A mean-standard deviation exposition of the theory of the firm under uncertainty

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A Mean-Standard Deviation Exposition of the Theory of the Firm under Uncertainty: A Pedagogical Note

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In his paper in this *Review*, Agnar Sandmo derived a set of major conclusions indicating that a competitive firm behaves differently under uncertainty than in a world of certainty. Hayne Leland extended the results to noncompetitive market structures. The purpose of this paper is to show that firms' behavior under uncertainty can be easily derived using a geometric approach based on the mean-standard deviation framework introduced by Harry Markowitz (1952, 1959) and extended by James Tobin.

Section I discusses briefly the major conclusions related to the firm's behavior under uncertainty and gives a general description of the approach followed in this paper. Section II introduces the firm's attitude toward risk. Section III describes a model of profit maximization under conditions of risk, using a mean-standard deviation-of-profit framework. In Section IV, the model is applied to derive geometrically the major conclusions of the theory of the firm under uncertainty stated in Section I.

I. The Theory of the Firm under Uncertainty

A. *General*

Traditional microeconomic theory assumes that under certainty and regardless of the market structure, a firm's objective is to maximize its profit for the given constraints. The optimal output is obtained at the point at which the firm's marginal cost equals its marginal revenue. However, if uncertainty prevails, there is no reason to believe, a priori, that this maximization principle will hold.

Both Sandmo and Leland have used the assumption that faced with uncertainty, the firm will maximize the expected value of its utility of profit. The introduction of a nonlinear utility function[1] permits the incorporation of the firm's attitude toward risk into the decision-making process. Sandmo assumes a subjective probability distribution of prices with the level of output and cost function known in advance, that is, under the firm's control. Consequently, since the firm is unable to influence the price distribution, it is considered a price taker, and Sandmo's model is valid only under condition of perfect competition. Leland's model is more general. It assumes a random demand function that allows us to handle noncompetitive structure, where a firm can fix the level of output and/or the price. Sandmo's conclusions are shown to be a special case of Leland's model when the market is competitive.

The major conclusions that follow from the theory of the firm operating under uncertainty are: (i) if a firm is risk averse[2] its optimal output is smaller than the certainty output; (ii) if a firm displays decreasing absolute risk aversion, its optimal output varies inversely with its fixed costs; (iii) if a competitive firm displays decreasing absolute risk aversion, it has an upward-sloping supply curve; (iv) if a firm is risk averse, an equilibrium exists, even under constant or decreasing marginal costs; (v) if a firm is

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[1] For linear utility functions, the firm will be maximizing expected profit which implies that the firm is indifferent to the magnitude of risk that is associated with the production under uncertainty: the firm is risk neutral.

[2] Risk aversion is defined in Section IIA.

risk averse, equilibrium requires the existence of positive profit.

In addition to the above conclusions derived by Sandmo and Leland, I will show that: (vi) if a firm displays decreasing absolute risk aversion, its optimal output varies inversely with its perceived level of risk; (vii) if a firm displays a decreasing absolute risk aversion, its optimal output varies inversely with its variable costs; (viii) under uncertainty the competitive firm will produce a higher output than the noncompetitive firm selling at the same price.

B. *Risk, Expected Profit, and Equilibrium: A Description of the Mean-Standard Deviation Approach*

In this paper I attempt to picture the firm's optimum output under price uncertainty in a mean-standard deviation-of-profit plane, with the standard deviation of profit considered as a proxy for risk. On this plane the firm's indifference map is drawn, a geometrical representation of its attitude toward risk. In Section III, I derive the relationship between expected profit and the standard deviation of profit (risk), which is called "the profit-opportunity locus," and drawn on the same plane. Equilibrium is found at the point where the profit-opportunity locus and the firm's highest indifference curve are tangent. From this equilibrium point it is shown that the corresponding level of optimum output under uncertainty can be easily derived.

II. The Firm's Attitude Toward Risk

A. *Definitions*

A firm is assumed to have a utility function of profit that displays positive marginal utility of profit. The firm's attitude toward risk is indicated by the change in its marginal utility when profit varies. A firm is said to be risk averse if its marginal utility of profit decreases with increasing profit, risk neutral if its marginal utility of profit is constant, and a risk seeker if its marginal utility of profit increases with increasing profit.[3] In this paper, firms are assumed to display risk aversion.

A risk-averse firm is said to display decreasing absolute risk aversion if its risk aversion decreases with increasing profit, constant absolute risk aversion if its risk aversion remains constant when profit changes, and increasing absolute risk aversion when its risk aversion increases with increasing profit.[4]

B. *Indifference Curves in the Mean-Standard Deviation-of-Profit Plane*

The expected utility of profit can be written as a function of the first two moments of the distribution of profit.[5] An indifference curve in the mean-standard deviation-of-profit plane is then represented by the locus of points for which the expected utility of profit remains constant. An indifference map is generated by varying the constant value of the expected utility of profit. In the mean-standard deviation-of-profit plane, the risk-averse firm has indifference curves with positive marginal rate of substitution between expected profit and risk. For the risk-neutral firm, the marginal rate of substitution is zero and for the risk-seeker firm it is negative.[6]

[3]Mathematically we have: (i) $U'(\pi) > 0$ and (ii) $U''(\pi) < 0$ for the risk-averse firm, $U''(\pi) = 0$ for the risk-neutral firm, and $U''(\pi) > 0$ for the risk-seeking firm where $U(\pi)$ is the firm's utility-of-profit function.

[4]The absolute risk-aversion function is written $r_A = -U''(\pi)/U'(\pi)$. Mathematically we have $r'_A > 0$ for the firm with decreasing absolute risk aversion, $r'_A = 0$ for the firm with constant absolute risk aversion, and $r'_A < 0$ for the firm with increasing absolute risk aversion.

[5]This assumption implies that profits are normally distributed. We exclude quadratic utility curves since they display increasing absolute risk aversion.

[6]Mathematically we have $EU(\pi) = f(E, \sigma)$, where E is the expectation operator and σ the standard deviation of profit. The marginal rate of substitution between expected profit (E) and risk (σ) is $dE/d\sigma = -(\partial EU/\partial\sigma)/(\partial EU/\partial E)$. The partial $\partial EU/\partial\sigma$ is negative for risk-averse firms, zero for risk-neutral firms, and positive for risk-seeking firms. The partial $\partial EU/\partial E$ is positive for the three cases. It follows that the marginal rate of substitution is positive for risk-averse firms, zero for risk-neutral firms, and negative for risk-seeking firms.

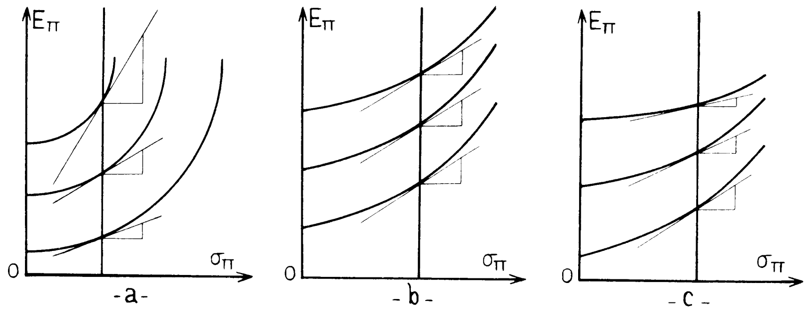

FIGURE 1

C. *The Type of Risk Aversion and the Shape of the Indifference Curves*

The type of risk aversion can be measured by the variation in the firm's marginal rate of substitution when risk is held constant. If, starting from a given level of risk, we move up to higher indifference curves, the same additional unit of risk requires increasing compensating expected profit, then the firm displays increasing absolute risk aversion: its marginal rate of substitution increases with profit for a given level of risk as illustrated in Figure 1a. If the marginal rate of substitution is constant, the firm displays constant absolute risk aversion as shown in Figure 1b. If the marginal rate of substitution decreases with increasing expected utility of profit, the firm displays decreasing absolute risk aversion as in Figure 1c.

III. The Model

A. *The Model under Perfect Competition*

Assume that

(1) $$p = \mu + e$$

where the prices (p) are stochastic and expressed as the sum of the expected price (μ) and a random element (e) with *constant* variance and zero expected value. The *ex ante* prices fluctuate around their expected value and the *ex post* price may differ from μ. Equation (1) implies that

(2) $$E(p) = \mu$$

(3) $$\sigma(p) = \sigma(e) = \text{constant}$$

where E is the expectation operator, $\sigma(p)$ the standard deviation of prices, and $\sigma(e)$ the standard deviation of the random element (e). Firms are further assumed to maximize the expected utility of their profit, that is,

(4) $$\text{Max}\, E[U(\pi)]$$

where U is the firm's utility function and π the level of profit. The firm's objective function (4) is to be maximized given the constraints, that is, the firm's revenues and costs expressed in a profit function.

The Cost Function. The cost function is assumed to be known with certainty and given by

(5) $$C(q) = V_1(q) + F$$

when (q) is the known level of output, C the total costs, V_1 the variable costs such as $V_1(0) = 0$, and F the fixed costs.

The Profit Function. The profit function (π) is given by

(6) $$\pi = TR - C = p \cdot q - V_1(q) - F$$

where TR is the firm's total revenues.

The Profit-Opportunity Locus. Using equation (6) we can obtain the expected profit and the standard deviation of profit from which the profit-opportunity locus is derived. From equation (6) we have

(7) $$E(\pi) = \mu \cdot q - V_1(q) - F$$

(8) $$\sigma(\pi) = q \cdot \sigma(p)$$

where $E(\pi)$ and $\sigma(\pi)$ are the expected profit and the standard deviation of profit, respectively. Equation (8) states that $\sigma(\pi)$, which is considered as a proxy for the risk faced by the firm operating under uncertainty, is proportional to the level of output (q). The constant factor is the standard deviation of prices. Since $\sigma(p)$ is known, and since the output q is under the firm's control, it follows that the firm can choose the level of risk $\sigma(\pi)$ it is willing to bear simply by varying the level of output. The profit-opportunity locus is obtained from equations (7) and (8). From equation (8) we have

$$q = \sigma(\pi)/\sigma(p) \tag{9}$$

Substituting in equation (7) we get

$$E(\pi) = \{(\mu/\sigma(p)) \cdot \sigma(\pi) - F\} - \{V_2(\sigma(\pi))\} \tag{10}$$

Note that V_2 is the variable cost function in terms of $\sigma(\pi)$ rather than q, and the constant $\sigma(p)$ enters in the coefficients of the function V_2.

B. *The Model under Imperfect Competition*

Under imperfect competition, we assume a demand function $p = f(q) + e$ in which the random disturbance is additive. Assuming that the firm is "quantity setting," the profit function becomes

$$\pi = \{f(q) + e\} \cdot q - V_1(q) - F \tag{11}$$

from which we derive

$$E(\pi) = f(q) \cdot q - V_1(q) - F \tag{12}$$

$$\sigma(\pi) = q \cdot \sigma(e) = q \cdot \sigma(p) \tag{13}$$

since $\sigma(p) = \sigma(e)$. It follows that

$$E(\pi) = h(\sigma(\pi)) - V_2(\sigma(\pi)) - F \tag{14}$$

where the revenue function $h(\sigma(\pi))$ satisfies the condition $h(0) = 0$. Equation (14) is the profit-opportunity locus under imperfect competition for the quantity-setting firm.

IV. Applying the Model: A Comparative Static Analysis

To prove the set of conclusions stated in Section IA, we must subject the model to a comparative static analysis. Starting from an initial equilibrium point, a change in one of the parameters, that is, the expected price μ, the risk $\sigma(p)$,[7] the fixed costs F, or the coefficients of the variable cost function V_2, will lead to a new equilibrium point. A comparison of the initial and final equilibria allows us to reach the desired conclusions.

A. *Comparative Output: Certainty versus Uncertainty* (Conclusion i: Figure 2)

Assuming perfect competition and uncertainty, equation (10) holds: expected profit is equal to the difference between a linear function of $\sigma(\pi)$ with slope $(\mu/\sigma(p))$ and intercept $(-F)$, and the transformed variable cost function V_2 which is assumed to display increasing marginal costs. (This assumption is relaxed in Section IVD.) In Figure 2, $E(\pi)$ is the shaded area between the straight line Ff and the curve V_2.

If certainty prevails, then the profit function becomes

$$\pi = (p \cdot q - F) - V_1(q) \tag{15}$$

In order to compare output under certainty to the uncertain output, p the price under certainty is set equal to μ, and equation (15) is rewritten as the difference between $(\mu \cdot q - F)$ and $V_1(q)$. This is graphed on the same diagram as equation (10) where the vertical axis reads profit π, instead of expected profit $E(\pi)$, and the horizontal axis reads (q) instead of $\sigma(\pi)$. We assume $\sigma(p)$ to be higher than one.[8] Both areas are reproduced in Figure 2 to give the exact shape of $E(\pi)$ and π as a function of $\sigma(\pi)$ and q, respectively, on the mean-standard deviation plane. Superimposing the indifference map on the same plane we can obtain the equilibrium point at A under uncertainty for a risk-averse firm. In the case of risk neutrality, equilibrium is found at point N, the maximum of the expected profit

[7]The risk was defined as $\sigma(\pi)$. However, according to equation (8) any change of the constant $\sigma(p)$ will affect risk as $\sigma(\pi)$.

[8]This assumption is made for expository reasons and does not affect the generality of the results.

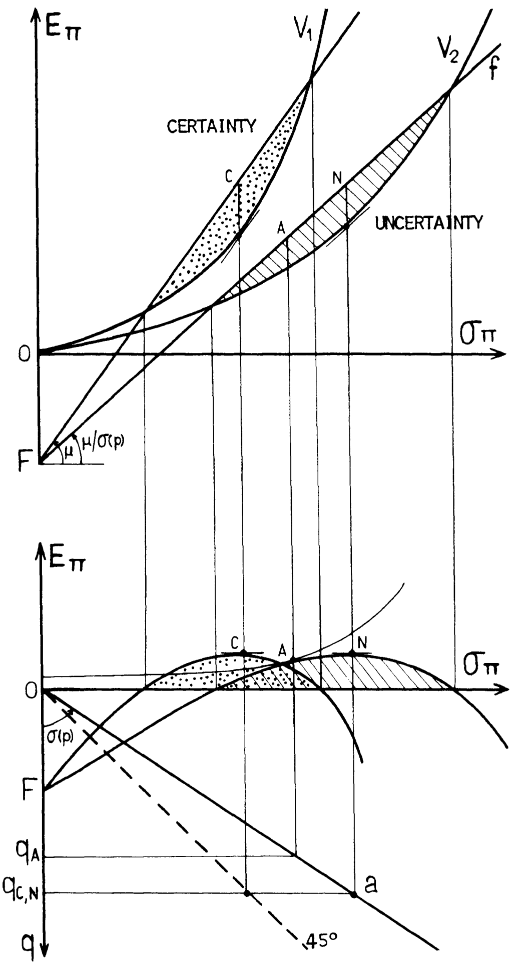

FIGURE 2

curve, since the risk-neutral firm maximizes expected profit. Equilibrium under certainty is found at point C, the maximum of the profit curve.

The Optimum Output. Output can be read along the q axis with output increasing when we move away from the origin downward. The certainty output (q_c) is found using the 45° line, since the $\sigma(\pi)$ axis is the output axis under certainty. Output under certainty and risk neutrality being equal (because risk-neutral firms maximize profit regardless of risk), we can derive the relevant Oa line under uncertainty since we have two points through which the line passes: the origin and point a. Once Oa is

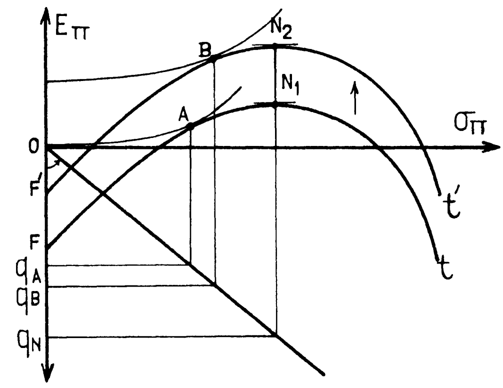

FIGURE 3

drawn we can find output under uncertainty and risk aversion. Since the equilibrium point A is to the left of point N, we obtain a smaller output q_A, which proves Conclusion i under perfect competition and increasing marginal costs. Output under uncertainty and risk aversion is smaller than output under risk neutrality or certainty. The result is independent of the shape of the indifference curve except that risk aversion prevails.

B. *Change in Fixed Costs under Uncertainty* (Conclusion ii: Figure 3)

Assuming perfect competition and increasing marginal costs (later these two assumptions will be relaxed) and referring to Figure 3, we can see that a reduction of

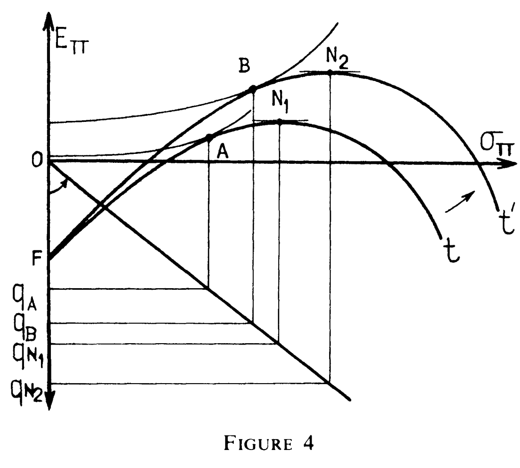

FIGURE 4

fixed costs from OF to OF' shifts up the profit-opportunity locus in a parallel fashion from Ft to $F't'$. Assuming that firms display decreasing absolute risk aversion, the indifference map is of the type described in Figure 1c. The equilibrium point moves from A to B and the corresponding level of output from q_A to q_B with $q_B > q_A$, output varies inversely with fixed costs. For the risk-neutral firm, the equilibrium point moves from N_1 and N_2 with the level of output q_N unaffected. In the case of certainty a similar result is easily derived.

C. *Change in Expected Price and the Supply Curve under Perfect Competition and Uncertainty* (Conclusion iii: Figure 4)

The following assumes that the firm revises its expectations about future prices. An increase in expected price from μ to μ' will rotate the straight line Ff around point F, leaving variable costs constant.[9] As a result, the profit-opportunity locus will shift from Ft to Ft' as indicated in Figure 4. Given decreasing absolute risk aversion, the equilibrium will move from A to B when expected price rises from μ to μ'. The corresponding level of optimum output increases from q_A to q_B. It follows that the decreasing absolute risk-averse competitive firm has an upward-sloping supply curve as in the case of certainty. The conclusion holds true for the risk-neutral firm.

D. *The Cases of Constant and Decreasing Marginal Costs* (Conclusions iv, v: Figures 5, 6)

Referring to Figure 5, observe that an equilibrium exists under constant marginal costs at point A to which correspond the optimum output q_A and the break-even point G.[10] When marginal costs decrease, we move to a new equilibrium point at B

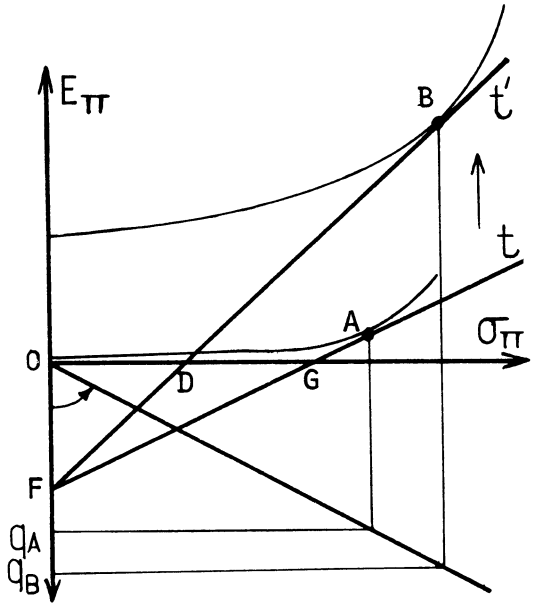

FIGURE 5

with a larger optimum output q_B and a new break-even point D. Under certainty, equilibrium does not exist: firms will maximize their level of output in order to achieve maximum profit (traditional break-even analysis).

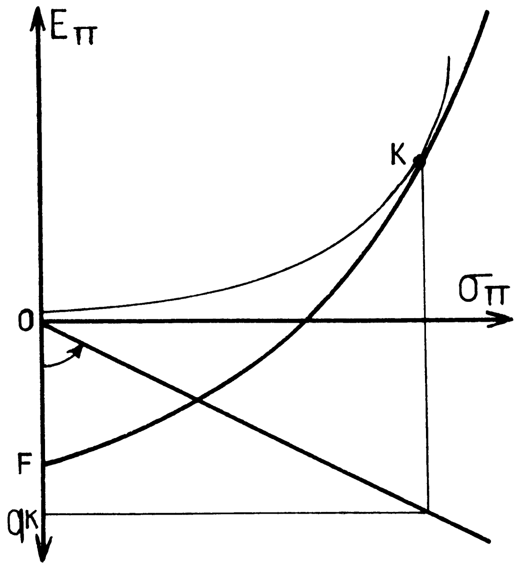

FIGURE 6

[9] Referring to Figure 2, we can see that an increase in expected price will rotate the straight line Ff counterclockwise around point F.

[10] In the case of constant marginal costs, the profit-opportunity locus is a straight line.

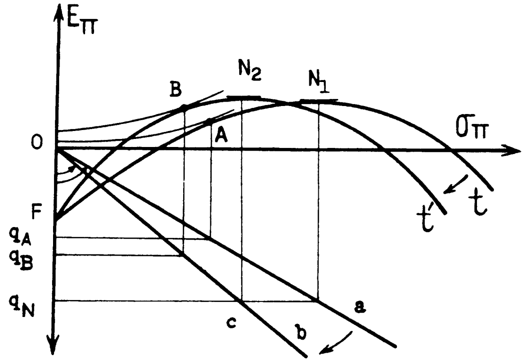

FIGURE 7

Referring to Figure 6, we see that equilibrium may also exist under decreasing marginal costs at point K, to which corresponds the optimum output q_K.[11] Again, under certainty, optimum output does not exist when marginal costs fall. Firms will maximize output.

Referring to Figures 5 and 6 we can see that expected positive profit is required for equilibrium to exist. When the firm expects loss, no equilibrium can be found since the indifference map is not defined in the lower quadrant. This proves Conclusion v.

E. *Change in the Level of Risk* (Conclusion vi: Figure 7)

The firm can revise its expectations about the distribution of future prices resulting in a revised standard deviation $\sigma(p)$, and therefore a change in the level of risk. Suppose $\sigma(p)$ is revised downward. The profit-opportunity locus will shift from Ft to Ft' in Figure 7 since both the line Ff and the transformed variable costs curve V_2 will rotate counterclockwise around point F as $\sigma(p)$ decreases.[12] The equilibrium point moves from A to B. The corresponding level of output should be derived with caution, since the slope $\sigma(p)$ now varies. Initially the line is Oa and the corresponding output q_A. When $\sigma(p)$ decreases we get a new line Ob which can be easily derived from Oa and the fact that for a risk-neutral firm the level of output q_N remains constant when $\sigma(p)$ changes. This allows us to obtain point c from which the line Ob and output q_B are derived with $q_B > q_A$. It follows that the firm with decreasing absolute risk aversion will increase (decrease) its level of output when risk is revised downward (upward).

F. *Change in Variable Costs* (Conclusion vii: Figure 8)

Changes in variable costs will leave the line Ff unaffected. Suppose variable costs are reduced, resulting in a rightward shift of the variable cost curve in Figure 2. In this case the profit-opportunity locus will shift upward, rotating around point F as shown in Figure 8. The equilibrium point will move from A to B for the decreasing absolute risk-aversion firm and from N_1 to N_2 for the risk-neutral firm. The level of output will increase from q_A to q_B in the first case and from q_{N_1} to q_{N_2} in the second. As a result of a reduction (rise) in variable costs, the firm with decreasing absolute risk aversion and the risk-neutral firm have increased (decreased) output.

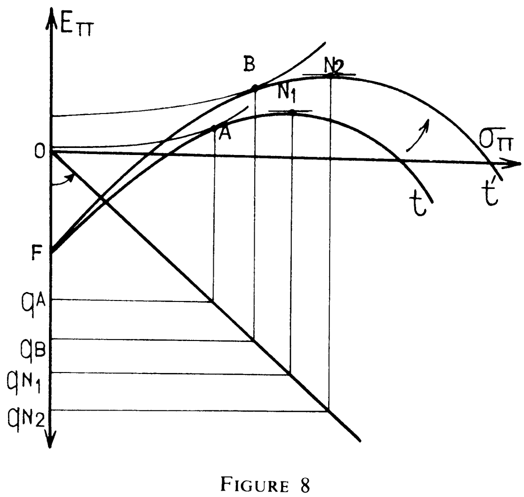

FIGURE 8

[11] In the case of decreasing marginal costs, the profit-opportunity locus is convex to the origin.

[12] Referring to Figure 2, we can see that a decrease in $\sigma(p)$ will rotate both the line Ff and the variable cost curve V_2 counterclockwise around point F resulting in a new profit-opportunity locus Ft' as shown in Figure 7.

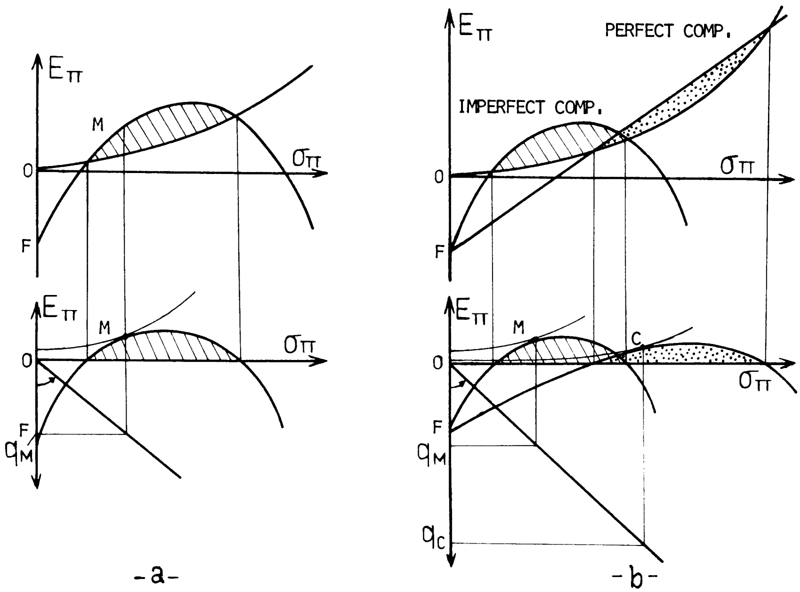

FIGURE 9

G. *Equilibrium under Uncertainty and Imperfect Competition and Comparison with Competitive Equilibrium* (Conclusion viii: Figure 9)

Under imperfect competition, equation (14) holds. Suppose $h(\sigma(\pi))$ is quadratic, that is, the firm faces a linear downward-sloping demand function. Then Figure 9a gives the expected profit area for a firm under noncompetitive market structure. We observe equilibrium at point M and the corresponding optimum level of output q_M. This derivation is similar to that of the competitive firm. It follows that output under uncertainty and imperfect market is smaller than under certainty. This justifies the generality of Conclusion i as well as Conclusions ii, iv, and v since they involve the cost function, which is independent of the market structure.

Comparative Output. Under imperfect competition the slope of the revenue curve $h(\sigma(\pi))$ at point F is larger than the slope of the revenue curve under perfect competition.[13] Referring to Figure 9b, which assumes a similar indifference map for both market structures, we see that the equilibrium point for the imperfect competition case is M under risk aversion, and C under perfect competition and risk aversion. It follows that output under imperfect competition is smaller than output under perfect competition, given risk aversion. A similar conclusion can be drawn for the case of risk

[13]Under imperfect competition expected total revenue is:

(a) $$TR = f\left[\frac{\sigma(\pi)}{\sigma(p)}\right] \cdot \frac{\sigma(\pi)}{\sigma(p)}$$

(b) $$\frac{dTR}{d\sigma(\pi)} = \frac{1}{\sigma(p)} \cdot f'\left[\frac{\sigma(\pi)}{\sigma(p)}\right] \cdot \frac{\sigma(\pi)}{\sigma(p)} + f\left[\frac{\sigma(\pi)}{\sigma(p)}\right] \cdot \frac{1}{\sigma(p)}$$

Therefore $$\lim_{\sigma(\pi)\to 0} \frac{dTR}{d\sigma(\pi)} = \frac{f(o)}{\sigma(p)}$$

where $f(o)$ is the price when output is zero. This limit price is certainly larger than the expected mean price, i.e., $f(o) > \mu$. Under perfect competition expected total revenue is:

neutrality. The level of expected profit corresponding to a given level of output will be higher under imperfect competition than under perfect competition.

(c) $TR = \frac{\mu}{\sigma(p)} \cdot \sigma(\pi)$ and $\frac{dTR}{d\sigma(\pi)} = \frac{\mu}{\sigma(p)}$

Therefore $\lim_{\sigma(\pi)\to 0} \frac{dTR}{d\sigma(\pi)} = \frac{\mu}{\sigma(p)}$.

Since $f(o) > \mu$ it follows that the slope of the revenue curve at point F is larger than the slope of the revenue curve under perfect competition.

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