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12 December 2007

Joint Meeting of the AMS - NZMS 2007
Victoria University of Wellington, New Zealand, December 12 - 15, 2007

A VIABILITY THEORY APPROACH TO A TWO-STAGE OPTIMAL CONTROL PROBLEM

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Abstract. A two-stage control problem is one, in which model parameters ("technology") might be changed at some time. An optimal solution to utility maximisation for this class of problems needs to thus contain information on the time, at which the change will take place (0, finite or never) as well as the optimal control strategies before and after the change. For the change, or switch, to occur the "new technology" value function needs to dominate the "old technology" value function, after the switch. We charaterise the value function using the fact that its hypograph is a viability kernel of an auxiliary problem and study when the graphs can intersect and hence whether the switch can occur. Using this characterisation we analyse a technology switching problem.

Keywords: value function, viability kernel, viscosity solutions
JEL:C6, C61, C69
MSC: 34H05, 34K35, 49J15, 49L25, 91B02, 91B62, 93C15,

Glossary

$U \subseteq \mathbb{R}^M$	set of control values
$f(\cdot,\cdot), \psi(\cdot,\cdot) : \mathbb{R}^N \times U \to \mathbb{R}^N$	system's dynamics, continuous in either argument (later assumed affine in first argument and indexed by technology)
$\phi(\cdot,\cdot,\cdot,\cdot) : \mathbb{R}^{N+3} \to \mathbb{R}^{N+3}$	auxiliary system's dynamics
$\Phi(\cdot), \Psi(\cdot)$	system's dynamics; set valued maps
$L : [0,\, T] \times \mathbb{R}^N \times U \to \mathbb{R}$	instantaneous utility (utility integrand); bounded function
$u(\cdot) : [0,\, T] \to\ U$	control; measurable function
$\mathcal{U}_{[t,T]}$	set of measurable controls on $[t, T]$ with values in U
$x(\cdot) : [0,\, T] \to \mathbb{R}^N$	state variable
$V^T(\cdot,\cdot) : [0,\, T] \times \mathbb{R}^N \to \mathbb{R}$	value function for T-horizon optimal control
$\underline{w}(\cdot,\cdot), \overline{w}(\cdot,\cdot)$	sub- and supersolution to Hamilton-Jacobi-Bellman equation
D, E, K	closed sets in $\mathbb{R}^N$
$\mathcal{NP}_D(x)$	set of proximal normals to D at $x \in D$
...	...
...	...

1. Introduction

The aim of this paper is three-fold. First, we want to establish which integrants of continuous-time optimal-control problems with affine dynamics can generate collections of value functions whose graphs can intersect. Second, we want to use the established result to prove existence, or its lack, of the switching time in a two-stage optimal control problem. Our third aim is to demonstrate applicability of recent results in viability theory regarding some equivalence between an optimal solution and a "viable" solution.

A two-stage control problem is one, in which model parameters might be modified at some time (see [27]). For example, a system's dynamics, which describes an accumulation process of pollution, may be altered through an installation of new filters. An optimal solution to utility maximisation in a problem of this kind needs contain information on the time, at which the change will take place (0, finite or never) as well as the optimal control strategies before and after the change. For the change, or switch, to occur the "new filter" value function needs to dominate the "old filter" value function, after the switch.

We will index the two-stage optimal control problems by a parameter responsible for the system's dynamics and characterise the corresponding value functions. If their graphs intersect for different parameter values then the switch exists and is optimal. We will charaterise the value functions using the fact that their hypographs are viability kernels of some auxiliary viability problems and study when the graphs can intersect.

What follows is a brief outline of what this paper contains. In Section 2, we describe a basic optimal control model, which we use in Section 5 to study a technology switching problem.[1] Basic viability results are presented in Section 3. In Section 4, a relationship between the viability kernel of an auxiliary problem and the basic finite-horizon optimal control problem are established. This result is then used to study the existence of a switching time in a technology selection problem. The concluding remarks summarise our findings.

2. Problem formulation

We consider a control system whose dynamics is given by:

$$\dot{x}(s) = f(x(s), u(s)) \tag{1}$$

where the state variable x belongs to $\mathbb{R}^N$, the control $u(\cdot) : [0, \infty) \to U \subset \mathbb{R}^M$ is a measurable function and $f : \mathbb{R}^N \times U \to \mathbb{R}^N$.

The optimal control problem

$$\max_{u(\cdot)} \int_t^T L\left(s, x(s; t, x, u(\cdot)), u(s)\right) ds \tag{2}$$

where $x(\cdot; t, x, u(\cdot))$ denotes absolutely continuous solutions to (1), with $0 < T < \infty$ and $L : [0,\, T] \times \mathbb{R}^N \times U \to \mathbb{R}$ is a given bounded function. We adopt

[1] According to our knowledge this will be the first application of viability theory to microeconomics. For macroeconomic applications refer to [13], [14], [15], [19] and [18]. For viability theory applications to environmental economics see [6], [16], [10] and [17]; for applications to financial analysis see [20] and the references provided there. .

the convention that $x(\cdot;t,x,u(\cdot))$ denotes the solution to (1) starting from $(t,x) \in [0,T] \times \mathbb{R}^N$.

If we denote the set of measurable controls on $[t,T]$ with values in U by $\mathcal{U}_{[t,T]}$ then the value function corresponding to the optimal control problem (1) and (2) is given by:

$$V^T(t,x) = \sup_{u \in \mathcal{U}_{[t,T]}} \int_t^T L\left(s, x(s;t,x,u(\cdot)), u(s)\right) ds \tag{3}$$

Our goal is to establish conditions allowing us to compare value functions that correspond to different system's dynamics $f(\cdot,\cdot)$, perhaps "indexed" by technologies. To do this, we will first characterise the value function (3) through a viability kernel or an equation of Hamilton-Jacobi-Bellman type.

We refer the reader to a result about viability characterisation obtained in [8] in a slightly different context. The result establishes that the epigraph of the minimal time to reach a set is a viability kernel of an auxiliary control process. Later, in Section 4, we will provide an analogous result for the optimisation problem (2), (1).

We will also use some available results for the well-known case of a Lipschitzian value function, [3], [4], [11]. In particular, under continuity assumptions on the system's dynamics and utility integrand the value function (3) is the unique Lipschitz viscosity solution[2] of the following equation:

$$\left\{ \begin{array}{c} \dfrac{\partial V}{\partial t}(t,x) + H\big(t,x,\dfrac{\partial V}{\partial x}(t,x)\big) = 0 \\ (t,x) \in [0,T) \times \mathbb{R}^N; \\ \text{with the final condition } V(T,x) = 0, \text{ for all } x \in \mathbb{R}^N \end{array} \right. \tag{4}$$

where the Hamiltonian[3] $H : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is:

$$H(t,x,p) = \max_{u \in U}(\langle p, f(x,u)\rangle + L(t,x,u)). \tag{5}$$

Our main result will enable us to compare value functions associated with different technologies. However, rather than obtaining $V(t,x)$ as a solution to the Hamilton-Jacobi equation (4), we will characterise the value function through a viability kernel of an auxiliary problem related to original optimal control problem. The approach to value function characterisation by viability kernels was dealt with in [2], [7], [12], [21], [24], [25]. Our use of this approach to economic problems' solution is novel.

In particular, our results will be based on the links between the viscosity supersolution, which we will define below, the value function's hypograph and the viability kernel of the auxiliary problem (see [2]).

[2]The viscosity solution of a partial differential equation is a continuous function that satisfies the equation and whose derivatives are considered in a generalised sense. See Section 3.3 for precise definitions.

[3]Notice that we depart from the traditional definition according to which the Hamiltonian will be the contents of brackets $(\cdot)$ in (5) (*i.e.*,"maximand") rather than the result of the maximisation, as we have define it.

3. Preliminaries

3.1. **Definitions, assumptions and notation.** We assume that the dynamics $f : \mathbb{R}^N \times U \to \mathbb{R}^N$ in equation (1) is a continuous function and satisfies:

$$\left\{ \begin{array}{c} ||f(x,u)|| \leq c_1(1+\|x\|) \\ ||f(x,u)-f(y,u)|| \leq c_1 \|x-y\| \end{array} \right. \quad \forall x,y \in \mathbb{R}^N, u \in U \tag{6}$$

where $c_1 > 0$ is constant; the control set U is a compact subset of $\mathbb{R}^N$. We can therefore describe the system's velocities at x as $f(x,U)$ where

$$f(x,U) = \{f(x,u), u \in U\} \quad \text{is a convex set} \quad \forall x \in \mathbb{R}^N. \tag{7}$$

It is well known that under (6), for every $(t,x) \in [0,\infty) \times \mathbb{R}^N$, the Cauchy Problem (CP):

$$\left\{ \begin{array}{c} \dot{x}(s) = f(x(s),u(s)) \text{ for almost every } s \in [t,\infty)\ ; \\ x(t) = x \end{array} \right. \tag{CP}$$

has an unique absolutely continuous solution denoted by $x(\cdot;t,x,u(\cdot))$.

We will also assume that $L : [0,\, T] \times \mathbb{R}^N \times U \to \mathbb{R}^N$ is continuous and satisfies :

$$\left\{ \begin{array}{c} ||L(t,x,u)|| \leq c_2(1+\|x\|) \\ ||L(t,x,u)-L(t,y,u)|| \leq c_2 \|x-y\| \end{array} \right. \quad \forall x,y \in \mathbb{R}^N, u \in U,\, t \in [0, \in T] \tag{8}$$

where $c_2 > 0$ is constant and

$$\forall x \in \mathbb{R}^N,\ t \in [0,\, T] \;\; L(t,x,U) = \{L(t,x,u), u \in U\} \text{ is convex.} \tag{9}$$

Later we will study an example where the function $f : \mathbb{R} \times U \to \mathbb{R}$ is linear in either variable and has the following form:

$$f(x,u) = \theta u - \mu x \tag{10}$$

with $\theta, \mu \in \mathbb{R}$ that can be associated with some technology and $L : t \in [0,\, T] \times \mathbb{R} \times U \to \mathbb{R}$

$$L(t,x,u) = e^{-\rho t} g(u,x) \tag{11}$$

where $g(u,x)$ is bounded, continuous and concave in each argument, decreasing in x; $\rho \in \mathbb{R}$.

3.2. **Viability theory.** Here we will present the notion of *viability-domain-with-a-target* introduced in [24]. We will characterise this set using the Viability Theorem provided in [7] (Theorem 2.3):

Proposition 1. *We assume that D and E are closed sets. Let us suppose that $\psi : \mathbb{R}^N \times U \to \mathbb{R}^N$ is a continuous function, Lipschitz in the first variable; furthermore, for every x we define set valued map $\psi(x,U) = \{\psi(x,u); u \in U\}$ which is supposed to be Lipschitz continuous with convex, compact, nonempty values.*

Then the two following assertions are equivalent[4]*:*

[4] Here $\mathcal{NP}_D(x)$ denotes the set of *proximal normals* to D at x i.e., the set of $p \in \mathbb{R}^N$ such that the distance of $x+p$ to D is equal to $||p||$.

(i)

$$\forall x \in D\backslash E, \quad \forall p \in \mathcal{NP}_D(x), \quad \min_u \langle \psi(x,u), p\rangle \leq 0 \tag{12}$$

respectively, $\max_u \langle \psi(x,u), p\rangle \leq 0$ *;*

(ii) *there exists* $u \in \mathcal{U}_{[t,T]}$ *such that (respectively, for all* $u \in \mathcal{U}_{[t,T]}$*) the solution of*

$$\begin{cases} \dot{x}(s) = \psi(x(s), u(s)) \text{ for almost every } s \\ x(t) = x \end{cases} \tag{13}$$

remains in D *as long as it does not reach* E*.*

Notice that the inequality $\min_u \langle \psi(x,u), p\rangle \leq 0$ in (12) means that there *exists* a control for which the system's velocity $\dot{x}$ "points inside" the set $D\backslash E$. Respectively, $\max_u \langle \psi(x,u), p\rangle \leq 0$ means that the system's velocity $\dot{x}$ "points inside" the set $D\backslash E$ for *all* controls from U.

When i) (or ii)) holds we say that D is a *viability domain* with target E (or, respectively, D is an *invariance domain* with target E) for the dynamics ψ. When $E = \emptyset$, then the proposition concerns the classical notion of viability (respectively, invariance) domain [2].

Definition 2. *Let* K *be a closed set. We call* viability kernel *in* K *with target* E*, for a dynamics* Ψ *denoted:*

$$\mathcal{V}iab_\Psi(K,\, E)$$

the largest closed subset of K*, which is a viability domain with target* E *for* Ψ*.*

It was proved (see for instance [1] and [24]) that $\mathcal{V}iab_\Psi(K, \emptyset)$ is also the set of x such that there exists $x(\cdot)$, a solution of

$$\dot{x}(s) \in \Psi(x(s)) \tag{14}$$

starting from x, which is defined on $[0, \infty)$ and $x(s) \in K$ for all $s \geq 0$. Respectively, $\mathcal{V}iab_\Psi(K, E)$ (*i.e.*, when $E \neq \emptyset$) is also the set of x such that there exists $x(\cdot)$, a solution of

$$\dot{x}(s) \in \Psi(x(s))$$

starting from x, which is defined on $[0, \tau)$ and $x(s) \in K$ for all $s \in [0, \tau)$ and if τ is finite then we have $x(\tau) \in E$.

Our conclusions regarding value functions will be based on the fact that the definition of a solution to a PDE of the type (4) gives some invariance properties of sets related to the value function (see Propositions 1 and 4). More precisely, the hypograph[5] of a supersolution to (4) is a viability domain in $[0,T] \times \mathbb{R}^{N+2}$ with some target for the auxiliary system's dynamics ϕ:

$$(t, x, z, r) \rightarrow \phi(t, x, z, r) = (1, f(x, U); L(x, U), 0)$$

[5]For $w : [0,T] \times \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ we have:
$Epi(w) := \{(t, x, r) \in [0,T] \times \mathbb{R}^N \times \mathbb{R} \mid w(t, x) \leq r\}$;
$Hypo(w) := \{(t, x, r) \in [0,T] \times \mathbb{R}^N \times \mathbb{R} \mid w(t, x) \geq r\}$.

and the epigraph of a subsolution is an invariance domain in $[0,T]\times\mathbb{R}^{N+2}$ with some target for the auxiliary system's dynamics $-\phi$:

$$(t,x,z,r)\to -\phi(t,x,z,r)=-(1,f(x,U);L(x,U),0).$$

In particular, we will exploit the fact that the *largest* closed viability domain (kernel) in $[0,T]\times\mathbb{R}^{N+2}$ for dynamics ϕ (with a target) is the hypograph of the *biggest* supersolution (value function) to the Hamiltion-Jacobi-Belleman equation (4).

We will use *Epi* for the epigraph and *Hypo* for the hypograph.

3.3. **Viscosity Solutions.** Let us define a viscosity solution to the first order Hamilton-Jacobi-Bellman equation (cf. [3] for instance):

Definition 3. *A* viscosity supersolution *of (4) is a lower semicontinuous (l.s.c.) function* $\overline{w}:[0,T)\times\mathbb{R}^N\to\mathbb{R}$ *if and only if*

$$\textit{for any } \varphi\in C^1 \textit{ and when}(t,x) \textit{ is a local minimum of } (\overline{w}-\varphi)\,,$$
$$\frac{\partial\varphi}{\partial t}(t,x)+H\big(t,x,\frac{\partial\varphi}{\partial x}(t,x)\big)\le 0\,.$$

A viscosity subsolution *of (4) is an upper semicontinuous (u.s.c.) function* $\underline{w}:(0,T]\times\mathbb{R}^N\to\mathbb{R}$ *if and only if*

$$\textit{for any } \varphi\in C^1 \textit{ and when}(t,x) \textit{ is a local maximum of } (\underline{w}-\varphi)\,,$$
$$\frac{\partial\varphi}{\partial t}(t,x)+H\big(t,x,\frac{\partial\varphi}{\partial x}(t,x)\big)\ge 0.$$

A viscosity solution *of (4) is a function which is both subsolution and supersolution (so, in particular, it is continuous).*

There are several different definitions of discontinuous viscosity solutions. In particular, Ishii's solutions (cf. [3]) are based on semicontinuous envelopes of functions; there are also Barron-Jensen-Frankowska's semicontinuous solutions ([3], [5]) for convex Hamiltonians) and Subbotin's minimax solution [26] (called bilateral solutions in [4]) see also [22]. We think that the definition that we use in this paper is perhaps the most appropriate for the study of our problem. In particular, we find that it enables us to adequately compare solutions to the Hamilton-Jacobi-Bellman equations.

To establish a link between the viscosity solutions and viability we will provide an equivalent definition of super- and subsolutions to (4) in terms of *proximal normals*[6]. The proof of the equivalence between the two definitions can be founded in [21] or [23]. Here, we quote the following proposition from [23] (Proposition 3.3).

Proposition 4. *A* viscosity supersolution *to (4) is a l.s.c. function* $\overline{w}$: $[0,T)\times\mathbb{R}^N\to\mathbb{R}$ *such that:*

$$\textit{for any } (p_t,p_x,p_r)\in\mathcal{NP}_{Epi(\overline{w})}\big(t,x,\overline{w}(t,x)\big),$$
$$p_t+H(t,x,p_x)\le 0$$

A viscosity subsolution for *(4) is an u.s.c. function* $\underline{w}:[0,T)\times\mathbb{R}^N\to\mathbb{R}$ *such that:*

$$\textit{for any } (p_t,p_x,p_r)\in\mathcal{NP}_{Hypo(\underline{w})}\big(t,x,\underline{w}(t,x)\big),$$
$$p_t+H(t,x,p_x)\big)\ge 0.$$

[6]Refer to footnote 4 and Proposition 1.

4. The Optimal Control Problem with Finite Horizon

This section is dedicated to the characterisation of the value function of a finite-horizon optimal control problem through the Hamilton-Jacobi-Bellman equation (4).

4.1. **The associated Mayer problem.** Consider the Bolza optimal control problem with the following value function:

(15)
$$V^T(t,x) = \sup_{u\in\mathcal{U}_{[t,T]}} \left\{ g\left(x(T;t,x,u(\cdot))\right) + \int_t^T L\left(x(s;t,x,u(\cdot)),u(s)\right) ds \right\}.$$

Function $g(\cdot)$ is a "scrap value" function at the final time T, which satisfies

$$\left\{ \begin{array}{c} |g(x)| \le c_2 \text{ for all } x\in\mathbb{R}^N, \\ g \text{ is upper-semicontinuous in } \mathbb{R}^N \end{array} \right. ; \tag{16}$$

$L:\mathbb{R}^N\times U\to\mathbb{R}$ satisfies (8), (9). If g is discontinuous then so is, in general, the value function $V^T(t,x)$.

We will consider the modified system's dynamics:

$$\dot{y}(t) = (f(x(t),u(t)); L(x(t),u(t)). \tag{17}$$

Here

$$y(\cdot;t,y,u(\cdot)) := (x(\cdot;t,x,u(\cdot)); z(\cdot;t,z,u(\cdot)))$$

is the solution of (17) starting at $(t,x,z) := (t,y) \in [0,T]\times\mathbb{R}^{N+1}$. The *set* of solutions starting at (t,y) will be denoted $S(t,y) := \{y(\cdot;t,y,u(\cdot)); u\in\mathcal{U}_{[t,T]}\}$.

Let $h:\mathbb{R}^{N+1}\to\mathbb{R}$, given by

$$h(y) := g(x) + z \quad \text{with} \quad y := (x,z)\,.$$

We define an associated Mayer problem as follows:

$$\text{Maximise } h\left(y(T;t,y,u(\cdot))\right) \tag{18}$$

over all absolutely continuous solutions of (17).

With the above notations, we define the following value function corresponding to problem (18) subject to (17):

$$W^T(t,y) = \sup_{u\in\mathcal{U}_{[t,T]}} h\left(y(T;t,y,u(\cdot))\right). \tag{19}$$

Note that the following relation is true[7]:

$$W^T(t,y) = V^T(t,x) + z\,.$$

We will study the properties of W^T; as a consequence, a characterisation for V^T will be obtained.

[7]Notice that $z(s) = z + \int_t^s L dt$ where z is some initial condition and

$$\begin{aligned} V^T(t,x) &= \sup_u [g(x(T)) + \int_t^T L dt] = \sup_u [g(x(T)) + z(T) - z] \\ &= \sup_u [h((x(T)), z(T)) - z] = \sup_u [h(y(T)) - z] = -z + \sup_u [h(y(T))] \end{aligned} \tag{20}$$

Before giving the main result of this section, we cite some classical results and recall the well known results when the function h is Lipschitz (for details, see [3], [4], [11]).

4.2. **Regularity and the Principle of Optimality.** We first recall some results concerning the regularity of W^T.

Lemma 5. *Suppose that (6), (7), (8), (9), hold true. Assume that h is upper semicontinuous. Then we have:*

(i) *(Existence of optimal control) There exists an optimal trajectory starting from each point $(t, y) \in [0, T] \times \mathbb{R}^{N+1}$ i.e. there exists $\bar{y}(\cdot) \in S(t, y)$ such that*

$$W^T(t, y) = h(\bar{y}(T; t, y, \bar{u}(\cdot))) \text{ for all } (t, y) \in [0, T] \times \mathbb{R}^{N+1}$$

(ii) *W^T is upper semicontinuous.*

Next we recall the Bellman Principle of Optimality, from which the Hamilton-Jacobi-Bellman PDE is derived, satisfied by the value function.

Proposition 6. *(Principle of Optimality) Let $g : \mathbb{R}^N \to \mathbb{R}$ be a bounded function and suppose that (6), (7), (9), (8) hold true. Then for all $(t, y) \in [0, T] \times \mathbb{R}^{N+1}$ and $\alpha > 0$ such that $t + \alpha \leq T$:*

$$W^T(t, y) = \sup_{u \in \mathcal{U}_{[t,T]}} W^T(t + \alpha, y(t + \alpha)). \tag{21}$$

4.3. **The Hamilton-Jacobi Partial Differential Equation for the Mayer problem.** Using the Principle of Optimality for the optimal control problem with a finite horizon (18), (17) we can prove that when the value function W^T is regular enough (e.g., u.s.c.) then this function is the viscosity solution in the sense of Definition 3 of the following PDE:

$$\begin{cases} \dfrac{\partial W^T}{\partial t}(t, y) + \bar{H}\big(y, W^T(t, y), \dfrac{\partial W^T}{\partial y}(t, y)\big) = 0 \\ \\ (t, y) \in [0, T) \times \mathbb{R}^{N+1}; \quad W^T(T, \cdot) = h(\cdot) \end{cases} \tag{22}$$

where the Hamiltonian $\bar{H} : \mathbb{R}^{N+1} \times \mathbb{R} \times \mathbb{R}^{N+1} \to \bar{\mathbb{R}}$ is:

$$\bar{H}(y, r, q) = \max_{u \in U} \langle q, (f(x, u), L(x, u)) \rangle\,. \tag{23}$$

Proposition 7. *If h is a Lipschitz function then W^T is the unique Lipschitz viscosity solution to (22) with the final condition $W^T(T, \cdot) = h(\cdot)$.*

This result, based on the Principle of Optimality, is classical (see [3], [4], [11]). Also, it is easy to check that the value function is Lipschitzian when h is Lipschitzian.

Remark 8. *We can verify that $\frac{\partial W^T}{\partial z} = 1$ for almost all $(t, y) \in [0, T] \times \mathbb{R}^{N+1}$ and as a consequence V^T is the unique Lipschitz viscosity solution to (4) with the final condition $V^T(T, \cdot) = g(\cdot)$.*

4.4. **The upper semicontinuous case for for the Mayer problem.** In this section we suppose that the function h is upper semicontinuous (u.s.c.). We have already said in Lemma 5 that the value function W^T is also upper semicontinuous if h is upper semicontinuous.

Following [21] we formulate a theorem, which says that the value function is the biggest[8] u.s.c. supersolution of (22).

Theorem 9. *If (6), (7) (8), (9) hold true then $Hypo(W^T)$ is viability kernel in $[0,T] \times \mathbb{R}^{N+2}$ with target $\{T\} \times Hypo(h)$ for the dynamics $(t,x,z,r) \to \phi(t,x,z,r) = (1, f(x,U), L(x,U), 0)$:*

$$Hypo(W^T) = \mathcal{V}iab_\phi\big([0,T] \times \mathbb{R}^{N+2}, \{T\} \times Hypo(h)\big)$$

As a consequence, the value function is the biggest upper semicontinuous subsolution to (22); furthermore, it verifies the final condition $W^T(T,\cdot) = h(\cdot)$.

Also notice that $Hypo(W^T)$ is a closed set because of the assumption on function h's upper semicontinuity. This helps comparisons between hypographs.

Now we can formulate the main result of this paper on the value functions' dominance implied by the corresponding Hamiltonians' dominance.

Proposition 10. *If $\bar{H}_1 \leq \bar{H}_2$ then $W_2 \leq W_1$. Similarly if $\bar{H}_1 \geq \bar{H}_2$ then $W_2 \geq W_1$.*

Proof. We will give the proof for the first part of the proposition; the second part can be proved in a similar manner. Also, our proof will finish when we have shown that $\bar{H}_1 \leq \bar{H}_2$ implies $Hypo(W_2) \subset Hypo(W_1)$ because the inclusion is trivially equivalent to $W_2 \leq W_1$.

We know from Proposition 7 and Remark 8 that the value functions W_i are viscosity solution of the following PDE:

$$\begin{cases} \dfrac{\partial W^T}{\partial t}(t,y) + \bar{H}\big(t, y, \dfrac{\partial W^T}{\partial y}(t,y))\big) = 0 \\ (t,y) \in [0,T) \times \mathbb{R}^{N+1}, \quad W_i^T(T,\cdot) = h_i(\cdot). \end{cases} \tag{24}$$

where the Hamiltonians $\bar{H}_i : \mathbb{R}^{N+3} \times \mathbb{R}^{N+3} \to \bar{\mathbb{R}}$ are given by:

$$\bar{H}_i(t,y,r,q) = \max_{u_i \in U_i} \langle q, (f_i(x,u), L_i(t,x,u)) \rangle \,.$$

Recall from Theorem 9 that

$$Hypo(W_i^T) = \mathcal{V}iab_{\phi_i}\big([0,T] \times \mathbb{R}^{N+2}, \{T\} \times Hypo(h_i)\big) \tag{25}$$

where

$$(t,x,z,r) \to \phi_i(t,x,z,r) = (1, f_i(x,U_i); L_i(x,U_i), 0)\,.$$

By Proposition 4 the property (25) is equivalent to:

$$\begin{cases} p_t + \bar{H}_i(t,y,p_y) = 0 \\ \text{for all } (p_t,p_y,p_r) \in \mathcal{NP}_{Hypo(W_i)}\big(t,y,W_i(t,y)\big) \end{cases} \tag{26}$$

[8]Biggest with respect to canonical order in the class of functions.

So, for the case of $i = 1, 2$, if $\bar{H}_1 \leq \bar{H}_2$ and (26) is satisfied for $i = 2$ then we have that

$$\left\{ \begin{array}{c} p_t + \bar{H}_1(t, y, p_y)) \leq p_t + \bar{H}_2(t, y, p_y)) = 0 \\ \text{for all } (p_t, p_y, p_r) \in \mathcal{NP}_{Hypo(W_2)}\big(t, y, W_2(t, y)\big) \end{array} \right. \tag{27}$$

hence $Hypo(W_2)$ is a viability domain for ϕ_1. So, we have that

$$\mathcal{V}iab_{\phi_2}\Big([0,T]\times\mathbb{R}^{N+2}, \{T\}\times Hypo(h_2)\Big) \subset \mathcal{V}iab_{\phi_1}\Big([0,T]\times\mathbb{R}^{N+2}, \{T\}\times Hypo(h_1)\Big)$$

because $Hypo(W_1)$ is viability kernel for ϕ_1. Consequently, $Hypo(W_2) \subset Hypo(W_1)$ and $W_2 \leq W_1$, which finishes the proof. $\square$

5. Technology switching problem

We consider control systems indexed by $i \in \{1, 2, \ldots, n\}$, n is finite, whose dynamics are given by:

$$\dot{x}_i(s) = f_i(x_i(s), u_i(s)) \tag{28}$$

where the state variable x_i belongs to $\mathbb{R}^N$, the control $u_i(\cdot) : [0, \infty) \to U$ is a measurable function and $f_i : \mathbb{R}^N \times U \to \mathbb{R}^N$.

The control problem consists of

$$\text{Maximise} \int_t^T L\left(x_i(s; t, x, u(\cdot)), u(s)\right) ds \tag{29}$$

over all absolutely continuous solutions of (28), where $x_i(\cdot; t, x, u(\cdot))$ denotes the solution of (28) starting from $(t, x) \in [0, \infty) \times \mathbb{R}^N$.

Here $L : \mathbb{R}^N \times U \to \mathbb{R}$ is a given bounded function. If we denote by $\mathcal{U}_{[t,T]}$ the set of measurable controls on $[t, T]$ with values in U, then the value function corresponding to the optimal control problem (1) and (2) is given by:

$$V_i(t, x) = \sup_{u \in \mathcal{U}_i(t)} \int_t^T L\left(x_i(s; t, x, u(\cdot)), u(s)\right) ds \tag{30}$$

$$\begin{aligned} W_i(t, y) &= \sup_{u \in \mathcal{U}_i(t)} \left(h_i(y_i(T; t, x, u(\cdot)))\right) \\ &= \sup_{u \in \mathcal{U}_i(t)} \left(z + \int_t^T L\left(x_i(s; t, x, u(\cdot)), u(s)\right) ds\right) \end{aligned} \tag{31}$$

We note that

$$W_i(t, y) = V_i(t, x) + z \text{ for all } (t, y) = (t, x, z) \in [0, T] \times \mathbb{R}^{N+1}$$

and that here $h_i(x, z) = z$. We conclude that comparing W_i (for different i) is equivalent to comparing V_i.

We examine an example where the result obtained in Proposition 10 enables us to compare the value functions of two related optimal control problems without solving them explicitly.

Example 11. *Consider an optimal control problem with linear dynamics indexed by "technology" $i = 1, 2$*

$$f_i(x, u) = \theta_i u - \mu_i x \tag{32}$$

and with the following concave utility function

$$L(t, x, u) = e^{-\rho t}(\ln u - \beta x). \tag{33}$$

Assess if a positive switching time between the technology's usage exists.

Here we have $\phi_i(t, x, z, r) = (1, \theta_i U - \mu_i x, e^{-\rho t}(\ln U - \beta x), 0)$. We aim to examine the viability kernels for the auxiliary dynamics associated with each technology. In other words, we want to see if a hypograph of the value function of one technology is included in the hypograph of the value function of the other technology. If so, we will conclude that there is no positive switching time between the use of the technologies.[9]

Because of Result 10 we can rely on the relationship between the Hamiltonians. Let us write the Hamiltonian (23) for technology i:

$$\begin{aligned}\bar{H}_i(t, y, r, q) &= \max_{u \in U} \langle q, (\theta_i u - \mu_i x, e^{-\rho t}(\ln u - \beta x)) \rangle \\ &= \langle q, (\theta_i u_i - \mu_i x, e^{-\rho t}(\ln u_i - \beta x)) \rangle .\end{aligned}$$

Here u_i is the maximiser, y, r, q are fixed, of dimensions $2, 1, 2$ respectively. Now, we can prove that if $\theta_1 < \theta_2$ and $\mu_2 \geq \mu_1$ then $\bar{H}_1 \leq \bar{H}_2$ so, we have an sufficient condition for the case where there is no switching.

Indeed we have that

$$\begin{aligned}\bar{H}_1(t, y, r, q) &= \max_{u \in U} \langle q, (\theta_1 u - \mu_1 x, e^{-\rho t}(\ln u - \beta x)) \rangle \\ &= \langle q, (\theta_1 u_1 - \mu_1 x, e^{-\rho t}(\ln u_1 - \beta x)) \rangle\end{aligned}$$

It is suficcient to find an u such that

$$\begin{aligned}\bar{H}_1(t, y, r, q) &= \max_{u \in U} \langle q, (\theta_1 u - \mu_1 x, e^{-\rho t}(\ln u - \beta x)) \rangle \\ &= \langle q, (\theta_1 u_1 - \mu_1 x, e^{-\rho t}(\ln u_1 - \beta x)) \rangle \\ &\leq \langle q, (\theta_2 u - \mu_2 x, e^{-\rho t}(\ln u - \beta x)) \rangle \\ &\leq \max_{u \in U} \langle q, (\theta_2 u - \mu_2 x, e^{-\rho t}(\ln u - \beta x)) \rangle \\ &= \bar{H}_2(t, y, r, q)\end{aligned}$$

If we denote by $q := (q_x, q_z)$ and by $\Gamma(u) := \langle q, (\theta_2 u - \mu_2 x, e^{-\rho t}(\ln u - \beta x)) \rangle - \langle q, (\theta_1 u_1 - \mu_1 x, e^{-\rho t}(\ln u_1 - \beta x)) \rangle$

$$\begin{aligned}\Gamma(u) &:= \langle q, (\theta_2 u - \mu_2 x, e^{-\rho t}(\ln u - \beta x)) \rangle - \langle q, (\theta_1 u_1 - \mu_1 x, e^{-\rho t}(\ln u_1 - \beta x)) \rangle \\ &:= q_x(\theta_2 u - \theta_1 u_1 + (\mu_1 - \mu_2) x) + q_z e^{-\rho t}(\ln u - \ln u_1)\end{aligned}$$

[9] Notice that we abstract from an implementation cost. However, if the inclusion is not proper (and/or the Hamiltonians equal for some arguments) we can argue that the switch will not occur because the "lazy" agent will have no incentive for a change.

we see that

$$\lim_{u\to\infty} \Gamma(u) := \infty \text{ if } q_x > 0$$

and that

$$\Gamma(u_1) \geq 0 \text{ if } q_x \leq 0,\ \theta_2 < \theta_2 \text{ and } \mu_2 \geq \mu_1$$

because $\theta_i, \mu_i, \beta, \rho, u, x$ are positive real numbers.

6. Concluding remarks

We have presented an approach suitable for the determination whether a "new" technology will replace the "old" technology. All the regulator needs to do is to compare the Hamiltonians of the optimal control problems formulated for each technology. We have seen that a linear dynamics and a simple concave utility function exclude such a possibility.

More generally, the approach presented in this paper help solve a two-stage optimal control problem by indicating when the problem will have no second stage.

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