + Jn_2^{car}(x_b)\tau - J\int_{\underline{x}}^{x_b} e_1(x)\left\{\dot{n}_1(x) - \dot{n}_1^{car}(x)\right\}dx - J\int_{x_b}^{x_r} e_2(x)\left\{\dot{n}_2(x) - \dot{n}_2^{car}(x)\right\}dx \\
& - Jz\left[\underline{x}\left\{n_1(\underline{x}) - n_1^{car}(\underline{x})\right\} + \int_{\underline{x}}^{x_b}\left\{n_1(x) - n_1^{car}(x)\right\}dx + \int_{x_b}^{x_r}\left\{n_2(x) - n_2^{car}(x)\right\}dx\right] \\
& - Ix_r + \omega_1\left[t_1^{car}(\underline{x}) - b\underline{x}\right] + \omega_2\left[t_2^{car}(x_b) - \frac{\delta_c n_2^{car}(x_b)}{s_c} - bx_b - \tau\right] \\
& + \vartheta_1\left[\kappa_1(\underline{x}) - b\underline{x} + a\underline{x} + e_1(\underline{x}) + f(n_1(\underline{x}) - n_1^{car}(\underline{x}))\right] \\
& + \vartheta_2\left[\kappa_2(x_b) - \frac{\delta_c n_2^{car}(x_b)}{s_c} - bx_b - \tau + ax_b + e_2(x_b) + f(n_1(\underline{x}) - n_1^{car}(\underline{x})) + f(n_2(x_b) - n_2^{car}(x_b))\right] \\
& + \varsigma\left[n_1(\underline{x}) - N\right]
\end{aligned}
\tag{A1}
$$

where $\lambda_i(x)$, $\varphi_h(x)$, $\mu_i(x)$ and $\theta_i(x), h \in \{1,2\}, i \in \{1,2,3\}$ are the Lagrangian multipliers for travel cost of road, travel cost of rail, the population and the number of car commuters at location x, respectively. ω, ϑ and ζ are respectively the Lagrangian multipliers of the boundary conditions of travel cost of road, rail and the total population at the edge of the CBD ($x = \underline{x}$). At location $x = x_r$, the total number of car commuters meets $n_2^{car}(x_r) = n_3^{car}(x_r)$. Because $n_1^{rail}(x_r) = 0$, $n_2(x_r) = n_3(x_r)$ for boundary condition. Other constraint conditions are $n_1(x_b) = n_2(x_b)$, $n_1^{car}(x_b) = n_2^{car}(x_b)$, $t_2^{car}(x_r) = t_3^{car}(x_r)$, $n_3(\overline{x}) = n_3^{car}(\overline{x}) = 0$ and $r_0(\underline{x}) = r_1(\underline{x}) = r_b$, where r_b is the rent of the boundary between the CBD and the residential area.

Policy variables and endogenous variables are summarized in Table A1. After integrating the Lagrangian by parts, we derive the first-order conditions of the Lagrangian.

Table A1 Policy variables and endogenous variables

Policy variables		Endogenous variables		Exogenous variables	
Car toll	τ	Transportation cost	$t^{car}(x)$, $\kappa(x)$	Total Population	N
Railway fare at x	$e(x)$	Total traffic volume	$n^{car}(\underline{x})$	Per capita income	y
		Traffic volume at x	$n(x)$, $n^{car}(x)$	Generalized cost of a car per distance	b
		Shadow price for unit-distance car cost	$\lambda(x)$	Marginal cost of railway per passenger distance	z
		Shadow price for unit-distance rail cost	$\varphi(x)$	Construction cost of railway per distance	I
		Shadow price for population density at x	$\mu(x)$	Travel time cost of railway per distance	a
		Shadow price for the number of car commuters at x	$\theta(x)$		
		Shadow prices for boundary conditions at $x=\underline{x}$	$\omega, \vartheta, \varsigma$		
		Utility	u		

The first order conditions are given by Eqs. (A2)-(A37). For simplicity, $r(t^{car}(x),\kappa(x),G,v(x))$ and $q(t^{car}(x),\kappa(x),G,v(x))$ are denoted as $r(x)$ and $q(x)$, respectively.

$$\frac{\partial L}{\partial e_1(x)}=0:\dot{\varphi}_1(x)-J\left(\dot{n}_1(x)-\dot{n}_1^{car}(x)\right)=0, \tag{A2}$$

$$\frac{\partial L}{\partial e_2(x)}=0:\dot{\varphi}_2(x)-J\left(\dot{n}_2(x)-\dot{n}_2^{car}(x)\right)=0, \tag{A3}$$

$$\frac{\partial L}{\partial t_1^{car}(x)}=0:-\dot{\lambda}_1(x)-\dot{\varphi}_1(x)+\mu_1(x)\dot{n}_1(x)\frac{\partial q_1(x)}{\partial t_1^{car}(x)}+\frac{\partial r_1(x)}{\partial t_1^{car}(x)}=0, \tag{A4}$$

$$\frac{\partial L}{\partial t_2^{car}(x)}=0:-\dot{\lambda}_2(x)-\dot{\varphi}_2(x)+\mu_2(x)\dot{n}_2(x)\frac{\partial q_2(x)}{\partial t_2^{car}(x)}+\frac{\partial r_2(x)}{\partial t_2^{car}(x)}=0, \tag{A5}$$

$$\frac{\partial L}{\partial t_3^{car}(x)}=0:-\dot{\lambda}_3(x)+\mu_3(x)\dot{n}_3(x)\frac{\partial q_3(x)}{\partial t_3^{car}(x)}+\frac{\partial r_3(x)}{\partial t_3^{car}(x)}=0, \tag{A6}$$

$$\frac{\partial L}{\partial n_1(x)}=0:\dot{\varphi}_1(x)g'(n_1(x)-n_1^{car}(x))-\left[\dot{\mu}_1(x)q(t_1^{car}(x),\kappa_1(x),G,u)+\mu_1(x)\left\{\frac{\partial q_1(x)}{\partial t_1^{car}(x)}\dot{t}_1^{car}(x)+\frac{\partial q_1(x)}{\partial \kappa_1(x)}\dot{\kappa}_1(x)\right\}\right]$$
$$+\left[\dot{\theta}_1(x)\Phi(\kappa_1(x))+\theta_1(x)\Phi'(\kappa_1(x))\dot{\kappa}_1(x)\right]+J\left\{\dot{e}_1(x)-z\right\}=0$$

, (A7)

$$\frac{\partial L}{\partial n_2(x)}=0:\dot{\varphi}_2(x)g'(n_2(x)-n_2^{car}(x))-\left[\dot{\mu}_2(x)q(t_2^{car}(x),\kappa_2(x),G,u)+\mu_2(x)\left\{\frac{\partial q_2(x)}{\partial t_2^{car}(x)}\dot{t}_2^{car}(x)+\frac{\partial q_2(x)}{\partial \kappa_2(x)}\dot{\kappa}_2(x)\right\}\right],$$
$$+\left[\dot{\theta}_2(x)\Phi(\kappa_2(x))+\theta_2(x)\Phi'(\kappa_2(x))\dot{\kappa}_2(x)\right]+J\left\{\dot{e}_2(x)-z\right\}=0$$

(A8)

$$\frac{\partial L}{\partial n_3(x)}=0:-\left[\dot{\mu}_3(x)q(t_3^{car}(x),G,u)+\mu_3(x)\frac{\partial q_3(x)}{\partial t_3^{car}(x)}\dot{t}_3^{car}(x)\right]=0, \tag{A9}$$

$$\frac{\partial L}{\partial n_1^{car}(x)}=0:-\dot{\varphi}_1(x)g'(n_1(x)-n_1^{car}(x))-\dot{\theta}_1(x)-J\left\{\dot{e}_1(x)-z\right\}=0, \tag{A10}$$

$$\frac{\partial L}{\partial n_2^{car}(x)}=0:-\dot{\varphi}_2(x)g'(n_2(x)-n_2^{car}(x))-\dot{\theta}_2(x)-J\left\{\dot{e}_2(x)-z\right\}=0, \tag{A11}$$

$$\frac{\partial L}{\partial n_3^{car}(x)}=0:\dot{\theta}_3(x)=0, \tag{A12}$$

$$\frac{\partial L}{\partial \kappa_1(x)}=0:\dot{\varphi}_1(x)+\mu_1(x)\frac{\partial q_1(x)}{\partial \kappa_1(x)}\dot{n}_1(x)-\theta_1(x)\dot{n}_1(x)\Phi'(\kappa_1(x))+\frac{\partial r_1(x)}{\partial \kappa_1(x)}=0. \tag{A13}$$

$$\frac{\partial L}{\partial \kappa_2(x)}=0:\dot{\varphi}_2(x)+\mu_2(x)\frac{\partial q_2(x)}{\partial \kappa_2(x)}\dot{n}_2(x)-\theta_2(x)\dot{n}_2(x)\Phi'(\kappa_2(x))+\frac{\partial r_2(x)}{\partial \kappa_2(x)}=0, \tag{A14}$$

$$\frac{\partial L}{\partial G}=0:-\frac{1}{JN}\left\{\begin{array}{l}\int_{\underline{x}}^{x_b}\mu_1(x)\dot{n}_1(x)\frac{\partial q_1(x)}{\partial t_1^{car}(x)}dx+\int_{x_b}^{x_r}\mu_2(x)\dot{n}_2(x)\frac{\partial q_2(x)}{\partial t_2^{car}(x)}dx+\int_{x_r}^{\overline{x}}\mu_3(x)\dot{n}_3(x)\frac{\partial q_3(x)}{\partial t_3^{car}(x)}dx\\+\int_{\underline{x}}^{x_b}\frac{\partial r_1(x)}{\partial t_1^{car}(x)}dx+\int_{x_b}^{x_r}\frac{\partial r_2(x)}{\partial t_2^{car}(x)}dx+\int_{x_r}^{\overline{x}}\frac{\partial r_3(x)}{\partial t_3^{car}(x)}dx\end{array}\right\}-1=0$$

, (A15)

$$\frac{\partial L}{\partial n_1(\underline{x})}=0:-\dot{\varphi}_1(\underline{x})g'(n_1(\underline{x})-n_1^{car}(\underline{x}))-\mu_1(\underline{x})q(t_1^{car}(\underline{x}),\kappa_1(\underline{x}),G,u)+\left[\dot{\mu}_1(\underline{x})q(t_1^{car}(\underline{x}),\kappa_1(\underline{x}),G,u)+\mu_1(\underline{x})\left\{\frac{\partial q_1(\underline{x})}{\partial t_1^{car}(\underline{x})}\dot{t}_1^{car}(\underline{x})+\frac{\partial q_1(\underline{x})}{\partial \kappa_1(\underline{x})}\dot{\kappa}_1(\underline{x})\right\}\right]+\theta_1(\underline{x})\Phi(\kappa_1(\underline{x})),$$
$$-\left[\dot{\theta}_1(\underline{x})\Phi(\kappa_1(\underline{x}))+\theta_1(\underline{x})\Phi'(\kappa_1(\underline{x}))\dot{\kappa}_1(\underline{x})\right]+Je_1(\underline{x})-J\dot{e}_1(\underline{x})-Jz\underline{x}+Jz+\left(\vartheta_1+\vartheta_2\right)f'(n_1(\underline{x})-n_1^{car}(\underline{x}))+\varsigma=0$$

(A16)

$$\frac{\partial L}{\partial n_1^{car}(\underline{x})}=0:\dot{\varphi}_1(\underline{x})g'(n_1(\underline{x})-n_1^{car}(\underline{x}))-\theta_1(\underline{x})+\dot{\theta}_1(\underline{x}) \quad , \tag{A17}$$
$$-Je_1(\underline{x})+J\dot{e}_1(\underline{x})+Jz\underline{x}-Jz-\left(\vartheta_1+\vartheta_2\right)f'(n_1(\underline{x})-n_1^{car}(\underline{x}))=0$$

$$\frac{\partial L}{\partial \kappa_1(\underline{x})} = 0: \varphi_1(\underline{x}) - \dot{\varphi}_1(\underline{x}) - \mu_1(\underline{x}) \frac{\partial q_1(\underline{x})}{\partial \kappa_1(\underline{x})} \dot{n}_1(\underline{x}) + \theta_1(\underline{x}) \dot{n}_1(\underline{x}) \Phi'(\kappa_1(\underline{x})) - \frac{\partial r_1(\underline{x})}{\partial \kappa_1(\underline{x})} + \vartheta_1 = 0, \tag{A18}$$

$$\frac{\partial L}{\partial t_1^{car}(\underline{x})} = 0: -\lambda_1(\underline{x}) + \dot{\lambda}_1(\underline{x}) - \varphi_1(\underline{x}) + \dot{\varphi}_1(\underline{x}) - \mu_1(\underline{x}) \frac{\partial q_1(\underline{x})}{\partial t_1^{car}(\underline{x})} \dot{n}_1(\underline{x}) - \frac{\partial r_1(\underline{x})}{\partial t_1^{car}(\underline{x})} + \omega_1 = 0, \tag{A19}$$

$$\frac{\partial L}{\partial e_1(\underline{x})} = 0: \varphi_1(\underline{x}) - \dot{\varphi}_1(\underline{x}) + J\left(\dot{n}_1(\underline{x}) - \dot{n}_1^{car}(\underline{x})\right) + \vartheta_1 = 0, \tag{A21}$$

$$\begin{aligned}
\frac{\partial L}{\partial n_1(x_b)} = 0: &-\varphi_1(x_b) g'(n_1(x_b) - n_1^{car}(x_b)) + \dot{\varphi}_1(x_b) g'(n_1(x_b) - n_1^{car}(x_b)) + \varphi_2(x_b) g'(n_1(x_b) - n_1^{car}(x_b)) - \dot{\varphi}_2(x_b) g'(n_1(x_b) - n_1^{car}(x_b)) \\
&+ \mu_1(x_b) q(t_1^{car}(x_b), \kappa_1(x_b), G, u) - \left[\dot{\mu}_1(x_b) q(t_1^{car}(x_b), \kappa_1(x_b), G, u) + \mu_1(x_b) \left\{ \frac{\partial q_1(x_b)}{\partial t_1^{car}(x_b)} \dot{t}_1^{car}(x_b) + \frac{\partial q_1(x_b)}{\partial \kappa_1(x_b)} \dot{\kappa}_1(x_b) \right\} \right] \\
&- \mu_2(x_b) q(t_2^{car}(x_b), \kappa_2(x_b), G, u) + \left[\dot{\mu}_2(x_b) q(t_2^{car}(x_b), \kappa_2(x_b), G, u) + \mu_2(x_b) \left\{ \frac{\partial q_2(x_b)}{\partial t_2^{car}(x_b)} \dot{t}_2^{car}(x_b) + \frac{\partial q_2(x_b)}{\partial \kappa_2(x_b)} \dot{\kappa}_2(x_b) \right\} \right] \\
&- \theta_1(x_b) \Phi(\kappa_1(x_b)) + \left[\dot{\theta}_1(x_b) \Phi(\kappa_1(x_b)) + \theta_1(x_b) \Phi'(\kappa_1(x_b)) \dot{\kappa}_1(x_b) \right] + \theta_2(x_b) \Phi(\kappa_2(x_b)) - \left[\dot{\theta}_2(x_b) \Phi(\kappa_2(x_b)) + \theta_2(x_b) \Phi'(\kappa_2(x_b)) \dot{\kappa}_2(x_b) \right] \\
&- J e_1(x_b) + J \dot{e}_1(x_b) + J e_2(x_b) - J \dot{e}_2(x) - Jz + Jz + \vartheta_2 g'(n_1(x_b) - n_1^{car}(x_b)) = 0
\end{aligned}$$

, (A22)

$$\begin{aligned}
\frac{\partial L}{\partial n_1^{car}(x_b)} = 0: &\varphi_1(x_b) g'(n_1(x_b) - n_1^{car}(x_b)) - \dot{\varphi}_1(x_b) g'(n_1(x_b) - n_1^{car}(x_b)) + \theta_1(x_b) - \dot{\theta}_1\left(x_b\right) + J e_1(x_b) - J \dot{e}_1(x_b) + Jz \\
&- \varphi_2(x_b) g'(n_1(x_b) - n_1^{car}(x_b)) + \dot{\varphi}_2(x_b) g'(n_1(x_b) - n_1^{car}(x_b)) - \theta_2(x_b) + \dot{\theta}_2\left(x_b\right) \\
&+ J\tau - J e_2(x_b) + J \dot{e}_2(x) - Jz - \frac{\delta_c}{s_c}(\omega_2 + \vartheta_2) - \vartheta_2 g'(n_1(x_b) - n_1^{car}(x_b)) = 0
\end{aligned}, \tag{A23}$$

$$\frac{\partial L}{\partial \kappa_1(x_b)} = 0: -\varphi_1(x_b) + \dot{\varphi}_1(x_b) + \mu_1(x_b) \frac{\partial q_1(x_b)}{\partial \kappa_1(x_b)} \dot{n}_1(x_b) - \theta_1(x_b) \dot{n}_1(x_b) \Phi'(\kappa_1(x_b)) + \frac{\partial r_1(x_b)}{\partial \kappa_1(\underline{x})} = 0, \tag{A24}$$

$$\frac{\partial L}{\partial \kappa_2(x_b)} = 0: \varphi_2(x_b) - \dot{\varphi}_2(x_b) - \mu_2(x_b) \frac{\partial q_2(x_b)}{\partial \kappa_2(x_b)} \dot{n}_2(x_b) + \theta_2(x_b) \dot{n}_2(x_b) \Phi'(\kappa_2(x_b)) - \frac{\partial r_2(x_b)}{\partial \kappa_2(\underline{x})} + \vartheta_2 = 0, \tag{A25}$$

$$\frac{\partial L}{\partial t_1^{car}(x_b)} = 0: \lambda_1(x_b) - \dot{\lambda}_1(x_b) + \varphi_1(x_b) - \dot{\varphi}_1(x_b) + \mu_1(x_b) \frac{\partial q_1(x_b)}{\partial t_1^{car}(x_b)} \dot{n}_1(x_b) + \frac{\partial r_1(x_b)}{\partial t_1^{car}(x_b)} = 0, \tag{A26}$$

$$\frac{\partial L}{\partial t_2^{car}(x_b)} = 0: -\lambda_2(x_b) + \dot{\lambda}_2(x_b) - \varphi_2(x_b) + \dot{\varphi}_2(x_b) - \mu_2(x_b) \frac{\partial q_2(x_b)}{\partial t_2^{car}(x_b)} \dot{n}_2(x_b) - \frac{\partial r_2(x_b)}{\partial t_2^{car}(x_b)} + \omega_2 = 0, \tag{A27}$$

$$\frac{\partial L}{\partial \tau} = 0: J n_1^{car}(x_b) - \omega_2 - \vartheta_2 = 0, \tag{A28}$$

$$\frac{\partial L}{\partial e_1(x_b)} = 0: -\varphi_1(x_b) + \dot{\varphi}_1(x_b) - J\left(\dot{n}_1(x_b) - \dot{n}_1^{car}(x_b)\right) = 0, \tag{A29}$$

$$\frac{\partial L}{\partial e_2(x_b)}=0:\varphi_2(x_b)-\dot{\varphi}_2(x_b)+J\left(\dot{n}_1(x_b)-\dot{n}_1^{car}(x_b)\right)+\vartheta_2=0, \tag{A30}$$

$$\begin{aligned}\frac{\partial L}{\partial t_2^{car}(x_r)}=0:&\{\lambda_2(x_r)-\lambda_3(x_r)\}-\dot{\lambda}_2(x_r)+\dot{\lambda}_3(x_r)+\varphi_2(x_r)-\dot{\varphi}_2(x_r)\\&+\mu_2(x_r)\frac{\partial q_2(x_r)}{\partial t_2^{car}(x_r)}\dot{n}_2(x_r)-\mu_3(x_r)\frac{\partial q_3(x_r)}{\partial t_3^{car}(x_r)}\dot{n}_3(x_r)+\frac{\partial r_2(x_r)}{\partial t_2^{car}(x_r)}-\frac{\partial r_3(x_r)}{\partial t_3^{car}(x_r)}=0\end{aligned}, \tag{A31}$$

$$\begin{aligned}\frac{\partial L}{\partial n_2(x_r)}=0:&\mu_2(x_r)q(t_2^{car}(x_r),\kappa_2(x_r),G,u)-\left[\dot{\mu}_2(x_r)q(t_2^{car}(x_r),\kappa_2(x_r),G,u)+\mu_2(x_r)\left\{\frac{\partial q_2(x_r)}{\partial t_2^{car}(x_r)}\dot{t}_2^{car}(x_r)+\frac{\partial q_2(x_r)}{\partial \kappa_2(x_r)}\dot{\kappa}_2(x_r)\right\}\right]\\&-\mu_3(x_r)q(t_3^{car}(x_r),G,u)+\left[\dot{\mu}_3(x_r)q(t_3^{car}(x_r),G,u)+\mu_3(x_r)\frac{\partial q_3(x_r)}{\partial t_3^{car}(x_r)}\dot{t}_3^{car}(x_r)\right]\\&+\{1-\Phi(\kappa_2(x_r))\}\theta_2(x_r)+\left[\dot{\theta}_2(x_r)\Phi(\kappa_2(x_r))+\theta_2(x_r)\Phi'(\kappa_2(x_r))\dot{\kappa}_2(x_r)\right]+\dot{\theta}_3(x_r)-\dot{\theta}_3(x_r)=0\end{aligned}, \tag{A32}$$

$$\frac{\partial L}{\partial \kappa_2(x_r)}=0:-\varphi_2(x_r)+\dot{\varphi}_2(x_r)+\mu_2(x_r)\frac{\partial q_2(x_r)}{\partial \kappa_2(x_r)}\dot{n}_2(x_r)-\theta_2(x_r)\dot{n}_2(x_r)\Phi'(\kappa_2(x_r))+\frac{\partial r_2(x_r)}{\partial \kappa_2(x_r)}=0, \tag{A33}$$

(A34)

$$\frac{\partial L}{\partial e_2(x_r)}=0:-\varphi_2(x_r)+\dot{\varphi}_2(x_r)-J\left(\dot{n}_2(x_r)-\dot{n}_2^{car}(x_r)\right)=0. \tag{A35}$$

$$\begin{aligned}\frac{\partial L}{\partial \overline{x}}=0:&\dot{\lambda}_3(\overline{x})t_3^{car}(\overline{x})+\lambda_3(\overline{x})\dot{t}_3^{car}(\overline{x})-\dot{\lambda}_3(\overline{x})t_3^{car}(\overline{x})-b\lambda_3(\overline{x})\\&+\mu_3(\overline{x})-\left[\dot{\mu}_3(\overline{x})q(t_3^{car}(\overline{x}),G,u)+\mu_3(\overline{x})\frac{\partial q_3(x_r)}{\partial t_3^{car}(x_r)}\dot{t}_3^{car}(\overline{x})\right]n_3(\overline{x})-\dot{\theta}_3(\overline{x})n_3(\overline{x})+\dot{\theta}_3(\overline{x})n_3^{car}(\overline{x})=0\end{aligned},$$

(A36)

$$\frac{\partial L}{\partial t_3^{car}(\overline{x})}=0:\lambda_3(\overline{x})-\dot{\lambda}_3(\overline{x})+\mu_3(\overline{x})\frac{\partial q_3(\overline{x})}{\partial t_3^{car}(\overline{x})}\dot{n}_3(\overline{x})+\frac{\partial r_3(\overline{x})}{\partial t_3^{car}(\overline{x})}=0, \tag{A37}$$

A.2 Shadow prices and derivation of Propositions

Arranging the first order conditions (A2) and (A10), we obtain:

$$\dot{\theta}_1(x)=-J\left(\dot{e}_1(x)-z\right)-J\left(\dot{n}_1(x)-\dot{n}_1^{car}(x)\right)g'(n_1(x)-n_1^{car}(x)). \tag{A38}$$

Arranging Eqs. (A2), (A13) and (A24), we obtain:

$$\varphi_1(x)=J\left(n_1(x)-n_1^{car}(x)\right)-J\left(n_1(x_b)-n_1^{car}(x_b)\right). \tag{A39}$$

Arranging Eqs. (A3), (A14) and (A33), we obtain

$$\varphi_2(x) = J\left(n_2(x) - n_2^{car}(x)\right). \tag{A40}$$

Arranging Eqs. (A13), (A18) and (A39), we obtain

$$\vartheta_1 = -\varphi_1(\underline{x}) = -J\left(N - n_1^{car}(\underline{x})\right) + J\left(n_1(x_b) - n_1^{car}(x_b)\right). \tag{A41}$$

Combining Eqs. (A3), (A30) and (A40) yields:

$$\vartheta_2 = -\varphi_2(x_b) = -J\left(n_1(x_b) - n_1^{car}(x_b)\right). \tag{A42}$$

Arranging Eqs. (A10), (A17), (A41), and (A42), we obtain

$$\theta_1(\underline{x}) = -J\left(e_1(\underline{x}) - z\underline{x}\right) + J\left(n_1(\underline{x}) - n_1^{car}(\underline{x})\right) f'(n_1(\underline{x}) - n_1^{car}(\underline{x})). \tag{A43}$$

Integrating Eq. (A38) from $x = \underline{x}$ to x with the use of Eq. (A43), we obtain

$$\theta_1(x) = J\begin{bmatrix} -\left(e_1(x) - zx\right) + \left(n_1(\underline{x}) - n_1^{car}(\underline{x})\right) f'(n_1(\underline{x}) - n_1^{car}(\underline{x})) \\ -\int_{\underline{x}}^{x} \left(\dot{n}_1(x) - \dot{n}_1^{car}(x)\right) g'(n_1(x) - n_1^{car}(x)) dx \end{bmatrix}. \tag{A44}$$

Combining Eqs. (A7), (A16), (A20), and (A41)-(A43), we obtain $\mu_1(\underline{x}) = 0$. Integrating Eq. (A7) from $x = \underline{x}$ to x with (A10), (A44) and $\mu_1(\underline{x}) = 0$, we obtain

$$\mu_1(x) = \frac{J}{q_1(x)}\begin{bmatrix} \begin{Bmatrix} \left(e_1(x) - zx\right) - \left(n_1(\underline{x}) - n_1^{car}(\underline{x})\right) f'(n_1(\underline{x}) - n_1^{car}(\underline{x})) \\ + \int_{\underline{x}}^{x} \left(\dot{n}_1(x) - \dot{n}_1^{car}(x)\right) g'(n_1(x) - n_1^{car}(x)) dx \end{Bmatrix} \{1 - \Phi(\kappa_1(x))\} \\ -\left\{\left(e_1(\underline{x}) - z\underline{x}\right) - \left(n_1(\underline{x}) - n_1^{car}(\underline{x})\right) f'(n_1(\underline{x}) - n_1^{car}(\underline{x}))\right\}\{1 - \Phi(\kappa_1(\underline{x}))\} \end{bmatrix}. \tag{A45}$$

Arranging Eqs. (A4), (A26) and (A39), we obtain

$$\lambda_1(x_b) = 0. \tag{A46}$$

Arranging Eqs. (A5), (A6), (A31) and (A40), we obtain $\lambda_2(x_r) = \lambda_3(x_r)$. Arranging Eqs. (A6), (A9), (A36), (A37) with this equality, we obtain

$$\lambda_2(x_r) = \lambda_3(x_r) = -\int_{x_r}^{\bar{x}} \frac{\partial r_3(x)}{\partial t_3^{car}(x)} dx = -J\int_{x_r}^{\bar{x}} \dot{n}_3(x) dx. \tag{A47}$$

Arranging Eqs. (A4), (A5), (A9), (A15), (A19), (A27), (A36), (A39), (A40), (A46)-(A47), yields $\omega_1 + \omega_2 = JN$. Arranging Eqs. (A4), (A19), (A28), (A41), and (A42) with this, we obtain:

$$\lambda_1(\underline{x}) = J\left(n_1^{car}(\underline{x}) - n_1^{car}(x_b)\right). \tag{A48}$$

Integrating Eq. (A4) from $x = \underline{x}$ to $x = x_b$ with the use of Eqs. (A39), (A45), (A46) and (A48), we obtain $\mu_1(x) = 0$. Arranging Eqs. (A2), (A13), (A44) with this, we obtain the following condition.

$$\theta_1(x) = 0. \tag{A49}$$

Arranging Eqs. (A3) and (A11), we obtain the following condition.

$$\dot{\theta}_2(x) = -J\left(\dot{e}_2(x) - z\right) - J\left(\dot{n}_2(x) - \dot{n}_2^{car}(x)\right) g'(n_2(x) - n_2^{car}(x)). \tag{A50}$$

Arranging Eqs. (A10), (A11), (A23), (A28), (A39), and (A40), we obtain:

$$\theta_1(x_b) + Je_1(x_b) = \theta_2(x_b) + Je_2(x_b) - J\left\{\tau - \frac{\delta_c n_1^{car}(x_b)}{s_c}\right\}. \tag{A51}$$

Combining Eqs. (A44) and (A51), we obtain the following condition.

$$\theta_2(x_b) = J\left[\begin{array}{l} -\left(e_2(x_b) - zx_b\right) + \left(n_1(\underline{x}) - n_1^{car}(\underline{x})\right) f'(n_1(\underline{x}) - n_1^{car}(\underline{x})) \\ -\int_{\underline{x}}^{x_b} \left(\dot{n}_1(x) - \dot{n}_1^{car}(x)\right) g'(n_1(x) - n_1^{car}(x)) dx + \left\{\tau - \dfrac{\delta_c n_1^{car}(x_b)}{s_c}\right\} \end{array}\right]. \tag{A52}$$

Integrating Eq. (A50) from $x = x_b$ to x with the use of Eq. (A52), we obtain

$$\theta_2(x) = J\left[\begin{array}{l} \left\{\tau - \dfrac{\delta_c n_1^{car}(x_b)}{s_c}\right\} - \left(e_2(x) - zx\right) + \left(n_1(\underline{x}) - n_1^{car}(\underline{x})\right) f'(n_1(\underline{x}) - n_1^{car}(\underline{x})) \\ -\int_{\underline{x}}^{x_b} \left(\dot{n}_1(x) - \dot{n}_1^{car}(x)\right) g'(n_1(x) - n_1^{car}(x)) dx - \int_{x_b}^{x} \left(\dot{n}_2(x) - \dot{n}_2^{car}(x)\right) g'(n_2(x) - n_2^{car}(x)) dx \end{array}\right]. \tag{A53}$$

Arranging Eqs. (A7), (A8), (A22), (A39), and (A40), we obtain the following condition.

$$\begin{aligned} &\mu_1(x_b) q(t_1^{car}(x_b), \kappa_1(x_b), G, u) + \theta_1(x_b)\left(1 - \Phi(\kappa_1(x_b))\right) \\ &= \mu_2(x_b) q(t_2^{car}(x_b), \kappa_2(x_b), G, u) + \theta_2(x_b)\left(1 - \Phi(\kappa_2(x_b))\right) - J\left\{\tau - \frac{\delta_c n_1^{car}(x_b)}{s_c}\right\}. \end{aligned} \tag{A54}$$

Integrating Eq. (A8) from $x = x_b$ to x with the use of Eqs. (A11), (A44), (A45), (A53) and (A54), we obtain

$$\mu_2(x) = \frac{J}{q_2(x)} \left[\begin{array}{l} \left\{ \tau - \frac{\delta_c n_1^{car}(x_b)}{s_c} \right\} \Phi(\kappa_2(x)) + \left[\begin{array}{l} \left(e_2(x) - zx\right) - \left(n_1(\underline{x}) - n_1^{car}(\underline{x})\right) f'(n_1(\underline{x}) - n_1^{car}(\underline{x})) \\ + \int_{\underline{x}}^{x_b} \left(\dot{n}_1(x) - \dot{n}_1^{car}(x)\right) g'(n_1(x) - n_1^{car}(x)) dx \\ + \int_{x_b}^{x} \left(\dot{n}_2(x) - \dot{n}_2^{car}(x)\right) g'(n_2(x) - n_2^{car}(x)) dx \end{array} \right] \{1 - \Phi(\kappa_2(x))\} \\ - \left\{ \left(e_1(\underline{x}) - z\underline{x}\right) - \left(n_1(\underline{x}) - n_1^{car}(\underline{x})\right) f'(n_1(\underline{x}) - n_1^{car}(\underline{x})) \right\} \{1 - \Phi(\kappa_1(\underline{x}))\} \end{array} \right]. \quad \text{(A55)}$$

Arranging the first order conditions (A5), (A27), (A28) and Eq. (A42), we obtain

$$\lambda_2(x_b) = \omega_2 + \vartheta_2 = J n_1^{car}(x_b). \quad \text{(A56)}$$

Integrating Eq. (A5) from $x = x_b$ to $x = x_r$ with the use of Eqs. (A40), (A55) and (A56), we obtain $\mu_2(x) = 0$. Arranging Eqs. (A3), (A14), (A53) with this, we obtain $\theta_2(x) = 0$. Eqs. (A44) and (A49) yields Proposition 2 (or Eq. (34)). Next, Eqs. (A53) and (A56) yield Proposition 3 (or Eq. (35)). Finally, (A49), (A54), and $\mu_2(x) = 0$ yield Proposition 4.

Regarding Proposition 1, we need another Lagrangian expressing the case in which the bottleneck is at the fringe of the CBD. We show the process in the online supplement and on the authors' website. But the derivation process is very similar to the above.

Appendix B. Empirical verification of Lemma 1 in Section 2.4

In section 2.4, we explored the relationship between the train overcrowding cost and opportunity cost and summarized the properties of the equilibrium as Lemma 1. Appendix B empirically verifies the relationship, taking Tokyu railway in Tokyo as an example.

We use the data of the train vehicle occupancies of trains on the Den-en-toshi Line and the Tōyoko Line, which pass a CBD station, Shibuya Station and Naka-meguro Station, respectively. The target time is between 8:00 a.m. and 8:30 a.m. because many offices start working at 9:00 a.m. The data is from October 2nd, 2018.

Firstly, we calculate train overcrowding costs per passenger based on the cost function in the benefit evaluation manual of railroad projects, provided by the Ministry of Land, Infrastructure, and Transportation. Next, we plot the overcrowding costs at each station interval, as in Fig. B.1. and B.2. On the horizontal axis, station names are shown. The leftmost station is the CBD station. So, the commuters go from the right stations to the left stations. The vertical axis shows the overcrowding costs at each station. The time written above each line shows when each train arrives at the CBD station.

To compare trains under the same conditions, in Den-en-toshi Line, we use data of only semi-express trains bound for Oshiage and Kiyosumi-shirakawa. On the Tōyoko Line, trains running during the target period are only commuter express and express trains. If the speed of trains is different, the origin and the destination patterns can be different. In particular, the commuter express being faster than the express, the area extending from the origin and the destination is wider. So, we only target the express trains.

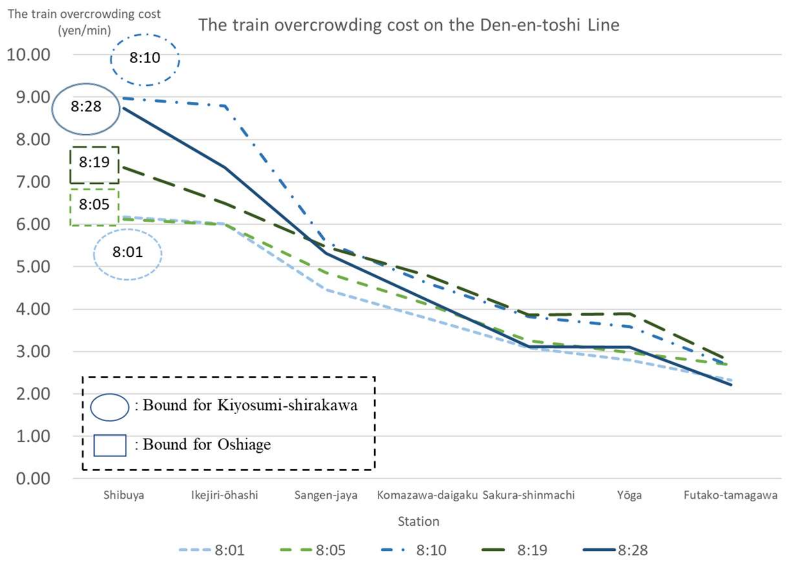

Fig. B.1. The train overcrowding cost on the Den-en-toshi Line

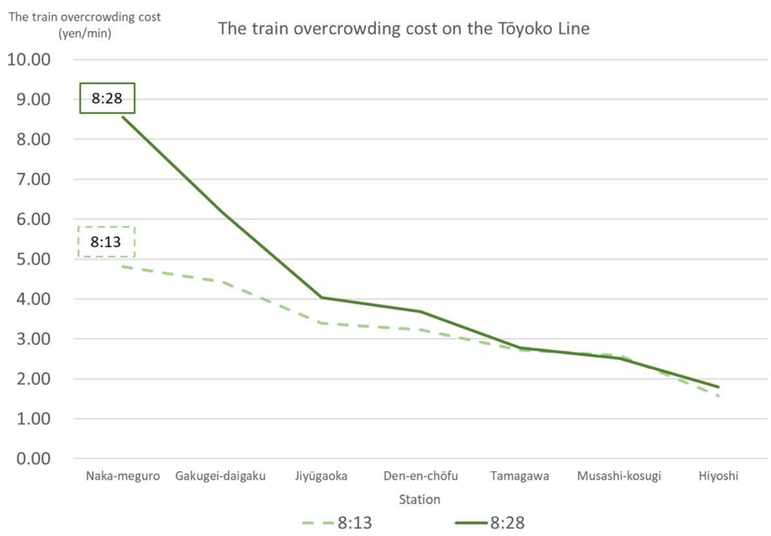

Fig. B.2. The train overcrowding cost on the Tōyoko Line

The properties in Lemma 1 are demonstrated as Figs. B.1. and B.2 in the following way.

Lemma 1 focuses on the two parts of the railway: the part between the CBD station and the station next to the CBD, and the part between the station next to the CBD station and farther stations. In Fig. B.1., we can regard the part between Ikejiri-ōhashi and Shibuya, and the part between Gakugei-daigaku and Naka-meguro as the two parts in Lemma 1. Trains arriving at Shibuya later basically have higher crowding costs, except for the train arriving at 8:28, if you separate the trains into two groups in terms of the destination station. The train arriving at Shibuya at 8:28 arrives at Otemachi at 8:47. As the vehicle occupancy of this train is less than that of the train arriving at 8:10, it is possible that some passengers on this train are late for work.

Looking at Fig. B.1., the crowding costs are similar when the trains run in suburbs, while they are very different when the trains are close to Shibuya. This property is very close to the characteristics demonstrated in Lemma 1. But, the difference in the crowding cost appears not only in the part between Shibuya and the station next to Shibuya (i.e., Ikejiri-ōhashi) but also in the part between Sangen-jaya and Ikejiri-ōhashi. This is probably because Ikejiri-ōhashi is also within the CBD. A similar property can be seen in

Fig. B.2, too. Actually, the characteristics in Lemma 1 are more clearly shown in Fig. B.2.

Appendix C. Parameter setting

As the cost related to cars, bottleneck capacity s_c and parameter δ_c are set as 11,500 households/hour and \$14.436/hour, respectively. As the cost related to railway, the capacity of a train c_{train} and the access cost Q are set as 576 persons and \$3.34 per trip.[9] The marginal cost of railway z_o is set as \$0.219/person/km.[10] The railway construction cost per km I is set as \$4,800,000/km/year.[11] The generalized costs of car b and railway a are set as 0.77(\$/km) and 0.80(\$/km), respectively, considering the average free flow speeds v^{car}=40(km/h) and v^{rail}=30(km/h).[12] Congestion cost parameter f and the end points of the probability density function a_1 and b_1 are set as 0.05\$/km/person, -110 and 170, respectively, so that the probability of car use is approximately 60(%), which indicates the probability of using cars along the railway in Sendai. The value of congestion parameter f differs across situations in order to analyze how much train overcrowding affects welfare gains.

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[9] With value of time $w=24.06$(\$/hour), the speed of walking $v^{walk}=60$(m/minute) and the distance to the station from home $\bar{d}=500$(m), we obtain Q as $\{24.06(\$/\mathrm{hour}) \times 1/60(\mathrm{hour/minutes}) \times [500(m) \div 60(\mathrm{m/minute})]\} = \$3.34$.

[10] With the running cost of Sendai subway at 9.5(billion yen/year) and the total travel distance on the subway at 0.435((billion person・km)/year), we obtain z_o as [9.5(billion yen/year)÷0.435((billion person・km)/year)]÷100(\$/yen)= \$0.219/(person・km)

[11] With the total construction cost at 240 (billion yen) and the total length of subway at 15 km, we obtain I as [240(billion yen)×0.03(1/year)]÷15(1/km)÷100(\$/yen)= \$4,800,000/(km・year).

[12] The generalized cost related to car commuting changes with the unit consumption of travel expenses φ. According to the Ministry of Land, Infrastructure, Transport and Tourism, (MLIT) (2008), when the speed of car v^{car} is 30 and 40 (km/hour), φ is respectively 0.18 and 0.17 (\$/km).

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