An attempt to derive the Risk Weight Function for the bank

Nguyen, Van Phuong

1 December 2019

An attempt to derive the Risk Weight Function for the bank

Phuong Van Nguyen[*]

Abstract

According to the Basel Accord II, one of the key factors in the Internal-Ratings Based (IRB) framework is the Risk Weight Function (RWF). Indeed, it uses four risk components including PD, LGD, EAD, and M as input to yield the capital requirement and thereby Risk-Weighted Asset (RWA). Given the extremely important role of the Risk Weight Function, in this project, we aim to derive it mathematically. At the end of this project, we yield the Risk Weight Function below.

$$\text{Capital Requirement (K)} = LGD * N\left[\frac{\text{Threshold} + \sqrt{\rho}N^{-1}(\alpha)}{\sqrt{1-\rho}}\right]$$

Keywords: Basel Accord II, Homogenous Loan Portfolio, Loss Distribution, Expected Loss, Unexpected Loss, Capital Requirement, Risk-Weight Functions.

[*]Corresponding email: `phuong.nguyen@economics.uni-kiel.de`

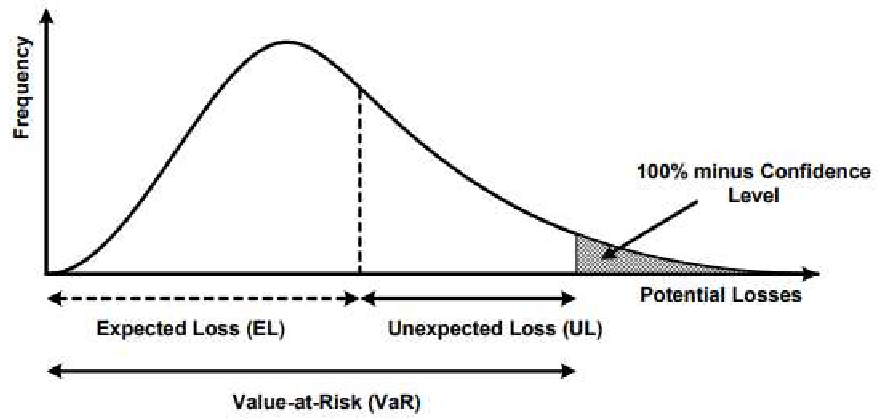
1 Why does the bank need to derive its RWF?

It is worth noting that the Basel-II IRB Risk Weight Function was developed from the Asymptotic Single Risk Factor (ASRF). For further detail, one might refer to an explanatory note on the Basel-II Risk Weight Function by BCBS (2005). Indeed, this function was originally developed by Gordy (2003), an economist of the Board of Governors of the Federal Reserve System. Thus, the bank might treat this model specification as its fundamental one to determine RWA, thereby CAR. However, it is worth noting that the use of the Basel-II IRB Risk Weight Function is not completely mandatory. The explanation for this issue is clearly revealed on page 4 of BCBS (2005) as follows.

"It should be noted that the choice of the ASRF for use in the Basel risk weight functions does by no means express any preference of the Basel Committee towards one model over others. Rather, the choice was entirely driven by above considerations. Banks are encouraged to use whatever credit risk models fit best for their internal risk measurement and risk management needs."

On the other hand, the primary role of the Risk Weight Function is to predict the **Unexpected Loss** (**UL**), thereby one can determine the capital requirement or economic capital allocation. Also, it is worth noting that the higher capital requirement, the higher the financial burden. Thus, the accuracy prediction for UL is particularly important to the bank. This is because it enables a given bank to determine its proper capital level not only fulfilling a Capital Adequacy Requirement (CAR) supervised by its Supervisor but also reducing the financial burdens.

Figure 1: The Likelihood of Losses of a given Portfolio



Given that a given bank would like to develop its own Risk Weight Function. This project presents our preliminary attempt fulfilling this assumption about this given bank's need. The rest of this paper is organized as follows. Section 2 presents the step-by-step procedure for deriving the Risk Weight Function. Section 3 gives conclusions.

2 Deriving the Risk Weight Function

2.1 Preliminary configurations

First, the following is to define several variables using to deriving the Risk Weight Function

- PD_i: Probabiltiy of Default of the ith client.
- EAD_i: Exposure At Default of the ith client.
- LGD_i: Loss Given Default of the ith client.

Second, we define that the ith client will suffer from the default if the value of his/her asset A_i falls below a specific threshold. Statistically speaking, the probability of default of the ith client is identical to the probability of the value of the ith client's asset A_i below a specific threshold as follows.

$$PD_i = Prob\Big(A_i < \text{Threshold}\Big) \tag{2.1.1}$$

Third, the **Expected Loss** (**EL**) of the ith client is therefore given below.

$$UL_i = PD_i * EAD_i * LGD_i \tag{2.1.2}$$

Fourth, since the bank has a huge number of clients, the total UL of its loan portfolio is computed as follows.

$$UL = \sum_i^m UL_i = \sum_i^m PD_i * EAD_i * LGD_i \tag{2.1.3}$$

If we adopt the assumption about the bank's homogeneous loan portfolio, its total UL in the expression (2.1.3) above will become as follows.

$$UL = m * PD * EAD * LGD = d * EAD * LGD \tag{2.1.4}$$

Fifth, we define the total probability of default of the bank's loan portfolio as follows.

$$d = m * PD \tag{2.1.5}$$

Afterward, plugging it into the expression (2.1.4) to obtain

$$UL = d * EAD * LGD \tag{2.1.6}$$

Sixth, the UL ratio is defined below.

$$lr = \frac{UL}{EAD} = \frac{d * EAD * LGD}{m * EAD} = \frac{d}{m} * LGD \tag{2.1.7}$$

Finally, our target is to compute the following probability.

$$P(lr \leq ul) = P\Big(\frac{d}{m} * LGD \leq ul\Big) = P\Big(d \leq \frac{ul * m}{LGD}\Big) \tag{2.1.8}$$

where ul donates the unexpected loss ratio of the bank's loan portfolio.

Indeed, the right-hand side of the expression (2.1.8) is the cumulative distribution. To compute this cumulative distribution, we have to find the distribution of d, the total probability of default of the bank's loan portfolio.

2.2 The innovation in value of asset

We assume that the innovation in the value of the ith client's asset is given below.

$$A_i = \sqrt{1-\rho} * \epsilon_i + \sqrt{\rho} * S \text{ where } \epsilon_i, S \sim N(0,1) \tag{2.2.1}$$

ϵ_i donates the specific or idiosyncratic risk, whereas S presents the systematic one.

Hey! What is the intuitively economical meaning of the formula above?

In this project, we skip the explanation for this question and give it on request. Based on the expression (2.2.1) above, one can easily prove that innovation in the value of asset is a normal distribution with the following moments.

$$\mu_{A_i} = 0 \text{ and } \sigma_{A_i} = 1 \text{ and } Cov(A_i, A_j) = \rho \tag{2.2.2}$$

We plug A_i in the expression (2.2.1) into the one (2.1.1) to yield the probability of default of the ith client as follows.

$$\begin{aligned} PD_i &= Prob\Big(A_i < \text{Threshold}\Big) \\ &= Prob\Big(\sqrt{1-\rho} * \epsilon_i + \sqrt{\rho} * S < \text{Threshold}\Big) \end{aligned} \tag{2.2.3}$$

Rearranging items in the expression above to yield

$$\begin{aligned} PD_i &= Prob\left(\epsilon_i < \frac{\text{Threshold} - \sqrt{\rho} * S}{\sqrt{1-\rho}}\right) \\ &= N\left(\frac{\text{Threshold} - \sqrt{\rho} * S}{\sqrt{1-\rho}}\right) \end{aligned} \tag{2.2.4}$$

Hey! Finally, the probability of default of the ith client is analytically computed, right? Based on the expression (2.2.4) above, it is identical to the probability of the idiosyncratic risk below the value of $\left(\frac{\text{Threshold}-\sqrt{\rho}*S}{\sqrt{1-\rho}}\right)$ or the cumulative standard normal distribution.

2.3 Binomial distribution and Law of Large Number

Based on the value of PD_i, we can define the following rule concerning if the ith client suffers from the default.

$$D_i = \begin{cases} 1, & \text{default} : PD_i \\ 0, & \text{no default} : 1 - PD_i \end{cases} \tag{2.3.1}$$

Thus, the total probability of default of the bank's loan portfolio in the expression (2.1.5) will become as follows.

$$d = \sum_{i}^{m} D_i \tag{2.3.2}$$

Since the variable D_i follows the Bernoulli process, the total probability of default of its loan portfolio follows the Binomial distribution $B(m, PD_i)$.

It is worth remembering that the bank's loan portfolio is a homogenous one so that we define PD_i as p. On the other hand, the bank has a huge number of clients. It implies m is very large so that the total probability of default of the bank's loan portfolio can be approximated to the Normal Distribution as follows.

$$d \sim N\Big(mp, mp(1-p)\Big) \tag{2.3.3}$$

Hey! Have a look at the expression (2.3.3), we almost reach our target, right?

It's time to recall our target which is described in the expression (2.1.8), such as.

$$P(lr \leq ul) = P\Big(d \leq \frac{m * ul}{LGD}\Big) \tag{2.3.4}$$

The right-hand side of the equation(2.3.4) is identical to the cumulative normal distribution in the expression (2.3.3) at the cut-off point of $\frac{m*ul}{LGD}$. Thus, the equation(2.3.4) can be tranformed to the following one.

$$P(lr \leq ul) = N\left(\frac{\frac{m*ul}{LGD} - mp}{\sqrt{mp(1-p)}}\right) \tag{2.3.5}$$

2.4 The Risk Weight Function

We re-arrange the items in the expression (2.3.5) above to yeild

$$P(lr \leq ul) = N\left[\sqrt{\frac{m}{p(1-p)}}\Big(\frac{ul}{LGD} - p\Big)\right] \tag{2.4.1}$$

Because of the property of the normal distribution and its cumulative function, the three potential outcomes can be obtained from the equation (2.4.1) as follows.

- If $\frac{ul}{LGD} > p$: Also since the bank has been serving a huge number of clients, say m goes to $(+\infty)$. Thus, $P(lr \leq ul) = N(+\infty) = 1$.
- If $\frac{ul}{LGD} = p$: $P(lr \leq ul) = N(0) = 1/2$.
- If $\frac{ul}{LGD} < p$: $P(lr \leq ul) = N(-\infty) = 0$.

Based on these analyses, the distribution of the probability of the expected loss rate below the unexpected loss rate or the cumulative function of the expected loss of the bank's loan portfolio at the cut-off point of ul, is given below.

$$P(lr \leq ul) = \begin{cases} 1, & \text{if} : \frac{ul}{LGD} > p \\ \frac{1}{2}, & \text{if} : \frac{ul}{LGD} = p \\ 0 & \text{if} : \frac{ul}{LGD} < p \end{cases} \tag{2.4.2}$$

Based on the rule (2.1.5), we are going to compute the expected value of this distribution.

$$\begin{aligned} E\Big[lr \leq ul\Big] &= E\Big[1_{\frac{ul}{LGD} > p}\Big] \\ &= Prob\Big[\frac{ul}{LGD} > p\Big] \end{aligned} \tag{2.4.3}$$

It is worth rememberring that the bank's loan portfolio is a homogenous one, $PD_i = p$, thus, $p = N\Big(\frac{\text{Threshold} - \sqrt{\rho} * S}{\sqrt{1-\rho}}\Big)$ (see the expression (2.2.4)). Afterward, we plug it into the equation (2.4.3) to obtain.

$$\begin{aligned} E\Big[lr \leq ul\Big] &= Prob\Big[\frac{ul}{LGD} > N\Big(\frac{\text{Threshold} - \sqrt{\rho} * S}{\sqrt{1-\rho}}\Big)\Big] \\ &= Prob\Big[N^{-1}\Big(\frac{ul}{LGD}\Big) > \frac{\text{Threshold} - \sqrt{\rho} * S}{\sqrt{1-\rho}}\Big] \\ &= Prob\left[S > \frac{\text{Threshold} - \sqrt{1-\rho} N^{-1}\Big(\frac{ul}{LGD}\Big)}{\sqrt{\rho}}\right] \end{aligned} \tag{2.4.4}$$

Finally, we obtain

$$E\Big[lr \leq ul\Big] = N\left[\frac{-\text{Threshold} + \sqrt{1-\rho} N^{-1}\Big(\frac{ul}{LGD}\Big)}{\sqrt{\rho}}\right] \tag{2.4.5}$$

According to the Basel Accord II (see page 6, BCBS (2005)), one typically choose the confidence level of 99.9 % or $\alpha = 0.001$. Statistically speaking,the cumulative distribution function of Techcombank's loss ratio at the cut-off point of ul is 0.999.

$$Prob\Big[lr \leq ul\Big] = 0.001 \tag{2.4.6}$$

Therefore, we can derive as follows.

$$\begin{aligned}
N\left[\frac{-\text{Threshold} + \sqrt{1-\rho}N^{-1}\Big(\frac{ul}{LGD}\Big)}{\sqrt{\rho}}\right] &= \alpha \\
\frac{-\text{Threshold} + \sqrt{1-\rho}N^{-1}\Big(\frac{ul}{LGD}\Big)}{\sqrt{\rho}} &= N^{-1}(\alpha) \\
-\text{Threshold} + \sqrt{1-\rho}N^{-1}\Big(\frac{ul}{LGD}\Big) &= \sqrt{\rho}N^{-1}(\alpha) \\
N^{-1}\Big(\frac{ul}{LGD}\Big) &= \frac{\text{Threshold} + \sqrt{\rho}N^{-1}(\alpha)}{\sqrt{1-\rho}}
\end{aligned} \tag{2.4.7}$$

Finally, we obtain our target, here is the unexpected loss ratio of the bank's loan portfolio

$$ul = LGD * N\left[\frac{\text{Threshold} + \sqrt{\rho}N^{-1}(\alpha)}{\sqrt{1-\rho}}\right] \tag{2.4.8}$$

The formula (2.4.8) is also treated as the **capital requirement (K)**. This is because it is to fulfill the **Unexpected Loss** as the recommendation of the Banking Committee on Banking Supervision.

3 Conclusion

The Basel Committee on Banking Supervision (BCBS) has been encouraging the banks all over the world to develop and use their own credit risk model, which fits best for their internal risk management and risk management needs (see BCBS (2004, 2005, 2006)). Thus, the primary purpose of our attempt is to demonstrate a potential idea about this encouragement of BCBS for the bank. On the other hand, we would like to argue that this preliminary attempt **must be developed further**. In particular, the two following international-standard model specifications must be referred.

- First, the Asymptotic Single Risk Factor (ASRF), which was developed by Gordy (2003), is served as the Basel-II IRB Risk Weight Function.
- Second, the Vasicek Loan Portfolio Distribution, which was developed by Vasicek (1987, 1991, 2002), is one of the fundamental models used by Moody's Analytics.

References

BCBS (2004). International Convergence of Capital Measurement and Capital Standards: A Revised Framework. *Bank for International Settlements Papers.*

BCBS (2005). An Explanatory Note on the Basel II IRB Risk Weight Functions. *Bank for International Settlements Papers.*

BCBS (2006). International Convergence of Capital Measurement and Capital Standards: A Revised Framework Comprehensive Version. *Bank for International Settlements Papers.*

Gordy, M. B. (2003). A risk-factor model foundation for ratings-based bank capital rules. *Journal of Financial Intermediation*, 12(3):199–232.

Vasicek, O. A. (1987). Probability of loss on loan portfolio. *KMV Corporation.*

Vasicek, O. A. (1991). Limiting loan loss probability distribution. *KMV Corporation.*

Vasicek, O. A. (2002). The distribution of loan portfolio value. *Published in Risk.*