A Dynamic Theory of the Declining Aggregated Labor Income Share: Intangible Capital vs. Tangible Capital

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Intangible Capital vs. Tangible Capital

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Abstract

Reports of the literature documenting the declining labor share of income have increased greatly in the past few years, which is opposed to one of the famous Kaldor (1961) "stylized facts" of growth. The declining labor income share has been observed since the 1980s in a number of countries, and especially in the United States. Recent studies have revealed the following five major driving forces of the declining labor share: (i) supercycles and boom-busts, (ii) rising and faster depreciation, (iii) superstar effects and consolidation, (iv) capital substitution and automation, and (v) globalization and labor bargaining power. We set up a two-sector optimal growth model with the R&D intermediate sectors. By integrating driving factors (ii) through (iv) above into the model, we demonstrate the long-term decline of the aggregated labor income share.
(131 words)

Key Words: capital intensity, elasticity of substitution, intangible capital stock, invented property and product capital (IPP), learning-by-doing technical progress, two-sector optimal growth model

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0. Introduction

Studies conducted in the past few years have increasingly augmented the literature documenting the declining labor share of income, which is opposed to one of the "stylized facts" of growth reported by Kaldor (1961). The labor income share decline has been observed since the 1980s in many countries, and especially in the United States. Recently, McKinsey Global Institute Discussion Paper (May, 2019) has surveyed studies of the literature examining factors driving the labor share decline, with categorization of the main driving factors as explained below.

- **Capital deepening, substitution, and automation** (Decline in prices of investment goods because of improvements in technology, particularly industrial robots and AI): IMF World Economic Outlook (2017), Karabarbounis and Neiman (2014), Elsby, Hobijn and Sahin (2013), Acemoglue and Restrepo (2018), and Lawrence (2015).
- **"Superstar" effects and consolidation** (Superstar firms are reaping rising shares of profits and value added): Autor et al. (2017) and Barkai (2017).
- **Globalization and labor bargaining power** (Increased trade competition and weak bargaining power of workers): Elsby, Hobijn and Sahin (2013), OECD Economic Outlook (2018), and IMF World Economic Outlook (2017).
- **Higher depreciation attributable to a shift to more intangible capital** (Greater use of capital in the form of intangibles and intellectual property products (IPP) capital): Koh et al. (2016) and Guiterrez (2017).
- **Supercycle and boom-bust** (Price supercycles in the energy and mineral sectors): Rognline (2015).

A noteworthy point is that, except for reports published by Koh et al. (2015), Barkai (2017), and Lawrence (2015), many reports have pointed out multiple factors driving the labor share decline. The main driving factors are not unique: they are multiple and are yet inconclusive.

Furthermore, the same working paper has presented re-examination of the five driving factors for the US economy based on the OECD STAN database from a macro–micro perspective. By ranking the five leading forces that have driven the recent capital share increase instead of those of the labor share decline, the report has indirectly described the main causes of the decline in labor share, as summarized in the following table.

<Table 1, here >

Although cyclical factors are the major driving forces, growth theory clarifies that leading driving factors (ii), (iii), and (iv) in the table are important. In fact, those factors jointly explain 56% of the decline in labor share. Factor (ii) was examined by Koh et al. (2016), who concluded that because of the transition to more intangible capital, especially in intellectual property and product (IPP) capital intensive economy, rising IPP depreciation and net IPP income have emerged. Factor (iii) is studied by Autor et al. (2017) among others. Technology and market conditions have facilitated the emergence of "superstar" firms with very high profit and a very low labor share. Factor (iv) is particularly examined by Karababounis and Neiman (2014). Decreased relative prices of capital goods because of IT technology and automation have induced firms to shift away from labor to capital.

Based on the discussion presented above, we set up a two-sector consumption goods and capital goods sectors – optimal growth model with intermediate goods sectors. Each sector's intermediate goods are produced by application of labor and tangible capital goods by Cobb–Douglass technologies with learning-by-doing technical progress. In contrast to tangible capital, assuming that intermediate goods become obsolete instantaneously and that their depreciation rate is therefore 100%, one might regard intermediate goods as intangible IPP capital goods. Furthermore, each final goods sector produces final goods with the sector's IPP capital and labor using Cobb–Douglass production technologies. Driving factor (ii) has been integrated into the model successfully. Driving factors (iii) and (iv) can also be integrated into the model as follows: By combining the intermediate sector with the final goods sector, the model can be recast as a standard two-sector optimal growth model with sector-specific total factor productivity (TFP) growth. Contrasted to the standard two-sector model with TFP growth studied by Takahashi (2017), the TFP growth rate is endogenously determined here. In Takahashi (2017), where a two-sector optimal growth model with a sector-specific TFP is set up and under the Cobb–Douglass technologies, it is demonstrated that each sector's optimal path converges to a sector-specific steady state. This property also holds here under the condition that the integrated consumption goods sector is more capital intensive than the capital goods sector. Given these circumstances, one can also demonstrate that, even if each intermediate sector's learning-by-doing technical activities were identical, the consumption goods sector's TFP growth rate could be greater than that of the capital goods sector. We also demonstrate that each sector's per-capita capital and output grow at the sector-specific growth rate determined by the sector's TFP.

Finally, we might conclude the following: First, the result implies that intangible capital input can be expected to replace labor input in both sectors in the long run because the price of intangible capital goods declines rapidly, not at the constant steady state wage rate. Secondly in the long run, the consumption goods sector with the lower labor share dominates the capital goods sector with the higher labor share in terms of the measure of efficient-unit value-added. Therefore the aggregated labor income share declines in the long run. Consequently, our model includes the major driving factors described above to explain the labor income share decline.

The paper is organized as follows: The next section presents the model and related assumptions. In Section 2, each sector's R&D process is solved explicitly. As described in Section 3, using the production possibility frontier, we integrate the model into a standard two-sector optimal growth model and solve it. In Section 4, the existence and uniqueness of the steady state are proved. In Section 5, saddle-point stability is presented. Section 6 explains the aggregated labor share decline. Section 7 concludes the paper.

1. Model and Assumptions

We introduce a sector specific R&D process into the Uzawa (1964) two-sector optimal growth model with Cobb–Douglas technologies. Each sector has its intermediate good sector in which a new technology is invented through a learning-by-doing process. This presents a sharp contrast to the model introduced by Ghiglino, Nishimura and Venditti (2017), where they assume that a part of labor of the "knowledge-intensive" sector is used as a kind of effort for the invention of a new technology. Before considering the two-sector case, we can consider a case with two sectors. Solving the sector's profit maximization problem and the market equilibrium conditions yields the integrated final good production function. The exact same argument can be applied to the remaining sector to obtain a similar integrated production function of the other sector. Using these two integrated functions, we set up an optimal growth problem similar to Uzawa's two-sector growth model. We demonstrate the existence of optimal steady states, the saddle-path stability around the optimal steady state.

We begin with competitive analysis of four labor markets. Based on those results, we set up the endogenous two-sector growth model with a Romer-type technical progress. For our analyses, the following market conditions are assumed.

Labor Market:

$$L = L_c + L_g = \left(L_{Yc} + L_{Mc}\right) + \left(L_{Yg} + L_{Mg}\right), \tag{1.1}$$

where

L : total labor supply, L_{Yi} : labor input for ith goods production as the final goods,

L_{Mi} : labor input for ith goods production as the intermediate goods,

where $i = c$: consumption goods sector, g : capital goods sector.

Production Functions in the Final-goods Sector:

$$Y_c = L_{Yc}^{\alpha_c} X_c^{1-\alpha_c}, \tag{1.2}$$

and

$$Y_g = L_{Yg}^{\alpha_g} X_g^{1-\alpha_g}. \tag{1.3}$$

Production Functions in the Intermediate-goods Sector:

$$X_c = A_c L_{Mc}^{\beta_c} K_c^{1-\beta_c}, \tag{1.4}$$

$$X_g = A_g L_{Mg}^{\beta_g} K_g^{1-\beta_g}. \tag{1.5}$$

As we have emphasized in Section 1, each sector's intermediate goods are produced by application of labor and tangible capital goods with Cobb–Douglass technologies embodied with newly invented technical progress. In contrast to tangible capital, assuming that intermediate goods become obsolete instantaneously and that their depreciation rate is therefore 100%, one might regard intermediate goods as intangible IPP capital goods. Furthermore, each final goods sector produces final goods with the sector's IPP capital and labor using Cobb–Douglass production technologies. Therefore, we may conclude that the driving factor (ii) has been integrated into the model successfully.

R&D Process (Learning by Doing):

$$\dot{A}_c = L_{Mc}^{\lambda_c} A_c^{\phi_c} \ (0 < \lambda_c < 1, 0 < \phi_c < 1), \tag{1.6}$$

$$\dot{A}_g = L_{Mg}^{\lambda_g} A_g^{\phi_g} \ (0 < \lambda_g < 1, 0 < \phi_g < 1). \tag{1.7}$$

Remark. Our R&D process is a Romer-type technical progress that was proposed originally by Jones (1995), who presented detailed discussions of this R&D process[2]. According to Jones (1995), ϕ represents the degree of externality across time in the R&D process; λ denotes the duplication externalities. The process contrasts to that proposed by Ghiglino, Nishimura and Venditti (2017), who assume the R&D process such as $\dot{A} = z(1-u)A - \eta A$. When $\eta = 0$ and $z(1-u) = L_M^{\lambda}$, their model coincides with ours.

The model considered here is summarized as a schematic representation in Figure 1.

<Figure 1, here>

Next we consider each sector's profit maximization problems.

Final-goods Sector Problem:

$$(*) \ \underset{(L_{Y_i}, X_i)}{Max} \ L_{Y_i}^{\alpha_i} X_i^{1-\alpha_i} - w_{Y_i} L_{Y_i} - p_i X_i$$

The first-order conditions of the expression above are the following: Note that the price of each final goods is normalized as one.

[2] Especially, Section III in Jones (1995) presents detailed discussion.

$$p_i = (1-\alpha_i) L_{Y_i}^{\alpha_i} X_i^{-\alpha_i}, \tag{1.8}$$

$$w_{Y_i} = \alpha_i L_{Y_i}^{\alpha_i - 1} X_i^{1-\alpha_i}, \tag{1.9}$$

where $i = c, g.$

Intermediate-goods Sector Problem:

$$(^{**})\ \underset{(L_{M_i}, K_i)}{Max}\ p_i A_i L_{M_i}^{\beta_i} K^{1-\beta_i} - w_{M_i} L_{M_i} - r_i K_i$$

The first-order conditions of the expression above are the following:

$$r_i = (1-\beta_i) p_i A_i L_{M_i}^{\beta_i} K^{-\beta_i}, \tag{1.10}$$

$$w_{M_i} = \beta_i p_i A_i L_{M_i}^{\beta_i - 1} K_i^{1-\beta_i}\ , \tag{1.11}$$

where $i = c, g.$

The market equilibrium condition is $w_{Y_c} = w_{M_c} = w_{Y_g} = w_{M_g}$, where

$$w_{Y_c} = \alpha_c L_{Y_c}^{\alpha_c - 1} \left(A_c L_{Mc}^{\beta_c} K_c^{1-\beta_c} \right)^{1-\alpha_c} = \alpha_c L_{Yc}^{\alpha_c - 1} A_c^{1-\alpha_c} L_{Mc}^{\beta_c(1-\alpha_c)} K_c^{(1-\alpha_c)(1-\beta_c)},$$

$$p_c = (1-\alpha_c) L_{Y_c}^{\alpha_c} \left(A_c L_{M_c}^{\beta_c} K^{1-\beta_c} \right)^{-\alpha_c} = (1-\alpha_c) L_{Y_c}^{\alpha_c} A_c^{-\alpha_c} L_{M_c}^{-\alpha_c \beta_c} K_c^{-(1-\beta_c)\alpha_c},$$

and

$$\begin{aligned} w_{M_c} &= p_c \beta_c A_c L_{M_c}^{\beta_c - 1} K_c^{1-\beta_c} = (1-\alpha_c) L_{Y_c}^{\alpha_c} A_c^{-\alpha_c} L_{M_c}^{-\alpha_c \beta_c} K_c^{-(1-\beta_c)\alpha_c} \beta_c A_c L_{M_c}^{\beta_c - 1} K_c^{1-\beta_c} \\ &= \beta_c (1-\alpha_c) L_{Y_c}^{\alpha_c} A_c^{1-\alpha_c} L_{M_c}^{\beta_c - 1 - \alpha_c \beta_c} K_c^{(1-\alpha_c)(1-\beta_c)}. \end{aligned}$$

From the equilibrium condition $w_{Y_c} = w_{M_c}$, it follows that

$$\alpha_c L_{M_c} = \beta_c (1-\alpha_c) L_{Y_c} \tag{1.12}$$

$$\Rightarrow \quad L_c = L_{Y_c} + L_{M_c} = \left[\frac{\alpha_c}{\beta_c (1-\alpha_c)} + 1 \right] L_{M_c}$$

$$\therefore L_{M_c} = \frac{\beta_c (1-\alpha_c)}{\alpha_c + \beta_c (1-\alpha_c)} L_c \quad \text{and} \quad L_{Y_c} = \frac{\alpha_c}{\alpha_c + \beta_c (1-\alpha_c)} L_c.$$

Similarly, from the equilibrium condition $w_{Y_g} = w_{M_g}$, it follows that

$$\alpha_g L_{M_g} = \beta_g (1-\alpha_g) L_{Y_g} \tag{1.13}$$

$$\Rightarrow \quad L_g = L_{Y_g} + L_{M_g} = \left[\frac{\alpha_g}{\beta_g(1-\alpha_g)} + 1 \right] L_{M_g}$$

$$\therefore L_{M_g} = \frac{\beta_g(1-\alpha_g)}{\alpha_g + \beta_g(1-\alpha_g)} L_g \quad \text{and} \quad L_{Y_g} = \frac{\alpha_g}{\alpha_g + \beta_g(1-\alpha_g)} L_g .$$

Each sector's labor input is a fixed proportional ratio to the total labor supply.

Based on the results presented above, the proportional property can be extended to the whole economy as shown below.

From (1.12) and (1.13), it follows

$$\frac{\alpha_c L_{M_c}}{\beta_c(1-\alpha_c)} = \frac{\alpha_g L_{M_g}}{\beta_g(1-\alpha_g) L_{Y_g}} = 1.$$

Then, by exchanging the terms, we obtain the constant ratio ξ as follows

$$\frac{\beta_g(1-\alpha_g) L_{Y_g}}{\beta_c(1-\alpha_c) L_{Y_c}} = \frac{\alpha_g L_{M_g}}{\alpha_c L_{M_c}} = \frac{\alpha_g(1-L_{Y_g})}{\alpha_c(1-L_{Y_c})} = \xi.$$

Solving the above relations yields

$$L_{Y_g} = \frac{\beta_c(1-\alpha_c)\xi}{\beta_g(1-\alpha_g)} L_{Y_c} \tag{1.14}$$

and

$$L_g - L_{Y_g} = \frac{\alpha_c \xi}{\alpha_g}(L_c - L_{y_c}). \tag{1.15}$$

From (1.15) and (1.13),

$$L_g - L_{Y_g} = \frac{\alpha_c \xi}{\alpha_g}(L_c - L_{Y_c}) \Rightarrow L_g - L_{Y_g} - \frac{\alpha_c \xi}{\alpha_g} L_c = -\frac{\alpha_c \xi}{\alpha_g} L_{Y_c}$$

$$\Rightarrow L_g - \left(\frac{\alpha_g}{\alpha_g + \beta_g(1-\alpha_g)} \right) L_g - \frac{\alpha_c \xi}{\alpha_g} L_c = -\frac{\alpha_c \xi}{\alpha_g} L_{Y_c}$$

$$\Rightarrow\left(\frac{\beta_g(1-\alpha_g)}{\alpha_g+\beta_g(1-\alpha_g)}\right)\left(L-L_c\right)-\frac{\alpha_c\xi}{\alpha_g}L_c=-\frac{\alpha_c\xi}{\alpha_g}L_{Y_c}$$

.

$$\Rightarrow L_{Y_c}=\left(\frac{\alpha_g\beta_g(1-\alpha_g)}{\alpha_c\xi\left(\alpha_g+\beta_g(1-\alpha_g)\right)}+1\right)L_c-\frac{\alpha_g\beta_g(1-\alpha_g)}{\alpha_c\xi\left(\alpha_g+\beta_g(1-\alpha_g)\right)}L=(\Phi+1)L_c-\Phi L.$$

Then from (1.14),

$$L_{Y_g}=\frac{\beta_c(1-\alpha_c)\xi}{\beta_g(1-\alpha_g)}L_{Y_c}=\frac{\beta_c(1-\alpha_c)\xi}{\beta_g(1-\alpha_g)}\left[(\Phi+1)L_c-\Phi L\right]=\Gamma\left[(\Phi+1)L_c-\Phi L\right].$$

Substituting this result and L_{M_g} into the labor equilibrium condition of the sector yields

$$L_g=L_{M_g}+L_{Y_g}=\frac{\beta_g(1-\alpha_g)}{\alpha_g+\beta_g(1-\alpha_g)}L_g+\Gamma\left[(\Phi+1)L_c-\Phi L\right]=\Lambda L_g+\Gamma\left[(\Phi+1)L_c-\Phi L\right]$$

$$=\Lambda L_g+\Gamma\left[(\Phi+1)(L-L_g)-\Phi L\right].$$

Solving the above equation with respect to L_g produces the following:

$$L_g=\frac{\Gamma}{1-\Lambda+\Gamma(\Phi+1)}L=DL \text{ and } L_c=L-L_g=(1-D)L.$$

Consequently, we demonstrated that each sector's labor input is also proportional to the total supply of labor: L. This property is important. In fact, it establishes that if the total labor supply grows at rate *n*, then each sector's labor input also grows at rate *n*.

2. Solving the R&D Process

We make the following assumption related to the total population growth rate.

Assumption 1. $\frac{\dot{L}}{L}=n\Rightarrow L=L_0e^{nt}\,(0<n<1).$

Considering the consumption sector only, the exact same argument can be applied to the capital goods sector denoted by index "g".

From the discussion in Section 1, substituting L_{M_c} into Eq. (1.6) gives

$$\dot{A}_c = \left(L_{M_c}^{\lambda_c}\right) A_c^{\phi_c} = \left(\frac{\beta_c(1-\alpha_c)}{\alpha_c+\beta_c(1-\alpha_c)} L_c\right)^{\lambda_c} A_c^{\phi_c} = \left(\frac{\beta_c(1-\alpha_c)}{\alpha_c+\beta_c(1-\alpha_c)} L_{c0} e^{nt}\right)^{\lambda_c} A_c^{\phi_c} = B^{\lambda_c} e^{\lambda_c nt} A_c^{\phi_c}.$$

We can solve the above differential equation explicitly:

By defining $z_c \equiv A_c^{1-\phi_c}$, the R&D process can be rewritten as the following differential equation:

$$\frac{dz_c}{dt} = (1-\phi_c) A_c^{-\phi_c} \frac{dA_c}{dt}.$$

Rewriting it further provides

$$\frac{dz_c}{dt} = (1-\phi_c) A_c^{-\phi_c} \frac{dA_c}{dt} = L_{M_c}^{\lambda_c} A_c^{\phi_c} (1-\phi_c) A_c^{-\phi_c} = L_{M_c}^{\lambda_c}(1-\phi_c) = (1-\phi_c) B^{\lambda_c} e^{\lambda_c nt}.$$

Integrating both sides of the equation above yields

$$z_c = \frac{(1-\phi_c)B^{\lambda_c}}{\lambda_c n} e^{\lambda_c nt} + d_c \qquad (d_c\text{:constant of integration}).$$

Then

$$z_{c0} = \frac{(1-\phi_c)B^{\lambda_c}}{\lambda_c n} + d_c \Rightarrow d_c = z_{c0} - \frac{(1-\phi_c)B^{\lambda_c}}{\lambda_c n}.$$

The initial conditions can be assumed as shown below.

Assumption 2. $z_{c0} = \dfrac{(1-\phi_c)B^{\lambda_c}}{\lambda_c n}$ or $A_{c0} = \left[\dfrac{(1-\phi_c)B^{\lambda_c}}{\lambda_c n}\right]^{\frac{1}{1-\phi_c}}$

Finally, we obtain the following solution:

$$A_c = \left[\frac{(1-\phi_c)B^{\lambda_c}}{\lambda_c n} e^{\lambda_c nt}\right]^{\frac{1}{1-\phi_c}} = \left[\frac{(1-\phi_c)B^{\lambda_c}}{\lambda_c n}\right]^{\frac{1}{1-\phi_c}} e^{\frac{\lambda_c n}{1-\phi_c}t} = A_{c0} e^{\gamma_{cA} t} \text{ where } \gamma_{cA} = \frac{\lambda_c n}{1-\phi_c}.$$

Applying the same logic to the investment sector yields

$$A_g = A_{g0} e^{\gamma_{gA} t} \text{ where } A_{g0} = \left[\frac{(1-\phi_g)B^{\lambda_g}}{\lambda_g n}\right]^{\frac{1}{1-\phi_g}} \text{ and } \gamma_{gA} = \frac{\lambda_g n}{1-\phi_g}.$$

Each sector's TFP growth rate depends on parameters λ_i, ϕ_i and n. In other words, it depends on the sector-specific R&D process and the total population growth rate. Jones

(1995) reported the same property.

3. Integrated Optimal Growth Problem

We redefine outputs as $Y_c = C$ and $Y_g = Y$ to avoid further complication of double indices.

$$C = L_{Yc}^{\alpha_c}\left(A_c L_{M_c}^{\beta_c} K^{1-\beta_c}\right)^{1-\alpha_c} = L_{Yc}^{\alpha_c} A_c^{1-\alpha_c} L_{M_c}^{\beta_c(1-\alpha_c)} K^{(1-\alpha_c)(1-\beta_c)}$$

$$= \left(\frac{\alpha_c}{\alpha_c + \beta_c(1-\alpha_c)}\right)^{\alpha_c} \left(\frac{\beta_c(1-\alpha_c)}{\alpha_c + \beta_c(1-\alpha_c)}\right)^{\beta_c(1-\alpha_c)} A_c^{1-\alpha_c} L_c^{\alpha_c+\beta_c(1-\alpha_c)} K_c^{(1-\alpha_c)(1-\beta_c)}$$

$$= D_c A_c^{1-\alpha_c} L_c^{\alpha_c+\beta_c(1-\alpha_c)} K_c^{(1-\alpha_c)(1-\beta_c)}$$

$$= D_c A_c^{1-\alpha_c} L_c^{\varepsilon_c} K_c^{1-\varepsilon_c},$$

whereas applying the same logic to the investment sector yields

$$Y = D_g A_g^{1-\alpha_g} L_g^{\varepsilon_g} K_g^{1-\varepsilon_g},$$

where $\varepsilon_i = \alpha_i + \beta_i(1-\alpha_i)$ for $i = c, g$.

Based on the arguments presented earlier, following Uzawa (1964), the following two-sector model can be set up with the consumption-goods and capital-goods sectors. Each sector integrates the final-good and the intermediate-good sectors as

$$C = D_c A_c^{1-\alpha_c} L_c^{\varepsilon_c} K_c^{1-\varepsilon_c}$$

and

$$Y = D_g A_g^{1-\alpha_g} L_g^{\varepsilon_g} K_g^{1-\varepsilon_g},$$

where

$$D_c = \left(\frac{\alpha_c}{\alpha_c + \beta_c(1-\alpha_c)}\right)^{\alpha_c} \left(\frac{\beta_c(1-\alpha_c)}{\alpha_c + \beta_c(1-\alpha_c)}\right)^{\beta_c(1-\alpha_c)},$$

and

$$D_g = \left(\frac{\alpha_g}{\alpha_g + \beta_g(1-\alpha_g)}\right)^{\alpha_g} \left(\frac{\beta_g(1-\alpha_g)}{\alpha_g + \beta_g(1-\alpha_g)}\right)^{\beta_g(1-\alpha_g)}.$$

Rewritten in terms of per-capita units, one obtains the following.

$$c = \frac{C}{L} = D_c A_c^{1-\alpha_c} \ell_c^{\varepsilon_c} k_c^{1-\varepsilon_c} = D_c \left(e^{\gamma_{cA} t}\right)^{1-\alpha_c} \ell_c^{\varepsilon_c} k_c^{1-\varepsilon_c}$$
$$= D_c e^{(1-\alpha_c)\gamma_{cA} t} \ell_c^{\varepsilon_c} k_c^{1-\varepsilon_c}$$

and

$$y = \frac{Y}{L} = D_g A_g^{1-\alpha_g} \ell_g^{\varepsilon_g} k_g^{1-\varepsilon_g}$$
$$= D_g e^{(1-\alpha_g)\gamma_{gA} t} \ell_g^{\varepsilon_g} k_g^{1-\varepsilon_g},$$

where $k_c = \frac{K_c}{L}, \ell_c = \frac{L_c}{L}, k_g = \frac{K_g}{L}$ *and* $\ell_g = \frac{L_g}{L}$.

Normalizing the output by each sector's rate of technical progress obtained in Section 2 gives

$$\tilde{c} = \frac{c}{e^{(1-\alpha_c)\gamma_{cA} t}} = D_c \ell_c^{\varepsilon_c} k_c^{1-\varepsilon_c} = D_c k_c^{\beta} \ell_c^{1-\beta} \quad (0 < \beta < 1)$$

and

$$\tilde{y} = \frac{y}{e^{(1-\alpha_g)\gamma_{gA} t}} = D_g \ell_g^{\varepsilon_g} k_g^{1-\varepsilon_g} = D_g k_g^{\alpha} \ell_g^{1-\alpha} \quad (0 < \alpha < 1).$$

Note that we define that $\alpha = 1 - \varepsilon_g$ and $\beta = 1 - \varepsilon_c$ to avoid further notational complications.

Assumption 3. Utility function u(・) is defined on $\mathbb{R}_{++}$ as the following standard form:

$$u(\mathrm{c}(t)) = u\left(C(t)/L(t)\right) = \frac{c(t)^{1-\sigma}}{1-\sigma} \text{ for } t \geq 0 \text{ and } \sigma > 0.$$

The objective function can be rewritten in terms of efficiency units as

$$\frac{c^{1-\sigma}}{1-\sigma} = \frac{\left[\tilde{c} e^{(1-\alpha_c)\gamma_{cA} t} e^{nt}\right]^{1-\sigma}}{1-\sigma} = e^{[(1-\alpha_c)\gamma_{cA}+n](1-\sigma)t} \left(\frac{\tilde{c}^{1-\sigma}}{1-\sigma}\right),$$

where we omit the time index from the variables for simplicity.

Solving the following problem (*) yields the production possibility frontier.

Lemma 1. Solving the following problem (*) yields the production possibility frontier (PPF) of both sectors as $\tilde{y} = T(\tilde{c}, k)$.

$$(*) \max \tilde{y} = D_g k_g^{\alpha} \ell_g^{1-\alpha} \quad \text{s.t. } \tilde{c} = D_c k_c^{\beta} \ell_c^{1-\beta},\ \ell_c + \ell_g = 1 \text{ and } k = k_c + k_g.$$

This equation can be written with parameters in the implicit function form of

$$T(\tilde{c}, k) = D_g \left[\frac{(1-\alpha)\beta}{\Delta} \right]^{1-\alpha} (k - k_c) = D_g \left[(1-\alpha)\beta \right]^{1-\alpha} \Delta^{(\alpha-1)} (k - k_c),$$

where $\Delta = (1-\alpha)\beta k + (\alpha - \beta) k_c,\ k_c = e(\tilde{c}, k),$

$$T_k = \alpha D_g \left[\frac{(1-\alpha)\beta}{\Delta} \right]^{1-\alpha} = \alpha D_g \left[(1-\alpha)\beta \right]^{1-\alpha} \Delta^{\alpha-1}, \tag{3.1}$$

And

$$T_{\tilde{c}} = -\frac{T_k}{\beta} D_g \left[\frac{\Delta}{\alpha(1-\beta)} \right]^{1-\beta} = -D_g^2 \frac{\alpha^{\beta}(1-\alpha)^{1-\alpha}}{\beta^{\alpha}(1-\beta)^{1-\beta}} \Delta^{\alpha-\beta}. \tag{3.2}$$

Remark. Note that due to the duality, T_k is the capital rental rate and $(-T_{\tilde{c}}^*)$ stands for the price of $\tilde{c}$.

Proof. Baier, Nishimura and Yano (1998) present the argument comprehensively.■

Actually,

$$\frac{\partial \Delta}{\partial \tilde{c}} = (\alpha - \beta) \frac{\partial k_c}{\partial \tilde{c}} \quad \text{and} \quad \frac{\partial \Delta}{\partial k} = (1-\alpha)\beta + (\alpha - \beta) \frac{\partial k_c}{\partial k}.$$

That is true because function $k_c = e(\tilde{c}, k)$ can be derived from solving the following relation expressed by the implicit function.

$$\left[(\alpha - \beta) k_c + \beta(1-\alpha) k \right]^{1-\beta} \tilde{c} = D_c \left[\alpha(1-\beta) \right]^{1-\beta} k_c$$

In that equation,

$$\Delta \equiv (\alpha - \beta) k_c + \beta(1-\alpha) k.$$

Because of this relation, we can derive following partial derivatives:

$$\begin{cases} \dfrac{\partial k_c}{\partial k} = \dfrac{\beta(1-\beta)(1-\alpha)\Delta^{-\beta}\tilde{c}}{D_c\left[\alpha(1-\beta)\right]^{1-\beta} - (1-\beta)(\alpha-\beta)\Delta^{-\beta}\tilde{c}} \\ \qquad\qquad\text{and} \\ \dfrac{\partial k_c}{\partial \tilde{c}} = \dfrac{\Delta^{1-\beta}}{D_c\left[\alpha(1-\beta)\right]^{1-\beta} - (1-\beta)(\alpha-\beta)\Delta^{-\beta}\tilde{c}} \end{cases} \tag{3.3}$$

The following relations are established:

$$\begin{cases} \dfrac{\partial \Delta}{\partial k} = (\alpha-\beta)\dfrac{\partial k_c}{\partial k} + \beta(1-\alpha) = \dfrac{\beta(1-\alpha)D_c\left[\alpha(1-\beta)\right]^{1-\beta}}{D_c\left[\alpha(1-\beta)\right]^{1-\beta} - (1-\beta)(\alpha-\beta)\Delta^{-\beta}\tilde{c}}, \\ \qquad\qquad \textit{and} \\ \dfrac{\partial \Delta}{\partial \tilde{c}} = (\alpha-\beta)\dfrac{\partial k_c}{\partial \tilde{c}} = \dfrac{(\alpha-\beta)\Delta^{1-\beta}}{D_c\left[\alpha(1-\beta)\right]^{1-\beta} - (1-\beta)(\alpha-\beta)\Delta^{-\beta}\tilde{c}}. \end{cases} \tag{3.4}$$

Differentiating T_k and $T_{\tilde{c}}$ with respect to k and $\tilde{c}$ again and substituting Eq. (3.3) yields

$$\begin{aligned} \frac{T_{\tilde{c}\tilde{c}}}{T_{\tilde{c}}} &= \alpha(\alpha-\beta)\Delta^{-1}\left(\frac{\partial k_c}{\partial \tilde{c}}\right) = \frac{(\alpha-\beta)^2\Delta^{-\beta}}{D_c\left[\alpha(1-\beta)\right]^{1-\beta} - (1-\beta)(\alpha-\beta)\Delta^{-\beta}\tilde{c}} \\ &= \frac{(\alpha-\beta)^2}{D_c\left[\alpha(1-\beta)\right]^{1-\beta}\Delta^{\beta} - (1-\beta)(\alpha-\beta)\tilde{c}}, \end{aligned} \tag{3.5}$$

$$\begin{aligned} \frac{T_{\tilde{c}k}}{T_{\tilde{c}}} &= (\alpha-\beta)\Delta^{-1}\left[\beta(1-\alpha) + (\alpha-\beta)\left(\frac{\partial k_c}{\partial k}\right)\right] = \frac{(\alpha-\beta)\beta(1-\alpha)D_c\left[\alpha(1-\beta)\right]^{1-\beta}\Delta^{-1}}{D_c\left[\alpha(1-\beta)\right]^{1-\beta} - (1-\beta)(\alpha-\beta)\Delta^{-\beta}\tilde{c}} \\ &= \frac{(\alpha-\beta)\beta(1-\alpha)D_c\left[\alpha(1-\beta)\right]^{1-\beta}}{D_c\left[\alpha(1-\beta)\right]^{1-\beta}\Delta - (1-\beta)(\alpha-\beta)\Delta^{1-\beta}\tilde{c}}. \end{aligned} \tag{3.6}$$

In addition, the following equation is obtained:

$$T_{kk} = \alpha(\alpha-1)D_g\left[(1-\alpha)\beta\right]^{1-\alpha}\Delta^{\alpha-2}\left\{\frac{\beta(1-\alpha)D_c\left[\alpha(1-\beta)\right]^{1-\beta}}{D_c\left[\alpha(1-\beta)\right]^{1-\beta} - (1-\beta)(\alpha-\beta)\Delta^{-\beta}\tilde{c}}\right\}. \tag{3.7}$$

Using the PPF, the representative household's problem over time can be written as the simple problem shown below.

$$(**)\begin{cases} Max \int_0^{\infty} u(\tilde{c})e^{-\rho t}dt \\ s.t.\ \dot{k} = T(\tilde{c},k) - \delta k \end{cases}$$

Therein, $\rho \equiv r - \left[(1-\alpha_c)\gamma_{cA}\right](1-\sigma) + n.$ Also note that r is the representative household's subjective discount rate and δ stands for the depreciation rate plus the rate of population.

Remark. A discrete version of the problem (**) was studied by Bosi et al. (2005). In contrast to our model with the Cobb–Douglass technologies, they assume endogenous labor and general neoclassical production technologies.

Assumption 4. $\rho \equiv r - \left[(1-\alpha_c)\gamma_{cA}\right](1-\sigma) + n > 0.$

The Hamiltonian of the problem (**) can be written as

$$\mathrm{H} = u(\tilde{c})e^{-\rho t} + \lambda\left[T\left(\tilde{c},k\right) - \delta k\right].$$

The first-order conditions of the problem are

$$\frac{\partial \mathrm{H}}{\partial \lambda} = \dot{k} \ \Rightarrow\ \dot{k} = g\left(\tilde{c},k\right) \equiv T(\tilde{c},k) - \delta k, \tag{3.8}$$

$$-\frac{\partial \mathrm{H}}{\partial k} = \dot{\lambda} \ \Rightarrow \dot{\lambda} = -\lambda\left[T_k(\tilde{c},k) - \delta\right], \tag{3.9}$$

$$\frac{\partial \mathrm{H}}{\partial \tilde{c}} = u' e^{-\rho t} + \lambda T_{\tilde{c}} = 0. \tag{3.10}$$

Because of the Inada conditions, all variables including capital stock "k" must be bounded. Therefore, the transversality conditions are expected to be satisfied automatically.

Differentiating (3.10) with respect to time "t" gives

$$-\rho e^{-\rho t}u''\left(\tilde{c}\right) + e^{-\rho t}u''\left(\tilde{c}\right)\dot{\tilde{c}} + \dot{\lambda}\,T_c + \lambda T_{cc}\,\dot{\tilde{c}} + \lambda T_{ck}\,\dot{k} = 0$$

or

$$\left[e^{-\rho t}u''+\lambda T_{\tilde{c}\tilde{c}}\right]\dot{\tilde{c}}=\rho e^{-\rho t}u'+\dot{\lambda}T_{\tilde{c}}-\lambda T_{\tilde{c}\tilde{c}}\dot{k} \qquad (3.11).$$

From (3.8)–(3.10), we obtain the following.

$$\begin{cases} i)\ \lambda T_{\tilde{c}\tilde{c}}=-\left(\dfrac{T_{\tilde{c}\tilde{c}}}{T_{\tilde{c}}}\right)u'e^{-\rho t} \\ \\ ii)\ \dot{\lambda}T_{\tilde{c}}=u'e^{-\rho t}\left[T_k-\delta\right] \\ \\ iii)\lambda T_{\tilde{c}k}\dot{k}=-\left(\dfrac{T_{\tilde{c}\tilde{c}}}{T_{\tilde{c}}}\right)u'e^{-\rho t}\left[T-\delta k\right] \end{cases} \qquad (3.12)$$

Then substitution of *i*) through *iii*) of (3.12) into (3.11) yields the following expressions.

$$\left[\left(\frac{u''}{u'}\right)-\left(\frac{T_{\tilde{c}\tilde{c}}}{T_{\tilde{c}}}\right)\right]\dot{\tilde{c}}=\rho-\left[T_k-\delta\right]+\left(\frac{T_{\tilde{c}k}}{T_{\tilde{c}}}\right)\left[T-\delta k\right]$$

$$=\left(\rho+\delta\right)-T_k+\left(\frac{T_{\tilde{c}k}}{T_{\tilde{c}}}\right)\left[T-\delta k\right]$$

Rewriting the equation above provides (3.13) as the final result.

$$\dot{\tilde{c}}=f\left(\tilde{c},k\right)=\left(\frac{1}{\left(\dfrac{u''}{u'}\right)-\left(\dfrac{T_{\tilde{c}\tilde{c}}}{T_{\tilde{c}}}\right)}\right)\left\{(\rho+\delta)-T_k+\left(\frac{T_{\tilde{c}k}}{T_{\tilde{c}}}\right)\left[T(\tilde{c},k)-\delta k\right]\right\} \qquad (3.13)$$

Differential equations (3.8) and (3.13) constitute the two-dimensional nonlinear differential equation system in the end.

4. Steady State

Eq. (3.8) and Eq. (3.13) give the following two-dimensional simultaneous nonlinear differential equation system shown below.

$$\begin{cases} \dot{k} = g\left(\tilde{c}, k\right) \equiv T(\tilde{c}, k) - \delta k, \\ \dot{\tilde{c}} = f\left(\tilde{c}, k\right) = \left(\dfrac{\left[(\rho+\delta) - T_k\right] + \left(\dfrac{T_{\tilde{c}k}}{T_{\tilde{c}}}\right)\left[T(\tilde{c}, k) - \delta k\right]}{\left(\dfrac{u''}{u'}\right) - \left(\dfrac{T_{\tilde{c}\tilde{c}}}{T_{\tilde{c}}}\right)} \right) \end{cases} \tag{4.1}$$

It is noteworthy that $u' = \tilde{c}^{-\sigma}$ and $u'' = -\sigma \tilde{c}^{-\sigma-1}$ imply that $\left(u''/u'\right) = -\sigma/\tilde{c}$.

The steady state $\left(\tilde{c}^*, k^*\right)$ can be defined as the solution of the following system:

$$\begin{cases} \dot{k} = g\left(\tilde{c}^*, k^*\right) = 0, \\ \dot{\tilde{c}} = f\left(\tilde{c}^*, k^*\right) = 0. \end{cases}$$

We demonstrate below the existence and uniqueness of the steady state.

Proposition 1. There exists a steady state $\left(\tilde{c}^*, k^*\right)$ of (4.1).

Proof. At the steady state, the following equations are expected to hold simultaneously.

$$T\left(\tilde{c}, k\right) = \delta k \tag{4.2}$$

$$T_k\left(\tilde{c}, k\right) = \rho + \delta \tag{4.3}$$

Combining the duality property Eq. (3.1) and Eq. (4.3) provides

$$\rho + \delta = \alpha D_g \left[(1-\alpha)\beta\right]^{1-\alpha} \Delta(*)^{\alpha-1}.$$

where (*) implies that it is evaluated at the steady state.

Solving this equation with respect to $\Delta(*)$ yields

$$\Delta(*) = \left[\frac{\rho+\delta}{\alpha D_g \left[(1-\alpha)\beta\right]^{1-\alpha}} \right]^{-\left(\frac{1}{1-\alpha}\right)} = (\rho+\delta)^{-\left(\frac{1}{1-\alpha}\right)} (\alpha D_g)^{\frac{1}{1-\alpha}} \left[(1-\alpha)\beta\right]$$

On the other hand, $\Delta(*) \equiv (\alpha-\beta) k_c^* + \beta(1-\alpha) k^*$ holds. Then by combining them, the following equation holds:

$$(\alpha-\beta)k_c^* + \beta(1-\alpha)k^* = (\rho+\delta)^{-\frac{1}{1-\alpha}}(\alpha D_g)^{\frac{1}{1-\alpha}}\left[(1-\alpha)\beta\right].$$

Solving the above equation w.r.t. k_c^*, we obtain

$$k_c^* = \frac{(\rho+\delta)^{-\frac{1}{1-\alpha}}(\alpha D_g)^{\frac{1}{1-\alpha}}\left[(1-\alpha)\beta\right]}{(\alpha-\beta)} - \frac{\beta(1-\alpha)}{(\alpha-\beta)}k^*.$$

Therefor it follows that

$$\begin{aligned} k_g^* = k^* - k_c^* = k^* - \left(\frac{(\rho+\delta)^{-\frac{1}{1-\alpha}}(\alpha D_g)^{\frac{1}{1-\alpha}}\left[(1-\alpha)\beta\right]}{(\alpha-\beta)} - \frac{\beta(1-\alpha)}{(\alpha-\beta)}k^* \right) \\ = -\frac{(\rho+\delta)^{-\frac{1}{1-\alpha}}(\alpha D_g)^{\frac{1}{1-\alpha}}\left[(1-\alpha)\beta\right]}{(\alpha-\beta)} + \frac{\alpha(1-\beta)}{(\alpha-\beta)}k^*. \end{aligned} \tag{4.4}$$

Since T_k^* stands for the capital rental rate, due to the constant returns to scale production property implies that $\tilde{y}^* = \left(\frac{T_k^*}{\alpha}\right)k_g^* = \left(\frac{\rho+\delta}{\alpha}\right)k_g^*$. Substituting (4.4) into this relation yields,

$$\begin{aligned} \tilde{y}^* &= \left(\frac{\rho+\delta}{\alpha}\right)\left\{ -\frac{(\rho+\delta)^{-\frac{1}{1-\alpha}}(\alpha D_g)^{\frac{1}{1-\alpha}}\left[(1-\alpha)\beta\right]}{(\alpha-\beta)} + \frac{\alpha(1-\beta)}{(\alpha-\beta)}k^* \right\} \\ &= -\frac{(\rho+\delta)^{-\frac{\alpha}{1-\alpha}}(\alpha D_g)^{\frac{1}{1-\alpha}}\left[(1-\alpha)\beta\right]}{\alpha(\alpha-\beta)} + \frac{(\rho+\delta)(1-\beta)}{(\alpha-\beta)}k^*. \end{aligned}$$

On the other hand, along the steady state $\tilde{y}^* = \delta k^*$ also holds due to (4.2). Thus we have finally,

$$-\frac{(\rho+\delta)^{-\frac{\alpha}{1-\alpha}}(\alpha D_g)^{\frac{1}{1-\alpha}}\left[(1-\alpha)\beta\right]}{\alpha(\alpha-\beta)} + \frac{(\rho+\delta)(1-\beta)}{(\alpha-\beta)}k^* = \delta k^*.$$

Solving this w.r.t. k^* yields

$$\left\{\frac{(\rho+\delta)(1-\beta)}{(\alpha-\beta)} - \delta\right\}k^* = \frac{(\rho+\delta)^{-\frac{\alpha}{1-\alpha}}(\alpha D_g)^{\frac{1}{1-\alpha}}\left[(1-\alpha)\beta\right]}{\alpha(\alpha-\beta)} \Rightarrow k^* = \frac{(\rho+\delta)^{-\frac{\alpha}{1-\alpha}}(\alpha D_g)^{\frac{1}{1-\alpha}}\left[(1-\alpha)\beta\right]}{\alpha\left[\alpha(1-\beta)\rho + \beta(1-\alpha)\delta\right]}.$$

In order to obtain $\tilde{c}^* > 0$, we use the implicit function:

$$\left[(\alpha-\beta)k_c + \beta(1-\alpha)k\right]^{1-\beta}\tilde{c} = D_c\left[\alpha(1-\beta)\right]^{1-\beta}k_c.$$

Solving it w.r.t. $\tilde{c}$,

$$\tilde{c}^* = \frac{D_c\left[\alpha(1-\beta)\right]^{1-\beta}}{\left[(\alpha-\beta)k_c^* + \beta(1-\alpha)k^*\right]^{1-\beta}} k_c^* = \frac{D_c\left[\alpha(1-\beta)\right]^{1-\beta}}{\Delta(*)^{1-\beta}} k_c^*$$

$$= \frac{D_c\left[\alpha(1-\beta)\right]^{1-\beta}}{\Delta(*)^{1-\beta}} - \left\{ \frac{(\rho+\delta)^{-\frac{1}{1-\alpha}}(\alpha D_g)^{\frac{1}{1-\alpha}}\left[(1-\alpha)\beta\right]}{(\alpha-\beta)} - \frac{\beta(1-\alpha)}{(\alpha-\beta)} k^* \right\}.$$

Substituting k^* into the above and simplifying yields

$$\tilde{c}^* = \frac{\left(\alpha D_g\right)^{\frac{1}{1-\alpha}} D_c\left(\rho+\delta\right)^{-(\frac{\beta}{1-\alpha})}\left[(1-\alpha)\beta\right]^{\beta}\left[\alpha(1-\beta)\right]\left\{\alpha\left[(1-\beta)\rho+(1-\alpha)\delta\right]-(1-\alpha)\beta\right\}}{\alpha(1-\beta)\left[(1-\beta)\rho+(1-\alpha)\delta\right]}$$

We complete the proof.■

To demonstrate the uniqueness, first Lemma 2 and Lemma 3 must be presented under the following two additional assumptions.

Assumption 5. $\alpha_g > \alpha_c$ and $\beta_g > \beta_c$.

Remark. Assumption 5 is not the usual capital intensity condition. By that assumption, the labor intensities of both the final goods and the intermediate sector in the capital goods sector are greater than those in consumption good sector. The first condition implies that the consumption final goods sector not only uses more intangible IIP capital intensive technologies but also in the intermediate sector of the consumption goods sector uses tangible capital intensive technologies. Because $\beta = 1-\varepsilon_c$ and $\alpha = 1-\varepsilon_g$, it follows that

$$\beta-\alpha = (1-\varepsilon_c)-(1-\varepsilon_g) = (\alpha_g-\alpha_c)+(\beta_g-\beta_c)+(\alpha_c\beta_c-\alpha_g\beta_g) \cong (\alpha_g-\alpha_c)+(\beta_g-\beta_c).$$

Assumption 4 implies that $\beta > \alpha$. In other words, the unified consumption goods final sector uses more tangible capital intensive technologies than the unified capital goods final sector does. That result is similar to the famous stability condition derived and then reported by Uzawa (1964).

Lemma 2. Under Assumption 4, the following five sign conditions hold:

$$1)\Delta(*) > 0,\ 2)\left[D_c\left[\alpha(1-\beta)\right]^{1-\beta} - (1-\beta)(\alpha-\beta)\Delta(*)^{-\beta}\tilde{c}\right] > 0,$$
$$3)T_k^* > 0,\ \ 4)T_{\tilde{c}}^* < 0,\ \text{ and } 5)T_{kk}^* > 0.$$

Proof.

1) From the definition of Δ,

$$\Delta(*) = (\alpha-\beta)k_c^* + \beta(1-\alpha)k^* = \alpha(1-\beta)k_c^* + \beta(1-\alpha)k_g^* > 0.$$

2) From 1) and Assumption 4, the result follows.

3) $T_k^* = \rho + \delta > 0.$

4) $T_{\tilde{c}} = -D_g^2 \dfrac{\alpha^{\beta}(1-\alpha)^{1-\alpha}}{\beta^{\alpha}(1-\beta)^{1-\beta}} \Delta(*)^{\alpha-\beta} < 0.$

5) From (3.7), 1) and 2), the result follows.

This completes the proof. ■

5. Saddle-point Stability

Linearization of the system at $\left(\tilde{c}^*, k^*\right)$, one can derive the following linear system of

$$\begin{pmatrix} \dot{k} \\ \dot{\tilde{c}} \end{pmatrix} = \begin{bmatrix} g_{\tilde{c}}^* & g_k^* \\ f_{\tilde{c}}^* & f_k^* \end{bmatrix} \begin{pmatrix} \tilde{c} - \tilde{c}^* \\ k - k^* \end{pmatrix},$$

where

$$g_{\tilde{c}}^* = \left.\frac{\partial g}{\partial \tilde{c}}\right|_{\left(\tilde{c}^*, k^*\right)} = \left.\frac{\partial T}{\partial \tilde{c}}\right|_{\left(\tilde{c}^*, k^*\right)} = T_{\tilde{c}}^*, \tag{5.1}$$

$$g_k^* = \left.\frac{\partial g}{\partial k}\right|_{\left(\tilde{c}^*, k^*\right)} = \left.\frac{\partial T}{\partial k}\right|_{\left(\tilde{c}^*, k^*\right)} - \delta = T_k^* - \delta, \tag{5.2}$$

$$f_{\tilde{c}}^* = \left.\frac{\partial f}{\partial \tilde{c}}\right|_{\left(\tilde{c}^*, k^*\right)} = \frac{-T_{kk}^* + T_{\tilde{c}k}^*}{\left[\left(\dfrac{u''}{u'}\right) - \left(\dfrac{T_{\tilde{c}\tilde{c}}^*}{T_{\tilde{c}}^*}\right)\right]}, \tag{5.3}$$

$$\begin{aligned} f_k^* = \left.\frac{\partial f}{\partial k}\right|_{\left(\tilde{c}^*, k^*\right)} &= \frac{\left\{-T_{kk}^* + \left(\dfrac{T_{\tilde{c}k}^*}{T_{\tilde{c}}^*}\right)[T_k^* - \delta]\right\}\left[\left(\dfrac{u''}{u'}\right) - \left(\dfrac{T_{\tilde{c}\tilde{c}}^*}{T_{\tilde{c}}^*}\right)\right]}{\left[\left(\dfrac{u''}{u'}\right) - \left(\dfrac{T_{\tilde{c}\tilde{c}}^*}{T_{\tilde{c}}^*}\right)\right]^2} \\ &= \frac{-T_{kk}^* + \left(\dfrac{T_{\tilde{c}k}^*}{T_{\tilde{c}}^*}\right)[T_k^* - \delta]}{\left[\left(\dfrac{u''}{u'}\right) - \left(\dfrac{T_{\tilde{c}\tilde{c}}^*}{T_{\tilde{c}}^*}\right)\right]}. \end{aligned} \tag{5.4}$$

Based on the well-known property, under our sign patterns, the following two conditions are expected to be satisfied for saddle-point stability in the steady state as

$$i) \quad g_{\tilde{c}}^* f_k^* - g_k^* f_{\tilde{c}}^* < 0,$$
$$ii) \quad \left(g_{\tilde{c}}^* + f_k^*\right)^2 - 4\left(g_{\tilde{c}}^* f_k^* - g_k^* f_{\tilde{c}}^*\right) > 0.$$

We will show saddle-point stability below.

Proposition 2. The steady state $\left(\tilde{c}^*, k^*\right)$ is saddle-point stable.

Proof. Note that should the condition *i*) hold, then the *ii*) automatically holds. Therefore, all we need to do is to establish *i*).

Calculating $g_{\tilde{c}}^* f_k^* - g_k^* f_{\tilde{c}}^*$ yields:

$$g_{\tilde{c}}^* f_k^* - g_k^* f_{\tilde{c}}^* = \frac{-T_{kk}^* T_{\tilde{c}}^* + T_{kk}^*[T_k^* - \delta]}{\left[\left(\dfrac{u''}{u'}\right) - \left(\dfrac{T_{\tilde{c}\tilde{c}}^*}{T_{\tilde{c}}^*}\right)\right]}.$$

From Lemma 2, $-T_{kk}^* T_{\tilde{c}}^* > 0$ holds. Also from (5.2), $T_k^* = \rho + \delta$ implies $[T_k^* - \delta] = \rho > 0$.

It follows that $T_{kk}^*[T_k^* - \delta] > 0$ holds. Therefore, the numerator is positive. Due to (3.5), the denominator turns to be negative.

Thus we finally have shown that

$$g_{\tilde{c}}^* f_k^* - g_k^* f_{\tilde{c}}^* < 0.$$

It implies that the steady state $\left(\tilde{c}^*, k^*\right)$ is saddle-point stable. ■

Remark. As we have demonstrated in Appendix, the global saddle-point stability will be established for some parameter values. The hardest part of numerical analysis is that we need to solve the following implicit function explicitly w.r.t. k_c:

$$\left[(\alpha - \beta)k_c + \beta(1-\alpha)k\right]^{1-\beta} \tilde{c} = D_c\left[\alpha(1-\beta)\right]^{1-\beta} k_c.$$

It is clear that we cannot obtain an explicit formula by solving for an arbitrarily given value of the coefficient β. Fixing $\beta = 0.5$, the above equation becomes a quadratic equation. And by giving some ad-hoc values of δ and γ, we have numerically shown the global saddle-point stability. Although our numerical analysis is very restricted, but we confirm viability of the global saddle-point stability.

6. Labor Income Share Decline

In this section, we unify all the results presented above and demonstrate that the aggregated labor income share declines in the long run.

Before that, we summarize our important results as explained below.

(P1) The optimal steady state output path $\left(\tilde{c}^*(t), y^*(t)\right)$ in terms of the original unit grows at its own growth rate and they are expressed as

$$\begin{cases} \tilde{c}^*(t) = \tilde{c}^* e^{(1-\alpha_c)\gamma_{cA}t}, \\ y^*(t) = y^* e^{(1-\alpha_g)\gamma_{gA}t}. \end{cases}.$$

Therefore, each steady state grows at a different growth rate: $(1-\alpha_i)\gamma_{iA}$ $(i = c, g)$.

(P2) The optimal steady state is saddle-point stable: each sector's optimal path locally converges to its own steady state.

(P3) Both Assumption 5 $(\beta > \alpha)$ and the Cobb–Douglas technologies imply that, along the optimal steady state, the consumption goods final sector has a lower labor income share than that of the capital goods final sector.

Because of (P1), even if $\gamma_{cA} = \gamma_{gA}$, it follows that $\alpha_g > \alpha_c \Rightarrow (1-\alpha_g)\gamma_{gA} < (1-\alpha_c)\gamma_{cA}$.

Therefore, along the optimal steady state, the value-added of the consumption goods sector dominates that of the capital goods sector in the long run. (P2) implies that each sector's optimal path converges to its own optimal steady state. Because $\beta > \alpha$ holds along the steady state because of (P3) and that also exhibits that the consumption goods sector's integrated capital income share is greater than that of the capital goods sector. Therefore, the value-added domination of the consumption goods final sector with the lower labor income share implies that the aggregated labor income share declines in the long run. Our result can be summarized as the following proposition.

Proposition 3. Under our assumptions, if $\gamma_{cA} \cong \gamma_{gA}$ or $(1-\alpha_g)\gamma_{gA} < (1-\alpha_c)\gamma_{cA}$ were to hold, then the aggregated labor income share could be expected to decline in the long run.

Remark. Each sector's labor share is closely related to its TFP growth rate. As shown in Figure 2, the sector with a higher TFP growth rate exhibits a lower labor income share. We observe the similar relation among other OECD countries. In terms of our variables, it implies that $\beta > \alpha \Leftrightarrow (1-\alpha_g)\gamma_{gA} < (1-\alpha_c)\gamma_{cA}$. Thus these empirical facts clearly support our claims, especially (P3).

7. Conclusion

We have demonstrated that setting up the endogenous TFP growth model based on the five major driving forces leads to an exhibition of the decline of the aggregated labor income share in the long run. Among other things, regarding the intermediate goods as intangible IIP capital goods, we successfully introduce the driving factor (ii). However, when it comes to the "superstar" firm factor, our model exhibits a severe defect: our two-sector model is not based on firm-level analysis, but on industry-level analysis. The recent firm-level empirical study reported by Kehrig and Vincent (2017) examining the U.S. manufacturing sector has documented two facts: an important reallocation of production towards hyper-productive plants and a downward adjustment of the labor share of those same plants over time. To study the micro-level mechanism of the labor share decline, it remains an urgent task to produce a multi-firm optimal growth model.

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Appendix

The following parameter values are assigned in order to implement the simulation work:

Parameter	*Explanation*	*Value*
α_c		0.2
α_g		0.9
β_c	$= (0.5 - \alpha_c) / (1 - \alpha_c)$	0.5
β_g		0.6
ε_c	$= \alpha_c + \beta_c (1 - \alpha_c)$	0.8
ε_g	$= \alpha_g + \beta_g (1 - \alpha_g)$	0.88
α	$= 1 - \varepsilon_g$	0.02
β	$= 1 - \varepsilon_c$	0.5 (fixed)
$\delta + n$	Depreciation rate + rate of population growth (n)	0.40
r	Representative household's discount rate	0.04
γ	TFP growth rate ($\gamma = \gamma_{cA} = \gamma_{gA}$)	0.40
σ	Coefficient of the utility function	2.5
ρ	$= r - (1 - \alpha_c)\gamma(1 - \sigma) + n$	0.58

We use Mathematica ver. 10.3 to solve the problem[3].

Note that we fix the value of β as 0.5. By so doing, the implicit function turns out to be the quadratic equation of k_c:

$$\left[(\alpha - \beta)k_c + \beta(1-\alpha)k\right]^{0.5} \tilde{c} = D_c \left[\alpha(1-\beta)\right]^{0.5} k_c$$

$$\Rightarrow (***) D_c \left[\alpha(1-\beta)\right]\left(k_c\right)^2 - (\alpha - \beta)\left(\tilde{c}\right)^2 k_c - \beta(1-\alpha)k\tilde{c} = 0$$

[3] The Mathematical code used here will be provided upon request.

Choosing the proper root, we can numerically solve the implicit function w.r.t. k_c as the explicit function of $\left(k,\tilde{c}\right)$. Substituting the result into $T(\tilde{c},k)$, Eq. (3.1), Eq. (3.5), Eq. (3.6) and Eq. (3.7), we obtain the numerically expressed differential equation system (4.1). First we draw the vector field of the dynamical system (4.1) to check the saddle-path stability. Then we solve the numerically expressed dynamical system (4.1) and derive the stable and unstable manifolds. The result is shown as Figure A.1.

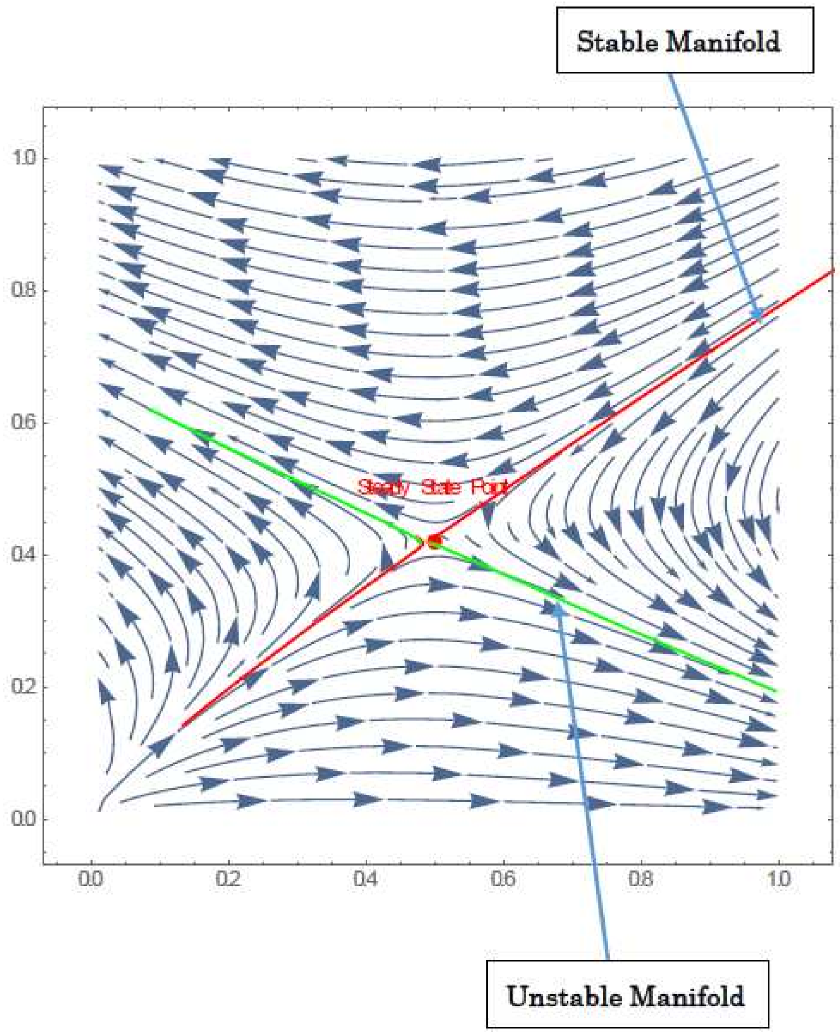

Figure A.1: Vector Fields, Stable and Unstable manifolds

Leading Forces	i) Supercycles and Boom-bust	ii) Rising and Faster depreciation	iii) Superstar effects and Consolidation	iv) Capital substitution and Automation	v) Globalization and Labor bargaining power
Weighted Contribution (%)	33	26	18	12	11

Table 1: Contribution of respective drivers to the capital share increase

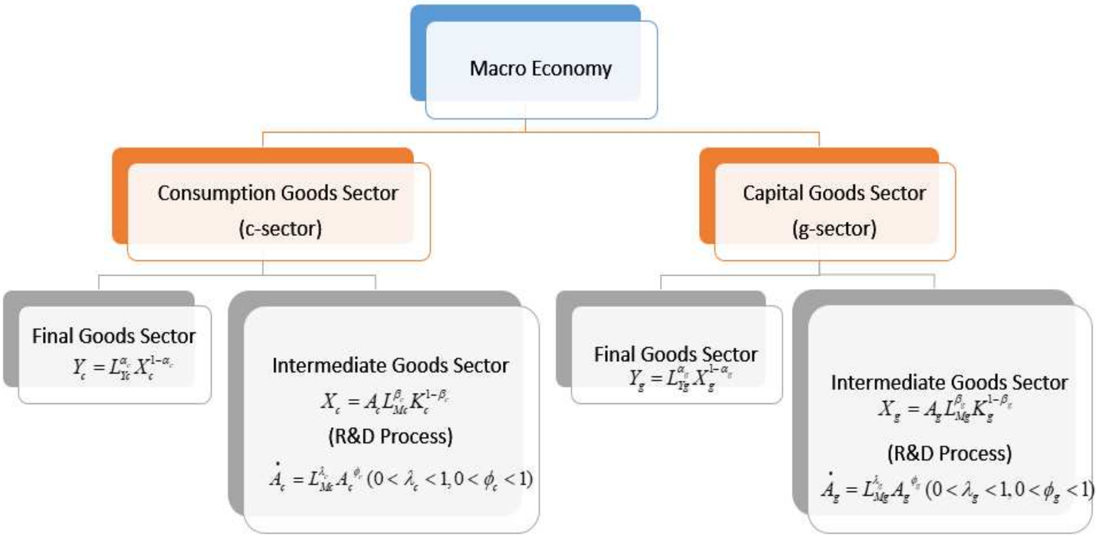

Figure 1: Tree diagram of the model.

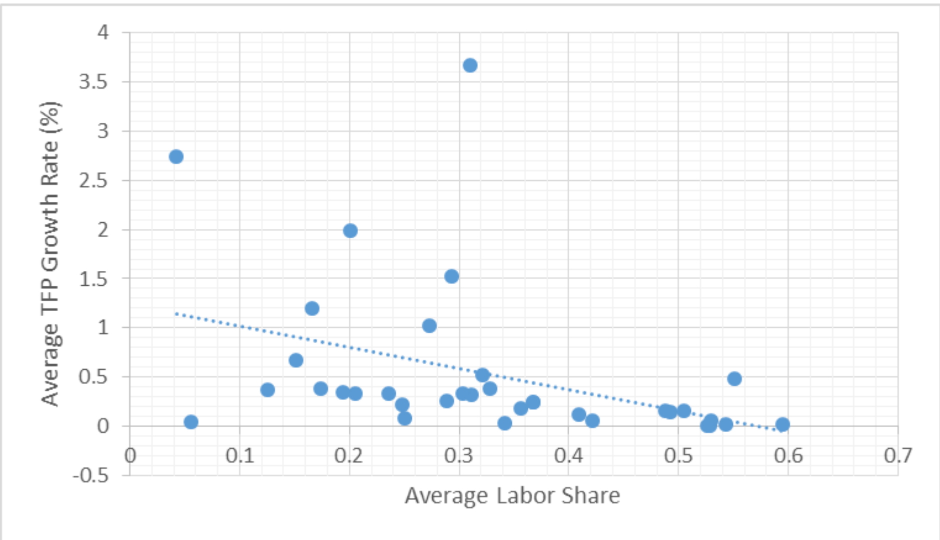

Figure 2: Average TFP Growth Rate and Labor Share from 1999 to 2010 in US: Corr. = - 0.39