Fuzzy Set Theory: A Primer

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Abstract

In the Classical Logic, the 'Law of the Excluded Middle' states that out of two contradictory propositions (where one proposition is the negation of the other) one must be true, and the other false, or that every proposition must either be true or false: it will not be possible to be and not to be the same thing; these truth values with a sharp demarcation may be represented by '1' for 'true' and '0' for 'false' and form the basis of the traditional set theory. Thus it cannot consider cases of vagueness. However, there are many concepts and predicates that are surrounded by vagueness or uncertainty. For example, consider the predicates such as 'tall' and 'nice', adverbs such as 'quickly' and even quantifiers such as 'many', all of which can be vague. So are the predicates 'developed' and 'poor', or the concepts of 'development' and 'poverty'. A well-known account of vagueness comes in terms of 'degree theory' that drops classical Logic. The degree theory proposes a gradual transition between 'perfect falsity' to 'perfect truth', so that there are more than two truth values, that is, an infinite number of truth values along a spectrum between perfect truth and perfect falsity. Thus truth comes in degrees. Fuzzy set logic, developed by the American mathematician Lotfi Aliasker Zadeh, seeks to quantify the degree of truth in borderline cases. Thus perfect truth may be represented by '1' and perfect falsity by '0', with borderline cases having a truth value anywhere between 0 and 1. The present paper provides an introduction to fuzzy set theory.

“Vagueness, … is my topic tonight ….. You will no doubt think that, in the words of the poet: “Who speaks of vagueness should himself be vague”. I propose to prove that all language is vague, and that therefore my language is vague, but I do not wish this conclusion to be one that you could derive without the help of the symbolism. I shall be as little vague as I know how to be if I am to employ the English language.”

– Bertrand Russel (1923: 84)

“Who can turn a can into a cane?
Who can turn a pan into a pane?
It’s not too hard to see,
It’s Silent E.

“Who can turn a cub into a cube?
Who can turn a tub into a tube?
It’s elementary
For Silent E.”

– Tom Lehrer's 1971 children’s song “Silent E”

Base step: A one day old human being is a child.

Induction step: If an n day old human being is a child, then that human being is also a child when it is $n + 1$ days old.

Conclusion: Therefore, a 36,500 day old human being is a child.

No child should work.
Every person is a child of someone.
Therefore, no one should work.

Driver: “Do I turn left?”
Passenger: “Right”.

Fuzzy Set Theory

A Primer

1. Introduction

Let us start with the famous ‘paradox of the heap’, more properly called *sorites paradox*; the word ‘sorites’ is a Greek word meaning heap; the paradox is attributed to the Megarian philosopher Eubulides of Miletus of 4th century BC. He was a pupil of Euclid of Megara, one of the pupils of Socrates and the founder of the Megarian school. Eubulides was a contemporary of Aristotle, and famous for his seven paradoxes. Two of these paradoxes are the heap (*sôritês*) paradox and the ‘bald man’ (*phalakros*) paradox

The first paradox involves a heap of sand, from which grains of sand are removed one by one. We know that when we remove a single grain, what remains is still a heap; remove one more, still a heap remains; one more, again one more, we will still have a heap; now what happens when we extend this logic as we remove more and more individual grains such that each time a grain of sand is removed we are still left with a heap of sand? When we are finally left with a single grain, the logic suggests that we still have a heap! Thus we are led to a contradiction. That is the paradox. But when can we say the heap ceased to be a heap and turned into a non-heap?

We have a number of similar paradoxes. Consider a bag of some sand grains that I can very easily lift. Add one more grain of sand, I can still lift that bag; one more, again one more; still I can lift it easily. If we continue with this logic, it can mean that I will be able to lift the bag of sand grains, even if it has tons of sand!

The other paradox, called the bald man (*phalakros*) paradox runs like this: suppose we have a friend with a head full of hair; removing a single hair will not turn him into a bald man; remove one more hair, again one more; he will still be non-bald. If we extend this logic repeatedly, what will happen? We will have to admit that our friend is still non-bald, even if there is only one hair finally on his head! A contradiction.

The *sorites paradox* was later used by the Greek philosophers as a method of dialectical argument, mainly by the Sceptics against the Stoics.

Such paradoxes are collectively called the continuum fallacy. It refers to vagueness or uncertainty surrounding many concepts and predicates and argues that two states or conditions cannot be considered distinct, because between them there exist different states in a continuum. For example, consider the predicates such as ‘tall’ and ‘nice’, adverbs such as ‘quickly’ and even quantifiers such as ‘many’, all of which can be vague (Keefe and Smith 1996, p 5). So are the predicates ‘developed’ and ‘poor’, or the concepts of ‘development’ and ‘poverty’. Before coming to these vague predicates/concepts, let us consider the vague predicate ‘tall’.

We know that certainly there are cases in which we can without doubt classify people as ‘tall’ and ‘short’, even though we do not have any exact borderline of height *h*, above which one is tall and below which one is short. At the same time, there are certain cases in which we cannot state that a person is *definitely* tall. In such cases, we may say that person is ‘borderline tall’. Also note that the sorites paradox too applies here. Suppose Ram is definitely tall. Now consider cutting away a millimeter of his height; he will still be tall. Cut away one more millimeter, again another millimeter, he will still be tall. Extending this logic leads us to the usual contradiction that he will still be tall, even after a sizeable height is cut away from him.

Similarly, the predicate ‘poor’ also is vague; so are the related predicates ‘extreme’ and ‘chronic’, used in the measurement of poverty. For example, take the case of a person, poor in income. We know that giving her one more paise (one-hundredth denomination of a rupee) will not make her non-poor. Give her one more paise, again one more paise, still she will not be non-poor. Extending this logic several times will lead us to the usual contradiction that she will still be poor, even after amassing a lot of income. Similarly, we might come across several cases in which we might classify a person as ‘borderline poor’. Moreover, the official use of poverty line to distinguish between the poor and the non-poor cannot be taken as exact, because it is not possible to find a sharp borderline between the poor and the non-poor. For example, take the poverty line accepted in India in terms of a benchmark daily per capita expenditure of Rs. 27 and Rs. 33 in rural and urban areas, respectively. Can we say a rural person having a daily per capita expenditure of Rs. 27.01 is non-poor? Can we so assert, even if we increase her income to Rs. 27.50 or even to Rs. 28 or 29? Thus the predicate ‘poor’ is as vague as ‘tall’. The same applies to the other predicates relevant to poverty measurement such as ‘extreme’ and ‘chronic’. Note that these are only a few among many vague predicates/concepts that social scientists usually study.

Thus a concept or predicate may be considered *vague*

(i) if it is found to be lacking in clarity, or
(ii) if there is uncertainty about the kind of objects belonging to that concept or having characteristics that correspond to that predicate (so-called ‘border-line cases’), or
(iii) if the Sorites paradox applies to the concept or predicate.

In explaining vagueness, so far we have considered examples having only one dimension. In the case of the predicate ‘tall’, we have taken height as the only dimension. We have explained baldness in terms of the number of hairs on the head of a person. And in the case of poverty, the only dimension we have considered is the amount of income one has. However, in the case of some concepts or predicates, we can find that multiple dimensions are relevant. For example, consider the predicate ‘nice’. We know that our friend Rani is very polite, sociable and generous, but is sometimes bad-tempered also. We may say that Rani is not *definitely* nice, but only borderline nice, considering all the relevant (multiple) dimensions.

So is the predicate ‘poor’, which is multi-dimensional as well as vague. A person may be poor not just in terms of income; she may be poor in health, education, etc. also. Now suppose that Rani has a good income, but is illiterate and chronically ill. She is classified as non-poor in terms of income, but as poor in terms of education and health, even if we allow for vagueness of ‘poor’ in these dimensions. Now can we consider her ‘poor’? Or suppose that Rani has a very low income, but is well educated and very healthy; thus she is poor in terms of income, but non-poor in health and education (after allowing for vagueness of ‘poor’ in these dimensions). Now can we consider her ‘poor’?

From a multi-dimensional viewpoint of poverty, it is not clear whether we can classify her to be poor or non-poor in these cases. So she may be classified as ‘borderline poor’ in both these cases. Thus the multi-dimensionality of poverty is thus relevant to its vagueness.

So far we have considered predicates associated with individuals; the same applies to society or country. When we say a country is poor, we acknowledge that both the multi-dimensionality and vagueness of poverty are relevant here too. And note that development is the opposite of poverty, and hence that concept also is vague and multi-dimensional.

2. Logic and Fuzzy Logic

Aristotle was the first to develop Logic in Europe; in order to devise a concise theory of logic, and later mathematics, he posited the three ‘Laws of Thought’. The third of these is the ‘Law of the Excluded Middle’, also known as the law of the excluded third (in Latin *principium tertii exclusi* or *tertium non datur* = no third (possibility) is given). The law states that out of two contradictory propositions (where one proposition is the negation of the other) one must be true, and the other false, or that every proposition must either be true or false: it will not be possible to be and not to be the same thing, so that the ‘principle of bivalence’ (which only allows for ‘true’ or ‘false’ statements) holds; these truth values with a sharp demarcation may be represented by ‘1’ for ‘true’ and ‘0’ for ‘false’ and form the basis of the traditional set theory. Thus it cannot consider cases of vagueness.

History shows that Parminedes, a pre-Socratic Greek philosopher, had proposed the first version of this law around 400 BC, against which Heraclitus had proposed that things could be simultaneously true and not-true – the first light on fuzziness. However, Plato is regarded as having laid the foundation of fuzzy logic, who indicated that there was a third region (beyond ‘true’ and ‘false’), where these opposites “tumbled about”. But this dimension of logic remained in the dark for many centuries, and it was only around 1920 that a systematic alternative to the bi-valued logic of Aristotle came to be proposed by Jan Łukasiewicz, a Polish logician and philosopher, who mathematically described a three-valued (or trinary) logic system in which there are three truth values indicating ‘true’, ‘false’ and some indeterminate third value, which might be translated as the term ‘possible’ and to which he assigned a numeric value between ‘true’ and ‘false’. Later, he explored four-valued logics, five-valued logics, and then declared that in principle it is possible to derive an infinite-valued logic.

In 1921, Emil L. Post, an American mathematician, formulated additional truth degrees with more-than-two truth values ($n \geq 2$, where n are the truth values). Later on, Łukasiewicz and Alfred Tarski together formulated a logic on n truth values where $n \geq 2$ and in 1932 Hans Reichenbach introduced a logic of infinite truth values where n approaches infinity.

Thus a well-known account of vagueness comes in terms of ‘degree theory’ that drops classical Logic. The degree theory proposes a gradual transition between ‘perfect falsity’ to ‘perfect truth’, so that there are more than two truth values, that is, an infinite number of truth values along a spectrum between perfect truth and perfect falsity. Thus truth comes in degrees.

Fuzzy set logic, developed by the American mathematician Lotfi Aliasker Zadeh (1965, 1975), seeks to quantify the degree of truth in borderline cases. Thus perfect truth may be represented by ‘1’ and perfect falsity by ‘0’, with borderline cases having a truth value anywhere between 0 and 1.

Now let us turn to the set theory.

3. Set Theory – Classical and Fuzzy

Classical set theory, also called *crisp* or naïve set theory, postulates that either an element belongs to the set or it does not. For example, for the set of integers, either an integer is even or it is not (that is, it is odd). Those objects that belong to a set are called its members. As objects we allow anything: numbers, people, other sets... If x is a member of A, then we write $x \in A$. (The symbol "$\in$" is a derivation of the Greek letter Epsilon, ‘ε’.)

The membership or characteristic function of a crisp set may be written as

$$\mu_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$$

Some sets may be described in words, for example:

A is the set whose members are the first four positive integers.

B is the set whose members are the colors of the Indian flag.

By convention, a set can also be defined by explicitly listing its elements between braces (curly brackets), for example:

$C = \{4, 2, 1, 3\}$

D = {saffron, white, green}

Notice that two different descriptions may define the same set. For example, for the sets defined above, *A* and *C* are identical, since they have precisely the same members. The shorthand $A = C$ is used to express this equality. Similarly, for the sets defined above, $B = D$.

This method of listing elements is called Roster method or Enumeration or Description method.

Another method of using mathematical notation for describing a set by indicating the properties that its members must satisfy is called set-builder (or rule) notation or set comprehension.

The simplest sort of set-builder notation is $\{x : P(x)\}$, where *P* is a predicate in one variable. This indicates the set of everything satisfying the predicate *P*, that is, the set of every object *x* such that $P(x)$ is true. Some authors use the pipe symbol | rather than **:** to indicate the conditional. For example:

- $\{x : x$ is a real number and $x > 0\}$ is the set of all positive real numbers;

  $\{x \mid x \in \mathrm{R} \,;\, x > 0\}$

- $\{k :$ for some natural number n, $k = 2n\}$ is the set of all even natural numbers;

$\{k \mid k = 2n \,;\, n \in \mathrm{N}\}$

Also note that all sets under consideration in a certain case are subsets of some 'bigger' set, called universal set and denoted as 'U'. We also allow sets to be infinite and empty.

There are four common operations on sets: union, intersection, negation, and inclusion, denoted by the symbols $\cup$, $\cap$, ~, and $\subset$, respectively. The first two are connectives, as they produce a new set from two or more sets under some given condition. Let us explain these operations using the two sets:

$A = \{4, 2, 1, 3\}$ and

$B = \{\text{saffron, white, green}\}$.

Union ($\cup$) combines two sets together; that is, it is the set of all things which are members of either *A* or *B* or both and corresponds to "or" in logic. With the two sets above,

$A \cup B = \{4, 2, 1, 3, \text{saffron, white, green}\}$.

Intersection ($\cap$) is the overlap between two sets; that is, it is the set of all things which are members of both *A* and *B* and corresponds to "and" in logic. The two sets above have no common elements and hence their intersection is empty. That is,

$A \cap B = \emptyset$, the symbol for the null or empty set.

Negation (~), corresponding to 'not', gives the complement of a set, which contains all elements in the universal set that are not in the given set. Note that its definition requires that we define our universe (universal set, U) under consideration. of discourse, represented by the 'universal' set *U*.

Suppose that for our Set *A* defined above,

$U = \{\text{all counting numbers up to } 10\}$.

Then $\sim A = \{5, 6, 7, 8, 9, 10\}$.

Note that $A \cup \sim A = U$; that is, “everything that is *A*, and everything that is not *A*, is everything.”

Also note that $A \cap \sim A = \emptyset$; that is, “nothing is in both *A* and not *A* at the same time.” This is the famous Law of the Excluded Middle, which the fuzzy set intersections do not generally obey.

Inclusion or containment ($\subset$) concerns whether a set includes/contains elements in another set. If every member of the set *A* is also a member of the set *B*, then *A* is said to be a subset of *B*, written $A \subset B$, and read as *A is contained in B*. Equivalently, we can write $B \supset A$, read as *B is a superset of A*, or *B includes A*, or *B contains A*. In the case of *A* and *B* given above, it is clear that neither set includes the other. However, given another set

$C = \{1, 2, 3, 4, 5, 6\}$, we have $A \subset C$.

Note that the asymmetry of inclusion is highly useful in examining relationships between empirical cases that are quite different from the correlations usually used in social sciences. Also note that inclusion and intersection have a special relationship:

when $A \subset B$, then $A \cap B = A$ and when $A \subset B$ and $B \subset A$, then $A = B$.

Now let us consider the set of vowels of the English letters,

$V = \{a, e, i, o, u\}$.

Since a letter is either a consonant or a vowel, logically the set of consonants is the complement of *V*, that is,

$C = {\sim}V$.

But, we know that in English, the letter y behaves strangely; it is sometimes a vowel and sometimes, a consonant. For example, in the word "my," y is a vowel, but in the word "your", it is not. Now the question is: Does y belong to set V or to set C? The answer is of course not definite as it belongs to both the sets, not just to any one. That is, the English letters cannot be classified into two mutually exclusive sets as implied by the dichotomy between vowels and consonants. This simply means that the letter y violates the Law of the Excluded Middle that is implied in the definition $C = {\sim}V$. Thus we have an issue of fuzziness here.

A **fuzzy set** is based on a classical set itself, but it is distinguished by its membership function that ranges from 0 to 1, that is, [0, 1], in contrast to that of the classical set constrained to either 1 or 0, that is, {0, 1}.

Formally, given a set X of elements $x \in X$, any fuzzy subset A of X is defined as follows:

$$A = \{x, \mu_A(x)\},$$

where $\mu_A(x)$: $X \rightarrow [0, 1]$, where [0, 1] is the interval of real numbers from 0 to 1, is called the *membership function* in the fuzzy subset A.

The value $\mu_A(x)$ indicates the degree of membership of x in A, that is to say, the degree of truth of $A(x)$. In other words, the membership function is an index of "set-hood" that measures the degree to which an object x is a member of a particular set.

Note that if $\mu_A(x) \in \{0, 1\}$, then A is an *ordinary* subset of X, which constrains the membership function to either 1 or 0.

If $\mu_A(x) \in [0,1]$, then A is a *fuzzy* subset of X, which allows the membership function to range from 0 to 1.

Also note that $\mu_A(x) = 0$ means that x does not belong to A, whereas $\mu_A(x) = 1$ means that x belongs to A completely. When $0 < \mu_A(x) < 1$, x partially belongs to A and its degree of membership in A increases in proportion to the proximity of $\mu_A(x)$ to 1. We can say that the membership function thus acts as a linear filter. In short, the main difference between classical set theory and fuzzy set theory is that the latter allows for partial set membership.

Let us consider an illustrative example in terms of the predicate 'young'. Here the universal set (P) includes all the people of different ages. We also define a fuzzy subset Y (for young) in terms of answers to the question 'to what degree is person x young?' Note that we have to assign a degree of membership in the fuzzy subset Y to each person in the universe. Obviously, the easiest way to do this is with a membership function based on the person's age. Thus we can define this membership function of a person x as follows:

$$\text{young}(x) = 1, \qquad \text{if age}(x) \leq 20,$$

$$= \frac{[30 - \text{age}(x)]}{30 - 20}, \quad \text{if } 20 < \text{age}(x) \leq 30,$$

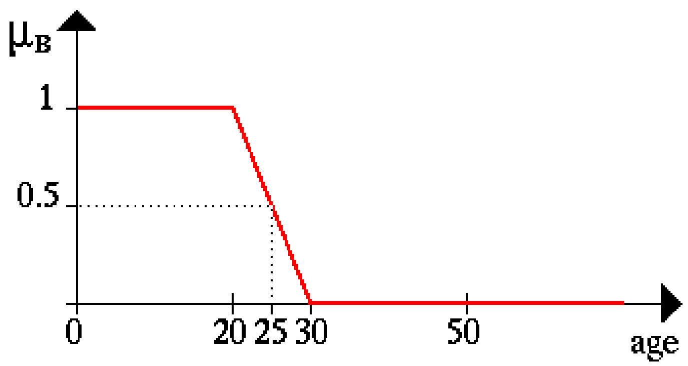
$$= 0, \qquad \text{if age}(x) > 30.$$

Given this definition, we can estimate some example values:

Person	Age (years)	Degree of youth
Ahalya	10	1
Anasuya	15	1
Arundati	21	0.9
Damayanti	25	0.5
Renuka	26	0.4
Sakuntala	28	0.2
Sita	35	0
Urmila	60	0

Accordingly, we can say that the membership value of Damayanti is 0.5 or that the degree of truth of the statement 'Damayanti is young' is 0.5.

A graph of this membership function appears as follows:



For another example, let us consider the concept of poverty. In our definition given below, membership in the set of poor people is 1 if daily per capita expenditure of a rural person is below Rs. 27, the degree of membership decreases linearly for expenditures ranging from Rs. 27 up to Rs. 50, and equals 0 for any income that exceeds Rs. 50.

$$\text{Poor}(x) = \begin{cases} 1, & \text{if } x < 27, \\ \frac{50-x}{50-27}, & \text{if } 27 \le x \le 50, \\ 0, & \text{if } x > 50. \end{cases}$$

Below we give some example values, based on this definition.

Person	Expenditure (Rs)	Degree of poverty
Ahalya	10	1
Anasuya	26	1
Arundati	28	0.96
Damayanti	30	0.87
Renuka	35	0.65
Sakuntala	45	0.22
Sita	53	0
Urmila	60	0

Note that the usual membership functions are not as simple as these cases. Hence next we consider the issue of measuring a membership function.

4. Measurement of Membership

We have already defined a membership function (in the fuzzy subset A of X) as

$$\mu_A(x):\ X \rightarrow [0, 1],$$

where [0, 1] is the interval of real numbers from 0 to 1, given a set X of elements $x \in X$.

Hence, $\mu_A(x) = 0$ if the element $x \in X$ does not belong to A, $\mu_A(x) = 1$ if x completely belongs to A and $0 < \mu_A(x) < 1$ if x partially belongs to A.

Following Chiappero-Martinetti (2000), let us suppose that the subset A defines the position of a country according to the degree of achievement of a given development dimension (such as income, education, health, etc,). In this case, a membership value equal to one identifies a condition of full achievement of a given development dimension, while a value equal to zero shows the reverse condition (of poverty). The intermediate values between 0 and 1 then describe gradual positions within the spectrum, an ascent from complete poverty to complete development. Thus we find that in order to define a membership function, we have

i) to define an appropriate arrangement of values on the basis of the different degrees of development ;

ii) to identify the two extreme conditions, that is, $\mu_A(x) = 1$ (full membership) and $\mu_A(x) = 0$ (non-membership) ; and

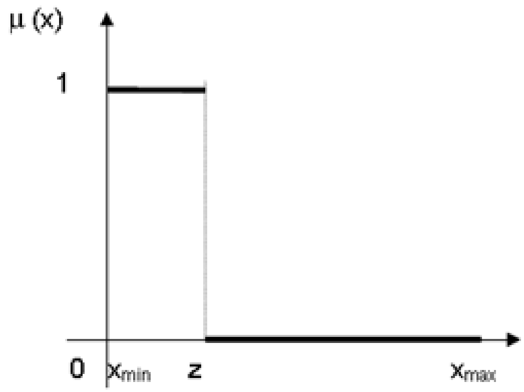
iii) to specify the membership values for all the other intermediate conditions.

Now, coming to the choice of a proper membership function, the main factors we have to consider are the application context and the kind of indicator that we want to describe. First let us consider the traditional approach of crisp set in terms of the usual poverty line.

traditional approach (i.e. poverty line)

$\mu(x) = 1$ if $0 \leq x \leq z$

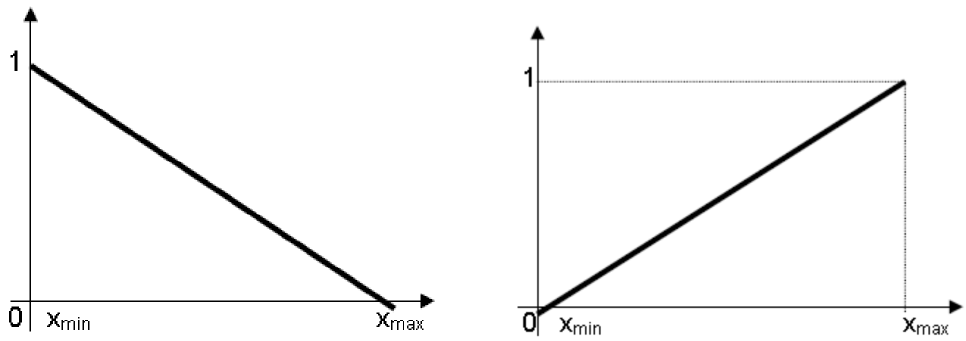
$\mu(x) = 0$ if $z \leq x \leq x_{max}$



In contrast to this, the fuzzy set theory suggests the following:

In the case of variables with equi-distributed values along an ordinal scale (values such as 1, 2, 3, etc., or 10, 15, 20, etc.), we have to employ the linear functions given below (Chiappero-Martinetti 2000).

Linear function



$$\mu(x) = \frac{X_{max} - X}{X_{max} - X_{min}} \qquad \mu(x) = \frac{X - X_{min}}{X_{max} - X_{min}}$$

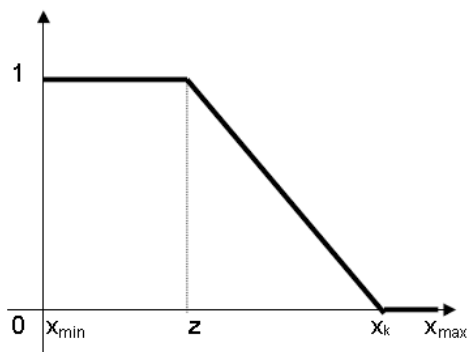
On the other hand, if it is possible to define, in clear terms, conditions of full membership (complete achievement) on one side and non-membership (total deprivation) on the other in the case of a given dimension of development such that it is possible to identify a given interval between these maximum and minimum threshold levels, then a trapezoidal function as shown below can be chosen (Chiappero-Martinetti 2000).

Trapezoidal function

$\mu(x) = 1$ if $x_{min} \leq x \leq z$

$\mu(x) = 0$ if $x_k \leq x \leq x_{max}$

$\mu(x) = \frac{x_k - x}{x_k - z}$ if $z \leq x \leq x_k$



If we have quantitative and qualitative variables with values that are not equi-distributed, then a sigmoid function, or Z-function, shown below, seems appropriate to describe the relevant membership function (Chiappero-Martinetti 2000).

Sigmoid (or logistic) function

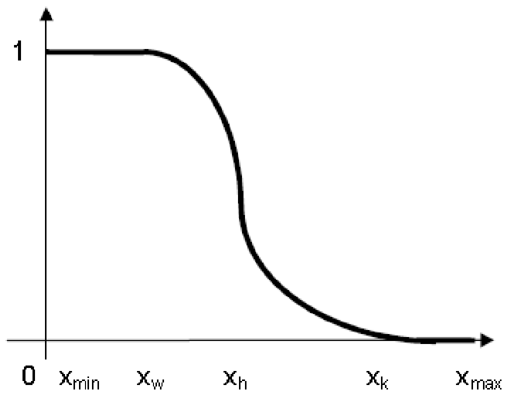
$x_k = \mu(x) = 0 \qquad x_h = \mu(x) = 0.5 \qquad x_w = \mu(x) = 1$

$\mu(x) = 1 \qquad \text{if } x_{min} \leq x \leq x_w$

$\mu(x) = 0 \qquad \text{if } x_k \leq x \leq x_{max}$

$$\mu(x) = 1 - \frac{1}{2}\left[\frac{x_w - x}{x_w - x_h}\right]^2 \qquad \text{if } x_w \leq x \leq x_h$$

$$\mu(x) = \frac{1}{2}\left[\frac{x_k - x}{x_k - x_h}\right]^2 \qquad \text{if } x_h \leq x \leq x_k$$



5. A Review of Literature

Application of fuzzy set theory to the economics of inequality and poverty

Amartya Sen appears to be the first to suggest that fuzzy set theory might be applied to the economics of inequality and poverty. Sen recognised (in his *On Economic Inequality*) that "the implicit notion of inequality that we carry in our mind is, in fact, much less precise ….. We may not indeed be able to decide whether one distribution *x* is more or less unequal than another, but we may be able to compare some other pairs perfectly well. …… There are reasons to believe that our idea of inequality as a ranking relation may indeed be inherently incomplete. If so, to find a measure of inequality that involves a complete ordering may produce artificial problems, because a measure can hardly be more precise than the concept it represents." (Sen 1973, p 5-6).

He has also made similar observations in the context of poverty. For example, in his *Poverty and Famines* he wrote that "while the concept of nutritional requirements is a rather loose one, there is no particular reason to suppose that the concept of poverty must itself be clear-cut and sharp. In fact, a certain amount of vagueness is implicit in both the concepts, and the really interesting question is the extent to which the areas of vagueness of the two notions, as commonly interpreted, tend to coincide. The issue, thus, is not whether nutritional standards are vague, but whether the vagueness is of the required kind." (Sen 1981, p 13).

Sen has indeed supported the use of fuzzy set theory and measures based on it, where precision of such measures is of importance. He writes: "A formal expression can be extremely precise without being at all a *precise representation* of the underlying concept to be captured. In fact, if that underlying concept is ambiguous, then the demands of precise representation call for *capturing* that ambiguity rather than replacing it by some different idea – precise in form but imprecise in representing what is to be represented. It is in this context that such mathematical structures as partial orderings, fuzzy sets, etc., have much to offer." (Sen 1989, p 317; italics as in the original).

While discussing inequality measurement, Kaushik Basu (1987) has argued that there are cases that fall between those where one can make a precise judgement and those where one could make no judgement at all – cases where one can only make an *imprecise* judgement. He writes: "Recent advances in the theory of fuzzy sets enable us to talk of human imprecisions in a meaningful way. And it is not difficult to maintain that a fuzzy binary relation captures our ambivalence in ranking states according to inequality better than a quasi-ordering" (Basu 1987, p. 276). It is on this basis that Basu has developed his axiomatic fuzzy set theoretic measure of inequality. One "interesting property" he has found is that "the conventional Gini-ranking is, in a sense, a best unfuzzy approximation of our fuzzy measure" (ibid,).

The use of fuzzy set theory has in no time naturally reached the area of poverty measurement with early contributions from Cerioli and Zani (1990) and Cheli and Lemmi (1995). As already explained above, fuzzy set theory as applied to poverty measurement, works in terms of a graded membership, suggesting a degree to which it is true that someone (or some household) is a member of the set of the poor. The membership function of the set of the poor is typically taken to lie on the [0,1] interval, with '0' suggesting definite non-membership, '1', definite membership and values between zero and one capturing the degree of membership. Formally, this 'membership function' maps an individual's (or household's) performance in terms of an indicator, or in terms of a set of indicators, on to a degree of membership of the set of the poor. The first study of fuzzy poverty measurement (Cerioli and Zani 1990) has employed both a simple linear membership function, based on income only, and also some variations of multi-dimensional fuzzy poverty membership functions. In the simple case, they have taken a level of income below which a person (or household) is counted as definitely income poor, and another level above which she is judged to be definitely not income poor. Between these two levels, the degree of membership of the set of the poor linearly decreases as income increases. In the case of one of the alternative multi-dimensional measures, Cerioli and Zani have suggested an ordinal ranking of levels of disadvantage for each of the dimensions considered. In each dimension there is some level below which a person (or household) is taken as definitely poor, and another level above which she is classified as definitely not poor. Between these levels, the degree of membership of the set of the poor person (for each dimension) is based on her position in the ordinal ranking. Once this is done for all the dimensions, they have explored various ways of weighting the dimensions of poverty to derive an aggregate measure in order to judge whether or not a person (or household) is definitely poor considering all the dimensions of poverty. It must be noted that as long as each dimension has positive weight, a person (or household) must qualify as definitely poor with a score of 1 on all dimensions in order to be counted as definitely poor aggregately with an overall score of 1.

However, this methodology of arbitrarily using two critical levels to define the range of levels of income or other indicators of fuzziness has been criticized by Cheli and Lemmi (1995). They have instead suggested an alternative 'Totally Fuzzy and Relative' (TFR) approach. In this approach, the cut-offs used to mark the relevant range of levels of each indicator of fuzzy dimension is determined by the distribution itself. Only those persons (or households) who are most (least) deprived in terms of the distribution of the relevant indicator (such as income) are definitely poor (not poor) in terms of that indicator. Between these levels, the degree of membership of the set of the poor in the relevant indicator is based on the distribution of the relevant indicator. A large number of studies have employed the TFR approach; for example, Chiappero-Martinetti (1994, 1996, 2000), Lelli (2001); Qizilbash (2002) and Clark and

Quizilbash (2003) use this method in order to analyse poverty or well-being in the context of Sen's capability approach.

Qizilbash (2002) has applied both the Cerioli-Zani and Cheli-Lemmi measures in the South African context, using data from the 1996 South African Census. The cutoffs used in this study for the fuzzy poverty measures are in line with the Cheli-Lemmi methodology. The worst-off (best-off) category in the sample is thus defined as definitely poor (definitely not poor) in each relevant dimension of the quality of life.

In another attempt to apply Sen's capability approach to the South African context, Stephan Klasen (1997; 2000) has used various indices as proxies for fourteen 'components' of his composite measure of deprivation (such as education, health, housing, nutrition, water, employment, safety, etc.). Each component is thought of as relating to some specific 'capability', with levels of achievement in terms of these components associated with a rank order number. Klasen has also included income as a component in his study. He has characterised the index which focuses only on the seven indices given above as a 'core deprivation index' (Klasen, 2000, p. 43). The choice of component indicators that Klasen has included in this index is motivated by the fact that they relate to capabilities listed in certain works by Amartya Sen (Klasen, 2000, p. 39).

Sara Lelli (2001) has been the first to try an empirical comparison, in the context of Sen's capability analysis, between the fuzzy sets approach (including the TFR) and factor analysis, a multivariate technique, preferred by some researchers; she has found that that the 'fuzzy aggregates' are insensitive to the choice of the form of the membership function. Lelli (2001, p. 25) has also found that both the methods (factor analysis and fuzzy sets) show that "income accounts only for a very limited part of the story and this should definitely be seen as a reason to follow multidimensional approaches like Sen's one."

Some intermediate contributions between Cerioli and Zani (1990) and the TFR approach are also present in the literature (for example, Dagum, Gambassi and Lemmi, 1992; Pannuzi and

Quaranta, 1995; Blaszczk-Przybycinska. 1992). Dagum and Costa (2004) have developed an approach similar to TFR leading to the so called Dagum's decomposition (also see Mussard and Pi-Alperin, 2005).

The methodological implementation of the TFR approach has since then developed in two directions. The first one is typified by the contributions of Cheli (1995), Cheli and Betti (1999) and Betti *et al*. (2004), using the method to analyse the fuzzy concept of poverty in terms of transition probability matrices in the dynamic context of two consecutive panel data sets. The second, with the contributions of Betti and Verma (1999, 2002, 2004) and Verma and Betti (2002), has focused more on capturing the multi-dimensional aspects, developing the concepts of 'manifest' and 'latent' deprivation to reflect the intersection and union of different dimensions. A latest advance of this method is given by Betti *et al*. (2005) that combines the above two developments in the form of an *Integrated Fuzzy and Relative* (IFR) approach to the analysis of poverty and social exclusion.

Cornelissen et al., (2000) have developed a few fuzzy mathematical models to assess sustainable development based on context-dependent economic, ecological, and societal sustainability indicators. Although a decision-making process regarding sustainable development is subjective, they argue that fuzzy set theory links human expectations about development, expressed in linguistic propositions, to numerical data, expressed in measurements of sustainability indicators. The fuzzy models thus developed provide a novel approach to support decisions regarding sustainable development.

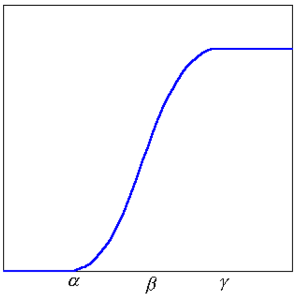
Von Furstenberg and Daniels (1991) and Baliamoune (2000) have applied fuzzy set theory to assess the degree of country compliance with the G-7 economic summit commitments. Baliamoune (2004) has been the first application of fuzzy-set theory to macroeconomic and social indicators of human well-being.

Buhong Zhenga and Charles Zheng (2015) propose to measure human development as a fuzzy concept. They stress that there exists a great deal of vagueness in quantifying a country's level of human development; one such source of vagueness is the weighting scheme embedded in the well-publicized UNDP's Human Development Index (HDI). They suggest to evaluate the resulting fuzziness in human development ranking with a truth value function. A truth value is simply a function of the probability that a randomly drawn bundle of weights will rank one country to have a higher HDI than another country. They derive simple and easily computable formulae for calculating the truth value. The method derived is equally applicable to fuzzy rankings with other composite indices.

Appendix:

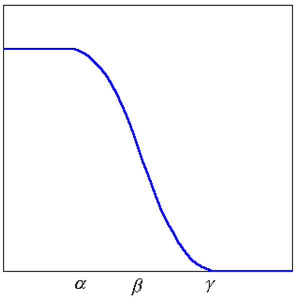
Some Other Membership Functions

(1) S function: monotonically increasing membership function



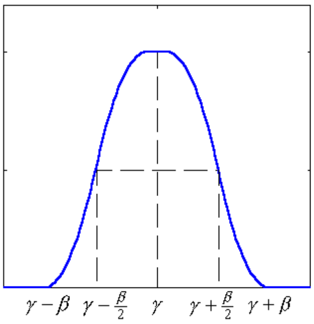
$$S(x;\alpha,\beta,\gamma)=\begin{cases} 0 & for\ x\le\alpha \\ 2(\frac{x-\alpha}{\gamma-\alpha})^2 & for\ \alpha\le x\le\beta \\ 1-2(\frac{x-\alpha}{\gamma-\alpha})^2 & for\ \beta\le x\le\gamma \\ 1 & for\ \gamma\le x \end{cases}$$

(2) Z function: monotonically decreasing membership function (Already given above)



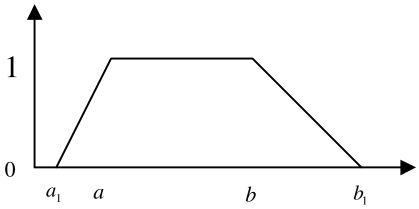
$$Z(x;\alpha,\beta,\gamma)=\begin{cases} 1 & for\ x\le\alpha \\ 1-2(\frac{x-\alpha}{\gamma-\alpha})^2 & for\ \alpha\le x\le\beta \\ 2(\frac{x-\alpha}{\gamma-\alpha})^2 & for\ \beta\le x\le\gamma \\ 0 & for\ \gamma\le x \end{cases}$$

(3) Π function: combines S function and Z function, monotonically increasing and decreasing membership function



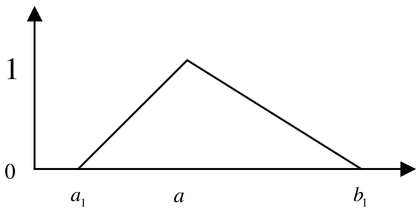
$$\Pi(x;\beta,\gamma)=\begin{cases} S(x;\ \gamma-\beta,\ \gamma-\frac{\beta}{2},\ \gamma) & for\ x\le\gamma \\ 1-S(x;\ \gamma,\ \gamma+\frac{\beta}{2},\ \gamma+\beta) & for\ x\ge\gamma \end{cases}$$

(4)Trapezoidal membership function: Piecewise continuous membership function



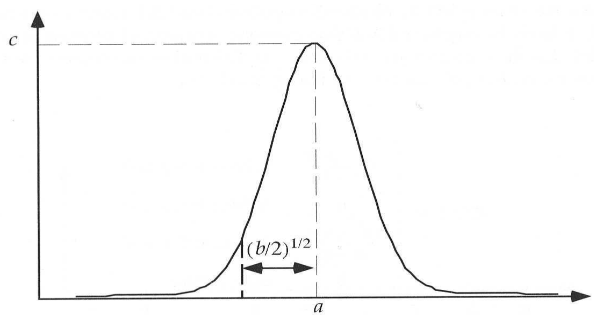
$$\mu_A(x)=\begin{cases} 0 & for\ x \le a_1 \\ \frac{x-a_1}{a-a_1} & for\ a_1 \le x \le a \\ 1 & for\ a \le x \le b \\ \frac{b_1-x}{b_1-b} & for\ b \le x \le b_1 \\ 0 & for\ b_1 \le x \end{cases}$$

(5) Triangular membership function



$$\mu_A(x)=\begin{cases} 0 & for\ x \le a_1 \\ \frac{x-a_1}{a-a_1} & for\ a_1 \le x \le a \\ \frac{b_1-x}{b_1-a} & for\ a \le x \le b_1 \\ 0 & for\ b_1 \le x \end{cases}$$

(6) Bell-shaped membership function



$$A(x) = ce^{-\frac{(x-a)^2}{b}}$$

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