# A Simple Macro-model of Covid-19 with Special Reference to India

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**Abstract**

Motivated by the prevailing severe situation in India, we extend the SIR(S) model of infectious diseases to incorporate demand dynamics and its interaction with Covid-19 spread. We argue that, on one hand, spread of Covid-19 creates panic among consumers and firms and negatively affects economic activity. On the other hand, economy activity intensifies the spread of the infection. Initially assuming that recovered individuals do not develop antibody and become susceptible again, we capture the interaction between economic activity and spread of the disease in a two-dimensional dynamical system. We show that a large fiscal expansion combined with measures to boost community health as well as improve health sector's capacity to provide critical care can at the same time improve the economy and control the spread of the disease. We also find a slightly counter-intuitive result that a fiscal contraction can, under certain conditions, increase output and reduce number of infected individuals. However, we show that such a policy also increases fragility in the system. Finally assuming that only a fraction of recovered individuals become susceptible to contracting the diseases again, we obtain richer dynamics in a three-dimensional dynamical system. The paper also highlights the important role of speed of adjustment in the goods market in stability properties of steady state including emergence of limit cycles.

Keywords: *Effective Demand, Animal spirits, Consumer Confidence, Cycles, Fiscal Policy, Public Health, Covid-19.*

JEL codes: *E12, E32, E62, H51*

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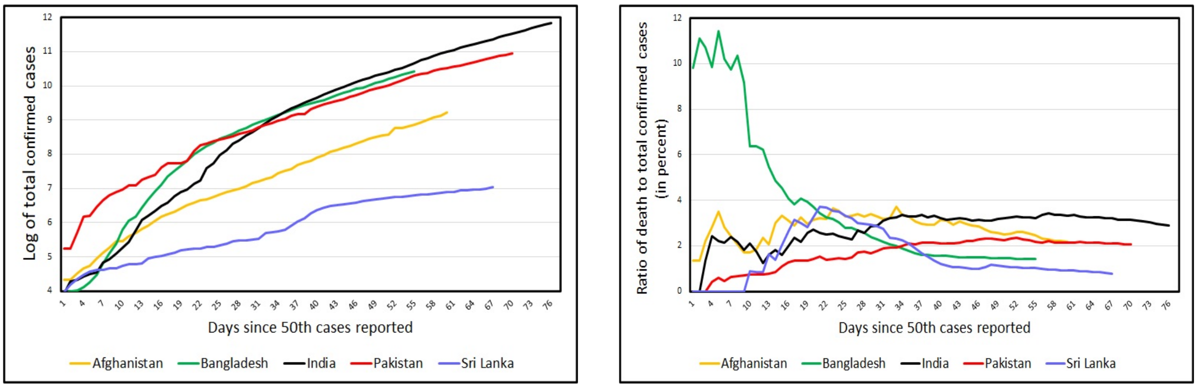
# 1 Introduction

We extend the SIR(S) (Susceptible, Infected and Recovered)[1] model of infectious diseases by introducing the economic activity for understanding the current severe situation and impending health and economic crisis in the context of India. Although this paper is motivated by Indian experience, our analysis is applicable for other countries as well. India is currently witnessing a rapid increase in Covid-19 infections while, at the same time, millions of people have lost their livelihoods because of a sudden and unprepared imposition of a lockdown by the government. Therefore along with controlling the covid19 spread, government also should focus on reviving the economy. In comparison with its neighbor countries (for instance Afghanistan, Bangladesh, Pakistan and Sri Lanka) India's performance in controlling the Covid-19 spread is not very satisfactory. Figure 1.1a presents the total number of cases since they first reported their 50th case while Figure 1.1b represents the ratio of death to cumulative number of confirmed cases in India and its neighbor countries.[2] Immense differences exist within the states as well. While Kerala is able to flatten the curve of cases, states like Maharashtra, Gujarat, Delhi, and Tamil Nadu have experienced an upward shoot in cases.




(a) Logarithm of the total confirmed cases of Covid-19 till 25th May, 2020.

(b) Ratio of death to total confirmed cases (in percent) of Covid-19 till 25th May, 2020.

Source: Computed using Our World in Data website.

Figure 1.1

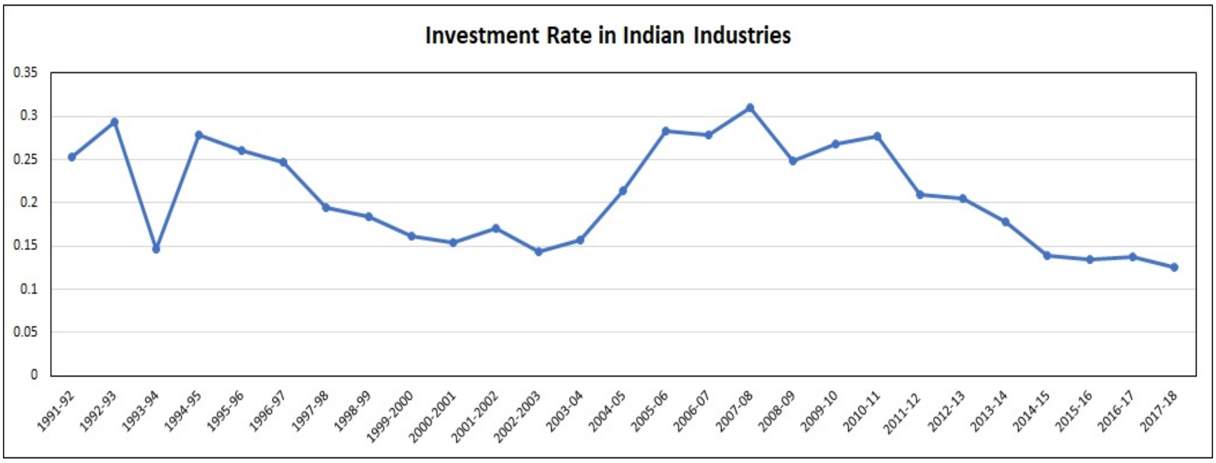
Because of the sudden imposition of lockdown by the central government, the supply chain of the economy broke down. The urban poor and the migrant workers are left with neither work nor food. With no transportation facilities available, migrant workers are left with

[1]The very last S indicates that there is a possibility that the recovered individuals may reenter the susceptible group again. See Hirsch et. al. (2004) for the basic SIR(S) model. Also see Kermack and McKendrick (1927), Baraun (1983, pp. 456-463), Murray (2002, pp. 315-327).

[2]Basu and Srivastava (2020a, 2020b, 2020c) analyze the relative performance of major South Asian nations in addressing the public health as well as economic repercussion of the Covid-19 pandemic. We follow the same procedure as Basu and Srivastava (2020a) for constructing Fingures 1.1a & 1.1b.

the only option of walking back to their native places. So far no satisfactory government measure has been taken for these migrant workers- not even proper distribution of food have been arranged despite the fact that sufficient stock of food grains is available with the centre. Total stock of food grains in the Central Pool is 56.939 million metric ton (MMT) out of which 24.7 MMT is wheat and 32.239 MMT is rice. These are substantially higher than the food grain stocking norms of 21.040 MMT for the quarter beginning from 1st April, 2020.[3]

As the Periodic Labour Force Survey of 2017-18 shows, unemployment rates in India stood at unprecedentedly high levels for both the urban as well as the rural sector and across gender even before the start of Covid19 spread. The MSMEs (micro, small and medium enterprises) are one of the most important sector of the economy and contribute about 30% of the country's gross domestic product (GDP), 40%–45% of exports, and employ about 30% of the labour force (about 114 million people). About 63 million unincorporated MSMEs are engaged in the non-agricultural sector, the majority of which are micro-enterprises in the informal sector.[4] The MSMEs were already incurring business losses and liquidity problems much before the lockdown owing to insufficient credit, delays in payments and unsold goods.[5] Despite the announcement of a few measures by the government "a survey of 5,000 MSMEs by the All India Manufacturers Organisation (AIMO) revealed that 71% of the enterprises could not pay salaries to their workers for March 2020 due to the lockdown".[6]




Rate of Investment = ratio of gross capital formation to the level of capital stock.
Source: Annual Survey of Industries 2017–18, author's calculations.

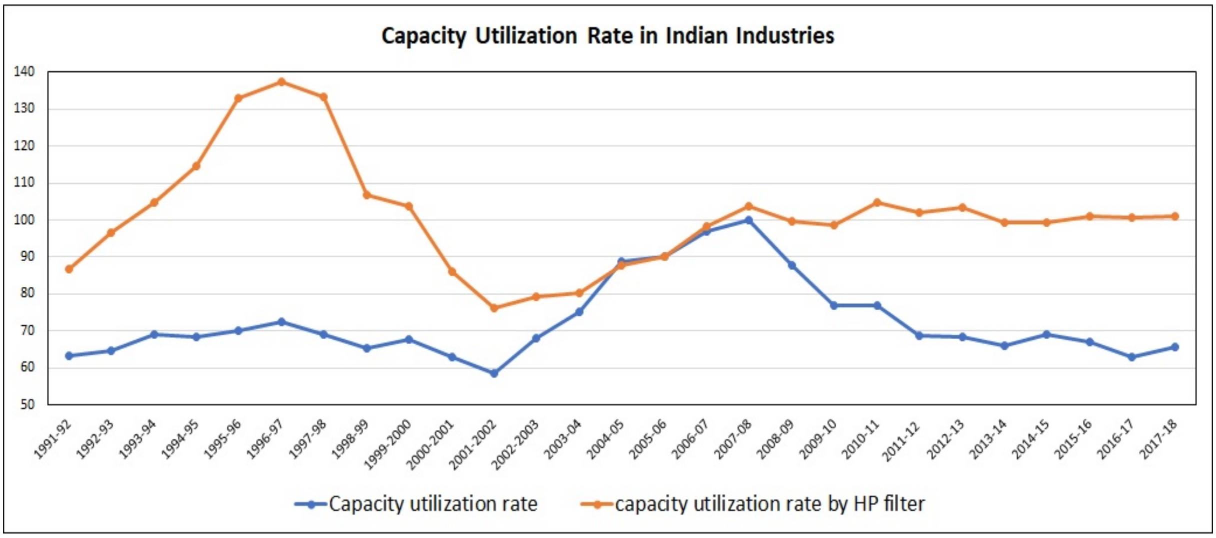
Figure 1.2: Rate of Investment in Indian Industries since 1991-92

[3]Monthly summary for the month of March, 2020; Department of Food and Public Distribution

[4]Using a structuralist urban dual economy model for a developing economy, Thakur and Guha (2019) argue that the informal sector in India, rather than being just a by-product of the formal sector, plays a crucial role in sustaining a certain kind of urban dynamics which ensures persistence of formal–informal duality.

[5]EPW editorials (2020)

[6]EPW editorials (2020)




Rate of capacity utilization = $\left\{\left(\frac{NVA}{K}\right) / \left(\frac{NVA}{K}\right)_{\text{at 2007-08}}\right\} * 100$; $NVA$ is the net value added and $K$ the level of capital stock. As $\frac{NVA}{K}$ was at its peak in 2007–08, we divide the NVA to K ratio by it.
Rate of capacity utilization through HP trend = $\left(\frac{NVA}{NVA_{\text{potential}}}\right) * 100$; potential $NVA$= Hodrick–Prescott trend of annual $NVA$, obtained with a smoothing parameter of 100.
Source: Annual Survey of Industries 2017–18, author's calculations.

Figure 1.3: Rate of Investment in Indian Industries since 1991-92

Another issue plaguing the Indian economy through out this decade is a continuous deterioration in the gross investment rate in Indian industries from 2010-11 (see Figure 1.2). On the one hand there is an uninterrupted slowdown in the Net Value Added (NVA) to capital stock ratio and, on the other hand, a stagnant actual to potential NVA since 2007-08 (see Figure 1.3). A year-on-year comparison of agricultural growth, as Anand and Azad (2019) points out using quarterly data, exhibits a striking decline since 2016–17.

The story is not satisfactory in the health sector either. For last two decades, the total health expenditure never exceeded beyond 4.26 percent of GDP. It constituted a merely 3.53 percent of GDP in 2017. Out-of-pocket expenditure in 2017 was 62.4 percent of the total health expenditure and 2.2 percent of GDP. Government expenditure on health in 2017 was only 0.96 percent of GDP. The doctor-patient ratio in India is much lower (0.8:1000) than the WHO recommendation of 1:1000. On the other hand merely 0.7 hospital-beds are available per 1000 population.[7] Only 59.55 percent of the population have the hand-washing facility.[8]

Focusing on the short run, we try to explore the consequences of the interaction between economic activity and the infection of the disease. Our analytical approach is divided into two parts: first part where we assume that once a person is recovered from the

[7] World Bank Data
[8] Our World in Data website

disease immediately enters the pool of susceptible individuals and the second part, where only a fraction of the the recovered individuals enter the pool of susceptible individuals. We assume that because of the interaction between susceptible and infected individuals the number of infected individuals at any point in time increases. Economic activity is a major site for spread of infection as infected persons may present themselves as asymptomatic and continue to participate in economic activity where they interact with susceptible persons. To account for this effect of economic activity, we assume the number of infected individuals increases proportionately with the level of output. At the same time, infection also creates panic among consumers and force them to cut down their spending on non-essential commodities, and similarly dampens the animal spirits of the investors. To incorporate these effects, we also assume that the economic activity is negatively affected by the number of infected individuals. Under certain conditions, we find a non-conventional result that a contractionary fiscal policy may help to reduce the number of infected individuals and may improve the economy slightly. However, as we show, it may increase the fragility in the system and a further contractionary fiscal policy may create instability in the entire system. Rather, a sufficiently large fiscal expansion along with some probable measures such as adequate isolation, adequate medical care, public health vigilance and control, mass-scale testing and segregating the infected individuals from others, a large scale use of PPE (Personal Protective Equipment) etc. can reduce the number of infected individuals and improve the economic situation. On the other hand, if the speed of adjustment of the goods market falls, an endogenous cycle of infected individuals and the employment level may emerge in the economy.

A richer dynamics is obtained when we assume that instead of all the recovered individuals only a fraction of them become susceptible to contracting the infection again. We show that a very high rate of infection or a very low recovery rate can destabilize the system. We also investigate whether the endogenous and perpetual cycles of infected individuals, susceptible individuals and the employment level can emerge in the three dimensional system or not. Speed of adjustment parameter of the goods market, under certain circumstances, plays an important role for ensuring stability in the system. We show that a better community health and a rise in the recovery rate from the disease raise the equilibrium level of output and the number of susceptible individuals, whereas the level of infected individuals falls. However, the effect of fiscal expansion on the equilibrium level of output is ambiguous. However, a proper policy-mix of a fiscal expansion along with all the measures that ensures a better community health and a rise in the recovery rate can lead to an increase in the steady state level of output and a fall in the level of infected individuals.

There are already a few papers that extend the canonical SIR model to understand the economic consequences of Covid-19. Eichenbaum, Rebelo and Trabandt (2020) investigate the interaction between economic decisions (such as purchasing consumption goods and working) and epidemics. The epidemic, as they argue, generates both the supply as well as the demand effects on economic activity which in turn can generate a large and persistent recession. As the economic agents are atomistic, they do not internalize the impact of their actions on the infection and death rates of other agents and therefore the competitive equilibrium of their model economy is not Pareto optimal. Bethune and Korinek (2020) analyze the trade-off between economic costs and epidemiological control and the externalities that arise when social and economic interactions transmit infectious diseases like Covid-19. Based on a SIS model they argue that the individually rational susceptible agents rationally reduce the level of their economic activity for reduction of the risk of infection. However, the rational infected agents, recognizing the fact that they have nothing to lose from further social interaction, do not internalize the social cost they impose on others. While the decentralized SIS economy converges to a steady state in which the disease is endemic, the social planner as it internalizes the infection externalities, by inducing infected agents to reduce their economic activity lowers the spread of the disease. Extending the analysis into a SIRS model they find similar kind of result. Considering a planner who wants to control the fatalities of a pandemic and minimize the output costs of the lockdown, Alvarez et al. (2020) analyze the optimal lockdown policy in a SIR model.

However, the above mentioned papers, unlike ours, do not adequately focus interaction between demand dynamics and spread of Covid-19.[9] Closest to our approach is that of Razmi (2020). However, our modeling strategy is different from him. Incorporating macroeconomic considerations in a SI (Susceptible-Infected) model of infectious diseases, Razmi (2020) analyzes the interaction between the economics and epidemiology. His focus is only on the employed workers. In each period a fraction of the new workers who enter into the workforce are infected while the rest become susceptible. Output is demand determined and interaction between the susceptible individuals and the economic activity leads to the occurrence of two different cases. In both these cases a unique stable steady state exists. Once the consumer (and investment) sentiment is allowed to depend on the state of viral spread, the instability may arise in system. Then, by introducing a ceiling on employment, he incorporates the supply-side constraint.

In our model, unlike Razmi (2020), rather than only employed workers, we divide the entire population of the country into the group of susceptible individuals and the infected one. Second, we consider a non-linear dynamics that allows the existence of multiple equilibria and opens the possibility of instability in the system even when the confidence

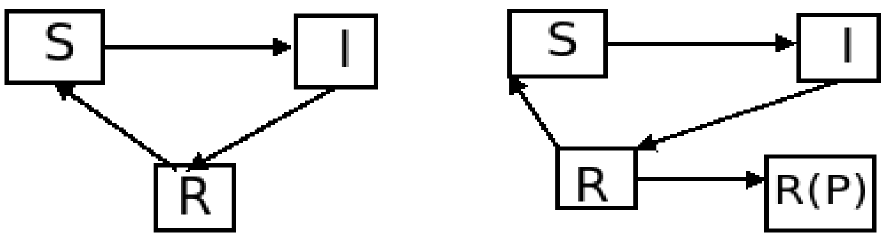
[9] Using a classical growth model with induced technical change, Michl and Tavani (2020) study the long-run consequences of temporary shocks such as Covid-19. However they do not employ the SIR(S) framework.

of the consumers and the animal spirit of the investors are not affected by the state of viral spread (we discuss this in Appendix A.1). Third, unlike Razmi (2020), under certain conditions, an endogenous cycle of infected individuals and the employment level (i.e. the limit cycle) emerges in the system. Finally, we convert the SI(S) model into a SIR(S) model and obtain a richer dynamics.

The rest of the paper is as follows. Section 2 sets up the basic model where we assume that there is no (or negligible number of) recovered people and the economy is purely demand-constrained. However as the number of infected individuals rises, people start to panic. Therefore the consumers' behavoiur and the animal spirits of investors are affected by the state of infection. In Section 3 we extend the model by incorporating a positive number of recovered people (a fraction of whom again returns to the susceptible class). Section 4 discusses emergence of limit cycles through the use of numerical simulations. Section 5 discusses some policy prescriptions and offers some concluding remarks.

## 2 The Model

One of the simplest model of infectious diseases like measles, malaria, chickenpox, mumps, smallpox, rubella, influenza and so on is the SIR model. In this kind of model the entire population ($N$) is divided into three groups which are mutually exclusive to each other. These are ($i$) the susceptible class ($S$) which consists of those individuals who are still not infected, but can catch the disease and become infected; ($ii$) the infected individuals ($I$) who have the disease and can transmit it; and ($iii$) the recovered individuals ($R$) who was infected by the disease and have recovered.




(a) when the whole $R$ immediately enters the pool of $S$

(b) when only a fraction of $R$ enters the pool of $S$ and the rest ($R(P)$) develops permanent immunity

Figure 2.1: SIR(S) model

While in case of most of many diseases such as measles and small pox, once an infected person recovers, she does not contract the infection again. There are, however, other diseases such as malaria and tuberculosis in which a recovered person can get infected again. So far there is no concrete evidence that once a person is infected by Covid19 she will not be infected in future. To begin our analysis, we are going to we assume once a person is recovered immediately enters the pool of susceptible individuals i.e. $\dot{R} = 0$ always holds (see Figure 2.1a). This assumption helps us to operate with a two dimensional system. We assume the total population $N(= S + I)$ is constant.[10] So rather than the SIR(S) model, the model that we present in this section is more of a SI(S) model.

In the standard SI(S) model, the rate of change of the infected number persons is $\dot{I} = \beta SI - \nu I$. This equation says that the number of infected persons at any point in time, increases by $\beta SI$ because of interaction between susceptible and infected individuals and decreases by $\nu I$ because a fraction of infected persons recovers. Since our objective is to understand how the dynamics of the economic activity interacts with the dynamics of spread of the diseases, we modify this equation to

$$\dot{I} = \beta SI + \theta Y - \phi - \nu I \tag{2.1}$$

where $\beta$, $\theta$, $\phi$, and $\nu$ are positive constants. $\beta$ is the infection rate. We are also assuming that the number of infected individuals at any point in time, increases because of interaction between susceptible and infected persons. However, we distinguish between economic and non-economic interactions and their implications for change in the number of infected persons. Economic activity is a major site for spread of infection as infected persons may present themselves as asymptomatic and continue to participate in economic activity where they interact with susceptible persons. To account for this effect of economic activity, we assume $I$ increases proportionately with output $Y$. Since economic activity is not the only trigger for interactions amongst infected and susceptible persons, we also include the term $\beta SI$ in the right-hand side of equation (2.1) but interpret it as increase in $I$ irrespective of economic activity. When it comes to decrease in the number of infected persons at any point in time, unlike the standard model, we interpret the fraction of infected persons who recover, $\nu$, as the effectiveness of health sector in providing timely aid and care to infected persons. The rate of increase in the number of infected persons may also slowdown because of stricter adherence to social distancing norms, diligent contact tracing and large-scale testing. Success of such policies ultimately boils down to community health of the society. We incorporate effect of community health in equation (2.1) through the positive constant $\phi$. We interpret $\phi$ as an index of community health, with a higher value of it indicating a better community health and, equation (2.1) then implies, a lower rate of increase in the number of infected persons.

[10] As a recovered person immediately joins the group of susceptible individuals, we are not separately categorising it.

As $S + I = N$ and $N$ is constant, $\dot{I} = -\dot{S}$ must hold. Hence discussing the dynamics of $I$ is sufficient. Inserting $S = N - I$ into equation (2.1) and after some rearrangement we get

$$\dot{I} = (\beta N - \nu)I - \beta I^2 - \phi + \theta Y. \tag{2.2}$$

Now let us focus on the goods market. There is excess supply of labour and no depreciation of capital in the economy. The production function is of Leontief type i.e.

$$Y = min\{\gamma_L L, \gamma_K K\} = L, \gamma_K = \frac{Y^P}{K} > \frac{Y}{K} \tag{2.3}$$

where $Y$ is real income or level of output, $L$ is total amount of labour employed, $K$ is the existing capital stock, $Y^P$ is the potential output level. So the actual output is below the potential output level. In other words, the economy operates at the low rate of capacity utilization. For simplicity we assume the labour productivity, $\gamma_L$ to unity. Therefore the output ($Y$) at any given point is equal to the labour employed ($L$). Note that $L \leq N$.

The actual level of output is demand-determined and is adjusted in accordance with the demand gap with a time lag. Rules and regulations are changing every now and then. Individuals' consumption habit is also changing. On the other hand reestablishment of the employment networks for the industries is also time consuming. Therefore we can expect the actual output is adjusting with a time lag. It can be written as

$$\dot{Y} = \rho[AD - Y] \tag{2.4}$$

Aggregate demand ($AD$) consists of the consumption demand ($C$), investment demand ($E$) and the government expenditure ($G$). Infection on the one hand creates panic among consumers and makes them reluctant of spending on non-essential commodities. It dampens the animal spirits of the investors on the other hand. However these relationship is not linear. Panic among individuals increases at an increasing rate with respect to the level of infected population $I$. Therefore we assume $C = \bar{c} + c_Y Y - c_I I^2$, $E = \bar{e} + e_Y Y - e_I I^2$, and $G = \bar{G}$. The aggregate demand ($AD$) is $C + E + G = a + bY - fI^2$. Where $a = \bar{c} + \bar{e} + \bar{G}$ is the autonomous part of the aggregate demand, $b = c_Y + e_Y$ is the induced part and $f = c_I + e_I$. Inserting the $AD = a + bY - fI^2$ into equation (2.4) we get

$$\dot{Y} = \rho[a - (1 - b)Y - fI^2] \tag{2.5}$$

We assume the steady state values of $I$ and $Y$ as $I^*$ and $Y^*$ where $I^*$ and $Y^*$ both are positive. To analyze the local stability of the long-run equilibrium, we linearize the system of differential equations (2.2) and (2.5) around the equilibrium and get

$$\begin{pmatrix} \dot{I} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \begin{pmatrix} I - I^* \\ Y - Y^* \end{pmatrix} \tag{2.6}$$

where the elements of the Jacobian matrix $\boldsymbol{J}$ are given by

$$J_{11} = \frac{\partial \dot{I}}{\partial I} = (\beta N - \nu) - 2\beta I \gtrless 0 \tag{2.7}$$

$$J_{12} = \frac{\partial \dot{I}}{\partial Y} = \theta > 0 \tag{2.8}$$

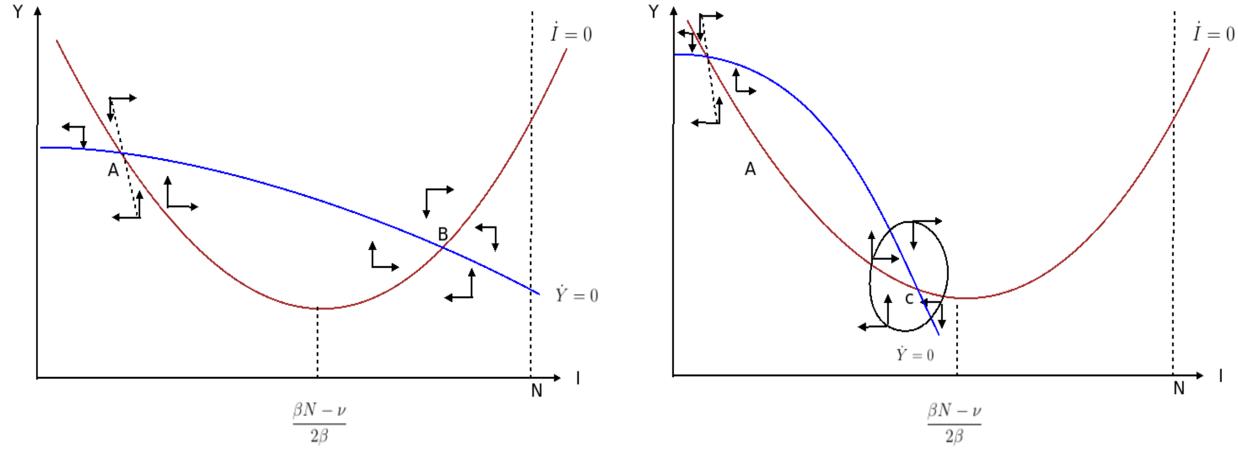
$$J_{21} = \frac{\partial \dot{Y}}{\partial I} = -2fI\rho < 0 \tag{2.9}$$

$$J_{22} = \frac{\partial \dot{Y}}{\partial Y} = -(1-b)\rho < 0 \tag{2.10}$$

All the above elements are evaluated at the short-run equilibrium. We omit this superscript "$*$" for convenience.

$J_{11}$ represents the effect of an increase in the number of infected individuals on a change in the number of infected individuals themselves. As more number of individuals are infected, *ceteris paribus,* the more is the chance that the susceptible individuals are also infected. At the beginning, $I$ is very small and therefore $(\beta N - \nu)$ dominates $2\beta I$. Consequently, equation (2.7) becomes positive. Through time, as the number of infected individuals rise, $2\beta I$ starts dominating $(\beta N - \nu)$ and therefore $J_{11}$ becomes negative. $J_{12}$ shows the effect of a rise in economic activity on the number of infected individuals. As the economic activity rises, more people gets in touch with others and maintaining physical distancing becomes difficult. Consequently, more people are infected. $J_{21}$ represents the effect of a rise in $I$ on the change in $Y$. As more number of people are infected, people panic and therefore the confidence of consumers and the animal spirits of the investors fall. This in turn, *ceteris paribus,* lowers the aggregate demand and hence the output level/ level of income. $J_{22}$ shows the effect of a rise in the level of output on a change in the output level itself. For the Keynesian stability condition to hold, we assume $J_{22} < 0$.

Figure 2.2 shows probable steady states of the model. Slope of the $\dot{I} = 0$ isocline, $\frac{dY}{dI}\Big|_{\dot{I}=0} = -\frac{\frac{\partial \dot{I}}{\partial I}}{\frac{\partial \dot{I}}{\partial Y}} = -\frac{J_{11}}{J_{12}} = -\frac{\{(\beta N-\nu)-2\beta I\}}{\theta} \gtrless 0$ according to whether $I \gtrless \tilde{I} = \frac{\beta N - \nu}{2\beta}$. Note that $\frac{d^2Y}{dI^2}\Big|_{\dot{I}=0} = \frac{2\beta}{\theta} > 0$. On the other hand, slope of the $\dot{Y} = 0$ isocline is $\frac{dY}{dI}\Big|_{\dot{Y}=0} = -\frac{\frac{\partial \dot{Y}}{\partial I}}{\frac{\partial \dot{Y}}{\partial Y}} = -\frac{J_{21}}{J_{22}} = -\frac{2fI}{(1-b)} < 0$. Note that $\frac{d^2Y}{dI^2}\Big|_{\dot{Y}=0} = -\frac{2f}{(1-b)} < 0$. Depending on the value of $f$ two cases may arise. When $f$ is relatively small we get multiple equilibria $A$ and $B$. Figure 2.2a depicts this scenario. However, if $f$ is relatively high, the $\dot{Y} = 0$ isocline bends fast and results equilibria $A$ and $C$. Figure 2.2b shows this. At the steady state $A$, as $I < \tilde{I} = \frac{\beta N - \nu}{2\beta}$,




(a) when $f$ is relatively small (b) when $f$ is relatively high

Figure 2.2: Probable steady states

$J_{11} > 0$. So is true for point $C$. On the other hand, as $I > \tilde{I} = \frac{\beta N - \nu}{2\beta}$ at $B$, $J_{11} < 0$. At point $A$ the slope of the $\dot{Y} = 0$ isocline is flatter than the $\dot{I} = 0$ isocline i.e. $0 > -\frac{J_{21}}{J_{22}} > -\frac{J_{11}}{J_{12}}$ which in turn (as $J_{12} > 0$ and $J_{22} < 0$) implies Det$(J)$ =$(J_{11}J_{22} - J_{12}J_{21}) < 0$. Hence point $A$ is a saddle point unstable. On the other hand, at point $B$ the slope of the $\dot{I} = 0$ isocline is greater than the slope of the $\dot{Y} = 0$ isocline i.e. $-\frac{J_{11}}{J_{12}} > 0 > -\frac{J_{21}}{J_{22}}$ which in turn (as $J_{12} > 0$ and $J_{22} < 0$) implies Det$(J)$ =$(J_{11}J_{22} - J_{12}J_{21}) > 0$. Further, the trace of the Jacobian matrix tr$(J) = J_{11} + J_{22} < 0$. Hence point $B$ is a stable steady state.[11] At point $C$ both the slopes are negative and the slope of the $\dot{I} = 0$ isocline is greater than the slope of the $\dot{Y} = 0$ isocline i.e. $0 > -\frac{J_{11}}{J_{12}} > -\frac{J_{21}}{J_{22}}$ which in turn (as $J_{12} > 0$ and $J_{22} < 0$) implies Det$(J)$ =$(J_{11}J_{22} - J_{12}J_{21}) > 0$. However, $J_{11} > 0$ here and therefore the sign of the trace is ambiguous. Its sign depends on the speed of the goods market adjustment parameter $\rho$. Tr$(J) = J_{11} + J_{22} \gtreqless 0$ depending on whether $\rho \lesseqgtr \hat{\rho} = \frac{(\beta N - \nu) - 2\beta I}{(1-b)}$. Consequently, point $C$ is a stable steady state if $\rho > \hat{\rho}$ and and unstable if $\rho < \hat{\rho}$. However, starting with a high value of $\rho$ if it falls to $\hat{\rho}$ the economy loses its stability and a limit cycle emerges due to Hopf bifurcation. As in our model $Y$ also denotes the employment level in the economy (see equation (2.3)), an endogenous and perpetual cycle of infected individuals and employment level emerges for a fall in $\rho$ to $\hat{\rho}$. The following proposition talks about the Hopf bifurcation.

**Proposition 1.** *For an appropriate value of the speed of adjustment parameter, $\rho$, the characteristic equation to (2.2) & (2.5) evaluated at the steady state $C$ has purely imaginary roots and for the same dynamical system, $\rho = \hat{\rho} = \frac{(\beta N - \nu) - 2\beta I}{(1-b)}$ provides a point of Hopf bifurcation.*

[11] Here as $J_{12} > 0$ and $J_{21} < 0$ so, $tr(J)^2 - 4Det(J) = (J_{11} - J_{22})^2 + 4J_{12}J_{21} \gtreqless 0$ and hence the steady state can be any of either a stable node or a stable spiral.

*Proof.* See in Appendix A.2. □

The effects of parametric changes can be shown by totally differentiating equations (2.2) and (2.5), which imply

$$\begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{bmatrix} dI \\ dY \end{bmatrix} = \begin{bmatrix} I^2 - IN \\ 0 \end{bmatrix} d\beta + \begin{bmatrix} I \\ 0 \end{bmatrix} d\nu + \begin{bmatrix} 1 \\ 0 \end{bmatrix} d\phi + \begin{bmatrix} -Y \\ 0 \end{bmatrix} d\theta + \begin{bmatrix} 0 \\ -\rho \end{bmatrix} da + \begin{bmatrix} 0 \\ I^2 \end{bmatrix} df \tag{2.11}$$

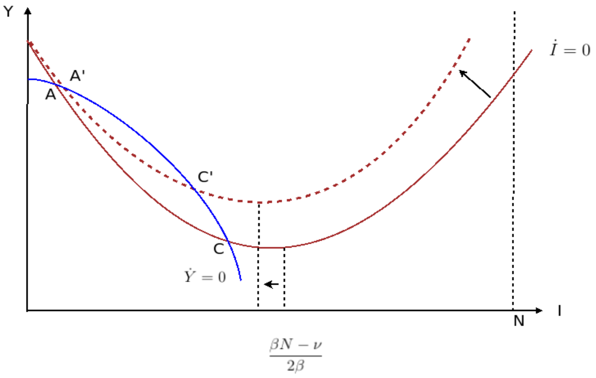
where $\mathbf{\Omega} = \begin{vmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{vmatrix}$.

The twin-damages in the form of demonetization and GST (Goods and Services Tax) have already harmed the economy.[12] The centre, despite repeated demand from the states, so far (since August, 2019) has not released the compensation due for states' revenue loss due to the introduction of GST. The Covid-19 pandemic on the one hand has added to the responsibilities of the state governments and on the other hand has dried up their revenues due to the lockdown.[13] These, along with job losses for millions of people, have contributed in lowering down the confidence level of consumers and the animal spirits of the investors. Consequently the value of $f$ is quite high. Therefore it is more likely that the Indian economy is currently at point $C$.

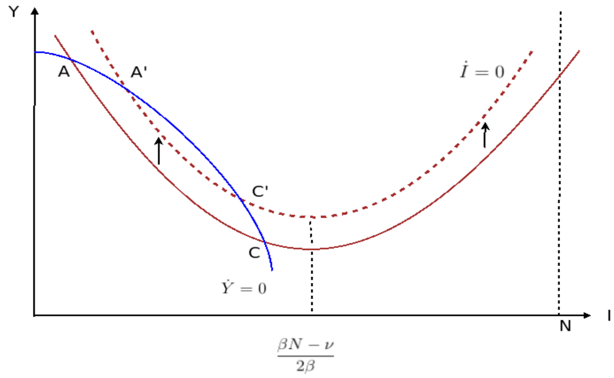
At $C$ as government expenditure increases, the equilibrium value of $I$ increases and $Y$ decreases (Figure 2.3d). This seems to be counter intuitive. The explanation is as follows. As $G$ increases, *ceteris paribus*, the aggregate demand of the economy rises and it pushes the $\dot{Y} = 0$ isocline upwards. For a given output level $Y$, at the old steady state $C$ the number of infected individuals are lower than required for satisfying the $\dot{I} = 0$ isocline. This lower level of $I$ puts upward pressure on the goods market through equation (2.9). As a result, output level starts rising initially. This rise in output level increases $I$ through equation (2.8). This rise in $I$ through equation (2.9) leads to a fall in the output level. What's more, as $I < \tilde{I} = \frac{\beta N - \nu}{2\beta}$ around $C'$, a rise in $I$ increases $I$ itself further (see equation (2.7)). Consequently, there is a further fall in the output level. This fall in output overcompensates the initial rise in $Y$. So, finally there is a rise in the number of infected people and a deterioration in the equilibrium output level.

However, for a sufficiently large rise in $G$, the $\dot{Y} = 0$ isocline is pushed upwards significantly and as a consequence the economy reaches to point $B$ where instead of a fall, the equilibrium level of output rises (see Figure 2.3d). This is because near $B$, $I$ is so high

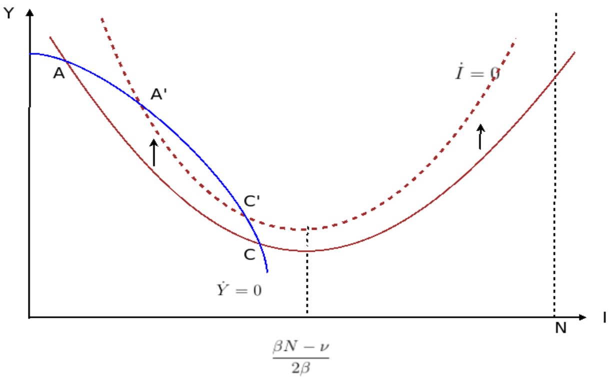
[12] Patnaik (2019).
[13] Patnaik (2020).



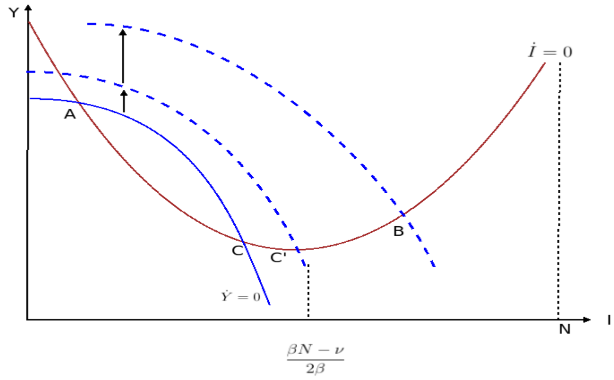

(a) Effect of a rise in $\nu$



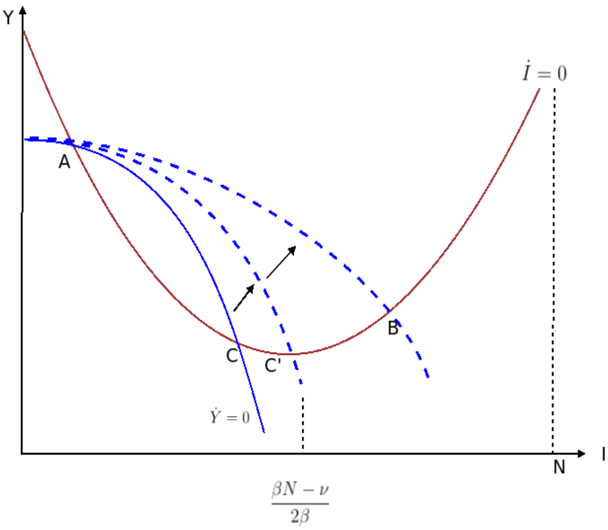

(b) Effect of a rise in $\phi$



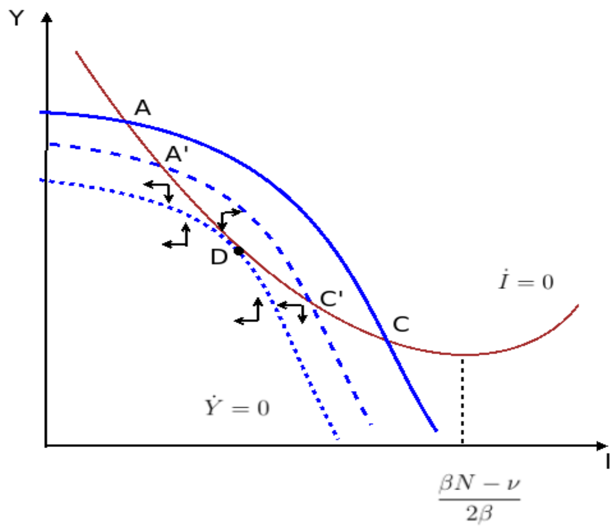

(c) Effect of a fall in $\theta$




(d) Effect of a rise in $a$




(e) Effect of a fall in $f$




(f) Effect of a fall in $a$

Figure 2.3: comparative statics diagram of various parameters

that $2\beta I$ dominates $(\beta N - \nu)$ and makes $J_{11} < 0$ i.e. a rise in $I$ reduces the level of it. Consequently, through equation (2.9) Y rises. Thus, while a small fiscal stimulus package is counterproductive, a sufficiently large fiscal stimulus can help the economy to get out of the severe situation (which is at point $C$).

Let us check what happens if government expenditure falls. As illustrated in Figure 2.3f, as government expenditure ($G$) falls (*ceteris paribus* it means a fall in $a$) the $\dot{Y} = 0$ isocline shifts downward. Consequently we reach to a new steady state $C'$ where $I$ falls and $Y$ rises. However, it does not mean that the government should implement a contractionary fiscal policy. Note that as $a$ falls the two equilibria $A$ and $C$ come closer and the system becomes more fragile in the sense that a shock which could previously be absorbed by the stable equilibrium $C$ is no more possible.[14] In other words as $A$ and $C$ come closer the stable region shrinks.[15] A further fall in $a$ causes the convergence of those two equilibria into an unique equilibrium $D$ which is unstable in nature (see Figure 2.3f ).[16] As a result, the instability arise in the system.

Although the Indian government (currently same is true for most of the countries) is more concerned about the level of its infected people, it does not necessarily mean that there should not be any fiscal expansion to boost the economy. Rather, a fiscal expansion along with all the measures that lowers the value of $\theta$ or $\beta$ and/or highers the value of $\nu$ or $\phi$ may leads to a rise in the steady state value of $Y$ leading $I$ unchanged. A proper policy-mix not only can increase $Y$ but can even decrease the sate of infected individuals.

A rise in $\nu$, for a given level of $I$, raises $Y$ and this rise in $Y$ depends on the level of $I$. On the other hand, because of a rise in $\nu$ the value of $\tilde{I}\left(= \frac{\beta N - \nu}{2\beta}\right)$ falls.[17] However, there is no change in the $\dot{Y} = 0$ isocline. As depicted in Figure 2.3a, for a rise in $\nu$, the system shifts from point $C$ to point $C'$ resulting a fall in the equilibrium level of infected individuals and a rise in output level. The intuition behind this is that as $\nu$ rises, through equation (2.2), $I$ falls. This fall in $I$ through equation (2.5) leads to a rise in $Y$. As $Y$ rises, there must be a rise in $I$. This rise in $I$ mitigates the initial fall in $I$. However, the final result is a fall in $I$ and a rise in $Y$. A simple algebra along with equation (2.11) yields $\frac{dI}{d\nu} = \frac{IJ_{22}}{\boldsymbol{\Omega}} < 0$ and $\frac{dY}{d\nu} = \frac{-IJ_{21}}{\boldsymbol{\Omega}} > 0$. The effectiveness of health sector in providing timely aid and care to infected persons increases in the recovery rate which in turn through equation (2.2) causes the equilibrium level of $I$ to fall. Due to overcrowding the epidemics builds up rapidly i.e. overcrowding increases the value of $\beta$.

[14] We find this kind of explanation in Jarsulic (1990, pp. 95)

[15] For more, see Appendix A.5. See Isaac and Kim (2013, pp. 264-65) for the concept of stable region.

[16] Here as the isocline are tangent, slopes must be equal i.e. $-\frac{J_{21}}{J_{22}} = -\frac{J_{11}}{J_{12}}$ which in turn implies $\text{Det}(J) = 0$.

[17] From equation (2.2) we get $Y\big|_{\dot{I}=0} = \frac{\beta I^2 - (\beta N - \nu)I + \phi}{\theta}$. Here $\frac{d}{d\nu}\left(Y\big|_{\dot{I}=0}\right) = \frac{I}{\theta} > 0$ and $\frac{d}{d\nu}\left(\frac{\beta N - \nu}{2\beta}\right) = \frac{-1}{2\beta} < 0$.

On the other hand a frequent hand wash, proper physical distancing and isolation lead to a fall in $\beta$. Note that the impact of a rise in $\beta$ is exactly the opposite of a rise in $\nu$.[18] Therefore we are not explicitly discussing it further.

A rise in $\phi$, the community health index, for a given level of $I$, raises $Y$.[19] For an increase in $\phi$, as Figure 2.3b depicts, the the vertical intercept of the $\dot{I} = 0$ isocline shifts upward while the slope remains unchanged.[20] Consequently, the economy shifts from $C$ to $C'$ ensuring a rise in the equilibrium output level and a fall in the number of infected individuals. The intuition behind this is that a proper information among people, a stricter adherence to social distancing norms, diligent contact tracing and large-scale testing (in other words, improvement in overall community health), through equation (2.2) lead to a fall in $I$. As $\frac{\partial \dot{Y}}{\partial I} < 0$, this fall in $I$ increases the level of output $Y$.[21]

In our model, $\theta$ capture the sensitivity of a change in $I$ while $Y$ changes. So the comparative static of a change in $\theta$ deserves some discussion. A mass-scale testing and segregating the infected individuals from others and a large scale use of PPE (not only by health workers but also workers of other sectors especially where maintaining physical distances is difficult[22]) can curb the value of $\theta$ and slow down the change in $I$. The short-run equilibrium effect of a change in $\theta$ on $I$ and $Y$ are $\frac{dI}{d\theta} = \frac{-YJ_{22}}{\boldsymbol{\Omega}} > 0$ and $\frac{dY}{d\theta} = \frac{YJ_{21}}{\boldsymbol{\Omega}} < 0$. As $\theta$ rises the $\dot{I} = 0$ isocline shifts downward whereas a fall in $\theta$ leads to an upward shift in the $\dot{I} = 0$ isocline (the fall in $\theta$ is shown Figure 2.3c). This is happening because $\frac{d}{d\theta}\left(\frac{dY}{dI}\bigg|_{\dot{I}=0}\right) = \frac{\{(\beta N - \nu) - 2\beta I\}}{\theta^2} \lesseqgtr 0$ according to whether $I \gtreqless \tilde{I} = \frac{\beta N - \nu}{2\beta}$ and $\frac{d}{d\theta}\left(\frac{d^2Y}{dI^2}\bigg|_{\dot{I}=0}\right) = -\frac{2\beta}{\theta^2} < 0$. Intuitively speaking, when $\theta$ rises, for a given level of $Y$ a higher level of $I$ is required (when $I > \tilde{I}$) to satisfy equation (2.2). On the other hand if $I < \tilde{I}$, for a rise in $\theta$ as smaller level of $I$ is required to satisfy equation (2.2). However, there is no change in the $\dot{Y} = 0$ isocline.

A fall in $\theta$ shifts the economy from $C$ to $C'$ ensuring a rise in the equilibrium output level and a fall in the number of infected individuals. The intuition behind this is that a fall in $\theta$ through equation (2.2) lead to a fall in $I$. As $\frac{\partial \dot{Y}}{\partial I} < 0$, this fall in $I$ increases the level of output $Y$. As $Y$ rises, there must be a rise in $I$. This rise in $I$ mitigates the initial fall in $I$. However, the final result is a fall in $I$ and a rise in $Y$.

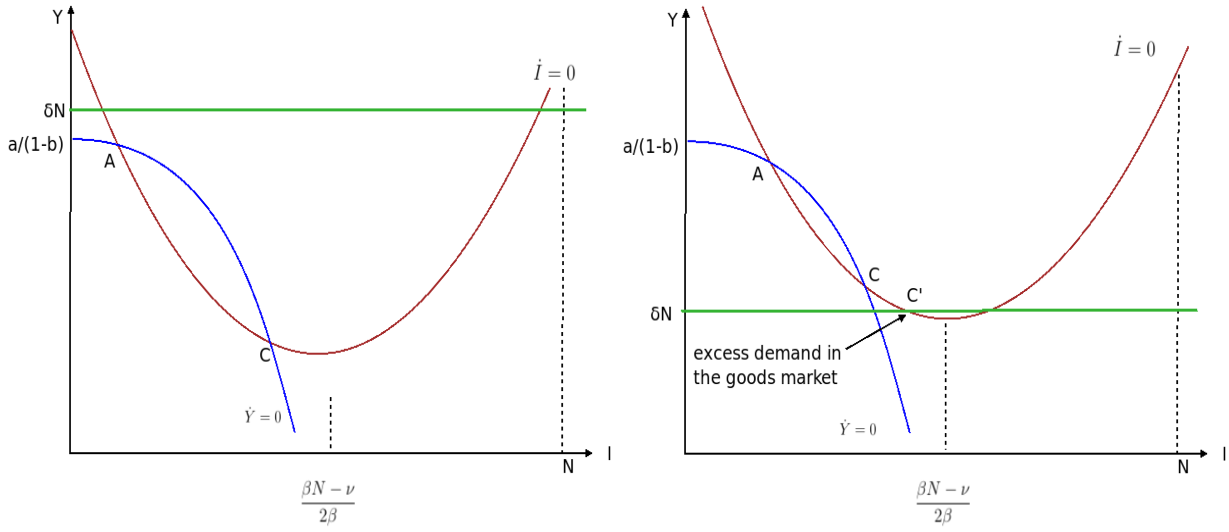
[18] From equation (2.11) we get $\frac{dI}{d\beta} = \frac{-I(N-I)J_{22}}{\boldsymbol{\Omega}} > 0$ and $\frac{dY}{d\beta} = \frac{I(N-I)J_{21}}{\boldsymbol{\Omega}} < 0$. Note that $\frac{d}{d\beta}\left(Y\big|_{\dot{I}=0}\right) = \frac{I^2 - NI}{\theta} < 0$ and $\frac{d}{d\beta}\left(\frac{\beta N - \nu}{2\beta}\right) = \frac{\nu}{2\beta^2} > 0$.

[19] From equation (2.2) we get $Y\big|_{\dot{I}=0} = \frac{\beta I^2 - (\beta N - \nu)I + \phi}{\theta}$. Here $\frac{d}{d\phi}\left(Y\big|_{\dot{I}=0}\right) = \frac{I}{\theta} > 0$

[20] Vertical intercept and the slope of the $\dot{I} = 0$ isocline are $Y\big|_{\dot{I}=0}^{I=0} = \frac{\phi}{\theta}$ and $\frac{dY}{dI}\bigg|_{\dot{I}=0} = \frac{2\beta I - (\beta N - \nu)}{\theta}$ respectively. Consequently, $\frac{d}{d\phi}\left(Y\big|_{\dot{I}=0}^{I=0}\right) = \frac{1}{\theta} > 0$ and $\frac{d}{d\phi}\left(\frac{dY}{dI}\bigg|_{\dot{I}=0}\right) = 0$.

[21] Here $\frac{dI}{d\theta} = \frac{-YJ_{22}}{\boldsymbol{\Omega}} > 0$ and $\frac{dY}{d\phi} = \frac{-J_{21}}{\boldsymbol{\Omega}} > 0$.

[22] Azad and Saratchand (2020)




(a) when $L < \delta N$ (b) when $L > \delta N$

Figure 2.4: When there is an employment ceiling

Recently based on the incidence of the infected individuals, doubling rate, the extent of testing and surveillance feedback, the Indian government has divided the districts into three zones: red, orange and green. They have allowed private and public sectors to be opened in less infected zones with maximum 33% workers. In other words there is a ceiling of maximum workers that can be employed. In our model let us assume that maximum a $\delta$ fraction of the population is allowed to work. In this simple 2-dimesional model that will be used as a ceiling. As long as $L \leq \delta N$ there is no change in our results (see Figure 2.4a). However, as Figure 2.4b depicts, if $L > \delta N$, there will always be an excess demand and as a result the goods market always will be in out of equilibrium.

These comparative static results are encapsulated in Table 2.1.

Table 2.1: Impact of changes in various parameters on $I^*$ and $Y^*$

| | At $C$ | | At $B$ | |
|---|---|---|---|---|
| | $I^*$ | $Y^*$ | $I^*$ | $Y^*$ |
| $\beta$ | + | − | + | − |
| $\nu$ | − | + | − | + |
| $\phi$ | − | + | − | + |
| $\theta$ | + | − | + | − |
| $a$ | + | − | + | + |
| $f$ | − | + | − | − |

# 3 SIR(S) model: when only a fraction of $R$ enters the pool of $S$

In the last section we assumed that infected persons upon recovery, immediately become susceptible to further infections. However, this may not be the only scenario as some of the recovered individuals may develop antibody. To account for this possibility, we now assume only a fraction of the recovered individuals join the pool of susceptible individuals (see Figure 2.1b). The main implication in terms of our formal analysis is that the dynamical system becomes a three-dimensional system as now, along with dynamics of $I$ and $Y$, we need to consider the dynamics of either $S$ or $R$. The reason why we only need the dynamics of either one of $S$ or $R$ is because $S + I + R = N$ is a constant. In this section we work with the following dynamical system.

$$\dot{I} = \beta SI - \nu I - \phi + \theta Y \tag{3.1}$$

$$\dot{Y} = \rho[a - (1-b)Y - fI^2] \tag{3.2}$$

$$\dot{S} = -\beta SI + \mu(N - S - I) \tag{3.3}$$

In the steady state $\dot{S} = \dot{I} = \dot{Y} = 0$. From equations (3.1), (3.2), and (3.3) we get simultaneous equations with respect to $I^*$, $Y^*$ and $S^*$ as

$$\beta S^* I^* - \nu I^* - \phi + \theta Y^* = 0 \tag{3.4}$$

$$a - (1-b)Y^* - fI^{*2} = 0 \tag{3.5}$$

$$-\beta S^* I^* + \mu(N - S^* - I^*) = 0 \tag{3.6}$$

We assume that there exists a unique set of positive values of $I^*$, $Y^*$, and $S^*$ that simultaneously satisfies equations (3.4), (3.5) and (3.6). The equilibrium level of infected individuals, susceptible individuals and the equilibrium level of output depend on $\beta$, $\nu$, $\mu$, $\theta$, $a$, $f$, $b$ and $N$. However, the steady state does not depend on the speed of adjustment parameters $\rho$.

To analyze the local stability of the equilibrium, we linearize the system of differential equations (3.1), (3.2), and (3.3) around the equilibrium and get

$$\begin{pmatrix} \dot{I} \\ \dot{Y} \\ \dot{S} \end{pmatrix} = \begin{pmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{pmatrix} \begin{pmatrix} I - I^* \\ Y - Y^* \\ S - S^* \end{pmatrix} \tag{3.7}$$

where the elements of the Jacobian matrix $\boldsymbol{J}$ are given by

$$J_{11} = \frac{\partial \dot{I}}{\partial I} = (\beta S - \nu) \gtrless 0 \tag{3.8}$$

$$J_{12} = \frac{\partial \dot{I}}{\partial Y} = \theta > 0 \tag{3.9}$$

$$J_{13} = \frac{\partial \dot{I}}{\partial S} = \beta I > 0 \tag{3.10}$$

$$J_{21} = \frac{\partial \dot{Y}}{\partial I} = -2\rho f I < 0 \tag{3.11}$$

$$J_{22} = \frac{\partial \dot{Y}}{\partial Y} = -(1-b)\rho < 0 \tag{3.12}$$

$$J_{23} = \frac{\partial \dot{Y}}{\partial S} = 0 \tag{3.13}$$

$$J_{31} = \frac{\partial \dot{S}}{\partial I} = -(\beta S + \mu) < 0 \tag{3.14}$$

$$J_{32} = \frac{\partial \dot{S}}{\partial Y} = 0 \tag{3.15}$$

$$J_{33} = \frac{\partial \dot{S}}{\partial S} = -(\beta I + \mu) < 0 \tag{3.16}$$

Equations (3.8)-(3.16) are all evaluated at the steady state. We once again do away with the superscript "$*$" for convenience. Most elements of the Jacobian matrix have the same interpretation as in Section 2 except a few. $J_{11}$ represents the effect of an increase in the number of infected individuals on a change in the number of infected individuals themselves. As more individuals get infected, *ceteris paribus,* the more is the chance that the susceptible individuals also gets infected. If there are a large number of susceptible individuals then $\beta S$ can outweigh the recovery rate, $\nu$ and consequently equation (3.8) becomes positive. Otherwise, the opposite happens. $J_{13}$ represents the effect of a rise in the pool of susceptible individuals on the pool of infected people. As $S$ rises, because of the interaction between it and $I$, the number of infected individuals rises. A rise in $I$ affects the change in the pool of susceptible individuals negatively (it is captured by $J_{31}$). As $I$ rises, because of the interaction between $S$ and $I$, the number of infected individuals rises. *Ceteris paribus,* this leads to a fall in $S$ by $\beta S$ units. Further, as $I$ rises, $R$ falls by $\mu$ units for any given value of $N$ and $S$. Together they imply that increases in $I$ negatively affects $\dot{S}$ by $(\beta S + \mu)$ units. Similarly, $J_{33}$ (the effect of an increase in the number of susceptible individuals on a change in the number of susceptible individuals themselves) is also negative. The characteristic equation of the Jacobian matrix (3.7) is given by

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0 \tag{3.17}$$

where $\lambda$ denotes a characteristic root. Coefficients of equation (3.17) are:

$$a_1 = -\text{tr}\boldsymbol{J} = -(J_{11} + J_{22} + J_{33}), \tag{3.18}$$

$$a_2 = \begin{vmatrix} J_{22} & 0 \\ 0 & J_{33} \end{vmatrix} + \begin{vmatrix} J_{11} & J_{13} \\ J_{31} & J_{33} \end{vmatrix} + \begin{vmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{vmatrix} = J_{22}J_{33} + J_{11}J_{33} - J_{13}J_{31} + J_{11}J_{22} - J_{12}J_{21}, \tag{3.19}$$

$$a_3 = -\text{Det}\boldsymbol{J} = -J_{11}J_{22}J_{33} + J_{12}J_{21}J_{33} + J_{13}J_{22}J_{31} \tag{3.20}$$

where $-a_1 = \text{tr}\boldsymbol{J}$ denotes the trace of $\boldsymbol{J}$; $a_2$ , the sum of the principal minors' determinants; and $-a_3 = \text{Det}\boldsymbol{J}$, the determinant of $\boldsymbol{J}$.

The necessary and sufficient condition for the local stability is that all characteristic roots of the Jacobian matrix must have negative real parts, which, from Routh–Hurwitz condition, is equivalent to $a_1 > 0$, $a_2 > 0$, $a_3 > 0$, and $a_1a_2 - a_3 > 0$. Let us investigate whether these inequalities hold. It helps to express$a_1$, $a_2$, $a_3$ and $a_1a_2 - a_3$ as functions of $\rho$ in the following manner.

$$a_1 \equiv a_1(\rho) = \underbrace{\{\nu + \mu + \beta(I - S)\}}_{\equiv A \gtrless 0} + \underbrace{(1 - b)}_{+}\rho = A + (1 - b)\rho. \tag{3.21}$$

$$a_2 \equiv a_2(\rho) = \underbrace{\left\{ \underbrace{\{\nu + \mu + \beta(I - S)\}}_{\equiv A \gtrless 0}\underbrace{(1 - b)}_{+} + \underbrace{2\theta f I}_{+} \right\}}_{\equiv C \gtrless 0}\rho + \underbrace{\{v(\beta I + \mu) + \beta\mu(I - S)\}}_{\equiv D \gtrless 0} = C\rho + D$$

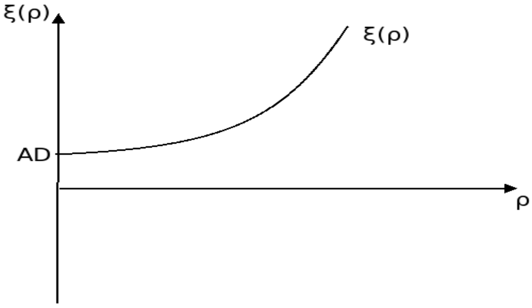
$$a_3 \equiv a_3(\rho) = \underbrace{\left[ \underbrace{(1 - b)}_{+}\underbrace{\{v(\beta I + \mu) + \beta\mu(I - S)\}}_{\equiv D \gtrless 0} + \underbrace{2\theta f I(\beta I + \mu)}_{+} \right]}_{\equiv F \gtrless 0}\rho = F\rho. \tag{3.22}$$

$$a_1a_2 - a_3 \equiv \xi(\rho) = \underbrace{(1 - b)C}_{+/-}\rho^2 + \underbrace{\{AC + (1 - b)D - F\}}_{+/-}\rho + \underbrace{AD}_{+/-} \tag{3.23}$$

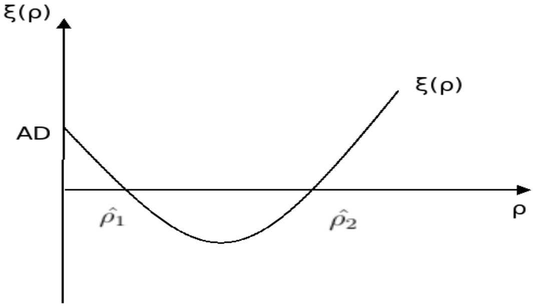
Along with the positive values of $a_1$, $a_2$, $a_3$, for achieving stability $\xi(\rho) = a_1a_2 - a_3$ also has to be positive. Figure 3.1 & 3.2 represents whether/when the sign of $\xi(\rho)$ is positive. For example, Figure 3.1a shows that irrespective of the size of $\rho$ (speed of adjustment parameter of the goods market), the economy is locally stable. On the other hand, Figure 3.1b shows that when $\rho < \hat{\rho}_1$ or $\rho > \hat{\rho}_2$ the economy is stable.

We obtain the following proposition related to stability.

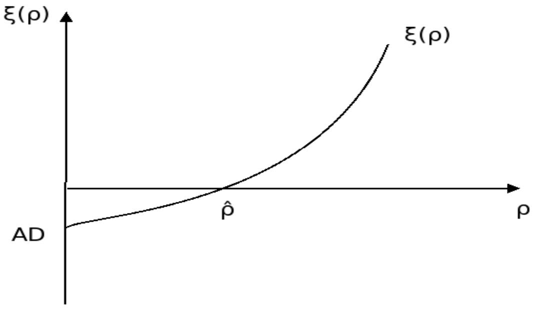
**Proposition 2.** *Suppose $A$, $D$, and $\{AC + (1 - b)D - F\}$ are positive. Then irrespective of the size of $\rho$, the steady state is locally stable.*



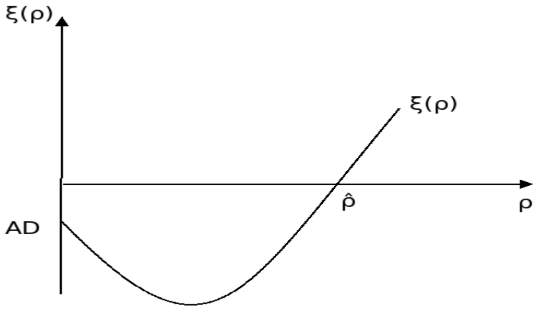

(a) $C > 0$, $AD > 0$ & $\{AC+(1-b)D-F\} > 0$




(b) $C > 0$, $AD > 0$ & $\{AC+(1-b)D-F\} < 0$

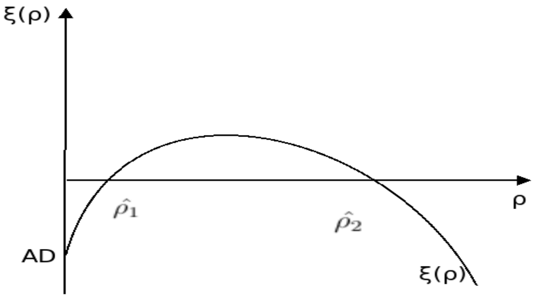



(c) $C > 0$,$AD < 0$ & $\{AC+(1-b)D-F\} > 0$

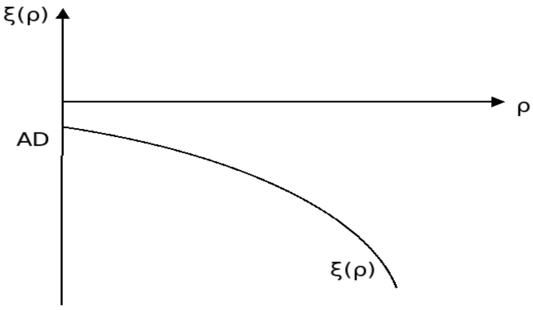



(d) $C > 0$, $AD < 0$ & $\{AC+(1-b)D-F\} < 0$

Figure 3.1: Diagram of $a_1a_2{-}a_3 \equiv \xi(\rho)$




(a) $C < 0$, $AD < 0$ & $\{AC+(1-b)D-F\} > 0$




(b) $C < 0$, $AD < 0$ & $\{AC+(1-b)D-F\} < 0$

Figure 3.2: Diagram of $a_1a_2{-}a_3 \equiv \xi(\rho)$

*Proof.* $A > 0$ implies $C > 0$ whereas a positive value of $D$ ensures $F > 0$. $A > 0$ ensures $a_1$ to be positive. On the other hand, as $C$ and $D$ both are positive $a_2$ must be positive. $F > 0$ ensures $a_3 > 0$. $AD$ and $(1-b)C$ are also positive. However the sign of $\{AC + (1-b)D - F\}$ is ambiguous. As here $\{AC + (1-b)D - F\} > 0$, irrespective of the value of $\rho$, $\xi(\rho)$ is positive (see Figure 3.1a). Hence the steady state is stable. Note that as $\frac{\partial a_1}{\partial \nu} = 1 > 0$, a sufficiently large fall in $\nu$ can make $a_1$ negative and consequently the system becomes unstable. On the other hand $\frac{\partial a_1}{\partial \beta} = (I - S)$ and as long as $I < S$, a sufficiently large rise in $\beta$ can make $a_1$ negative. Therefore a low infection rate $(\beta)$ is conducive for the achievement of the condition $a_1 > 0$ for local stability. □

Proposition 2 is be obtained under two circumstances. The first one is when the number of Infected individuals ($I$) is higher than the number of susceptible individuals ($S$). In that case $(I - S)$ is positive and ensures $A$ as well as $D$ to be positive. The second possible scenario is when $I < S$ (which is so far the case and let's hope this to hold in future as well) but the infection rate ($\beta$) is weak, and the recover rate ($\nu$) is very high. A sufficiently weak $\beta$ and a high value of $\nu$ may make $A = \{\nu + \mu + \beta(I - S)\} > 0$. On the other hand, a low value of $\beta$ and a high value of $\nu$ leads $\upsilon(\beta I + \mu)$ to dominate $\beta\mu(I - S)$. Consequently $D$ becomes positive. Proper information and awareness among people through conventional and social media, a frequent hand wash, proper social distancing and isolation, availability of PPE kits among health workers as well as those workers for whom maintaining social distance is not possible etc. have bearing on the community health parameter $\phi$ but can also reflect in a lower infection rate $\beta$.

Similar to the model in Section 2, the speed of adjustment in the goods market ($\rho$) has important implications for stability here as well. This parameter in certain circumstances can also lead to the emergence limit cycles in the system. These possibilities are described in Propositions 3 below.

**Proposition 3.** ($i$) *Suppose $A$ and $D$ are positive and $\{AC + (1 - b)D - F\} < 0$. Then limit cycles occur when the speed of adjustment of the goods market equals to some critical values.*

($ii$) *Suppose $A > 0$, $D < 0$, $F > 0$ and $\rho > \frac{-D}{C}$. Then a limit cycle occurs when the speed of adjustment of the goods market equals to some critical value.*

($iii$) *Suppose $A < 0$, $C > 0$, $D < 0$, $F > 0$ , $\rho > \frac{-A}{(1-b)}$ and $\rho > \frac{-D}{C}$. Then limit cycles occurs when the speed of adjustment of the goods market equals to some critical value.*

($iv$) *Suppose $A < 0$, $C > 0$, $D > 0$, $F > 0$ , and $\rho > \frac{-A}{(1-b)}$. Then limit cycles occurs when the speed of adjustment of the goods market equals to some critical value.*

($v$) *Suppose $A < 0$, $C < 0$, $D > 0$, $F > 0$ , $\rho > \frac{-A}{(1-b)}$ and $\rho < \frac{-D}{C}$. Then $\{AC + (1 - b)D - F\} < 0$, irrespective of the size of $\rho$, ensures the economy to be locally unstable whereas for $\{AC + (1 - b)D - F\} > 0$, limit cycles occur when the speed of adjustment of the goods market equals to some critical values.*

*Proof.* See Appendix A.3. □

Proposition 3($i$) says that when the speed of adjustment parameter of the goods market is neither two large nor too small then, the limit cycles may emerge (see Figure 3.1b). Intuitively, when $\rho$ is very small (i.e. sufficiently close to zero), the goods market is not adjusted and the analysis of the dynamic system therefore boils down to the analysis of the subsystem that consists of $I$ and $S$. Suppose initially $Y$ deviates from its equilibrium position and is now higher than its equilibrium value. A rise in $Y$ through equation (3.9) raises $I$. This rise in $I$ in turn through equation (3.14) leads to a fall in $S$. A fall in this $S$ through equation (3.10) causes a fall in $I$. This fall in $I$, through equation (3.14), increases $S$ which in turn through equation (3.10) increases $I$. This cycle goes on until the initial steady state is reached. In other words, the dynamics of the subsystem of $I$and S are stable. Second, when $\rho$ is sufficiently large, the goods market is adjusted instantaneously and therefore the effect of the goods market on the dynamical system does not last. In this case, too, the analysis of the dynamic system amounts to the analysis of the subsystem that consists of $I$ and $S$. Finally, when $\rho$ takes an intermediate value, the goods market adjusts with a lag. The goods market therefore has a lasting effect on the dynamical system and, accordingly, cyclical fluctuations occur.

Proposition 3($ii$) shows that a weaker speed of adjustment parameter of the goods market causes destabilization in the system. Suppose that the system is originally at a steady state and there is a sudden rise in the number of infected individuals $I$. In this situation, $I$ receives a positive feedback from equation (3.8)[23], and therefore a rise in $I$ increases it further. This is a direct destabilizing effect. A rise in $I$ through equation (3.11) decreases $Y$ which in turn through equation (3.9) leads to a fall in $I$. This is an indirect but stabilizing effect. For a a sufficiently small or weak $\rho$, the indirect stabilizing effect is weak and is dominated by the direct destabilizing effect. Hence the system becomes unstable. On the other hand when $\rho$ is sufficiently large or strong, the indirect effect dominates the direct effect and stabilizes the system. When $\rho$ is equal to a critical value $\hat{\rho}$ (see Figure 3.1c and/or 3.1d) in the intermediate range then a limit cycle ensues. A similar kind of destabilization mechanism is obtained for Proposition 3($iv$) as well.[24] On the other hand, the intuition behind emergence of limit cycles in Proposition 3($iii$) is the same as in Proposition 3($i$). Note that except for the case related to Proposition 3($v$), in all of the above cases (i.e. cases related to Propositions 3$(i) - (iv)$) a higher speed of adjustment parameter of the goods market stabilizes the system. Government cooperation for facilitating the reestablishment of the employment networks for the industries as well as reestablishment of the forward and backward linkages can boost the speed of adjustment parameter of the goods market and hence under certain circumstances can ensure stability in the system. Appendix A.6 shows all the possible combinations related to the signs of $A$, $C$, $D$, and $F$ and the possible results.

Table 3.1 summarizes the main results of the comparative statics analysis assuming that the equilibrium is locally stable. Here we restrict ourselves to an intuitive discussion of some of these results. We provide all details of these calculations in Appendix A.4.

[23] Here $D < 0$ which in turn ensures $(\beta S - \nu) > 0$.
[24] Note that here $A < 0$ which in turn ensures $(\beta S - \nu) > 0$.

Table 3.1: Impact of changes in various parameters on $I^*$ and $Y^*$

| | $I^*$ | $Y^*$ | $S^*$ |
|---|---|---|---|
| $\beta$ | $+$ | $-$ | $-$ |
| $\nu$ | $-$ | $+$ | $+$ |
| $\phi$ | $-$ | $+$ | $+$ |
| $\theta$ | $+$ | $-$ | $-$ |
| $a$ | $+$ | $+/-$ | $-$ |

A rise in $\phi$, the community health index, raises the equilibrium level of output (and hence employment level) and the number of susceptible individuals $S$, whereas $I$ falls. We get a similar kind of result for a rise in $\nu$. Thus an effective health sector in providing timely aid and care to infected persons, a stricter adherence to social distancing norms, diligent contact tracing and large-scale testing can decrease the level of infected individuals and increase the output level. Note that we obtained a qualitatively similar results in Section 2 as well. A rise in government expenditure increases the equilibrium level of $I$ and decreases the equilibrium level of $S$. However, its effect on the equilibrium level of output is ambiguous and depends on whether $D = [\nu(\beta I + \mu) + \beta\mu(I - S)] > 0$ or not. A very high recovery rate $\nu$, (as long as $I < S$) a small infection rate $\beta$, and/or a lower level of susceptible individual $S$ are conducive for $D$ to be positive. Consequently, an expansionary fiscal policy leads to a rise in the equilibrium output level. On the other hand, for a smaller value of $\nu$, a higher value of $\beta$ and/or a higher level of $S$ exactly the opposite happens.

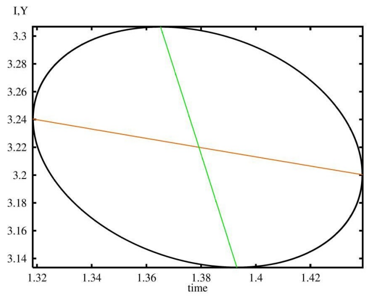
# 4 Numerical simulations

In this section, using some numerical examples we show that limit cycles can occur. Numerical values of parameters along with values of initial conditions and steady state values are provided in Table 4.1. Based on SIRD model, Chatterjee et. al (2020) study the COVID-19 infection in India and find the rate of infection, $\beta = 0.2945$ per day, rate of recovery, $\nu = 0.073$ per day and the death rate equals to 0.0028 per day. We set $\nu = 0.075$ which we think is a decent approximation for India. Bose and Bhanumurthy (2015) calculated the value of the capital expenditure multiplier as 2.45. We set $b = 0.6$ which ensures the autonomous multiplier to be 2.5. For each case, we draw the solution path from $t = 0$ to $t = 200$.

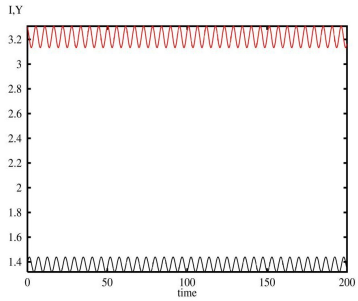
Figure 4.1a shows cyclical patterns in the $(I, Y)$ plane for Proposition 1. Figure 4.1b shows the transitional dynamics of the infected individuals (black colour) and the economic output (red colour). Considering $\rho = 0.8 > \hat{\rho}$ we get the transitional dynamics for the

Table 4.1: List of parameters, initial values, and equilibrium values

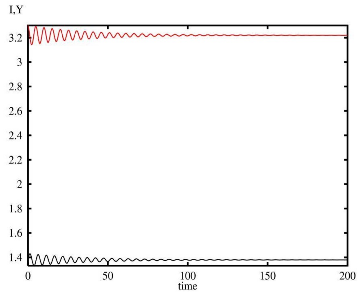
| | Parameters | | | | | | | | | | Initial Values | | | Equilibrium Values | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $\beta$ | $N$ | $\nu$ | $\phi$ | $\theta$ | $a$ | $b$ | $f$ | $\mu$ | $\hat{\rho}$ | $I(0)$ | $Y(0)$ | $S(0)$ | $I^*$ | $Y^*$ | $S^*$ |
| Pr. 1 | 0.03 | 14 | 0.0712 | 3 | 0.8 | 3 | 0.6 | 0.9 | - | 0.66507 | 1.38 | 3.3 | - | 1.3792 | 3.22 | - |
| Pr. 3($ii$) | 0.03 | 14 | 0.075 | 3 | 0.7 | 3 | 0.6 | 0.9 | 0.5 | 0.6828 | 1.4 | 4 | 13 | 1.2871 | 3.7726 | 11.802 |
| Pr. 3($i$) | 0.029 | 14 | 0.2 | 3 | 0.8 | 3 | 0.6 | 0.9 | 0.1 | 0.1589 | 1.2 | 3.6 | 9.6 | 1.3097 | 3.6408 | 9.1972 |




(a) Solution paths in $(I, Y)$ plane: an stable limit cycle occurs




(b) Transitional dynamics of proposition 1 (when $\rho = \hat{\rho}$ =0.66507)



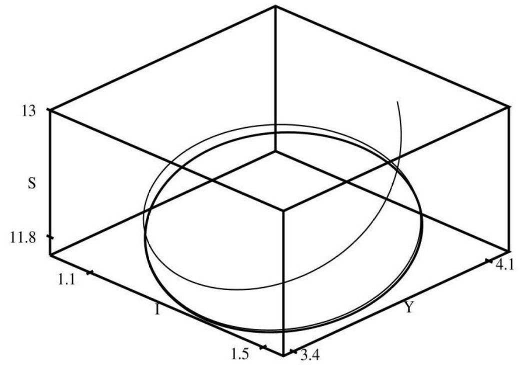

(c) Transitional dynamics of Proposition 1 (when $\rho = 0.8 > \hat{\rho}$)

Figure 4.1: Diagrams of numerical analysis of Proposition 1

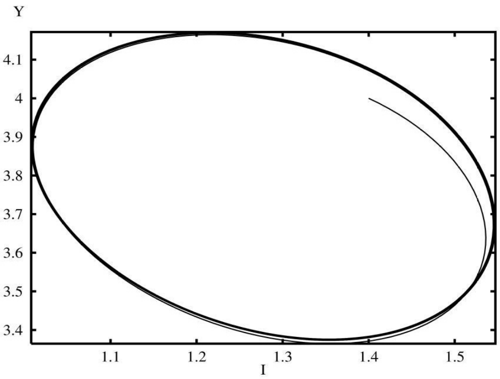
stable steady state in Figure 4.1c. Note that the solution path is not a perfect closed orbit in the sense that for an initial condition close to the long-run equilibrium the solution path converges to the equilibrium whereas for the initial condition further away from the long-run equilibrium, the solution path diverges from the equilibrium. As a result, we confirm that in this numerical example, the sub-critical Hopf bifurcation occurs and the periodic solution is unstable.

For Proposition 3($ii$) we get $A = 0.259553 > 0$, $C = 1.7255672 > 0$, $D = -0.117327525 < 0$, $F = 0.756165953 > 0$, and $[AC + (1-b)D - F] = -0.35522082 < 0$ and $0.6828 = \hat{\rho} > \frac{-D}{C} = 0.067993599$. So all the conditions required for Proposition 3($ii$) are satisfied. Figure 4.2a displays the the Hopf bifurcation in a three-dimensional space. Figures 4.2b, 4.2c, and 4.2d show cyclical patterns in the $(I, Y)$, $(I, S)$, and $(Y, S)$-planes. In the $(I, Y)$, and $(Y, S)$ planes clockwise cycles emerge. However, the pattern is not very clear in $(I, S)$ plane. Finally, Figure 4.2e shows the transitional dynamics of $I$, $Y$, and $S$. Black, red, and orange colour represent $I$, $Y$ and $S$ dynamics respectively. Considering $\rho > \hat{\rho}$ we get the transitional dynamics for the stable steady state in Figure 4.2f.

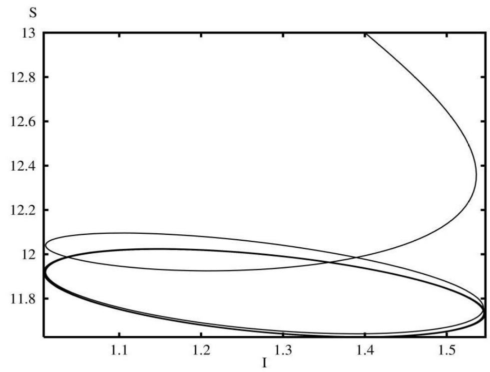
For Proposition 3($i$) we get $A = 0.712625 > 0$, $C = 1.914473 > 0$, $D = 0.00472251 > 0$, $F = 0.200582076 > 0$, and $[AC + (1-b)D - F] = -0.06226296 < 0$. So all the conditions required for Proposition 3($i$) are satisfied. Figure 4.3a displays the the Hopf bifurcation in a three-dimensional space. Figures 4.3b, 4.3c, and 4.3d show cyclical patterns in the $(I, Y)$, $(I, S)$, and $(Y, S)$-planes. In the all three planes i.e. $(I, Y)$, $(Y, S)$ and $(I, S)$-planes clockwise cycles emerge. Finally, Figure 4.3e shows the transitional dynamics of



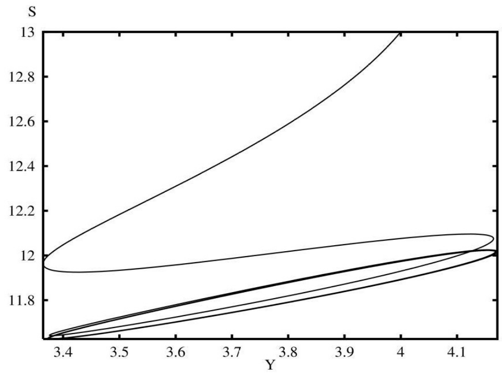

(a) Solution paths in $(I, Y, S)$ plane



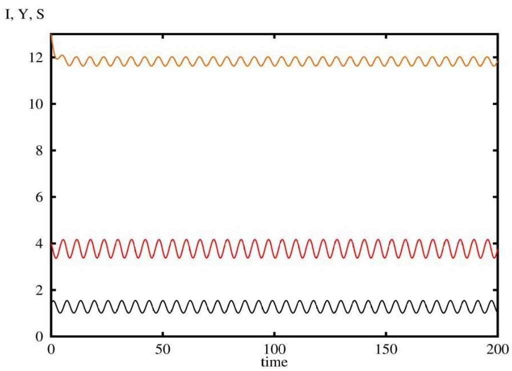

(b) Solution paths in $(I, Y)$ plane



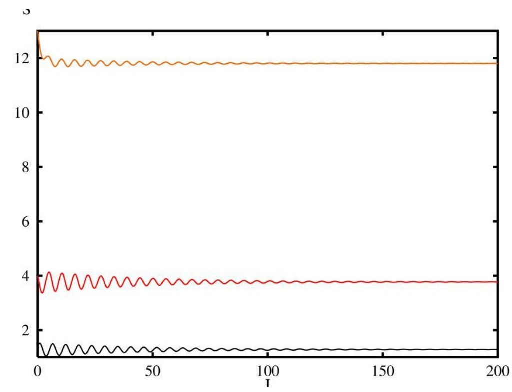

(c) Solution paths in $(I, S)$ plane




(d) Solution paths in $(Y, S)$ plane

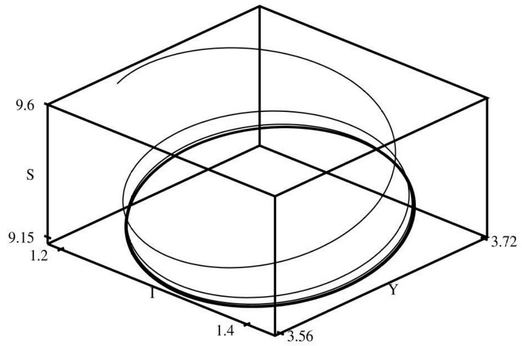



(e) Transitional dynamics of $II$ part of Proposition 3$(ii)$ when $\rho = \hat{\rho} = 0.6828$

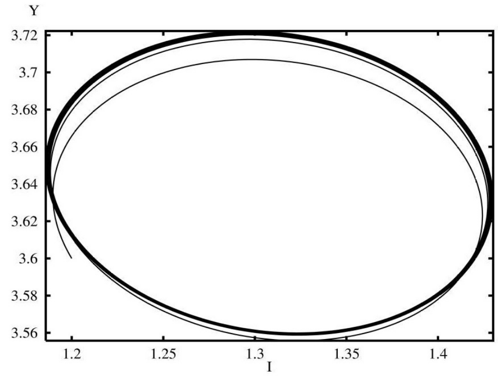



(f) Transitional dynamics of part of Proposition 3$(ii)$ (when $\rho > \hat{\rho}$).

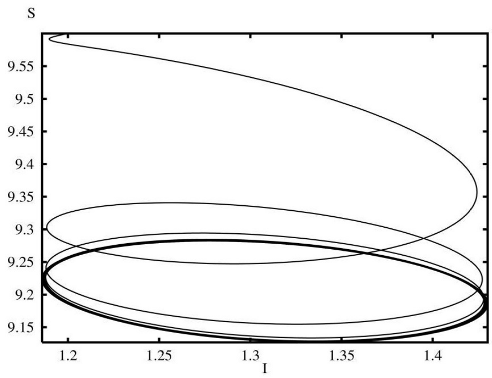
Figure 4.2: Diagrams of numerical analysis of Proposition 3$(ii)$



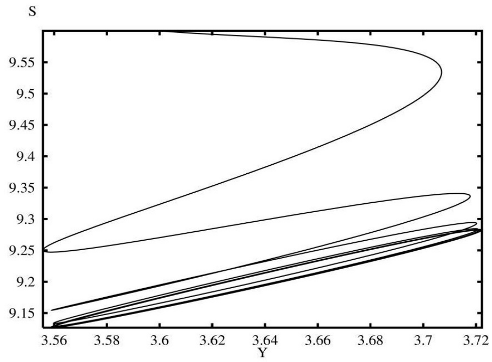

(a) Solution paths in $(I, Y, S)$ plane



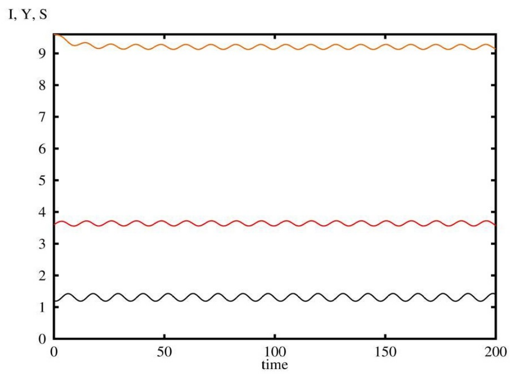

(b) Solution paths in $(I, Y)$ plane



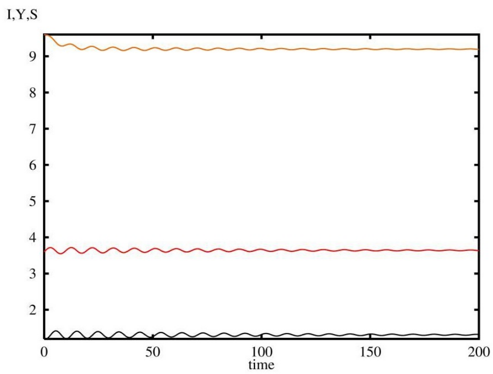

(c) Solution paths in $(I, S)$ plane




(d) Solution paths in $(Y, S)$ plane




(e) Transitional dynamics of Proposition 3($i$) when $\rho = \hat{\rho}_1 = 0.1589$




(f) Transitional dynamics of Proposition 3($i$) (when $\rho = 0.22 > \hat{\rho}$).

Figure 4.3: Diagrams of numerical analysis of Proposition 3($i$)

$I$, $Y$, and $S$. Considering $\rho = 0.22 > \hat{\rho}$ we get the transitional dynamics for the stable steady state in Figure 4.3f.

# 5 Conclusion

Considering the economic activity and its interaction with infectious disease we extended the SIR(S) model of infectious diseases and tried to explain the current severe situation and impending health and economic crisis in the context of India. In Section 2, we assumed that a recovered person immediately loses immunity and enters the pool of susceptible individuals. We found the occurrence of multiple equilibria- one of which is unstable. The pervasiveness of the disease creates panic among consumers and force them to cut down the spending on non-essential commodities. It dampens the animal spirits of the investors. Therefore the economic activity is affected by the number of infected individuals. Under certain conditions, we find a counterintuitive result that a contractionary fiscal policy may help to reduce the number of infected individuals and may improve the economy (and hence improve the labour employment). However, it makes the system more fragile. A further contractionary fiscal policy may destabilize the entire system altogether. Rather, a massive fiscal expansion along with some probable measures such as improvement in effectiveness of health sector in providing timely aid and care to infected persons, public health vigilance and control, stricter adherence to social distancing norms, diligent contact tracing and large-scale testing and segregating the infected individuals from others, a large scale use of PPE etc. can reduce the number of infected individuals and improve the economic condition. On the other hand, under certain conditions, if the speed of adjustment parameter of the goods market falls to a certain level, an endogenous cycle of infected individuals and the employment level emerges in the economy. A richer dynamics is obtained in Section 3 where we assume that instead of all the recovered individuals, only a fraction of them enters the pool of susceptible individuals. We discussed the stability of the system. We showed that a sufficiently high rate of infection or a sufficiently low recovery rate can destabilize the system. We also found that an endogenous and perpetual cycle of infected individuals, susceptible individuals and the employment level can emerge in the three dimensional system. Government cooperation for facilitating the reestablishment of the employment networks as well as reestablishment of the forward and backward linkages among industries can boost the speed of adjustment parameter of the goods market and hence can ensure stability in the system. We showed that a better community health and a rise in the recovery rate raise the equilibrium level of output and the number of susceptible individuals, whereas the level of infected individuals falls. However, the effect of a fiscal expansion on the equilibrium level of output is ambiguous. Here too, a proper policy-mix of fiscal expansion along with all the measures that ensures a better community health and a rise in the recovery rate can lead to a rise in the output level and a fall in the level of infected individuals. Finally, using some numerical examples we showed the occurrence of limit cycles.

Needless to say, the analytical framework in this chapter is subject to few limitations. First, our model is focused on the short run. Analyzing the impact of Covid-19 pandemic on the economy from the long run perspective would be an interesting exercise. Second, we did not focus on the health sector (explicitly) in our analysis. Introduction of a health sector and taking into account its feedback effect on the economy from both the short-run as well as the long run perspective would be also an interesting extension of the model. These issues are, however, left for future research.

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# A Appendix

## A.1 When the Infection does not have any impact on the aggregate demand.

Here $f = 0$ and therefore inserting it in equation (2.9) we get $J_{21} = \frac{\partial \dot{Y}}{\partial I} = 0$. Although the analysis of $\dot{I} = 0$ isocline remains unchanged, $\dot{Y} = 0$ isocline is now independent of the value of $I$. As the dynamics of the infected individuals is in a non-linear form, the phase diagram (Figure A.1) shows the existence of two equilibria $A$ and $B$. At the steady state $A$, as $I < \tilde{I} = \frac{\beta N - \nu}{2\beta}$, $J_{11} > 0$. Therefore at $A$ the determinant of the Jacobian matrix $\text{Det}(\boldsymbol{J}) = (J_{11}J_{22} - J_{12}J_{21}) = J_{11}J_{22} < 0$. Thus $A$ is a saddle point unstable steady state. However, as $I > \tilde{I} = \frac{\beta N - \nu}{2\beta}$ at $B$, $J_{11} < 0$. Therefore at $B$, $\text{Det}(\boldsymbol{J}) = J_{11}J_{22} > 0$ and the trace of the Jacobian matrix $\text{tr}(\boldsymbol{J}) = (J_{11} + J_{22}) < 0$. Hence $B$ is a stable steady state.[25]

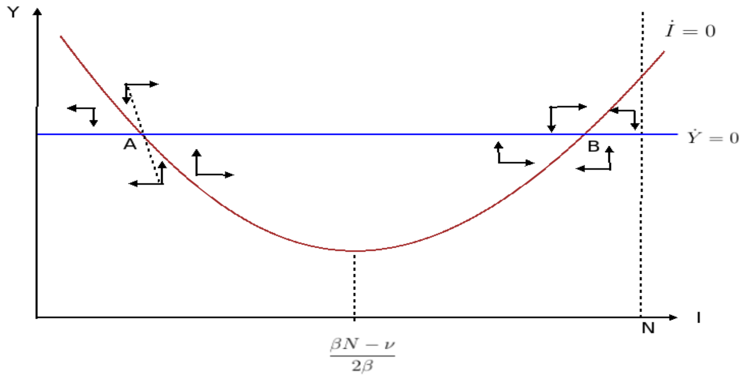



Figure A.1

## A.2 Proof of Proposition 1

*Proof.* The characteristic equation to (2.2) & (2.5) is

$$\lambda^2 + (-tr(J))\lambda + Det(J) = 0.$$

A necessary condition of the Hopf bifurcation for complex roots is $Det(J) > 0$, which is satisfied at $C$. The trace of the Jacobian matrix can be made either positive or negative by appropriately selecting the value of $\rho$ while leaving the other parameters constant. To see this, notice that $tr(J) = J_{11} + J_{22} = (\beta N - \nu) - 2\beta I - (1-b)\rho$. Hence when $\rho = \hat{\rho} = \frac{(\beta N - \nu) - 2\beta I}{(1-b)} > 0$ ( $\because$ here $\{(\beta N - \nu) - 2\beta I\} > 0$), the following equation holds exactly:

$$tr(J) = 2 * \text{Re}\lambda = [(\beta N - \nu) - 2\beta I - (1-b)\rho] = 0$$

where $tr(J)$ is the trace of $J$ and $\text{Re}\mu$ is the real part of its characteristic roots. As the determinant of the Jacobian matrix is positive, the product of the roots is positive in a neighborhood of the equilibrium, assuring $\text{Im}\lambda \neq 0$. Now differentiating the trace of the Jacobian matrix with respect to $\rho$ and then evaluating it at $\rho = \hat{\rho}$ we get

$$\frac{\partial(\frac{tr(J)}{2})}{\partial \rho}\bigg|_{\rho=\hat{\rho}} = \frac{-(1-b)}{2} < 0$$

[25] Here as $J_{12} > 0$ and $J_{21} = 0$ so, $tr(J)^2 - 4Det(J) = (J_{11} - J_{22})^2 > 0$ and hence the steady state is a stable node.

So the trace is smooth, differentiable and monotonically decreasing in the speed of adjustment parameter, $\rho$. The trace disappears at $\rho = \hat{\rho}$. From the preceding discussion, all conditions for Hopf bifurcation are satisfied at $\rho = \hat{\rho}$ $\square$

## A.3 Proof of Proposition 3

*Proof.* **Proposition 3(*i*) :** $A > 0$ implies $C > 0$ whereas a positive value of $D$ ensures $F > 0$. $AD$ and $(1-b)C$ are also positive. However the sign of $\{AC + (1-b)D - F\}$ is ambiguous. Here as $\{AC + (1-b)D - F\} < 0$, $a_1a_2 - a_3 \equiv \xi(\rho) = \underbrace{(1-b)C}_{+}\rho^2 + \underbrace{\{AC + (1-b)D - F\}}_{-}\rho + \underbrace{AD}_{+} \gtreqless 0$. The quadratic function $\xi(\rho)$ is convex and its intercept is positive (see Figure 3.1b). Equation $\xi(\rho) = 0$ has two positive real roots: $\hat{\rho}_1$ and $\hat{\rho}_2$. For $\rho \in (0, \hat{\rho}_1)$, we have $a_1 > 0$, $a_2 > 0$, $a_3 > 0$ and $a_1a_2 - a_3 > 0$; for $\rho \in (\hat{\rho}_1, \hat{\rho}_2)$, we have $a_1 > 0$, $a_2 > 0$, $a_3 > 0$ and $a_1a_2 - a_3 < 0$; and for $\rho > \hat{\rho}_2$, we have $a_1 > 0$, $a_2 > 0$, $a_3 > 0$ and $a_1a_2 - a_3 > 0$. Hence the Hopf bifurcation occurs at $\rho = \hat{\rho}_1$ and $\rho = \hat{\rho}_2$ (see Figure 3.1b). Indeed, at $\rho = \hat{\rho}_1$ and $\rho = \hat{\rho}_2$, we have $a_1 > 0$, $a_2 > 0$, $a_3 > 0$ and $a_1a_2 - a_3 = 0$. Further, $\frac{\partial\xi(\rho)}{\partial\rho} = 2(1-b)C\rho + \{AC + (1-b)D - F\}$ and so $\frac{\partial\xi(\rho)}{\partial\rho}\Big|_{\rho=\hat{\rho}_1} = [\underbrace{2(1-b)C\hat{\rho}_1}_{+} + \underbrace{\{AC + (1-b)D - F\}}_{-}] < 0$ and $\frac{\partial\xi(\rho)}{\partial\rho}\Big|_{\rho=\hat{\rho}_2} = \{\underbrace{2(1-b)C\hat{\rho}_2}_{+} + \underbrace{\{AC + (1-b)D - F\}}_{-}\} > 0$. Thus, all conditions of Hopf bifurcation are satisfied. As a result, there exists a continuous family of non-constant, periodic solutions of the system around $\rho = \hat{\rho}_1$ and $\rho = \hat{\rho}_2$.

**Proposition 3(*ii*) :** $D < 0$ and $\rho > \frac{-D}{C}$ together imply $a_2 > 0$. $A > 0$ ensures $a_1 > 0$ and $F > 0$ ensures $a_3 > 0$.

However the sign of $\{AC+(1-b)D-F\}$ is ambiguous. When $\{AC+(1-b)D-F\} > 0$, $a_1a_2 - a_3 \equiv \xi(\rho) = \underbrace{(1-b)C}_{+}\rho^2 + \underbrace{\{AC + (1-b)D - F\}}_{+}\rho + \underbrace{AD}_{-} \gtreqless 0$. The quadratic function $\xi(\rho)$ is convex and at $\rho = \hat{\rho}$, $a_1a_2 - a_3 \equiv \xi(\rho) = 0$ (see Figure 3.1c). Further, $\frac{\partial\xi(\rho)}{\partial\rho} = 2(1-b)C\rho + \{AC + (1-b)D - F\} > 0$ and so $\frac{\partial\xi(\rho)}{\partial\rho}\Big|_{\rho=\hat{\rho}} > 0$. Thus, all conditions of Hopf bifurcation are satisfied. As a result, there exists a continuous family of non-constant, periodic solutions of the system around $\rho = \hat{\rho}$.

Now consider $\{AC + (1-b)D - F\} < 0$. Then $a_1a_2 - a_3 \equiv \xi(\rho) = \underbrace{(1-b)C}_{+}\rho^2 + \underbrace{\{AC + (1-b)D - F\}}_{-}\rho + \underbrace{AD}_{-} \gtreqless 0$. The quadratic function $\xi(\rho)$ is convex and at $\rho = \hat{\rho}$, $a_1a_2 - a_3 \equiv \xi(\rho) = 0$ (see Figure 3.1d). Further, $\frac{\partial\xi(\rho)}{\partial\rho} = 2(1-b)C\rho + \{AC + (1-b)D - F\}$

and $\frac{\partial \xi(\rho)}{\partial \rho}\Big|_{\rho=\hat{\rho}} > 0$. Thus, all conditions of Hopf bifurcation are satisfied. As a result, there exists a continuous family of non-constant, periodic solutions of the system around $\rho = \hat{\rho}$.

**Proposition 3*(iii)* :** $A < 0$ and $\rho > \frac{-A}{(1-b)}$ together imply $a_1 > 0$. $D < 0$ and $\rho > \frac{-D}{C}$ together imply $a_2 > 0$. $F > 0$ ensures $a_3 > 0$.

$a_1 a_2 - a_3 \equiv \xi(\rho) = \underbrace{(1-b)C}_{+}\rho^2 + \underbrace{\{AC + (1-b)D - F\}}_{-}\rho + \underbrace{AD}_{+} \gtreqless 0$. The quadratic function $\xi(\rho)$ is convex and its intercept is positive (see Figure 3.1b). Equation $\xi(\rho) = 0$ has two positive real roots: $\hat{\rho}_1$ and $\hat{\rho}_2$. For $\rho \in (0, \hat{\rho}_1)$, we have $a_1 > 0$, $a_2 > 0$, $a_3 > 0$ and $a_1 a_2 - a_3 > 0$; for $\rho \in (\hat{\rho}_1, \hat{\rho}_2)$, we have $a_1 > 0$, $a_2 > 0$, $a_3 > 0$ and $a_1 a_2 - a_3 < 0$; and for $\rho > \hat{\rho}_2$, we have $a_1 > 0$, $a_2 > 0$, $a_3 > 0$ and $a_1 a_2 - a_3 > 0$. Hence the Hopf bifurcation occurs at $\rho = \hat{\rho}_1$ and $\rho = \hat{\rho}_2$ (see Figure 3.1b). Indeed, at $\rho = \hat{\rho}_1$ and $\rho = \hat{\rho}_2$, we have $a_1 > 0$, $a_2 > 0$, $a_3 > 0$ and $a_1 a_2 - a_3 = 0$. Further, $\frac{\partial \xi(\rho)}{\partial \rho} = 2(1-b)C\rho + \{AC + (1-b)D - F\}$ and so $\frac{\partial \xi(\rho)}{\partial \rho}\Big|_{\rho=\hat{\rho}_1} = [\underbrace{2(1-b)C\hat{\rho}_1}_{+} + \underbrace{\{AC + (1-b)D - F\}}_{-}] < 0$ and $\frac{\partial \xi(\rho)}{\partial \rho}\Big|_{\rho=\hat{\rho}_2} = \{\underbrace{2(1-b)C\hat{\rho}_2}_{+} + \underbrace{\{AC + (1-b)D - F\}}_{-}\} > 0$. Thus, all conditions of Hopf bifurcation are satisfied. As a result, there exists a continuous family of non-constant, periodic solutions of the system around $\rho = \hat{\rho}_1$ and $\rho = \hat{\rho}_2$.

**Proposition 3*(iv)* :** $A < 0$ and $\rho > \frac{-A}{(1-b)}$ together imply $a_1 > 0$. $C > 0$ and $D > 0$ together ensure $a_2 > 0$ and $F > 0$ ensures $a_3 > 0$.

However the sign of $\{AC + (1-b)D - F\}$ is ambiguous. When $\{AC + (1-b)D - F\} > 0$, $a_1 a_2 - a_3 \equiv \xi(\rho) = \underbrace{(1-b)C}_{+}\rho^2 + \underbrace{\{AC + (1-b)D - F\}}_{+}\rho + \underbrace{AD}_{-} \gtreqless 0$. The quadratic function $\xi(\rho)$ is convex and at $\rho = \hat{\rho}$, $a_1 a_2 - a_3 \equiv \xi(\rho) = 0$ (see Figure 3.1c). Further, $\frac{\partial \xi(\rho)}{\partial \rho} = 2(1-b)C\rho + \{AC + (1-b)D - F\} > 0$ and so $\frac{\partial \xi(\rho)}{\partial \rho}\Big|_{\rho=\hat{\rho}} > 0$. Thus, all conditions of Hopf bifurcation are satisfied. As a result, there exists a continuous family of non-constant, periodic solutions of the system around $\rho = \hat{\rho}$.

Now consider $\{AC + (1-b)D - F\} < 0$. Then $a_1 a_2 - a_3 \equiv \xi(\rho) = \underbrace{(1-b)C}_{+}\rho^2 + \underbrace{\{AC + (1-b)D - F\}}_{-}\rho + \underbrace{AD}_{-} \gtreqless 0$. The quadratic function $\xi(\rho)$ is convex and at $\rho = \hat{\rho}$, $a_1 a_2 - a_3 \equiv \xi(\rho) = 0$ (see Figure 3.1d). Further, $\frac{\partial \xi(\rho)}{\partial \rho} = 2(1-b)C\rho + \{AC + (1-b)D - F\}$ and $\frac{\partial \xi(\rho)}{\partial \rho}\Big|_{\rho=\hat{\rho}} > 0$. Thus, all conditions of Hopf bifurcation are satisfied. As a result, there exists a continuous family of non-constant, periodic solutions of the system around $\rho = \hat{\rho}$.

**Proposition 3*(v)* :** $A < 0$ and $\rho > \frac{-A}{(1-b)}$ together imply $a_1 > 0$. $C < 0$, $D > 0$ and $\rho < \frac{-D}{C}$ together imply $a_2 > 0$. $F > 0$ ensures $a_3 > 0$.

When $\{AC+(1-b)D-F\} < 0$, irrespective of the value of $\rho$, $\xi(\rho)$ is negative (see Figure 3.2b). Hence the steady state is unstable. On the other hand when $\{AC+(1-b)D-F\} > 0$,$a_1a_2 - a_3 \equiv \xi(\rho) = \underbrace{(1-b)C}_{-}\rho^2 + \underbrace{\{AC+(1-b)D-F\}}_{+}\rho + \underbrace{AD}_{-} \gtreqless 0$. The quadratic function $\xi(\rho)$ is concave and its intercept is negative (see Figure 3.2a). Equation $\xi(\rho) = 0$ has two positive real roots: $\hat{\rho}_1$ and $\hat{\rho}_2$. For $\rho \in (0, \hat{\rho}_1)$, we have $a_1 > 0$, $a_2 > 0$, $a_3 > 0$ and $a_1a_2 - a_3 < 0$; for $\rho \in (\hat{\rho}_1, \hat{\rho}_2)$, we have $a_1 > 0$, $a_2 > 0$, $a_3 > 0$ and $a_1a_2 - a_3 > 0$; and for $\rho > \hat{\rho}_2$, we have $a_1 > 0$, $a_2 > 0$, $a_3 > 0$ and $a_1a_2 - a_3 < 0$. Hence the Hopf bifurcation occurs at $\rho = \hat{\rho}_1$ and $\rho = \hat{\rho}_2$ (see Figure 3.2a). Indeed, at $\rho = \hat{\rho}_1$ and $\rho = \hat{\rho}_2$, we have $a_1 > 0$, $a_2 > 0$, $a_3 > 0$ and $a_1a_2 - a_3 = 0$. Further, $\frac{\partial\xi(\rho)}{\partial\rho} = 2(1-b)C\rho + \{AC+(1-b)D-F\}$ and so $\frac{\partial\xi(\rho)}{\partial\rho}\Big|_{\rho=\hat{\rho}_1} = [\underbrace{2(1-b)C\hat{\rho}_1}_{-} + \underbrace{\{AC+(1-b)D-F\}}_{+}] > 0$ and $\frac{\partial\xi(\rho)}{\partial\rho}\Big|_{\rho=\hat{\rho}_2} = \{\underbrace{2(1-b)C\hat{\rho}_2}_{-} + \underbrace{\{AC+(1-b)D-F\}}_{+}\} < 0$. Thus, all conditions of Hopf bifurcation are satisfied. As a result, there exists a continuous family of non-constant, periodic solutions of the system around $\rho = \hat{\rho}_1$ and $\rho = \hat{\rho}_2$. □

## A.4 Effects of parametric changes

We examine the effect of changes in the fiscal policy, the infection rate ($\beta$), the recovery rate ($\nu$), and $\theta$ (the responsiveness of the number of infected individuals ($I$) due to a rise in the economic activity ($Y$)) on the steady-state values of $I$, $Y$, and $S$. We only consider the stable steady state here.

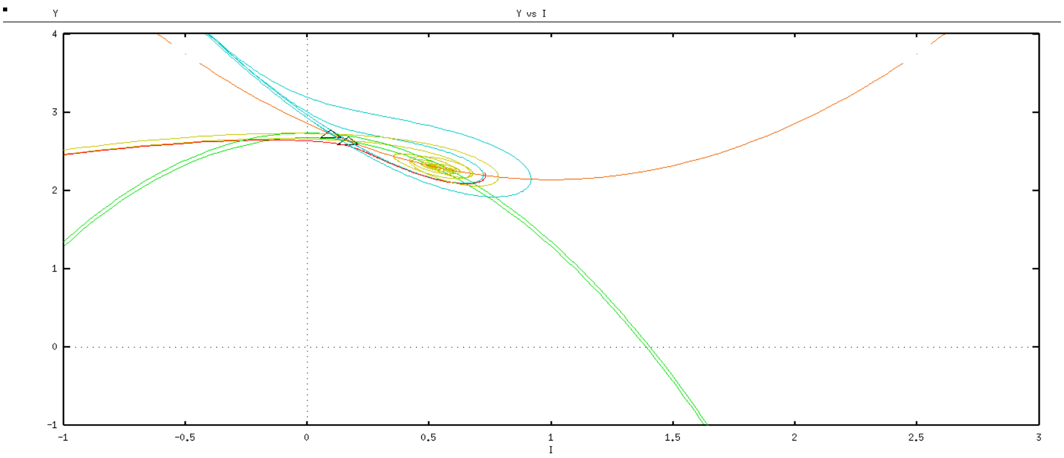
The effects of parametric changes can be shown by totally differentiating equations (3.1), (3.2) and (3.3), which imply

$$\begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & 0 \\ J_{31} & 0 & J_{33} \end{bmatrix} \begin{bmatrix} dI \\ dY \\ dS \end{bmatrix} = \begin{bmatrix} -SI \\ 0 \\ SI \end{bmatrix} d\beta + \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} d\nu + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} d\phi + \begin{bmatrix} -Y \\ 0 \\ 0 \end{bmatrix} d\theta + \begin{bmatrix} 0 \\ -\rho \\ 0 \end{bmatrix} da \tag{A.1}$$

Note that for stability, $a_3 = -\text{Det}(J)$ must be positive. Consequently we assume $\text{Det}(J) < 0$. From equation (A.1) we get $\frac{dI}{d\beta} = \frac{-SIJ_{22}J_{33}}{\text{Det}(J)} = \frac{-SI(1-b)(\beta I+\mu)\rho}{\text{Det}(J)} > 0$; $\frac{dY}{d\beta} = \frac{2\rho fSI^2\mu}{\text{Det}(J)} < 0$; $\frac{dS}{d\beta} = \frac{SI[(1-b)(\nu+\mu)\rho+2\theta\rho fI]}{\text{Det}(J)} < 0$; $\frac{dI}{d\nu} = \frac{I(1-b)(\beta I+\mu)\rho}{\text{Det}(J)} < 0$; $\frac{dY}{d\nu} = \frac{-2\rho fI^2(\beta I+\mu)}{\text{Det}(J)} > 0$; $\frac{dS}{d\nu} = \frac{-I(1-b)(\beta S+\mu)\rho}{\text{Det}(J)} > 0$; $\frac{dI}{d\phi} = \frac{(1-b)(\beta I+\mu)\rho}{\text{Det}(J)} < 0$; $\frac{dY}{d\phi} = \frac{-2\rho fI(\beta I+\mu)}{\text{Det}(J)} > 0$; $\frac{dS}{d\phi} = \frac{-(1-b)(\beta S+\mu)\rho}{\text{Det}(J)} > 0$; $\frac{dI}{d\theta} = \frac{-Y(1-b)(\beta I+\mu)\rho}{\text{Det}(J)} > 0$; $\frac{dY}{d\theta} = \frac{2YfI(\beta I+\mu)\rho}{\text{Det}(J)} < 0$; $\frac{dS}{d\theta} = \frac{Y(1-b)(\beta S+\mu)\rho}{\text{Det}(J)} < 0$; $\frac{dI}{da} = \frac{-\rho\theta(\beta I+\mu)}{\text{Det}(J)} > 0$; $\frac{dY}{da} = \frac{-\rho[\nu(\beta I+\mu)+\beta\mu(I-S)]}{\text{Det}(J)} \gtrless 0$; according to whether $[\nu(\beta I+\mu)+\beta\mu(I-S)] \gtrless 0$; and $\frac{dS}{da} = \frac{\rho\theta(\beta S+\mu)}{\text{Det}(J)} < 0$.

## A.5 A fall in $a$ reduces the area of the invariant set

Because of a contractionary fiscal policy, the region of invariant set becomes smaller. If we assume $\beta = 0.5$, $N = 5$, $\nu = 1.5$, $\phi = 2$, $\theta = 0.7$, $a = 1.78$, $b = 0.35$, $f = 0.9$, and $\rho = 0.9$, at the steady state $A$ we get $I^* = 0.096841$, $Y^* = 2.7255$ and at the stable steady state $C$ we get $I^* = 0.58377$ and $Y^* = 2.2666$. However, as $a$ falls to 1.74, at the new steady state $A'$ we get $I^* = 0.16725$, $Y^* = 2.6382$ and at the stable steady state $C'$ we get $I^* = 0.51337$ and $Y^* = 2.312$. Clearly, compared to the initial stable steady state $C$, at the new stable steady state $C'$ the steady state value of $Y$ is higher and $I$ is lower. However, due to a fall in $a$, as Figure A.2 shows, area of the invariant set becomes smaller. Consequently, the system becomes more fragile.




Note that the blue lines represent the invariant set. As we see, a fall in $a$ reduces the area of the invariant set.

Figure A.2: Invariant set

## A.6 Table of all possible cases

Table A.1: All possible cases

| | $A$ | $C$ | $D$ | $F$ | Results | | $A$ | $C$ | $D$ | $F$ | Results |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | + | + | + | + | See Proposition 2 or 3($i$) | 9 | − | + | + | + | See Proposition 3($iv$) |
| 2 | + | + | + | − | Not possible | 10 | − | + | + | − | Not possible |
| 3 | + | + | − | + | See Proposition 3($ii$) | 11 | − | + | − | + | See Proposition 3($iii$) |
| 4 | + | + | − | − | Unstable | 12 | − | + | − | − | Unstable |
| 5 | + | − | + | + | Not possible | 13 | − | − | + | + | See Proposition 3($v$) |
| 6 | + | − | + | − | Not possible | 14 | − | − | + | − | Not possible |
| 7 | + | − | − | + | Not possible | 15 | − | − | − | + | Unstable |
| 8 | + | − | − | − | Not possible | 16 | − | − | − | − | Unstable |