The distribution of the average of log-normal variables and exact Pricing of the Arithmetic Asian Options: A Simple, closed-form Formula

Alghalith, Moawia

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The distribution of the arithmetic average of log-normal variables and exact pricing of the arithmetic Asian options: A simple, explicit formula

Moawia Alghalith

Abstract: We overcome a long-standing obstacle in statistics. In doing so, we show that the distribution of the arithmetic, continuous average of log-normal variables is log-normal. Furthermore, we offer a breakthrough result in finance. In doing so, we introduce a simple, exact and explicit formula for pricing the arithmetic Asian options. The pricing formula is as simple as the classical Black-Scholes formula.

Keywords: Applied probability, the distribution of the average of log-normals, arithmetic Asian option, the Black-Scholes formula.

MSC: 60E05

1 Introduction

A long-standing obstacle in statistics is the determination of the distribution of the sum of log-normal variables. This paper overcomes this obstacle and shows that the distribution of the sum of log-normal variables is log-normal.

To our knowledge, there is no explicit formula for pricing Arithmetic Asian options. Recent literature used orthogonal polynomial expansions to approximate the distribution of the arithmetic average. Examples include Willems (2019) and Asmussen et al (2016). Some of the literature used Edgeworth expansions to approximate the distributions (see, for example, Li and Chen (2016)). Gambaro et al (2020) used a tree method for dicrete Asian options. Carsaro et al (2019) adopted a computional method. Cui et al (2018) used approximations. Others such as Aprahmiam and Maddah (2015) used the Gamma distribution approach. Some studies relied on Monte Carlo simulations. Examples include Lapeyre et al (2001) and Fu et al (1999). Others adopted a numerical approach. Examples include Linetsky (2004), Cerny and Kyriakou (2011), and Fusai et al (2011). Curran (1994) used the geometric mean to estimate the arithmetic mean.

The literature on pricing the arithmetic Asian options has two main features in common. First, it relies on approximations. Secondly, it largely adopts (very) complex methods. Consequently, this paper overcomes these two limitations. In this paper, we use a pioneering approach to pricing the arithmetic Asian options in continuous time. In doing so, we present an exact (yet very simple) method. Particularly, we show that the price of the arithmetic Asian option is exactly equivalent to the price of the European option with an earlier (known) expiry. The pricing formula is as simple as the classical Black-Scholes formula.

2 The method

The arithmetic average of the price underlying asset $S(u)$ over the time interval $[t, T]$ is given by

$$A_t = \frac{\int_t^T S(u)\, du}{T - t},$$

where t is the current time and T is the expiry time. So that, using the Black-Scholes assumptions, $EA_t = E\frac{\int_t^T S(u)du}{T-t} = \frac{e^{r(T-t)}-1}{r(T-t)} S(t)$, where r is the risk-free rate of return. By the mean value theorem for integrals, $E\frac{\int_t^T S(u)du}{T-t}$

$= ES\left(\hat{t}\right)$, where $\hat{t}$ is a time such that $t < \hat{t} < T$, and $ES\left(\hat{t}\right) = e^{r\left(\hat{t}-t\right)} S\left(t\right).$ This implies that $\frac{e^{r(T-t)}-1}{r(T-t)} = e^{r\left(\hat{t}-t\right)}$. We can solve for $\hat{t}-t$ as follows

$$\hat{t} - t = \frac{\ln\left(\frac{e^{r(T-t)}-1}{r(T-t)}\right)}{r}.$$

Thus $\hat{t}$ is known. For example, if $T-t=1$ and $r=.01$, $\hat{t}-t = \frac{\ln\left(\frac{e^{.01}-1}{.01}\right)}{.01} = .498$. We also show that A_t is log-normal and a stock price[1]. Since A_t is a price and $ES\left(\hat{t}\right) = EA_t$, the variance of A_t is $S\left(t\right)^2 e^{2r\left(\hat{t}-t\right)} \left(e^{\sigma^2\left(\hat{t}-t\right)} - 1\right).$ Thus the Black-Scholes formula can be directly and exactly applied. That is, the price of the Asian option (expiring at time T) is given by

$$C\left(t\right) = e^{-r\left(\hat{t}-t\right)} E\left[S\left(\hat{t}\right) - K\right]^{+} = e^{-r\left(\hat{t}-t\right)} E\left[A_t - K\right]^{+},$$

where K is the strike price. Clearly, this is the price of a European option with expiry $\hat{t}$. Thus, the price of the arithmetic Asian option (with expiry time T) is equal to the price of the equivalent European option with expiry time $\hat{t}$. This explains why the Asian option is cheaper than its European counterpart.

[1] See the appendix for the proofs.

Needless to say, the pricing formula for an arithmetic Asian call with expiry time T is

$$C(t,s) = sN(d_1) - e^{-r(\hat{t}-t)} KN(d_2),$$

where s is the current price, $d_1 = \frac{1}{\sqrt{\sigma^2(\hat{t}-t)}} \left[\ln(s/K) + (r+\sigma^2/2)(\hat{t}-t)\right]$, $d_2 = d_1 - \sqrt{\sigma^2(\hat{t}-t)}$, and σ is the volatility of the return rate of the underlying asset.

Practical example:

If $r = .05$, $T = 1$, $\sigma = .2$, $s = K = \$100$, then $\hat{t} = .502$ and thus the option price is $C(t) = \$6.91$.

Appendix.

Proof of A_t is log-normal.

Consider the stock price, $S(T) - s = \int_0^T dS(t)$, where $s \equiv S(0)$; squaring both sides yields

$$(S(T))^2 + s^2 = 2sS(T) + \left(\int_0^T dS(t)\right)^2 = 2sS(T) + \sigma^2 \int_0^T (S(t))^2\, dt \quad (1)$$

since $(dS(t))^2 = \sigma^2 (S(t))^2\, dt$. The left-hand-side of (1) is clearly log-normal (a lognormal plus a constant), and the right-hand-side of the equation is a sum of lognormal variables; therefore, the sum (or average) of log-normal variables is log-normal.

We can also present the sum without the constant s^2 by differentiating both sides of (1) with respect to r

$$\frac{\partial \left(S\left(T\right)\right)^2}{\partial r} = 2s\frac{\partial S\left(T\right)}{\partial r} + \sigma^2 \int_0^T \frac{\partial \left(S\left(t\right)\right)^2}{\partial r} dt$$

clearly the left-hand-side of the above equation is log-normal, and the right-hand-side of the equation is a sum of log-normal variables.

We can also show that the integral alone is log-normal; dividing both sides of the above equation by $S\left(T\right)$ yields

$$2TS\left(T\right) = 2Ts + \sigma^2 \int_0^T \frac{\partial \left(S\left(t\right)\right)^2}{S\left(T\right)\partial r} dt,$$

differentiating twice w.r.t. r

$$2T\frac{\partial^2 S\left(T\right)}{\partial r^2} = \sigma^2 \int_0^T \frac{\partial^2 X}{\partial r^2} dt,$$

where $X \equiv \frac{\partial(S(t))^2}{S(T)\partial r}$; the left-hand-side of the above equation is log-normal, and the right-hand-side of the equation is a sum of log-normal variables.$\square$

Proofs of A_t is a stock price.

1. Let $\int S(u)\,du \equiv I$, then

$$\frac{dI}{du} = S(u)$$

$du = (T-t)/n$; therefore

$$\frac{ndI}{T-t} = \frac{I}{T-t} = A_t = S(u)\,\square$$

2. The simplest and intuitive proof is that the time continuity implies that the average price A_t is a price on the interval $[S(t), S(T)]$. To be more precise, each (random) price at a specific time is an interval-valued (an interval of all possible outcomes of the price). Thus the elements of $[S(t), S(T)]$ are (vertical) intervals, then the time continuity guarantees the existence of a vertical interval of outcomes on $[S(t), S(T)]$, but the vertical interval is a price at a specific time. So the difference between a random variable and a non-random variable is that the random variable is interval-valued, and thus the mean-value theorem can be applied in the same way

to non-random variables if we view the elements of $[S(t), S(T)]$ as interval-valued.

3. The outcomes of A_t are the averages of paths and therefore they are outcomes (realizations) of prices. That is, each outcome is in the form $S(t)\, e^{\left(r-\frac{1}{2}\sigma^2\right)u+\Omega_1}$, where Ω is an outcome of a Brownian motion; thus it can be expressed as $A_t = S(t)\, e^{\left(r-\frac{1}{2}\sigma^2\right)u+W(u)}$; otherwise it will not be possible, using the price probability density, to obtain $EA_t = S(t)\, e^{r\left(\hat{t}-t\right)}$.

Conclusion:

In sum, this paper offers two ground-breaking contributions. The first one is in mathematical statistics (the distribution of the arithmetic average of log-normal variables). The second one is in finance (an explicit, simple formula for the price of the arithmetic Asian options). The first contribution will have a great impact on statistics since it will have so many applications in the future. Furthermore, there is a big practical advantage. In practice, the choice of the discrete times to be included in the average is arbitrary and controversial. The industry can avoid this problem altogether by trading continuous-average options (using our formula).

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