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May 2020

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October 2019

Abstract: Without the assumption on the factor linear growth equation and keeping other assumptions in the endogenous growth theory, we prove the growth rate and interest rate endogenous, and then we give general conditions for the existence and uniqueness of the growth rate. Under the condition of the constant returns to scale, the growth rate of every variable and interest rate are constant in the steady state. In addition, we give primary analyses on the stochastic economy with growth.

Key words: Endogenous Growth, Existence, Uniqueness, equilibrium

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1. Introduction

The Endogenous Growth Theory pioneered by Lucas (1988) and Romer (1990) makes the growth rate endogenous and emphasizes the unique growth rate for every variable. In the theory, the growth rate of one of factors is pre-expressed by the linear equation, and the growth rate exists in the compact convex set. Because the domain of the convex set of growth rate is positive, the positive growth rate can be solved by the dynamic optimal conditions. To guarantee the uniqueness of the growth rate, Lucas (1988) and Romer (1990) both pre-assume that one production factor grows with the linear growth equation[3]. After optimization of the variable of the linear growth equation, the theory can obtain the unique the growth rate. In other words, to ensure the existence and uniqueness of the growth rate, the endogenous growth theory relies on the definition of the exogenous linear growth equation of a production factor, that's why Jones (1995) regards it semi-endogenous.

The following researches go mainly in two directions. The first one follows deterministic framework as Lucas and Romer's framework, the dynamic equation of one factor is defined semi-endogenous linear, such as Jones (1995), Peretto (1998), Segerstrom (1998), Antonio, Salvador and Manuel (1999), Huffman (2007), and Kaboski (2009). The second direction follows the stochastic growth framework setup by Grossman and Helpman (1991), Aghion and Howitt (1992). Taking Aghion and Howitt's work as an example, they assume the arrival of new innovation in the leading firm stochastic, and the expectation of the innovation allows the productivity to increase fixed γ times in every period exogenously, similar works are Aghion, et al. (2001), Acemoglu, et al. (2015), Kerr (2018), and Grossman and Helpman (2018). However, Kortum (1997) checks that when arrivals of innovation with Pareto, exponential, and uniform distributions, the correlation between the arrival of new innovation and productivity fails to reconcile with assumptions in Grossman and Helpman (1991), Aghion and Howitt (1992).

Lucas and Moll (2014) supposes that individuals can increase their productivities through learning effect (individuals with lower level productivities can search and learn from those with higher level productivities, not vice versa). When the individual's income expectation of the individual is the mean-field game process, and the individual's productivity increment through the learning effect is inexhaustible, the growth rate of productivity is non-decreasing. Thus Lucas and Moll (2014) must assume the growth of productivity exists, and it is unchangeable in the balanced growth path analysis. Similar works in recent stochastic growth studies, such as Figuières et al. (2013), Akcigit et al. (2016), Lentz and Mortensen (2016), Akcigit and Kerr (2018) also assume the growth rate exist and can be parameterized.

In addition, the uniqueness of the growth rate should be studied in the endogenous growth theory. Since the uniqueness of the growth rate can ensure the uniqueness of the

[3] Lucas (1988) allows human capital (h_a) grow following: $\dot{h}_{a,t} = (1-\mu_t)\cdot\delta\cdot h_{a,t}$ with $\mu_t, \delta > 0$. Romer (1990) assumes the aggregate stock of designs (A) evolves as: $\dot{A}_t = A_t \cdot \delta \cdot H_{A,t}$ with $\delta > 0$, $H_{A,t} > 0$.

interest rate, it guarantees the steady state Arrow-Debreu allocations non-zero and constant in the infinite horizon (Stokey and Lucas with Prescott, 1989) when given proper initial values.

In summary, the endogenous growth theory is not fully complete. The theory presumes the exogenous positive growth rates of one factor so as to ensure the existence of the growth rate along the balanced path. Relaxing the assumption that growth rate of one factor is exogenously given, this paper proves when the factor is in-exhaustive, its growth rate exists and is unique. This paper is organized as follow: the section I is the introduction; the second II provides a complete endogenous growth model; the section III is the extension; section IV gives the primary analysis on the convergence of growth rate in the stochastic environment; section V demonstrates general conditions on the existence and uniqueness of the growth rate; and section VI is the conclusions.

II. the Complete Endogenous Growth Model

In this section, we setup the analysis in the infinite horizon framework, and make the growth rate and interest rate endogenous. We relax the presumption of the exogenous growth equation of the factor, and provide a proof on the existence and uniqueness of the growth rate as well as those of the interest rate under the condition of the factor reusable.

1. Labor

The representative family has l_t number of members at period *t* with each member endowed with one-unit labor. The family allocates her total labor (l_t) among final goods production and human capitals accumulation, respectively $l_{1,t}$, $l_{2,t}$. At the period *t*, the number of family's members should be finite, defined $\overline{L_t}$, therefore $\overline{L_t} = l_t$. If the family member increases with the growth rate n_t at period *t*, the family member at period *t*+1, $\overline{L}_{t+1}$ is $(1+n_t)\cdot\overline{L}_t$. We define n_t constant (the analyses of the situation when n_t is stochastic will be provided in Section IV).

2. Skills and Human Capital Accumulation

Skills are free to access in every period, and the stock of skills per capita in period *t*-1 (S_{t-1}) is combined with labor $l_{2,t}$ in the period *t* to form the skilled labor ($S_{t-1}\cdot l_{2,t}$) in the period *t*. The stock of the skills at the period *t* is dependent on the skill labors at period *t*-1 and the job training (J_t) at the period *t*. Skills formation mechanism at the period *t* can be shown as following:

$$S_t = J_t^{\nu}\cdot\left(S_{t-1}\cdot l_{2,t}\right)^{1-\nu} \qquad 0<\nu<1 \qquad (1)$$

Where ν represents the elasticity of skills with respect to job training. In addition, J_t, $S_{t-1}\in R^+$, $t = 1,2,3,\ldots$. It is worth to mentioning that skills, which are very similar to knowledge, are reusable as human capitals in Lucas (1988) and designs in Romer (1990). The stock of human capitals per capita at the period *t* can be shown as the following equation.

$$H_t = H_{t-1}^{\theta}\cdot S_t^{1-\theta} \qquad 0<\theta<1 \qquad (2)$$

Where $1-\theta$ is the elasticity of human capitals with respect to skills at period *t*.

Similarly, assume $H_{t-1} \in R^{+}, t = 1,2,3, \ldots$. Obviously, equation (2) is the production of human capitals with the constant returns to scale. Strictly speaking, the equal signs in equation (1) and (2) should be replaced by "$\leqq$", when both equations are interpreted as production frontiers of skills and human capitals at period *t*.

3. Final Goods Production

There are many homogenous firms that produce final goods by using capital (K_t), human capital (H_t) and labor ($l_{1,t}$), and the final goods production is also constant returns to scale. The final good production function is as following:

$$F(K_t, H_t, l_{1,t}) = K_t^{\alpha} \cdot (H_t \cdot l_{1,t})^{1-\alpha} \qquad 0 < \alpha < 1 \qquad (3)$$

Where α is the output elasticity of capitals. The production function in here is similar as Romer (1990), where part of human capitals goes into the final good production. Because of the constant returns to scale, we can obtain $F = r \cdot K + w \cdot Hl_1$, where r and w are interest rate and wage, respectively. As in Romer (1990), Human capitals are not exclusive between the final goods production and their accumulation (The exclusive situation as in Lucas (1988)[4] will be discussed in the section III). It is worth to mentioning that skills entering into the final goods production at the period *t* are still available in their accumulation as equation (1) at the period *t*+1.

4. Family

The representative family maximizes her long term utility function, and the utility at the period t is $U(C_t, l_{1,t}, l_{2,t})$. The maximization of the family is given as following:

$$MaxE\sum_{t=0}^{\infty} \beta^{t} U(C_t, l_{1,t}, l_{2,t}) \qquad 0 < \beta < 1 \qquad (4)$$

where β represents the family's discount factor. Assuming a constant population growth rate (*n*), then we have $\beta = \rho \cdot (1+n)$, where ρ is the family member's discount rate. Let $U(C_t, l_{1,t}, l_{2,t}) = lnC_t - \gamma_1 lnl_{1,t} - \gamma_2 lnl_{2,t}$ and $\gamma_1, \gamma_2 > 0$, $C_t, l_{1,t}, l_{2,t} \in R^{+}$. In addition, though labors provided by the family between the final goods production and the human capital accumulation are thought homogeneous, the family's preferences to labors allocated into these two sections are different. For example, if the family prefers labors in the human capital accumulation to those labors in the final goods production, γ_1 should be larger than γ_2. the budget constraint of the family is as following:

$$K_{t+1} = F(K_t, H_t, l_{1,t}) + (1-\delta)K_t - C_t - J_t \qquad (5)$$

Where δ is the rate of capitals depreciation, and $0 < \delta < 1$.

5. Competitive Markets

Markets of final goods, capitals and human capitals are competitive. Because the

[4] In Lucas (1988), firms can only use μ human capitals supplied by the family, and $1-\mu$ human capitals will be used in the human capitals accumulation. That is to say, human capitals should be exclusively allocated between the final goods production and human capitals accumulation. In Romer (1990), Designs, as the in-exhaustive factor, will be used in the final goods production and its accumulation simultaneously.

final goods production function is homogeneous of the first degree, the competitiveness of markets makes all final goods aggregated. Under the condition of competitive markets, firms are price takers and their profits are zero, and family owns all products.

6. The Equilibrium

Optimization problem of the representative family can be shown as following:

$$Max \sum_{t=0}^{\infty} \beta^t U(C_t, l_{1,t}, l_{2,t})$$

$$\begin{aligned} &K_{t+1} = F(K_t, H_t, l_{1,t}) + (1-\delta)K_t - C_t - J_t \\ &H_t = H_{t-1}^{\theta} \cdot S_t^{1-\theta} \\ &S_t = J_t^{\nu} \cdot (S_{t-1} \cdot l_{2,t})^{1-\nu} \\ &Y_t = F(K_t, H_t, l_{1,t}) = K_t^{\alpha} \cdot (H_t \cdot l_{1,t})^{1-\alpha} \\ &C_t > 0,\ J_t > 0, S_t > 0, H_t > 0, K_{t+1} > 0,\ \ t = 0,1,2,\ldots \end{aligned} \tag{P1}$$

The Lagrange function of the above problem can be represented as following:

$$L(C_t, K_t, H_t, l_{1,t}, J_t, \lambda_{1,t}, \lambda_{2,t}, \lambda_{3,t}) = \sum_{t=0}^{\infty} \beta^t \Big\{ lnC_t - \gamma_1 lnl_{1,t} - \gamma_2 lnl_{2,t} - \lambda_{1,t}\big[K_{t+1} - F(K_t, H_t, l_{1,t}) - (1-\delta)K_t + C_t + J_t\big] - \lambda_{2,t}\big[H_t - H_{t-1}^{\theta} \cdot S_t^{1-\theta}\big] - \lambda_{3,t}\Big[S_t - J_t^{\nu} \cdot \big(S_{t-1} \cdot l_{2,t}\big)^{1-\nu}\Big]\Big\} \tag{6}$$

the first order conditions are:

$$\frac{\partial L}{\partial C_t} = \frac{1}{C_t} - \lambda_{1,t} = 0 \tag{7}$$

$$\frac{\partial L}{\partial l_{1,t}} = -\gamma_1 \frac{1}{l_{1,t}} + (1-\alpha)\frac{Y_t}{l_{1,t}}\lambda_{1,t} = 0 \tag{8}$$

$$\frac{\partial L}{\partial l_{2,t}} = -\gamma_2 \frac{1}{l_{2,t}} + (1-\nu)\frac{S_t}{l_{2,t}}\lambda_{3,t} = 0 \tag{9}$$

$$\frac{\partial L}{\partial K_t} = \lambda_{1,t}\left[\alpha\frac{Y_t}{K_t} + (1-\delta)\right] - \lambda_{1,t-1}\frac{1}{\beta} = 0 \tag{10}$$

$$\frac{\partial L}{\partial H_t} = \lambda_{1,t}(1-\alpha)\frac{Y_t}{H_t} - \lambda_{2,t} + \lambda_{2,t+1}\beta\theta\frac{H_{t+1}}{H_t} = 0 \tag{11}$$

$$\frac{\partial L}{\partial J_t} = -\lambda_{1,t} + \lambda_{3,t}\nu\frac{S_t}{J_t} = 0 \tag{12}$$

Equation (7), (8), (9), and (12) are conditions on the static equilibrium, and equation (10) and (11) are the intertemporal conditions for the dynamic equilibrium. In the following part, we will prove the existence and uniqueness of the growth rate, and obtain other variables' relationship in the steady state by using the unique growth rate.

Define the growth rate of the variable X_t as $g_{X,t+1} = X_{t+1}/X_t$. Therefore, X_t increases when $g_X > 1$; X_t converges to some non-zero value when $g_X = 1$; and X_t converges to 0 when $g_X < 1$. The problem (P1) obviously does not allow the occurrence of the third case, thus we let $g_X \geq 1$ and assume $g_{X,t+1} \leq \bar{g}$ where $\bar{g}$ is a real number far greater than 1. Then $g_{X,t+1} \in [1, \bar{g}] \equiv A$, where A is a compact convex set.

6.1 The Static Equilibrium

Because labors of the representative family are homogeneous, the marginal utilities of labors used in final goods production and human capitals accumulation must be equal. Thus we have $\gamma_1/l_1 = \gamma_2/l_2$. When the growth rate of family members remains constant, the total labor supplied by the family can be normalized as one ($l_1 + l_2 = 1$). In addition, firms pay $\alpha \cdot Y$ for capitals and $(1-\alpha) \cdot Y$ for the $H \cdot l_1$. Labors $l_1$and labor l_2 in family can get $\gamma_1 \cdot (1-\alpha) \cdot Y/(\gamma_1 + \gamma_2)$ and $\gamma_2 \cdot (1-\alpha) \cdot Y/(\gamma_1 + \gamma_2)$ payment, respectively.

6.2 The Dynamic Equilibrium

In this part, we will show that the growth rate of every variable is same. The first step is to obtain the dynamic relationships of variables by using first order conditions. Based on equation (8) and (9), we can obtain:

$$\frac{Y_t}{C_t} = \frac{\gamma_1}{1-\alpha} \tag{13}$$

Since γ_1 and α are constant, the growth rate of Y_t and C_t must be equal: $g_Y = g_C$. Then equations (9), (12) and (13) imply:

$$\frac{J_t}{C_t} = \frac{\gamma_2 . v}{1-v} \tag{14}$$

Similarly, both γ_2 and v are constant, implying $g_J = g_C$. Equation (13) and (14) imply:

$$\frac{Y_t}{J_t} = \frac{\gamma_1}{1-\alpha} \cdot \frac{1-v}{\gamma_2 . v} \tag{15}$$

Equation (11) implies:

$$\lambda_{1,t}(1-\alpha)Y_t - \lambda_{2,t}H_t + \lambda_{2\cdot t+1}H_{t+1}\beta\theta = 0 \tag{16}$$

Define $M_{t+1} = \lambda_{2,t+1}H_{t+1}$ and $M_t = \lambda_{2,t}H_t$. The definition of equation (2) and Problem (P1) imply that $\lambda_{2,t} > 0, t = 1,2,\ldots$, and $\lambda_{2,t}H_t \in R^+, t = 1,2,\ldots$, therefore, M_{t+1} is the continuous operator defined on R^+. According to Theorem 18.E in Zeidler (1990, p68-69), equation (16) implies there is a unique fixed point in equation (16), such as $M_t = M_{t+1} = M$, so $M = \gamma_1/(1-\beta\theta)$.

In the next part, we prove the growth rate is unchangeable. According to equation (10), we obtain

$$\frac{C_{t+1}}{C_t} = \beta \cdot \left[\alpha Y_{t+1} \,/\, K_{t+1} + (1-\delta)\right] \tag{17}$$

Combining with equation (13) and (17), we can obtain

$$\frac{g_{Y,t} - \beta(1-\delta)}{g_{Y,t-1} - \beta(1-\delta)} = \frac{g_{Y,t}}{g_{K,t}} \tag{18}$$

In addition, equation (5) indicates:

$$\frac{g_{K,t+1} - (1-\delta)}{g_{K,t} - (1-\delta)} = \frac{g_{Y,t}}{g_{K,t}} \tag{19}$$

Now we get critical conditions to prove growth rates of all variables equal and unchangeable.

Proof:

The proof has two parts, the first part shows the situation of the same growth rates of capital and output is true; the second part shows the situation of different growth rates of capital and output is wrong.

Situation 1. if $g_{K,t} = g_{Y,t}$, we can obtain that $g_{K,t} = g_{Y,t} = g_{Y,t-1} = g$ by equation (18). it is straightforward that capitals and output have the same unchangeable growth rate. Then equation (13), (14), and (15) imply $g_{Y,t} = g_{K,t} = g_{C,t} = g_{J,t} = g$. In addition, equation (1) implies $g_{S,t} = g_{J,t} = g$, and equation (3) implies $g_{Y,t} = g_{K,t} = g_{H,t} = g$. Therefore, all variables' growth rate are same when $g_{K,t} = g_{Y,t}$. Moreover, since the consumption of the family grows at the constant rate, the long term utility will converge.

Situation 2. Assume $g_{K,t} \neq g_{Y,t}$, then equation (18) and (19) can be rewritten as:

$$g_{Y,t} - \frac{g_{Y,t}}{g_{K,t}} g_{Y,t-1} = \beta(1-\delta)\frac{g_{K,t} - g_{Y,t}}{g_{K,t}} \qquad (A.1)$$

$$g_{K,t+1} - \frac{g_{Y,t}}{g_{K,t}} g_{K,t} = \beta(1-\delta)\frac{g_{K,t} - g_{Y,t}}{g_{K,t}} \qquad (A.2)$$

Define $g_{Y,t}/g_{K,t} = \Delta_t$, $(1-\delta)(g_{K,t} - g_{Y,t})/g_{K,t} = \Gamma_t$ and put them into equations of $(A.1)$ and $(A.2)$. After eliminating terms including Γ_t, we obtain

$$g_{Y,t} = \beta g_{K,t+1} + \left(\prod_{j=1}^{t} \Delta_{t+1-j}\right)\left(g_{Y,0} - \beta g_{K,1}\right) \qquad (A.3)$$

if $g_{Y,0} = \beta g_{K,1}$ in $(A.3)$ is true, the equation $g_{Y,t} = \beta \cdot g_{K,t+1}, t = 1,2,3, \ldots$ holds. (Appendix I and II show analyses under the condition of $g_{Y,t} \neq \beta g_{K,t+1}$). Taking $g_{Y,t} = \beta \cdot g_{K,t+1}$ into equation (18), we obtain the following equation,

$$(1-\beta) g_{Y,t} g_{K,t} = \beta(1-\delta)\left(g_{K,t} - g_{Y,t}\right), \qquad (A.4)$$

Since $g_{Y,t} > 0$ and $g_{K,t}>0$, the left side of equation (A4) is positive, then $g_{K,t} > g_{Y,t}$. Define $\kappa = Sup\,\{g_{Y,t}/g_{K,t}, t = 1,2, \ldots\}$, and know that $\kappa < 1$. Furthermore, the growth rate of any variable lies in $A = [1, \bar{g}]$ as previously discussed. Define the norm $\rho\,(x,y) = |x - y|$, where $x, y \in A$. It is then implied that (A, ρ) is a complete metric space. Equation (18) $g_{Y,t} = \kappa\left[g_{Y,t-1} - \beta(1-\delta)\right] + \beta(1-\delta)$, defines a contract mapping $L : A \to A$ with modulus κ which satisfies $0 < \kappa < 1$. According to the contraction mappings theorem, there must exist $g_{Y,t-1} = g_{Y,t} = g_Y$. Similarly, equation (19) also implies a contraction mapping on $g_{K,t}$ and $g_{K,t+1}$, and $g_{K,t+1} = g_{K,t} = g_K$. Solving fixed points of $g_{Y,t}$ and $g_{K,t}$, we can obtain $g_Y = \beta(1-\delta) < 1$, $g_K = (1-\delta) < 1$, and $g_Y, g_K \notin A$, which contradict to the fact that growth rates of capital and output should be in $A = [1, \bar{g}]$. Therefore, the situation of $g_{K,t} \neq g_{Y,t}$ and $0 < \delta < 1$ must be false.

In the economic interpretation, $g_Y = \beta(1-\delta) < 1$ and $g_K = (1-\delta) < 1$ mean that the output and capital decrease and converge to zero, which causes other variables converge to zero. Under such circumstance, the maximal value of the family's objective does not exist when the consumption converges to zero. The present value of utility approaches to negative infinity when the period runs to the infinity (the discount value of objective function in the steady state will be negative infinity when the consumption converges to the zero). In the other words, the maximal value of the long term utility in the situation $g_{K,t} \neq g_{Y,t}$ is less than that of the long term utility in the situation $g_{K,t} = g_{Y,t}$, so the rational family will not choose the situation of $g_{K,t} \neq g_{Y,t}$. In conclusion, it is impossible for the situation with $g_{K,t} \neq g_{Y,t}$ to exist, and the following equation must hold true.

$$g_{J,t} = g_{S,t} = g_{H,t} = g_{Y,t} = g_{K,t} = g_{C,t} = g \tag{20}$$

Furthermore, since $g_{Y,t} = g_{K,t} = g$ and the interest rate at period *t* is $r_t = \alpha Y_t / K_t$, it follows that the interest rate in steady state is also invariant: $r_{t+1} = r_t = r > 0$. So steady state prices at each period exist and be invariant, and the Arrow-Debreu allocations in the steady state share the same equilibrium. Therefore, we obtain Proposition 1. Q.E.D.

Proposition 1. Given conditions that skills are free to reuse and the human capitals are not exclusive between the production of final goods and the human capitals accumulation. If the accumulation of human capitals, the accumulation of skills, and the production of final goods are all constant returns to scale, all variables' growth rates converge to the same and constant value, and the interest rate converges as well.

We can obtain the steady state growth rate and interest rate as following:

$$g = \beta \left[\alpha \cdot \frac{(1-\delta)(1-\beta)}{D} + (1-\delta) \right] \tag{21}$$

where $D = \left[\beta\alpha + \frac{(1-\alpha)}{\gamma_1} + \frac{\gamma_2 \upsilon (1-\alpha)}{\gamma_1 (1-\nu)} - 1 \right]$. The steady state interest rate as following:

$$r = \alpha \cdot \frac{Y}{K} = \alpha \cdot \frac{(1-\delta)(1-\beta)}{D} \tag{22}$$

and the linear relationships of other variables in the steady state can be shown as following:

$$\frac{Y}{J} = \frac{\gamma_1 (1-\nu)}{\gamma_2 \upsilon (1-\alpha)} \tag{23}$$

$$\frac{Y}{K} = \frac{(1-\delta)(1-\beta)}{D} \tag{24}$$

$$\frac{J}{S} = g^{1-\upsilon/\upsilon} \cdot \left(\frac{\gamma_2}{\gamma_1 + \gamma_2} \right)^{\frac{\upsilon - 1}{\upsilon}} \tag{25}$$

$$\frac{Y}{H} = \frac{\gamma_1 (1-\nu)}{\gamma_2 \upsilon (1-\alpha)} \cdot \left(\frac{\gamma_2}{\gamma_1 + \gamma_2} \right)^{\frac{\upsilon - 1}{\upsilon}} \cdot g^{\frac{(1-\upsilon)(1-\theta)-\upsilon}{(1-\theta)\upsilon}} \tag{26}$$

$$P_1 = \frac{\gamma_1 (1-\nu)}{\gamma_2 \upsilon} \cdot \left(\frac{\gamma_2}{\gamma_1 + \gamma_2} \right)^{\frac{\upsilon - 1}{\upsilon}} \cdot g^{\frac{(1-\upsilon)(1-\theta)-\upsilon}{(1-\theta)\upsilon}} \tag{27}$$

$$P_2 = \frac{\gamma_2 (1-\nu)}{\gamma_2 \upsilon} \cdot \left(\frac{\gamma_2}{\gamma_1 + \gamma_2} \right)^{\frac{\upsilon - 1}{\upsilon}} \cdot g^{\frac{(1-\upsilon)(1-\theta)-\upsilon}{(1-\theta)\upsilon}} \tag{28}$$

Where P_1 and P_2 represent payments for every unit labor of l_1 and l_2, respectively.

Proposition 2. The steady state growth rate is larger than 1.

Proof:

According to equations (13) -(15) and (23) -(26), every two variables shares linear equation in the steady state, the equation (5) can be represented as following:

$$g = (\frac{H}{K})^{1-\alpha} {l_1}^{1-\alpha} - m \qquad (A.7)$$

where m is a positive constant consisting of $C/K, J/K$ and $1-\delta$. Since $g \geq 1$, and l_1 is a constant smaller than 1, it's easy to check $K/H < 1$.Rewritting the equation (3) as $Y/H = (K/H)^{\alpha} {l_1}^{1-\alpha}$, it implies $Y/H \leq 1$.

Based on the equation (23), the inequality $\gamma_1(1-\nu)/\gamma_2 \upsilon(1-\alpha) > 1$ must hold true. Because g lies in $[1, \bar{g}]$, and $\gamma_2/(\gamma_1+\gamma_2)$ is smaller than 1, the inequality $g^{(\upsilon-1)/\upsilon} \cdot [\gamma_2/(\gamma_1+\gamma_2)]^{(\upsilon-1)/\upsilon} > 1$ must hold true. Put these two inequalities into the equation (26), the following inequality is true:

$$g^{\frac{-1}{(1-\theta)}} < 1 \qquad (A.8)$$

because of the condition of $0 < \theta < 1$, g must be a constant larger than 1. Q.E.D.

7. Calibration

We define $\bar{g} = g - 1$ and use the interest rate expression of equation (22) to calibrate. In addition, the range of parameters' values need to be determined. According to the equation (13) and (15), we have $\gamma_1 > 1-\alpha$, $\gamma_1(1-\nu) > \gamma_2 \upsilon(1-\alpha)$and $0 < D < 1$. Together they imply that $[\gamma_2 \upsilon + (1-\upsilon)]/\gamma_1(1-\nu) > (1-\alpha\beta)/(1-\alpha)$ and $\gamma_1 > (1-\alpha)/(1-\alpha\beta)$. Thus we set $\alpha = 0.33$, $\beta = 0.98$, $\delta = 0.05$, $\nu = 0.30$, $\gamma_2 \in [0.45, 0.55]$, and γ_1 to be 1, 0.99, or 0.98 to obtain Figure 1, which shows trajectories of $\bar{g}$ and r.

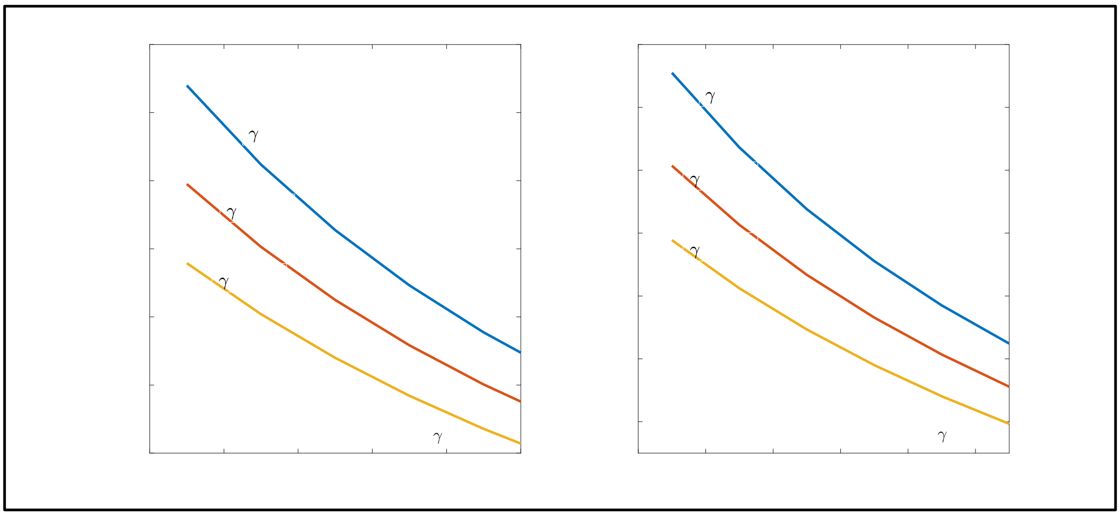

Figure 1. Trajectories of the growth rate, $\bar{g}$, and the interest rate, r.

III. Extension

The idea in the above section shows human capitals can be used simultaneously in the final goods production and human capitals accumulation, which shows the nonexclusive property of human capitals. In this section, we analyze the situation that human capitals are exclusive as in Lucas (1988), in which family will allocate her human capitals between the final goods production and human capitals accumulation.

Let human capitals in period t-1enter the economy in period t have no depreciation (the depreciation rate of human capital does not affect the result). The representative family allocates $(1-\mu_t)\cdot H_{t-1}$ of human capitals in the final goods production, the rest of human capitals, $\mu_t H_{t-1}$, will be used in their accumulation in period t. Therefore, the family optimization problem as following:

$$MaxE\sum_{t=0}^{\infty}\beta^t U(C_t, l_{1,t}, l_{2,t}) \quad 0<\beta<1$$

$$\begin{aligned} &K_{t+1}=F(K_t,H_t,l_{1,t})+(1-\delta)K_t-C_t-J_t \\ &H_t=(\mu_t H_{t-1})^{\theta}\cdot S_t^{1-\theta} \\ &S_t=J_t^{\nu}\cdot(S_{t-1}\cdot l_{2,t})^{1-\nu} \\ &Y_t=F(K_t,H_t,l_{1,t})=K_t^{\alpha}\cdot((1-\mu_t)\cdot H_{t-1}\cdot l_{1,t})^{1-\alpha} \\ &l_t=l_{1,t}+l_{2,t} \end{aligned} \tag{P2}$$

The first order conditions of problem (P2) are very similar to those of problem (P1), except the additional condition on μ_t and the condition on H_t, which are shown as following:

$$\frac{\partial L}{\partial \mu_t}=-\lambda_{1,t}\frac{(1-\alpha)Y_t}{1-\mu_t}+\lambda_{2,t}\cdot\theta\frac{H_t}{\mu_t}=0 \tag{29}$$

$$\frac{\partial L}{\partial H_t}=\lambda_{1,t+1}\beta\frac{(1-\alpha)Y_{t+1}}{H_t}-\lambda_{2,t}+\lambda_{2,t+1}\cdot\beta\theta\frac{H_{t+1}}{H_t}=0 \tag{30}$$

Other first order conditions remain same. We start to analyze from the equation (30), similarly, let $M_{t+1}=\lambda_{2,t+1}H_{t+1}$ and $M_t=\lambda_{2,t}H_t$, $M_t\in R^+, t=1,2,...$, according to Zeidler (1990), there exists a unique fixed point $M=\beta\gamma_1/(1-\beta\theta)$. Then we can rewrite equation (30) as following:

$$\frac{\gamma_1}{1-\mu_t}=\theta\frac{M}{\mu_t} \tag{31}$$

Equation (31) implies $\mu_t/(1-\mu_t)=\theta/(1-\beta\theta)$, and it shows that $\mu_t=\mu=\theta/[1+(1-\beta)\theta]$. Therefore, the optimal share of human capitals allocated in accumulation is also constant. Similarly, we can show that all variables grow with the same constant rate:

$$g_{J,t}=g_{S,t}=g_{H,t}=g_{Y,t}=g_{K,t}=g_{C,t}=g \tag{32}$$

In addition, equation (22) implies $\alpha Y_t/K_t=r_t$. Combining with equation (32), we can obtain $r_{t+1}=r_t=r$. Therefore, we obtain the following proposition.

Proposition 3. Given conditions that skills are free to reuse and human capitals are exclusive in the final good production and human capitals accumulation. If the accumulation of human capitals, the accumulation of skills, and the production of final goods are all constant returns to scale, all variables' growth rates converge to the same and constant value, and the interest rate converges as well.

IV. Stochastic Growth Analysis

In this section, we analyze the stochastic economy with growth. Stochastic process can be introduced into economy as two situations. In the first situation, one variable in economy is stochastic with the stationary intertemporal transitional probability, which is the fundamental assumption in the dynamic stochastic general equilibrium. In the second situation, the supply of labor can be stochastic, and we assume the stochastic labor supply in every period has identical numerical characteristics of the stochastics.

All variables in our framework are endogenous, so we introduce the new random variable, φ_t, into the equation (1) and rewrite equation as following:

$$S_t = \varphi_t J_t^{\nu} \cdot \left(S_{t-1} \cdot l_{2,t}\right)^{1-\nu} \quad (33)$$

the φ_t can be regarded as an exogenous shock of the productivity of skills. Similarly, let g_X be the steady growth rate of variable X_t. In addition, we define the deviate of the growth rate of variable X_t as $\tilde{g}_{X,t} = ln(g_{x,t}/g_X)$. Let $ln\varphi_t$ be a stochastic recursive sequence which satisfies $ln(\varphi_{t+1}/\varphi_t) = \rho_1 \cdot ln(\varphi_t/\varphi_{t-1}) + \varepsilon_t$, and $\varepsilon_t \sim (0, \delta^2)$, $|\rho_1| < 1$. Then equation (33), (2), and (3) imply the deviate of the growth rates of the stock of skill accumulation, human capitals accumulation, and production output are:

$$\tilde{g}_{S_t} = \tilde{g}_{\varphi_t} + v\tilde{g}_{J_t} + (1-v)\tilde{g}_{S_{t-1}} \quad (34)$$

$$\tilde{g}_{H_t} = \theta\tilde{g}_{H_{t-1}} + (1-\theta)\tilde{g}_{S_t} \quad (35)$$

$$\tilde{g}_{Y_t} = \alpha\tilde{g}_{K_t} + (1-\alpha)\tilde{g}_{,\tilde{g}_{H_t}} \quad (36)$$

Based on equation (5), we have:

$$g_{K,t+1}\tilde{g}_{K,t+1} = mr\tilde{r}_t \quad (37)$$

where $m = [1-(1-\alpha)/\gamma_1 - \gamma_2 v(1-\alpha)/(\gamma_1(1-v)]/\alpha$. In addition, according to conditions: $\alpha(Y_t g_{Y,t+1})/(g_{K,t}K_{t+1}) = r_{t+1}$, $Y_{t-1}g_{Y,t} = Y_{t-1} \cdot \bar{g}_{Y,t} \cdot (1+\tilde{g}_{Y_t})$, and $K_{t-1}g_{K,t} = K_{t-1}\bar{g}_{K,t}(1+\tilde{g}_{K_t})$, we can obtain that $\left(1+\tilde{g}_{Y_t}\right) = (1+\tilde{g}_{K_t}) \cdot (1+\tilde{r}_t)$, which can be simplified as following:

$$\tilde{g}_{Y_t} - \tilde{g}_{K_t} = \tilde{r}_t \quad (38)$$

Combining equation (36) and (37), we can obtain

$$g\tilde{g}_{K,t+1} = mr[\alpha-1)\tilde{g}_{K_t} + (1-\alpha)\tilde{g}_{H_t}] \quad (39)$$

Define state variable vector as $Z_t = (\tilde{g}_{S_t}, \tilde{g}_{H_t}, \tilde{g}_{K_{t+1}})^T$ and the exogenous shock vector as $\Phi = (ln(\varphi_t/\varphi_{t-1}), 0, 0)$. Then we can express equations (34)- (39) as the matrix form $E_t[FZ_t + GZ_{t-1} + L\Phi_t + M\Phi_{t-1}] = 0$ where:

$$F = \begin{bmatrix} 1 & 0 & -v \\ \theta - 1 & 1 & 0 \\ 0 & 0 & g \end{bmatrix} \quad (40)$$

$$G = \begin{bmatrix} v - 1 & 0 & g \\ -\theta & 0 & 0 \\ 0 & mr(\alpha - 1) & mr(1 - \alpha) \end{bmatrix} \quad (41)$$

$$L = [-1 \quad 0 \quad 0]^T \quad (42)$$

$$M = [0 \quad 0 \quad 0]^T \quad (43)$$

$$E\Phi_t = N \cdot \Phi_{t-1}, \quad N = [\rho \quad 0 \quad 0]^T \quad (44)$$

In addition, the system formed by equation (34) - (39) can be expressed as $Z_t = PZ_{t-1} + Q\Phi_t$ where $P = F^{-1}G$, and Q satisfies $Vvec(Q) = -vec(LN + M)$ with $V = N^T \otimes F$. Based on parameters in the previous section, we have the eigenvalues of P:

$$eig(P) = \begin{bmatrix} -0.67 & 0 & 0 \\ 0 & -0.23 & 0 \\ 0 & 0 & 0.02 \end{bmatrix} \quad (45)$$

all state variables eigenvalues' absolute values in the system less than 1, and the control variables and state variables are linear. Therefore, the dynamics of all variables in this system will converge to steady states when the exogenous shock, φ_t, goes into the economy and makes variables impulse response.

As mentioned in the section II, labor supply of the family can be stochastic. Such scenario is similar to the assumption in t Mirrless (1971) , Werning (2002), Golosov et al. (2003), Kocherlakota (2005), Albenesi and Sleet (2006), Golosov and Tsyvinski (2006), and Golosov et al. (2016). However, the difference between our scenario and above works' scenario is that, in our analysis, the family makes optimal allocations, and there is no central planner who can design the tax table to affect allocations of the stochastic family labor supply. Therefore, the labor supply expectation can be normalized as $E(l_1 + l_2) = 1$ in our scenario. Define the variance of the stochastic labor supply as $D(l_1 + l_2) = \sigma^2$. The family will allocate labor supply according to $E(l_1/l_2) = \gamma_1 \cdot (1 - \alpha)/[\gamma_2 \cdot (1 - \alpha)]$. Following similar steps in section II, it can be concluded that the expected growth rates of variables and the expected interest rate behave similarly to those in the deterministic economy.

V. General Conditions on Existence and Uniqueness of the Growth Rate

Section II and section III give mathematical functions of skills, human capitals accumulation, production, and utility, and show the unique growth rate and interest rate. In this section, we provide general conditions on the existence and uniqueness of the growth rate in the endogenous growth theory.

Similarly, the growth rate of family members is constant, without loss generality, we assume the population growth rate is zero, so let $l_{1,t} + l_{2,t} = 1$, $t = 1,2,3, \ldots$; and define the utility function as $U(C_t, l_{1,t}, l_{2,t})$, $C_t \in R^+$, $l_{1,t}, l_{2,t} \in L$, L is a compact convex set in R^+. U is bounded and $U' > 0$, $U'' < 0$ for every variable in the utility function. We provide additional assumptions as followings：

Assumption 1. functions of skill accumulation, human capitals accumulation, and the final goods production are homogeneous of degree one on the first two variables, and let f_i, where $i = 1, 2, 3$, represent the production functions for skills accumulation, human capitals accumulation and final goods, respectively. In particular, the function of skills accumulation satisfies: $f_1 : S \times J \times L \to S$ and $S_t \in S \subset R^+, J_t \in J \subset R^+$, the function of human capitals accumulation satisfies: $f_2 : H \times S \to H$ and $H_t \in H \subset R^+, S_t \in S \subset R^+$; and the function of the final goods production satisfies: $f_3 : K \times H \times L \to R^+$ and $K_t \in K \subset R^+, H_t \in H \subset R^+, l_{1,t} \in L$.

Assumption 2. $f_i(0,\cdot,\cdot) = 0$, and $f_i(\cdot\,,0,\cdot) = 0$ for $i = 1,3$; $f_i(0,\cdot\,) = 0$, and $f_i(\cdot\,,0) = 0$ for $i = 2$.

Assumption 3. f_i is twice differentiable, and satisfies: $f_i'(x,\cdot) > 0$, $f_i'(\cdot, x) > 0$, $f_i''(x,\cdot) < 0$, $f_i''(\cdot, x) < 0$, for $i = 1,2,3$.

Assumption 4. $f_1$satisfies Inada conditions: $\lim_{x\to 0} \partial f_1/\partial\, x \to \infty$ and $\lim_{x\to\infty} \partial f_1/\partial x \to 0, x \in S, x \in J$, and the condition: $\partial\,[f_1(1/z, a)\ \cdot z]/\partial z > 0$.

Based on the above assumption, we can prove

$$g_{S,t+1} = g_{J,t+1} = g_{S,t} = g \qquad (46)$$

Proof:

Let $l_{1,t}^*$ and $l_{2,t}^*$ be optimal values which keep the utility function maximal. Because the utility function is strictly concave for every variable, and $l_{1,t}$, $l_{2,t} \in L$, there is unique value for $l_{1,t}^*$ and $l_{2,t}^*$, respectively. The homogeneity of f_1 implies: $g_{S,t+1} S_t / J_{t+1} = f_1(S_t/J_{t+1}, 1)$. We define $S_t/J_{t+1} = 1/z_t$, then $g_{S,t+1} = f_1(1/z_t, 1) \cdot z_t \overset{\text{def}}{=} m(1/z_t) \cdot z_t$. According to the assumption 3, we can obtain $g_{S,t+1}/dz_t > 0d$and

$$-m'\left(\frac{1}{z_t}\right) \cdot \left(\frac{1}{z_t}\right) + m\left(\frac{1}{z_t}\right) > 0 \qquad (47)$$

and integrate inequality (46), we obtain $\int_\Omega (1/z_t)/m(1/z_t) d\mu > \int_\Omega n_t d\mu$ and $1/z_t \in \Omega = (0, \infty)$, which implies $m(1/z_t) > 1/z_\text{t}$. Therefore, $g_{S,t+1} > 1$. Twice derivative of $g_{S\,,t+1}$ with respect to dz_t is

$$\frac{d^2 g_{S,t+1}}{d{z_t}^2} = m''\left(\frac{1}{z_t}\right) \cdot \left(\frac{1}{z_t}\right)^3 > 0 \qquad (48)$$

According to the definition $f_1(1/z_t, 1) \overset{\text{def}}{=} m(1/z_t)$ and the assumption 3, we obtain $m'' > 0$ which implies $df_1^2(1/z_t, 1)/(dz_t)^2 > 0$, $g_{S,t+1} = f_1(1, z_t)$, $df_1'(1, z_t)/dz_t > 0$, and $df_1''(1, z_t)/(dz_t)^2 < 0$. In addition, based on the assumption 2 and 4, we can obtain a unique z_t which makes $m(1/z_t) \cdot z_t = g_{S,t+1} = f_1(1, z_t)$. Therefore, g_S exists and is unique, and then we can obtain the equation $g_{S,t+1} = g_{S,t} = g$. Figure 2 provides a graphic explanation on the existence and uniqueness of the growth rate.

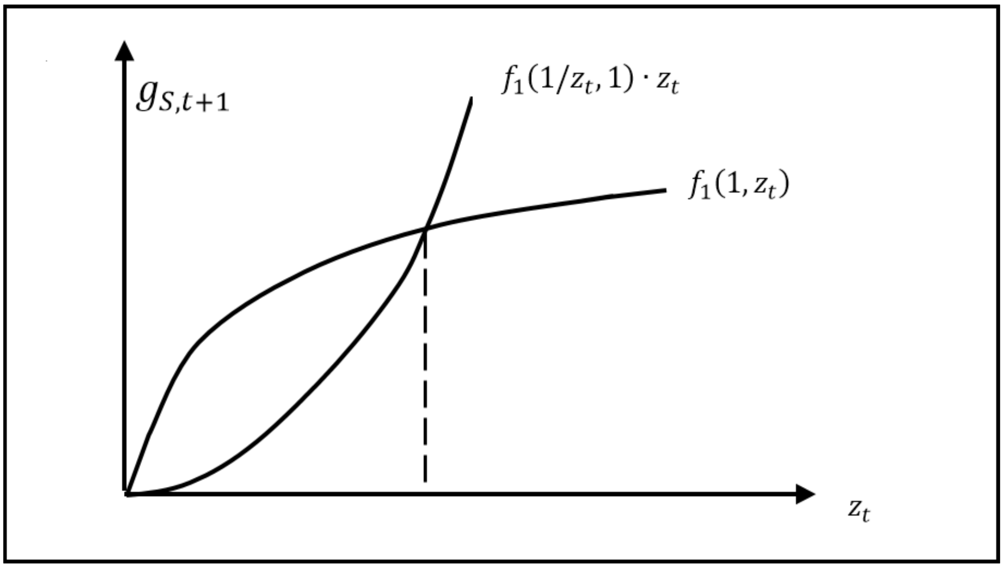

Figure 2. The Existence and Uniqueness of $g_{S,t+1}$

Since $g_{S,t+1} = g_{S,t} = g$ and z is uniquely determined, it is obvious for the equation $g_{S,t+1} = g_{J,t+1} = g_{S,t} = g$ to be true. Similarly, we can prove the equation $g_{J,t} = g_{S,t} = g_{H,t} = g_{K,t} = g_{Y,t} = g$ to be true. The equation of the family resource constraint implies $g = Y_t/K_t + (1-\delta) - C_t/K_t - J_t/K_t$. And because Y_t/K_t and J_t/K_t are both constant in the steady state, the following equation to be true,

$$g_{J,t} = g_{S,t} = g_{H,t} = g_{Y,t} = g_{K,t} = g_{C,t} = g \qquad (49)$$

Additionally, let $Y/H = f_3(K/H, 1)$, then $df_3(K/H,1)/d(K/H) = \partial f_3(K,H)/\partial(K) = r_t$, which implies that $df_3(K/H,1)/d(K/H)$ is monotonous, therefore, $f_3(K/H,1)/d(K/H) = f_3'|_{K/H}$ is constant in the steady state, so r_t in the steady state must be invariant as well.

The next part shows the form of the utility function. Firstly, we prove that the growth rates of Lagrange multipliers in the optimization are the same constant. Secondly, we use the constant growth rates of Lagrange multipliers to show that the utility function is the power function with respect to C_t. The general first order conditions are shown as following:

$$\frac{\partial L}{\partial C_t} = U'_{c_t} - \lambda_{1,t} = 0 \qquad (50)$$

$$\frac{\partial L}{\partial K_t} = \lambda_{1,t} \cdot [f_3'(K_t,\cdot) + (1-\delta)] - \lambda_{1,t-1} \cdot \frac{1}{\beta} = 0 \qquad (51)$$

$$\frac{\partial L}{\partial H_t} = \lambda_{1,t} \cdot f_3'(\cdot, H_t) - \lambda_{2,t} + \lambda_{2,t+1} \cdot \beta \cdot f_2'(H_t,\cdot) = 0 \qquad (52)$$

$$\frac{\partial L}{\partial J_t} = -\lambda_{1,t} + \lambda_{3,t} \cdot f_1'(S_t,\cdot) = 0 \qquad (53)$$

Based on the homogeneity of f_i in Assumption 1, F'_{H_t}, and S'_{J_t} are constants, thus, $g_{\lambda_{1,t}} = g_{\lambda_{3,t}}$. According the equation (51), the condition of $g_{\lambda_{1,t}} = g_{\lambda_{3,t}} = g_{\lambda}$ holds true, in which g_{λ} is a constant. The equation (52) can be transformed as following:

$$g_{\lambda_{1,t}} = g_{\lambda_{2,t}} \cdot \frac{1 + g_{2,t+1} \cdot \beta \cdot H'_{H_{t+1}}}{1 + g_{2,t} \cdot \beta \cdot H'_{H_t}} \tag{54}$$

If $g_{\lambda_{1,t}} = g_\lambda > g_{\lambda_{2,t}}$, $\exists\, t$, inequality $g_{\lambda_{2,t}} < g_{\lambda_{2,t+1}}, \forall\, t+i, i = 1,2,3, \ldots$ must be true which means $g_{\lambda_{2,t}}$ converges to $g_{\lambda_{1,t}}$ for $t+i$, $i = 1,2,3, \ldots$; Similarly, the situation of $g_{\lambda_{1,t}} = g_\lambda < g_{\lambda_{2,t}}$, $\exists t$, $g_{\lambda_{2,t}}$ converges to $g_{\lambda_{1,t}}$ for $t+i$, $i = 1,2,3, \ldots$. The solution of the difference equation (54) with respect to $g_{2,t}$ and $g_{2,t+1}$ has a global stability point. We have the conclusion: $g_{\lambda_{1,t}} = g_{\lambda_{2,t}} = g_{\lambda_{3,t}} = g_\lambda$. Let $(\lambda_{1,t+\Delta t} - \lambda_{1,t})/\lambda_{1,t} = \Delta t \cdot g_\lambda$, the following equation holds true,

$$\frac{\lambda_{1,t+\Delta t} - \lambda_{1,t}}{\lambda_{1,t}} = \frac{U''(C_t) \cdot C_t \cdot g}{U'(C_t)} = g_\lambda \tag{55}$$

We can obtain the equality $U''(C_t) \cdot C_t / U'(C_t) = g_\lambda / g$, and let $U''(C_t) \cdot C_t / U'(C_t) = -\sigma$. Integrating the equality $U''(C_t) \cdot C_t / U'(C_t) = -\sigma$, the utility function with respect to C_t is the power function. Especially, the utility function has the constant relative risk aversion coefficient σ. The relationship between the growth rate and the interest rate in the steady state satisfies $g^{-\sigma}/\beta - 1 + \delta = r$.

Example 1

Let $U(C_t, l_{1,t}, l_{2,t}) = C_t^{\,1-\sigma}/(1-\sigma) \cdot l_{1,t}^{\ \gamma_1} \cdot l_{2,t}^{\ \gamma_2}$, where $\sigma > 1$, $0 < \gamma_1, \gamma_2 < 1$; and keep other functions as those of in section II. We can find out that all ratios of two variables are constant which implies all variables share the same steady state growth rate. The equation with respect to the steady state growth rate is as following:

$$Dg^{-\sigma} - \alpha\beta g + C = 0 \tag{56}$$

where $D = 1 - (1-\alpha)/\gamma_1 - \gamma_2 v(1-\alpha)/\gamma_1(1-v)$, $C = \beta(\alpha - D)(1-\delta)$. The equation (56) is the high order equation with respect to g. It's difficulty to get the explicit solution if $\sigma \neq 2$. In here, we assign $\alpha = 0.33$, $\beta = 0.98$, $\delta = 0.05$, $v = 0.35$, $\gamma_1 = 1$ and $\gamma_2 = 0.45$ to calculate the trajectory of $Dg^{-\sigma} - \alpha\beta g + C$. The figure with respect to $\sigma = 2, 3, 4, 5$, respectively is shown as following:

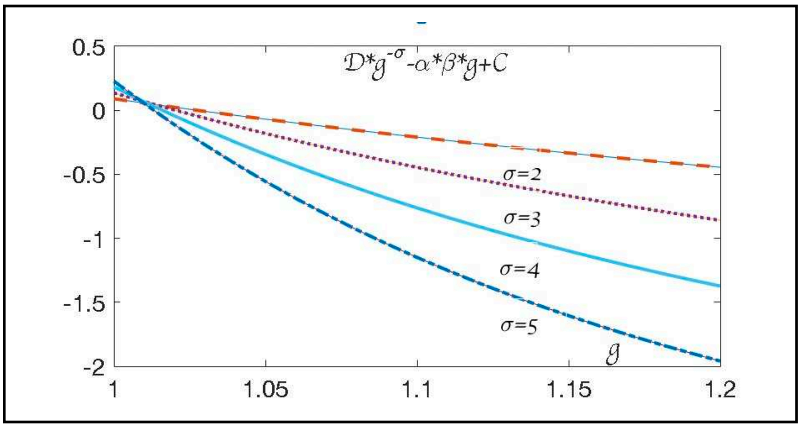

Fig 3. the trajectory of $Dg^{-\sigma} - \alpha\beta g + C$

The Fig 3 shows that trajectories of $Dg^{-\sigma} - \alpha\beta g + C$ are all monotonous under conditions of $\sigma = 2, 3, 4, 5$, respectively. All growth rates are larger than 1 when the condition of $Dg^{-\sigma} - \alpha\beta g + C = 0$ holds.

Example 2

Let the long term utility function be Epstein-Zin utility function as $U(C_t, l_{1,t}, l_{2,t}) = \{[{C_t}^{1-\sigma}/(1-\sigma) \cdot {l_{1,t}}^{\gamma_1}/\gamma_1 \cdot {l_{2,t}}^{\gamma_2}/\gamma_2]^{\phi} + [\sum_{t=1}^{\infty} \beta^t U(C_{t+1}, l_{1,t+1}, l_{2,t+1})]^{\phi}\}^{1/\phi}$, and keep other functions as those of in section II. The steady state growth rate satisfies the equation as following:

$$Dg^{(\phi-\phi\sigma-1)} - \alpha\beta^{1-\phi} g + E = 0 \qquad (57)$$

where D is the same as that in the example 1, $E = \beta^{1-\phi}(\alpha - D)(1-\delta)$. In here, we assign $\alpha = 0.33$, $\beta = 0.98$, $\delta = 0.05$, $\nu = 0.35$, $\gamma_1 = 1$, $\gamma_2 = 0.45$ and $\sigma = 0.35$ to calculate the trajectory of $Dg^{(\phi-\phi\sigma-1)} - \alpha\beta^{1-\phi} g + E$. The figure 4 shows the different trajectories under conditions of $\phi = -2, -3, -4, -5$ as following:

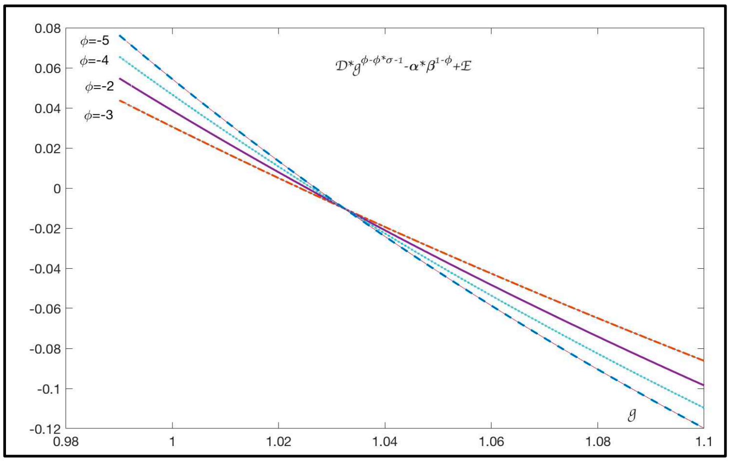

Fig 4. the trajectory of $Dg^{(\phi-\phi\sigma-1)} - \alpha\beta^{1-\phi} g + E$

The Fig 4 shows that trajectories of $Dg^{(\phi-\phi\sigma-1)} - \alpha\beta^{1-\phi} g + E$ are all monotonous under conditions of $\phi = -2, -3, -4, -5$, respectively. All growth rates are larger than 1 when the condition of $Dg^{(\phi-\phi\sigma-1)} - \alpha\beta^{1-\phi} g + E = 0$ holds.

VI. Conclusions

We make the growth rate and interest rate endogenous without using the assumption that one of factors' dynamics should be consistent with the exogenous linear growth equation. By keeping other assumptions in the endogenous growth theory, we prove that the growth rate of every variable and interest rate exist, and are unique by using the fixed point theorem. This is to say, the endogenous growth theory is complete without the assumption of in-exhaustive factor with exogenous linear growth function. Given proper coefficients, the growth rate and interest rate are calibrated. In the steady state, all variables share the same and constant growth rate, and the Arrow-Debreu allocations in every period can be explicitly expressed exponentially by the constant growth rate and initial values. In addition, we give primary analyses on the stochastic economy with growth and briefly discuss the convergence of variables. The last part of our paper discusses the general conditions on the existence and uniqueness of the growth in the steady state.

Appendix I

In the appendix I, we use the recursive method to analyze the growth rates in the situation of $g_{Y,t} \neq \beta \cdot g_{K,t+1}$. Putting definitions of $g_{Y,t}/g_{K,t} = \Delta_t$ and $(1-\delta)(g_{K,t} - g_{Y,t})/g_{K,t} = \Gamma_t$ into equations $(A.1)$ and $(A.2)$, these two equations can be rewritten by the recursive forms as following:

$$g_{Y,t} = \Delta_t g_{Y,t-1} + \beta \Gamma_t \qquad (B.1)$$

$$g_{K,t+1} = \Delta_t g_{K,t} + \Gamma_t \qquad (B.2)$$

The above two equations can be transformed recursively as following:

$$g_{Y,t} = (\prod_{j=1}^{t-1} \Delta_{t+1-j}) g_{Y,1} + \beta\left[\sum_{i=1}^{t-1}(\prod_{j=1}^{i} \Delta_{t-j})\Gamma_{t-i} + \Gamma_t\right] \qquad (B.3)$$

$$g_{K,t+1} = (\prod_{j=1}^{t-1} \Delta_{t+1-j}) g_{K,1} + \sum_{i=1}^{t-1}(\prod_{j=1}^{i} \Delta_{t-j})\Gamma_{t-i} + \Gamma_t \qquad (B.4)$$

combine $(B.3)$ and $(B.4)$ to eliminate $\sum_{i=1}^{t}(\prod_{j=1}^{i} \Delta_{t+1-j})\Gamma_{t-i} + \Gamma_t$, and obtain

$$g_{Y,t} = \beta g_{K,t+1} + \left(\prod_{j=1}^{t-1} \Delta_{t-j}\right)(g_{Y,1} - \beta g_{K,2}) \qquad (B.5)$$

$\Delta_{t+1-j} > 0$, $j = 1,2,3 \ldots, t$ in the equation $(B.5)$. If $g_{Y,1} \neq \beta g_{K,2}$ holds, then inequality $g_{Y,t} \neq \beta g_{K,t+1}$ must be true. The following is the analysis under the condition of $g_{Y,1} \neq \beta g_{K,2}$. The situation 2 of the proof in section II is the analysis under the condition of $g_{Y,1} = \beta g_{K,2}$.

According to equations (18) and (19), the following equation must hold true:

$$\frac{g_{Y,t} - \beta g_{K,t+1}}{g_{Y,t-1} - \beta g_{K,t}} = \frac{g_{Y,t}}{g_{K,t}} \quad , \quad t = 1,2,3, \ldots, \qquad (B.6)$$

Obviously, the equation$(B.6)$ should satisfy the condition of $g_{Y,t-1} - \beta g_{K,t} \neq 0, t = 1,2,3, \ldots$, and multiplying both sides of equation $(B.6)$ by $(g_{Y,t-1} - \beta g_{K,t})$ and $g_{K,t}$, we can obtain the following equation:

$$g_{Y,t}\left(g_{K,t} - g_{Y,t-1}\right) = \beta g_{K,t}\left(g_{K,t+1} - g_{Y,t}\right), \quad t = 1,2,3, \ldots, \qquad (B.7)$$

Defining $g_{K,t} - g_{Y,t-1} = N_t$, the above equation can be transformed as following:

$$g_{Y,t} N_t = \beta g_{K,t} N_{t+1} \qquad (B.8)$$

Equation $(B.8)$ shows the correspondence $m: N_t \to N_{t+1}$. Now we should prove the lemma 1 to show the correspondence $m: N_t \to N_{t+1}$ is monotonous.

Lemma 1: Under the condition of $g_{Y,t-1} - \beta g_{K,t} \neq 0$, the ratio of $g_{Y,t}/\beta g_{K,t}$, $t = 1,2,3, \ldots$ is larger than 1 or smaller than 1.

Proof:

Since $g_{Y,t}$ and $g_{K,t}$ both belong to $A = [1, \bar{g}]$, $g_{Y,t}/g_{K,t} > 0$. According to $(B.6)$, if the condition of $g_{Y,t-1} - \beta g_{K,t} > 0$ (< 0), $\exists\, t$ is true, the inequality $g_{Y,t} - \beta g_{K,t+1} > 0$ (< 0) for $\forall\, t$ must be true.

Q.E.D.

Thus, under the condition of $g_{Y,t-1} - \beta g_{K,t} \neq 0$, no matter whether the $g_{Y,t-1} - \beta g_{K,t} > 0$ or < 0, the correspondence $m: N_t \to N_{t+1}$ is monotonous.

To use the Brouwer Fixed Point Theorem in Zeidler (1986, p52), we firstly prove the next two lemmas. Let $N_t \in B, t = 1,2,3,\ldots$, lemma 2 shows the set B is a nonempty, convex, compact subset of R, and lemma 3 shows the correspondence $m: N_t \to N_{t+1}$ is continuous.

Lemma 2: the set B is a nonempty, convex, compact subset of R.

Proof: By the condition of $g_{Y,t-1}$, $g_{K,t} \in A = [1, \bar{g}]$, $Sup\, N_t = \bar{g} - 1$, $Inf\, N_t = 1 - \bar{g}$, so the B is bounded. Obviously, $Sup\, N_t$ and $Inf\, N_t \in B$, so B is closed, thus B is compact set. Since $g_{Y,t-1}$ and $g_{K,t}$ are both real numbers, N_t is real number, so B is nonempty. Let $B = A - A = \{g_{K,t} - g_{Y,t-1} | g_{K,t} \in A, g_{Y,t-1} \in A,\}$ and $0 < \lambda < 1$.

$$(1-\lambda)N' + \lambda N = (1-\lambda)\left(g'_{K,t} - g'_{Y,t-1}\right) + \lambda(g_{K,t} - g_{Y,t-1})$$

$$= (1-\lambda)g'_{K,t} + \lambda g_{K,t} - (1-\lambda)g'_{Y,t-1} + \lambda g_{Y,t-1} \qquad (B.12)$$

By the convexity of A, if $g_{K,t}, g'_{K,t}, g_{Y,t-1}, g'_{Y,t-1} \in A$, $(1-\lambda)g'_{K,t} + \lambda g_{K,t} \in A$, $(1-\lambda)g'_{Y,t-1} + \lambda g_{Y,t-1} \in A$, $(1-\lambda)N' + \lambda N \in A - A$, $A - A$ is a convex set.

Q.E.D.

Lemma 3. the correspondence $m: N_t \to N_{t+1}$ is continuous.

Proof: A is bounded, and $g_{K,t} \neq 0$, $g_{Y,t}/\beta g_{K,t}$ is bounded. Let $L = Sup\,\{g_{Y,t}/\beta g_{K,t},\quad t = 1,2,3\ldots,\}$, the distance between LN' and LN, $d(LN', LN) \to 0$ when the distance between N' and N, $d(N', N) \to 0$.

Q.E.D.

Now we can use the Brouwer Fixed Point Theorem to show m has fixed point N^*. Because $m: N_t \to N_{t+1}$ is monotonous, and m has fixed point N^*, there is only one fixed point $N^* = 0$ which implies $g_{K,t} = g_{Y,t-1}$.

Appendix II

In the Appendix II, we show all variables share the same growth rate under the condition of $g_{K,t+1} = g_{Y,t}$. Putting the condition of $g_{K,t} = g_{Y,t-1}$ into the equation (3), we can obtain:

$$lng_{K,t+1} = \alpha lng_{K,t} + (1-\alpha)lng_{H,t} \qquad (B.9)$$

where $lng_{K,t+1}$, $lng_{H,t} \in [0, ln\bar{g}]$, $t = 1,2,3,\ldots$, and recursively transform the equation $(B.9)$, the following equation should be true:

$$lng_{K,t} = \alpha^t lng_{K,0} + \sum_{i=0}^{t} \alpha^i (1-\alpha) lng_{H,t} \qquad (B.10)$$

the first term of right side converges to 0 as $t \to \infty$. Because $g_{X,t}, t = 1,2,3,\ldots$ lies in the set of $[0, ln\bar{g}]$, and $[0, ln\bar{g}]$ is the subset of the positive real number set R^+, the following inequality is true:

$$0 \leq \sum_{i=0}^{t} \alpha^i (1-\alpha) lng_{H,t} \leq \left| \sum_{i=0}^{t} \alpha^i (1-\alpha) ln\bar{g} \right|$$
$$= (1-\alpha^t) ln\bar{g} < ln\bar{g} \qquad (B.11)$$

Since $\left|\alpha^i(1-\alpha)lng_{H,t}\right| \leq \left|\alpha^i(1-\alpha)ln\bar{g}\right|$, $i = 1,2,3,\ldots$, and $\sum_{i=0}^{t} \alpha^i (1-\alpha) ln\bar{g}$ converges to the $ln\bar{g}$, $\sum_{i=0}^{t} \alpha^i (1-\alpha) lng_{H,t}$ converges as $t \to \infty$. Thus we can obtain the equality $g_{K,t} = g_{K,t+1}$ as $t \to \infty$. With the analytical step in the situation 1 of the section II, the equation of $g_{J,t} = g_{S,t} = g_{H,t} = g_{Y,t} = g_{K,t} = g_{C,t} = g$ must hold true under the condition of $g_{K,t} = g_{Y,t-1}$.

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