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Abstract

In this paper we want to describe a model of the dynamics of the Bitcoin cryptocurrency system. We can define a duality in these dynamics: Bitcoin mostly behaves as a deterministic system and in some time intervals, much shorter, it enters a stochastic regime. In particular, using Poincaré's recurrence theorem, it was possible to study when the transition from one regime to another occurs. Furthermore, by applying our hypothesis to real data it was possible to explain a reason why the Bitcoin system is affected by such a "high volatility".

Keyboards: Ergodic Theory, Bitcoin, Finance, Deterministic, Stochastic

JEL codes: C44, E37, G17, F17

MSC: 62P05, 37K99, 91G60, 37A60, 82D99

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1 Introduction

Bitcoin is a cryptocurrency introduced by S. Nakamoto [14] in 2009 based on the idea that no central government could have control over the currency in circulation. The blockchain is the fulcrum of these systems and is essentially a register in which the data of the owners of the currency are entered, transactions occur in an encrypted manner. Essentially, it is a data structure consisting of a list of transaction blocks linked together so that each one refers to the previous one in the chain. Each block in the blockchain is identified by a hash generated using the SHA256 cryptographic algorithm on the block header. A block is a data structure that aggregates transactions to include them in the public register [1].

Over time, the growing popularity of this cryptocurrency has grown exponentially also following the bubbles that have been created since 2011. Basically, the mathematical structure behind this cryptocurrency is linked to the architecture of mining and to encryption. From an architectural point of view, the integrity of the blockchain network is guaranteed through consensus algorithms such as *Proof-of-Work* (PoW) and *Proof-of-Stake* (PoS), that solves the Byzantine Generals Problem [7]: complexity is linked precisely to this type of problem [10]. On the other hand, from the point of view of cryptography, the mathematical complexity is linked to the parabolic equations underlying the cryptographic functions (as in the case of the SHA256 algorithm), which represent the backbone of the blockchain. Or again, given the presence of Options on Bitcoin, the completely mathematical aspect occurs when trying to create a price dynamics model for these options [2].

In [9], we tried to create an affinity between a cryptocurrencies system and statistical mechanics; in this way we have defined how (although not knowing the Hamiltonian of the system), it is possible to represent a cryptocurrencies system as a microcanonical ensemble whose Boltzmann entropy can be determined. What we want to propose in this paper is the use of the Ergodic Theory to model the dynamics of the Bitcoin system. We consider the Poincaré's recurrence theorem [6] and we show that Bitcoing system behaves mostly as a deterministic (albeit chaotic) system but, in some time intervals, it acts as a stochastic system. The Poincaré Theorem is used in relation to the problem of the stability of the solutions of the restricted three-body problem by extending its previous notion of stability: according to Poisson, a system will be considered stable if it returns infinitely often in positions arbitrarily close to the initial position. So the theorem implies that a system with three degrees of freedom that conserves volume has an infinite number of solutions that are Poisson-stable. What we want to demonstrate is how it is possible to apply Ergodic Theory to a still under-studied system, e.g. Smith [3] did something similar with the pool table, highlighting how, after a finite amount of time, the system returns to a state arbitrarily close to its initial state. The paper structure is the following: in section 2 we recalled the theoretical assumptions underlying the ergodic theory giving the proof of Poincaré's recurrence theorem; in section 3 we defined on which theoretical basis we came to the conclusion that we can assume the existence of a dualism between the regimes that describe the Bitcoin dynamics, and we consider a dataset of real prices; finally in section 4 some conclusions are drawn.

2 The Ergodic theory

The ergodic problem arises from Boltzmann's ideas on statistical mechanics and it is studied and generalized especially by von Neumann and Birckhoff [11]. The original ergodic hypothesis was the following: *the constant energy surface is composed of a huge (but finite) quantity of cells, which can be numbered; during the temporal evolution a trajectory passes through all the cells thus providing the possibility of replacing an average over time with an average in the phase space.*

With this assumption, the time average and the expectation of an observable are the same [8]. In particular, we can define M as the phase space of a system, $f : M \to \mathbb{R}$ an observable of this system, $T : M \to M$ the time evolution such that if $x \in M$ is the initial state and the measurements of the observable f are given by $f(x), f(T(x)), ..., f(T^k(x)), ...$; the time average is given by

$$\frac{\sum_{k=0}^{n-1} f(T^k x)}{n} \tag{1}$$

while the space average of the observable is

$$\int_X f \, d\mu \tag{2}$$

Boltzmann ergodic hypothesis defines that for almost every initial state $x \in M$ the time averages of any observable f converge, as time tends to infinity, to the space average of f.

Modern ergodic theory is considered a branch of measure theory with objectives far beyond the original Boltzmann problem. In particular, Birkhoff got a result that holds in general for any measure preserving transformation (while the Boltzmann ergodic hypothesis is not true for any measure preserving transformation). Let $T : M \longrightarrow M$ an endomorphism preserving transformation, and let $(M, \mathcal{N}, \mu)$ a σ-finite measure space, Birkhoff's ergodic theorem [4] states that for any $f \in L^1(M, \mathcal{N}, \mu)$ the limit

$$\bar{f}(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x)) \tag{3}$$

exists for μ-almost every $x \in M$. Moreover, $\bar{f} \in L^1(M, \mathcal{N}, \mu)$ and if $\mu(X) < \infty$, then

$$\int_X f \, d\mu = \int_X \bar{f} \, d\mu \tag{4}$$

For a measurable $(T_t)_{t \geq 0}$, Birkhoff's ergodic theorem states that for any function $f \in L^1(M, \mathcal{N}, \mu)$ the limit

$$\bar{f}(x) = \lim_{t \to \infty} \frac{1}{t} \sum_{k=0}^{n-1} f(T_t(x)) \tag{5}$$

exists a.e. with the same properties as f.

Ergodic theory, in addition to the field of dynamic systems, also applies to other systems that may be of interest in probability, geometry, number theory or even economics, as defined by Peters [12] and Day et al. [5].

2.1 The Poincaré Recurrence Theorem

The Poincaré recurrence theorem is one of the basic theorems of ergodic theory; it states that certain systems will, after a sufficiently long but finite time, return to a state arbitrarily close to, or exactly the same as, their initial state [6].

First, let's define an μ-invariant map.

Definition 1. *Let M be a non empty-set, $\mathcal{N}$ a $\sigma-$algebra on M, μ a finite measure on M respect to the $\sigma-$algebra $\mathcal{N}$ and $T : M \to M$ a $\mathcal{N}$-measurable then T is called $\mu-$invariant if for all $A \in \mathcal{N}$ we have that:*

$$\mu(f^{-1}(A)) = \mu(A) \tag{6}$$

We can define the theorem as following.

Theorem 1. *Let M be a non-empty set, $\mathcal{N}$ a $\sigma-$algebra on M, μ a finite measure on M respect to the $\sigma-$algebra $\mathcal{N}$ and $T : M \to M$ a $\mathcal{N}$-measurable and $\mu-$invariant map on M, then for every $U \in \mathcal{N}$ such that $\mu(U) > 0$, exists $\bar{U} \subseteq U$, $\mathcal{N}-$measurable set such that $\mu(\bar{U}) = 0$ and for all $x \in U \setminus \bar{U}$ exists $n \in \mathbb{N} \setminus \{0\}$ such that*

$$T^n(x) \in U. \tag{7}$$

Proof. We fix $U \in \mathcal{N}$ such that $\mu(U) > 0$ and we define $\bar{U} \subseteq U$ in the following way:

$$\bar{U} := \{x \in U \mid T^n(x) \notin U,\ \forall n \in \mathbb{N} \setminus \{0\}\}.$$

$\bar{U}$ is a $\mathcal{N}-$measurable set because T is $\mathcal{N}$-measurable and by definition of $\bar{U}$ we can say that

$$\bar{U} = \bigcap_{n=1}^{\infty} (T^{-n}(M \setminus U) \cap U).$$

Thus we introduce the following sequence $\{U_n\}_{n \in \mathbb{N} \setminus \{0\}}$ of $\mathcal{N}-$measurable sets of M defined in this way:

$$U_n := T^{-n}(\bar{U}) \quad \forall n \in \mathbb{N} \setminus \{0\}.$$

Then for all $n, m \in \mathbb{N} \setminus \{0\}$, such that $n \neq m$, we want to check that

$$U_n \cap U_m = \emptyset. \tag{8}$$

Using an absurd argument and supposing that exists $n, m \in \mathbb{N} \setminus \{0\}$ such that $n < m$ and $x_0 \in M$ such that $x_0 \in U_n \cap U_m$ we can conclude that $y_0 = f^n(x_0) \in \bar{U}$ and $T^{m-n}(y_0) = T^m(x_0) \in M$ and this is absurd by definition of $\bar{U}$. To conclude the proof we just use $\sigma-$additivity of μ and the $\mu-$invariance of T in the following way:

$$1 = \mu(M) \geq \sum_{n=1}^{\infty} \mu(U_n) = \sum_{n=1}^{\infty} \mu(\bar{U})$$

and so $\mu(\bar{U}) = 0$. □

3 Problem formulation

What we want to verify using ergodic theory is that the Bitcoin system behaves "semi-deterministic" at certain times and stochastic at others. In [9] we defined how the different economic subjects are fully described by 2 variables, $\{x_i, y_i\}$; which indicate respectively the ability to buy and the ability to sell of a certain financial asset. We also defined how the price of the different cryptocurrencies is the synthesis of the two variables described above.

However, in this model we make a simplification compared to the previous one: our phase space becomes the price of Bitcoins. The idea behind this model is that there is an upper bound (which depends on the standard deviation of prices) of recurrence of the Poincaré theorem referring to price dynamics.

The assumption of this upper bound is made since from empirical evidence we note that the Bitcoin system does not follow the same behavior as a deterministic system in the sense of physics (where it is known through experiments that an upper bound does not exist); for us it is the existence of this upper bound that determines the "chaotic" behavior of prices (generating a "deterministic oscillation"). When this threshold is exceeded according to our model, we are out of the deterministic phase and we have entered a new phase that we can assume to be stochastic, until the situation is re-stabilized by verifying that we have returned back below the upper bound. As we can see in the figure 1, in small time intervals (e.g. from 14:29 - 15:11) the system in place behaves in a stochastic way. From the graph we can also see that in most of the times the system under examination is in a deterministic regime and is forced in small time intervals to pass into a stochastic regime, passing as soon as possible to a deterministic type system. In this case it is as if the inertia of the system tended to determinism.

So according to our model, the daily Bitcoin dynamics will present different deterministic and some stochastic phases. In particular, these dynamics can be uniquely identified by k **price endomor-**

Figure 1: Extract of the Bitcoin price of 21/5/2020, kitco.com

phisms and j **SDE with constant coefficients**. So we can divide a day of Bitcoin dynamics into $k+j+1$ time instants $t_0, ..., t_{k+j+1}$ which divide the day into $k+j$ time intervals (t_i, t_{i+1}) which can be deterministic or stochastic. The k deterministic intervals are associated with k endomorphisms $T_1, ..., T_k : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ which determine dynamics in the associated range; while the j stochastic intervals are associated with j pairs of coefficients (μ_j, σ_j) with which the solution price distribution of the linear constant coefficient SDE of the associated range is determined

$$dX = \mu_j \, dt + \sigma_j \, dW \tag{9}$$

3.1 Dataset

We consider the evolution of the Bitcoin system in a discrete way with 1 minute steps. We used the prices on 20/5/2020 and 21/5/2020 for a total of 1442 observations for each day, from midnight to midnight. These prices are derived from the Bitfinex exchanger and refer to the USD [1]. The difficulty of this type of analysis lies precisely in the construction of the dataset as the exchangers do not collect the 1 minute prices in a list (e.g. as happens for the daily prices in an annual interval), but they give the possibility of being able to analyze the graph interactive where a

[1] kitco.com/bitcoin-price-chart-usd/

different price corresponds to the different time points. What we did in this case was precisely analyzing the different time points corresponding to the various minutes of the day, transcribing them and subsequently carrying out the data analysis.

3.2 Numerical results

In this section we will show how, through the use of real prices of Bitcoin, the dynamics previously modeled faithfully reflects reality. The analysis carried out in this case do not consider the "edge effects", that is, they do not take into account what happens at the ends of each time series. The reference set of Poincaré's theorem (U in theorem 1) is the interval of center the average of prices with radius the standard deviation $[\mu-\sigma, \mu+\sigma]$. This interval was chosen based on the evidence of the data since in general there could be a coefficient that multiplies the standard deviation, making the interval $[\mu - k \cdot \sigma, \mu + k \cdot \sigma]$. E.g. we tested the intervals by placing $k = 1, \frac{1}{3}, \frac{3}{4}$, but $k = 1$ it was the most suitable since we considered 6 hours intervals in calculating the average; therefore, as the dynamics study interval varies, the optimal k can change.

What we are going to do is to compare the time series of Bitcoin prices with the interval created previously, breaking this series into 6 hours time intervals. When a price falls within the range, a Boolean value corresponding to True will be assigned to the corresponding minute; otherwise, a False value will be assigned to that minute. At this point we will have a long sequence of True or False values of which we will count the consecutive False until the sequence is interrupted by a True value. E.g. our reference range is constructed as $[9010, 9030]$ and the price at the time $t = 0$ is equal to 9020. In this case we can say how the price corresponding to that instant falls in the "mark" interval that minute with a value of True. Continuing, at the time $t = 1$ we notice how the price is 9008; in this case, it is not within the range and the corresponding instant in time will be "marked" with a False value. At this point, False values will be counted until a True value is encountered.

The upper bound for this time series, since we considered 6 hours for the calculation of the mean, was found to be optimal half of the standard deviation $(\frac{\sigma}{2})$. In the following figures we can see how the time series of prices behaves when the dynamic changes from deterministic to stochastic. The tables will consist of 3 columns indicating the minute, the price at that moment and the Boolean value indicating whether or not that price belongs to the previously defined range.

Minute	Price	Belong
11:13	9737,9	True
11:14	9723,2	False
11:15	9675	False
11:16	9674,9	False
⋮	⋮	⋮
11:58	9415	False
11:59	9400,3	False
12:00	9338,9	True

(a) Corresponding chart

Figure 2: Bitcoin price extract 20/5 from 11:13 to 12:00

Minute	Price	Belong
16:20	9527,3	True
16:21	9532,8	False
16:22	9569,8	False
16:23	9564,1	False
⋮	⋮	⋮
17:17	9565,5	False
17:18	9565,5	False
17:19	9559,8	True

(a) Corresponding chart

Figure 3: Bitcoin price extract 20/5 from 16:20 to 17:19

Minute	Price	Belong
4:39	9395,1	True
4:40	9369,1	False
4:41	9268,9	False
4:42	9329	False
⋮	⋮	⋮
5:14	9389	False
5:15	9390,6	False
5:16	9414,9	True

(a) Corresponding chart

Figure 4: Bitcoin price extract 21/5 from 4:40 to 5:16

Minute	Price	Belong
14:28	8975,8	True
14:29	8968,7	False
14:30	8961,3	False
14:31	8955,9	False
⋮	⋮	⋮
15:10	8951,1	False
15:11	8968,9	False
15:12	9003,8	True

(a) Corresponding chart

Figure 5: Bitcoin price extract 21/5 from 14:28 to 15:12

From this analysis it turns out that as long as there is no consecutive False series with length greater than the upper bound imposed, then there exists a function from $T : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ which manages the price dynamics from $t = 0$ to the beginning of the False list. So we can say that as soon as a list of False values starts, we are in the stochastic phase.

At the end of a stochastic phase, to determine when a subsequent one would begin, it would be necessary to average further the prices following the beginning of the False list, thus creating a new interval based on the average and standard deviation of these prices and carry out the previous analysis. For this reason, based on the results obtained, we can affirm that the "high volatility" of Bitcoins is nothing more than the transition from a deterministic regime to a stochastic one.

4 Conclusions

In this paper we have proposed a model of the dynamics of Bitcoin prices and we have applied it to real data with temporal steps of 1 minute length, recognizing a dualism between deterministic and stochastic regime. Basically, the Bitcoin system has a deterministic dynamic which is, at various times, broken by the stochastic dynamic. To understand how the system passes from one regime to another we use the Poincaré's theorem and we assume the existence of an upper bound of recurrence.

The next step is to continue in the wake of the study of the dynamics of Bitcoin by introducing a new type of stochastic dynamics based on a finite set of functions that determine the dynamics of the system.

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