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Abstract

In their seminal paper, Grossman and Shapiro (1984) assume that it is not profitable for a firm to deviate to the *supercompetitive* price of Salop (1979). In this paper, it is shown that this assumption is violated if, roughly, each firm reaches less than half of all consumers unless it is a duopoly. This implies that most of the simulations in Grossman and Shapiro (1984) are not actually equilibria. More importantly, this implies that for their equilibrium to exist nearly all consumers must receive at least one ad. For example, with more than four firms in the market, at least 96% of the consumers *must* receive at least one ad, and the percentage increases with the number of firms.

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1 Introduction

In the seminal Grossman and Shapiro (1984) paper on informative advertising in an oligopoly, though it holds locally, it is not shown that its competitive equilibrium holds globally. In particular, it is assumed "that the parameters of the problem are such that it is [not] profitable to reduce price to absorb a neighbor[1] (what Salop (1979) terms the supercompetitive region)..." and noted that without the assumption the equilibrium may not exist. The reason for the assumption may be because demand is discontinuous at the supercompetitive price, and so the assumption simplifies their analysis. It is known that the discontinuity often makes obtaining conditions for when a firm would not deviate to the supercompetitive price intractable, *e.g.*, in a random-utility (Perloff and Salop, 1985) model of advertising Christou and Vettas (2008) note this discontinuity and for this reason, are unable to obtain existence conditions in that model. While this issue is known, it is not well-explored as previous work that has examined existence (e.g., Tirole (1988), Christou and Vettas (2008)) have focused instead on whether a deviation to the (higher) monopoly price is profitable. Deviation to the supercompetitive price also may not be well explored because with complete information it is not a profitable deviation in Grossman and Shapiro (1984) like in the complete information Salop (1979) model on which it is based. Further, with incomplete information, deviating to the supercompetitive price increases demand proportionately less than it does with complete information, which may reinforce any presumption that such a deviation is unlikely to be profitable.

In this paper, a condition for firms not to deviate to the supercompetitive price is derived. If there are three or more firms in the market and the fraction of consumers contacted by any one firm is roughly less than one-half, then the competitive equilibrium proposed in Grossman and Shapiro (1984) does not exist. This condition implies that for their equilibrium to exist with more than three firms, almost all consumers must receive at least one ad in equilibrium. For example, with more than four firms in the market, more than ninety-

[1]That is, the price such that all consumers that also receive an ad from the firm's closest rival buy from the firm instead.

six percent of consumers must receive at least one ad (ninety-nine percent with more than six), and more than eighty percent must receive an ad from at least two firms. As the number of firms increases, these fractions increase going to one in the limit as the number of firms goes to infinity. In addition, this condition implies that most of the simulations in Grossman and Shapiro (1984, Appendix) are not in fact equilibria as a firm would deviate to the supercompetitive price. It is also shown that given the parameters in the simulations (this is not a general statement), there can never be an equilibrium with two or three firms, and with four firms the fraction of consumers who must receive at least one ad from each firm is greater than one-half. Finally, that the competitive equilibrium in Grossman and Shapiro when there are a large number firms requires nearly complete information may help to reconcile the result found in Creane and Manduchi (2019) that with a continuum of firms, advertising is always socially insufficient whereas with Grossman and Shapiro's large n assumption advertising is always socially excessive, which mirrors the excessive entry in the complete information Salop (1979) model.

In the next section, Grossman and Shapiro's 1984 general model is outlined with their first-order conditions for the competitive equilibrium presented.[2] Section 3 establishes a condition for firms to deviate from the competitive equilibrium by choosing the supercompetitive price. To do this, the demand at the supercompetitive price is first derived in subsection 3.2. Proposition 1, that an equilibrium cannot exist if the fraction of consumers reached by any one firm is less than half and the number of firms is large, is then presented. An example is presented showing that this implies that the benchmark case presented in Grossman and Shapiro (1984, Appendix) is not an equilibrium. Finally, it is shown that the parameters of the simulations imply that there cannot be an equilibrium if there are only two or three firms in the market.

[2]In the main body of Grossman and Shapiro (1984) it is assumed that n is sufficiently large so that $1-(1-\phi)^{n-1}$ could be treated as zero, that is, as they acknowledge, they are assuming that the probability that a consumer receives no ads from all but one firm is zero (a stronger condition than every consumer receives at least one ad). In this paper, the general model is considered, that is *without* making the assumption that $1-(1-\phi)^{n-1}$. In Appendix C considers the model *with* this assumption to show that a similar result to Proposition 1.

2 The Grossman and Shapiro (1984) model

Consumers are modeled as in Salop (1979): the possible product space is the unit circle on which a unit mass δ of consumers are uniformly distributed. Each consumer has unit demand and their location indicates their most preferred product location, for which they have dollar value v. For a product located distance z from the consumer, he incurs disutility of dollar value tz, where t is the per-unit "transportation" cost, so given price p his surplus is $v - tz - p$. He only learns about a product (its location on the circle) and its price if he receives an ad from the firm. There is a variation of the standard covered market assumption (Grossman and Shapiro, 1984, Fn.9), that v is large enough so that all consumers buy in equilibrium.

Firms have marginal cost of production c, fixed costs F, and the n firms are evenly located on the circle. For advertising, a firm chooses ϕ, the fraction of the population reached with cost function $\delta A(\phi;\alpha)$, where α is a shift parameter. Firms simultaneously and independently set price p and advertising reach ϕ. As Grossman and Shapiro (1984) are looking for a pure-strategy symmetric equilibrium, the representative firm chooses p and ϕ given the prices and advertising of the other firms ($\overline{p}$ and $\overline{\phi}$) to maximize profit (Grossman and Shapiro, 1984, Eq.8, their notation):

$$\pi(p,\phi) = (p-c)\delta x(p,\phi) - F - \delta A(\phi), \tag{1}$$

where, to make the role of the other firms' prices clearer, demand ($x(p,\phi)$) is explicitly (Grossman and Shapiro, 1984, Eq.6)

$$x(p,\phi) = \phi \left\{ \frac{\delta(\overline{p}-p)}{t}[1-(1-\overline{\phi})^{n-1}] + \frac{\delta}{n\overline{\phi}}[1-(1-\overline{\phi})^{n}] \right\}. \tag{2}$$

(Their explicit derivation is summarized in section 3.1 below.)

In the main body of the text, Grossman and Shapiro (1984) assume at this point that $(1-\overline{\phi})^{n-1} = 0$ to simplify the derivation of the equilibrium, but in their Appendix, the three conditions for the symmetric pure-strategy equilibrium without this assumption, that is, for the general model, are given. The general model is considered here, that is, without assuming that $(1-\overline{\phi})^{n-1}=0$. (In Appendix C this restricted model is considered with a similar result.) Substituting equation 2 into the profit expression (1), differentiating with respect to p, setting to zero, multiplying through by $t/\delta\phi$, and then imposing symmetry (*i.e.*, when $\overline{p}=p$ and $\overline{\phi}=\phi$), yields (Grossman and Shapiro, 1984, Eq. A1)

$$(p-c)[1-(1-\phi)^{n-1}]=\frac{t}{n\phi}[1-(1-\phi)^{n}]. \tag{3}$$

Differentiating profit with respect to ϕ, setting to zero, multiplying through by $\phi n/\delta$, and then imposing symmetry, yields (Grossman and Shapiro, 1984, Eq. A2)

$$(p-c)[1-(1-\phi)^{n}]=n\phi A_{\phi}(\phi,\alpha). \tag{4}$$

Finally, there is the entry condition (Grossman and Shapiro, 1984, Eq. A3)

$$(p-c)[1-(1-\phi)^{n}]=n[f+A(\phi,\alpha)], \tag{5}$$

where Grossman and Shapiro (1984) have substituted $f=F/\delta$ as the population is normalized, the integer constraint on n is ignored, and demand (evaluated at $p=\overline{p}$ and $\phi=\overline{\phi}$) is $x(p,\phi)=\delta[1-(1-\phi)^{n}]/n$.

From equation (3) the equilibrium price as a function of the equilibrium level of advertising and number of firms is (this is also noted in Christou and Vettas (2008))

$$p(\phi,n)=c+\frac{t}{\phi n}\frac{1-(1-\phi)^{n}}{1-(1-\phi)^{n-1}}. \tag{6}$$

As expected, this price is decreasing in ϕ and n as there is greater competition.[3] This price when $n=2$ is the same as in Tirole's 1988 duopoly model of advertising. Finally, this is

[3]For decreasing in n: $1/n$ and $\frac{1-(1-\phi)^{n}}{1-(1-\phi)^{n-1}}$ are decreasing in n and positive; hence their product is decreasing in n. For decreasing in ϕ the same logic follows for $n>1$.

the optimal price even without imposing symmetry in ϕ as the firm's ϕ enters linearly in the derivative with respect to p. That is, (6) is the optimal price given all the other firms are setting their advertising reach to ϕ, independent of the firm's own (strictly positive) ϕ. An implication (formally in Lemma 1 below) is that in looking for a deviation, only deviations in price need be examined.

The variable profits (*i.e.*, gross of advertising and enter costs), given equilibrium ϕ, are

$$\breve{\pi}(\phi) = (p-c)\delta x(p,\phi) = \frac{t\delta}{\phi n^2}\frac{1-(1-\phi)^n}{1-(1-\phi)^{n-1}}[1-(1-\phi)^n]. \tag{7}$$

3 Deviating to the supercompetitive price

In considering deviation to the supercompetitive price, as the firm chooses price and advertising simultaneously, it may at first glance seem that the examination of deviating to the supercompetitive price requires including deviations in advertising. However, because of the structure of profit in (1), only deviations in price need be checked as a firm's ϕ enters variable profit ($(p-c)\delta x(p,\phi)$) linearly and in the convex cost function $A(\phi)$. To be precise:

Lemma 1 *If it is (not) profitable to deviate in price alone, then it is (not) profitable to deviate to in price (p) and corresponding optimal advertising reach (ϕ) given the deviation price.*

Although obvious, for completeness a proof is provided in Appendix A. Thus, with the lemma, only the profitability of a deviation in price to the supercompetitive price needs to be checked.

In the Salop (1979) model with linear transportation cost (and complete information), which Grossman and Shapiro (1984) is built on, the supercompetitive price equals marginal cost, so it cannot be a profitable deviation if equilibrium profits are strictly positive. Thus, the next item to check is whether with incomplete information, the supercompetitive price may be less than or equal to marginal cost, and if so under what restrictions. It turns out that in this model with incomplete information, the supercompetitive price is always greater than marginal cost and so could be a profitable deviation.

Lemma 2 *In the Grossman and Shapiro (1984) model, the supercompetitive price is always greater than marginal cost.*[4]

Proof. The supercompetitive price is the price at which the firm captures all of its nearest neighbor's demand: $p^{super} = p(\phi, n) - t/n$. For the equilibrium price above, it is

$$\begin{aligned} p^{super} - c = p(\phi, n) - \frac{t}{n} - c &= c + \frac{t}{\phi n} \frac{1-(1-\phi)^n}{1-(1-\phi)^{n-1}} - \frac{t}{n} - c \\ &= \frac{t}{n} \frac{1-(1-\phi)^n}{\phi[1-(1-\phi)^{n-1}]} > 0, \end{aligned}$$

where the inequality follows as $\phi \in (0,1)$ and $1-(1-\phi)^n > \phi[1-(1-\phi)]^{n-1}$. ■

With these preliminaries established, next the firm's profit at the supercompetitive price is derived. To do this, first the demand at the supercompetitive price must be derived as the demand expression (2) from Grossman and Shapiro (1984) does not hold at this price as there is a discontinuous jump up in demand as price decreases. Before doing this, the derivation of the demand expression (2) from Grossman and Shapiro (1984) is briefly summarized to provide a template for deriving demand at the supercompetitive price in subsection 3.2.

3.1 The derivation of demand in Grossman and Shapiro (1984)

To derive a firm's demand Grossman and Shapiro (1984) consider the probabilities that any consumer a firm reaches is also reached by another firm(s), and the consumer's choice at that point. To do this, consumers are partitioned into groups ($k = 1, 2...n$), where k indicates the firm's ranking in surplus among all firms to the consumers, e.g., group 1 consumers would choose the firm even if they knew the location of all the firms. Setting this firm's address at

[4] The next discontinuous jump occurs when price drops to $p(\phi, n) - 2t/n$. It can be shown that such a price is above marginal cost only if $\phi < 1/2$, and so cannot strengthen the results in Proposition 1.

0 and moving away from the firm clockwise to its neighbor on the right there is a location z_1 such that, given the firm's price p and its rivals' price $\overline{p}$, the consumer located at z_1 is indifferent between the two firms:

$$v - tz_1 - p = v - t\left(\frac{1}{n} - z_1\right) - \overline{p},$$

i.e., all consumers in $(-z_1, z_1)$ prefer the firm's price and product over any rival.[5] Solving for z_1 obtains

$$z_1 = \frac{\overline{p} - p}{2t} + \frac{1}{2n},$$

and so, since the total fraction of consumers is length $(-z_1, z_1)$, the total number of consumers in the first group is (Grossman and Shapiro, 1984, Eq.2)

$$N_1 = 2\delta z_1 = \frac{\delta(\overline{p} - p)}{t} + \frac{\delta}{n}. \tag{8}$$

To find the consumers who, with complete information, find firm 0's product the second most preferred follows a similar logic. Moving to the right (so second most preferred after the firm located at $1/n$), and finding the consumer indifferent between firm 0 and firm $2/n$ we have

$$z_2 = \frac{\overline{p} - p}{2t} + \frac{1}{n},$$

so, the number of consumers in the second group is $n_2 = \delta/n$ and "repeating this procedure" yields $z_k = (\overline{p} - p)/2t + k/2n$ and so (Grossman and Shapiro, 1984, Eq. 3)

$$N_k = \frac{\delta}{n}. \tag{9}$$

[5] In Grossman and Shapiro (1984, Fn. 8) it is assumed that the choice set is $p \in (\overline{p} - t/n, \overline{p} + t/n)$ so that the indifferent consumer is between the firm and its nearest rival, *i.e.*, there are no discontinuities.

The last n^{th} group has the remaining consumers, or (Grossman and Shapiro, 1984, Eq.4)

$$N_n = \frac{\delta}{n} - \frac{\delta(\overline{p} - p)}{t}. \tag{10}$$

Note that $\sum_{k=1}^{n} N_k = 1$.

Demand then is

$$x(p, \phi) = N_1\phi_1 + N_2\phi_2 + ... + N_n\phi_n,$$

where ϕ_k is the probability of making a sale to a consumer in that group. Any consumer reached in group 1 by firm 0, buys regardless of what other ads they receive and so, the probability $\phi_1 = \phi$. Group 2 consumers buy if they receive an ad from firm 0 and no ad from their most preferred, which occurs with probability $(1 - \overline{\phi}) = \phi_2$, *etc.*, or more generally (Grossman and Shapiro, 1984, Eq.5):

$$\phi_k = \phi(1 - \overline{\phi})^{k-1} \qquad k = 1, 2, ...n. \tag{11}$$

Using equations (8-11) and summing yields (Grossman and Shapiro, 1984, Eq.6), equation (2) here:

$$x(p, \phi) = \frac{\delta\phi(\overline{p} - p)}{t}[1 - (1 - \overline{\phi})^{n-1}] + \frac{\delta\phi}{n\overline{\phi}}[1 - (1 - \overline{\phi})^n]. \tag{2}$$

Again, this demand only holds for $p \in (\overline{p} - t/n, \overline{p} + t/n)$ and so cannot be used for calculating profits at the supercompetitive price $\overline{p} - t/n$ as there is a discrete increase in quantity demanded at that price.

3.2 Demand at the supercompetitive price ($p = \overline{p} - t/n$)

The derivation of demand at the supercompetitive price follows the template of subsection 3.1.[6] As before, consumers are partitioned into groups ($k = 1, 2...n$), where k indicates the firm's ranking in surplus among all firms to the consumers. The key difference is the size of

[6] It is straightforward from this to derive demand for $p \in (\overline{p} - 2t/n, \overline{p} - t/n]$, but for brevity, it is omitted as in this range the supercompetitive price maximizes profits.

group 1 – the consumers who would buy with complete information from the firm. As the firm decreases its price from above the supercompetitive price to the supercompetitive price, there is a discrete increase in demand: at that price, a consumer located at the rival firm's location $1/n$ receives the same surplus from either firm. However, since transportation costs are linear that means that any consumer to the right (clockwise) from firm $1/n$ also receives the same surplus from either firm 0 or firm $1/n$. For example, with complete information firm 0 captures all of firm $1/n$'s consumers up to the consumer who is indifferent between firm $1/n$ and firm $2/n$. This means that the for $0 \leq z \leq 2/n$, that is consumers between firm 0 and firm $2/n$, the best alternative to firm 0 is firm $2/n$. Thus, in this case, z_1 is defined by (recalling that p is the firm's price and $\overline{p}$ is its rival's price, so $\overline{p} - t/n$ is the supercompetitive price)

$$v - tz_1 - p = v - tz_1 - (\overline{p} - t/n) = v - t\left(\frac{1}{n} - z_1\right) - \overline{p}.$$

Solving for z_1 obtains $z_1 = 3/2n$ and so, the total number of consumers in the first group is

$$N_1 = 2\delta z_1 = \frac{3\delta}{n}. \tag{12}$$

The expression makes intuitive sense as the supercompetitive price with complete information captures all the consumers of the firm's nearest neighbors, so it has three times the sales.

Finding the consumers who, with complete information, find firm 0's product the second most preferred is as before except that the consumer is further from firm 0. Moving to the right (so second most preferred after the firm located at $2/n$), and finding the consumer indifferent between firm 0 and firm $3/n$ we have

$$z_2 = \frac{\overline{p} - p}{2t} + \frac{3}{2n} = \frac{2}{n},$$

where the second equality is from the firm setting the supercompetitive price. This also makes sense as the surplus firm 0 is creating for consumers is the same as firm $1/n$, and since firm $1/n$ is setting the same price as firm $3/n$, the indifferent consumer between those two is located at $2/n$. So, the number of consumers in the second group is $N_2 = \delta/n$. By similar logic, $z_k = (\overline{p} - p)/2t + k/2n + 1/2n = k/2n + 3/2n$.

$$N_k = \frac{\delta}{n}. \tag{13}$$

In this case, unlike the demand in Grossman and Shapiro (1984), the firm is setting a specific price and as the first group had $3/n$ of the circle (capturing the entirety of its neighbors' consumers and not just half their space), there are only $n-2$ groups. Note that $\sum_{k=1}^{n-2} N_k = 1$.

Demand then is

$$x(p, \phi) = N_1\phi_1 + N_2\phi_2 + ... + N_{n-2}\phi_{n-2},$$

where ϕ_k is the probability of making a sale to a consumer in that group. Any consumer reached in group 1 by firm 0, buys regardless of what other ads they receive and so, the probability $\phi_1 = \phi$. Group 2 consumers buy if they receive an ad from firm 0 and no ad from their most preferred, which occurs with probability $(1 - \overline{\phi}) = \phi_2$, *etc.*. That is, the probabilities are as before as the logic for the groups is the same as before. More generally

$$\phi_k = \phi(1 - \overline{\phi})^{k-1} \qquad k = 1, 2, ...n-2. \tag{14}$$

Using equations (12-14) and summing yields the demand for the firm deviating to the supercompetitive price,

$$x(\overline{p} - t/n, \phi) = \frac{\delta}{n}\left[1 + 2\phi - (1-\phi)^{n-2}\right]. \tag{15}$$

The expression is intuitive: in the complete information case ($\phi = 1$), when a firm deviates from the competitive price set by all firms to the supercompetitive price, it captures all of its two closest neighbors' demand – its sales go from $1/n$ of the market to $3/n$ of the market. Another intuitive check on the expression is when $n = 3$; then the expression becomes $\delta\phi$

– the firm captures all consumers that it reaches.[7] One thing to note is that, unlike with complete information, deviating to the supercompetitive price does not triple demand (RHS (15) v RHS (2) evaluated at $p = \overline{p}$). It does not even increase it by $\phi 2/n$ (the probability of reaching a consumer also reached by the nearest two rivals) except in the limit when $\phi = 1$. Hence, in this aspect, deviation to the supercompetitive price is not as attractive as in the complete information. The reason is that when the firm reaches a consumer that is also reached by its two closest rivals, there is a chance that that consumer would not have bought from its rival to begin with. That is, if the consumer is far away from the closest rival, then there is a chance that this consumer received an ad from yet another firm whose product is very close to this consumer's most preferred.[8]

3.3 Gain from the supercompetitive price (existence)

Using the demand at the supercompetitive price derived in (15), the firm's variable profit (gross of advertising and entry costs) from deviating to this price given the rivals are setting $\overline{p}$ is

$$\pi^{super} = (\overline{p} - t/n - c)x(p - t/n, \phi) = (\overline{p} - t/n - c)\frac{\delta}{n}\left[1 + 2\phi - (1-\phi)^{n-2}\right].$$

Given the equilibrium price (6), variable profit from deviating to the supercompetitive price is

$$\pi^{super}(\phi, n) = \frac{t}{n}\left(\frac{1-(1-\phi)^n}{\phi[1-(1-\phi)^{n-1}]} - 1\right)\frac{\delta}{n}\left[1 + 2\phi - (1-\phi)^{n-2}\right]. \tag{16}$$

Subtracting candidate equilibrium variable profit (7) from deviation variable profit (16)

[7] Likewise, when $n = 4$ the firm captures all of the consumers that receive an ad from it except for the consumers in the furthest firm's turf (1/4) when they also receive an ad from that firm (ϕ).

[8] Interestingly, in all of the examples of Grossman and Shapiro (1984), at the equilibrium price, a firm captures $1/n^{th}$ of the market to at least the 0.001 approximation. The reason is that in all of the examples the firm's advertising reaches a far larger fraction of the market than it would obtain with complete information (at least four times $1/n$). And, so, there is a chance that a far-away consumer it reaches buys from it, which cannot happen with complete information.

yields

$$t\delta \frac{(1-\phi)\left[(2\phi-1)(1-\phi)^{n-2}+1\right]\left[(1+2\phi)-(1-\phi)^{n-2}\right]-\left[1-(1-\phi)^{n}\right]^{2}}{\phi n^{2}\left[1-(1-\phi)^{n-1}\right]} \tag{17}$$

Expression (17) can be positive, that is, deviating to the supercompetitive can increase profits. As the denominator is positive, the condition turns on the sign of $(1-\phi)[(2\phi-1)(1-\phi)^{n-2}+1][(1+2\phi)-(1-\phi)^{n-2}]-[1-(1-\phi)^{n}]^{2}$. Note that for $\phi \in (0,1)$, this numerator as n goes to infinity equals $\phi(1-2\phi)$, Therefore, for any $0<\phi<0.5$, there exists n' large enough such that for all $n>n'$, (17) will be negative (and hence the candidate competitive equilibrium in Grossman and Shapiro (1984) not an equilibrium when n is large enough). For any $n>2$, the condition is a ϕ only slightly less than 1/2.

Proposition 1 *In Grossman and Shapiro (1984) for any* $0<\phi<0.5$*, for sufficiently large n, the Grossman and Shapiro (1984) competitive equilibrium does not exist. For small n, if* $n \geq 3$*, then for* $\phi = 0.4679$*, their competitive equilibrium does not exist. As n increases, this critical* ϕ *increases approaching* $1/2$ *in the limit.*

Proof.

The first part was established already above. For the second part, the sign of (17), deviation profit less candidate equilibrium profit, is the same as the expression

$$(1-\phi)\left[(2\phi-1)(1-\phi)^{n-2}+1\right]\left[(1+2\phi)-(1-\phi)^{n-2}\right]-\left[1-(1-\phi)^{n}\right]^{2}. \tag{18}$$

If (18) is increasing in n, then for any ϕ and n such that deviation is profitable, it is profitable for any greater n. Differentiating (18) with respect to n yields

$$-2ln(1-\phi)(1-\phi)^{n-1}\phi\left[1-2\phi+(1-\phi)^{n-2}(1-(1-\phi)(2-\phi))\right]. \tag{19}$$

As $ln(1-\phi)<0$, the sign of expression (19) turns on the sign of $[1-2\phi+(1-\phi)^{n-2}(1-(1-\phi)(2-\phi))]$. The expression $(1-\phi)^{n-2}(1-(1-\phi)(2-\phi))$ may be negative, and its magnitude is decreasing in n. Thus, when $\phi < 1/2$ so that $1 - 2\phi > 0$, if $[1 - 2\phi + (1 - \phi)^{n-2}(1 - (1 - \phi)(2 - \phi))] > 0$ when $n = 3$, it is so for any greater n, *i.e.*, deviation profits are greater for any greater n. The roots of $[1 - 2\phi + (1 - \phi)^{n-2}(1 - (1 - \phi)(2 - \phi))]$ in ϕ when $n = 3$ are $\{0, 2 \pm \sqrt{2}\}$. For $\phi \in (0, 2 - \sqrt{2})$, the expression is positive. As $1/2 < 2 - \sqrt{2}$, the expression $[1 - 2\phi + (1 - \phi)^{n-2}(1 - (1 - \phi)(2 - \phi))]$ is positive for $n \geq 3$ and $\phi < 1/2$: (19) is positive, that is, (18) is increasing in n for $n \geq 3$. Thus, if expression (18) is positive, *i.e.*, deviation profits are greater than candidate equilibrium profits, for some n when $n \geq 3$ and $\phi < 1/2$, it is so for any greater n. Solving for roots of (16), deviations profits less candidate equilibrium profits, in ϕ when n equals 3 obtains the condition 0.4679. ■

Remark 1 *Solving for roots of (16) shows that the ϕ needed such that the Grossman and Shapiro (1984) competitive equilibrium quickly approaches* $1/2$*: If $n = 4$, then $\phi < 0.4946$; if $n = 5$, then $\phi < 0.4988$.*

Remark 2 *It can be shown that deviation profits are increasing in p at the supercompetitive price. That is, locally, the supercompetitive price is more profitable than any lower price and so at least locally if it is profitable, then it is the best response. To do this requires the derivation of the demand and so profits for $p \in [p(\phi, n) - 2t/n, p(\phi, n) - t/n]$, which follows straightforwardly from the demand and profit derivations for the supercompetitive price. These derivations have been omitted for brevity.*

The intuition is straightforward. First, lower advertising reach ϕ raises the equilibrium price as there is less competition and so raises the marginal profit on the supercompetitive profit, making a deviation down to the supercompetitive price more attractive. Second, while an increase in n reduces the equilibrium price, the firm need not reduce its price relatively as much (t/n) to set the supercompetitive price and still capture all the consumers who are its nearest rivals' "turf." Further, the percentage increase in demand by deviating to the supercompetitive price is increasing in n (that is $x(p - t/n, \phi)/x(p, \phi)$ is increasing in n).

Thus, when ϕ is low and n is high, firms find it profitable to deviate to the supercompetitive price.

When $\phi > 1/2$ and n is large, though the competitive equilibrium may exist, this also implies that the competitive equilibrium with four or more firms requires most consumers to receive at least one ad. For example, if there are six firms, the probability any given consumer receives zero ads at the minimum ϕ necessary is less than 2 percent ($(1-\phi)^6 = 1/64$). The one-half condition is similar but different from that found with a continuum of sellers in Creane and Manduchi (2019). There, a necessary condition is that the probability a given consumer receives at least one advertisement is at least one-half, while here requires one-half is the probability that a given consumer receives at least one advertisement from *each* firm and so the probability a given consumer receives at least one advertisement is at least $1-(1-1/2)^n$. Thus, the requirement for an equilibrium there is less restrictive.

A different indication of the condition's relevance is that five of the eight examples of equilibria in the appendix of Grossman and Shapiro (1984) are not equilibria. The following shows this for the benchmark case in the appendix of Grossman and Shapiro (1984).

Example 1 *In the Appendix of Grossman and Shapiro (1984) it is assumed that* $v = 250$ $t = 250$ *and* $c = 50$. *For the "Base case," it is further assumed that* $f = 0.75$ *and* $\alpha = 3$, *so that the equilibrium entry, price and advertisement are* $n = 14, p = 87, \phi = 0.48$, *which as rounded values do approximately satisfy the equations A1-A3 in Grossman and Shapiro (1984), equations (3-5) above, as well as the solution for price above (6). In particular, profit gross of entry and advertising costs (LHS 5) is* $(p-c)[1-(1-\phi)^n]\frac{\delta}{n} \approx 2.643\delta$. *In this case, the supercompetitive price would be* $p - t/n = 87 - 250/14 \approx 69.143$ *and so profit at the supercompetitive price is* $(p - t/n - c)\frac{\delta}{n}\left[1 + 2\phi - (1-\phi)^{n-2}\right] \approx 2.679\delta$ *and so a firm would deviate to the supercompetitive price.*[9]

On the other hand, for the duopoly case (examined in Tirole (1988)) a firm would never deviate to the supercompetitive price.

[9]If one uses the precise values that satisfy A1-A3 in Grossman and Shapiro (1984), rather than the rounded numbers given in Grossman and Shapiro (1984), the equilibrium profits gross of entry and advertising costs are ≈ 2.657291, while deviation profits are ≈ 2.708301.

That ϕ must be greater than 1/2 for an equilibrium to exist when n is large raises an issue with the Grossman and Shapiro's 1984 argument for the approximation used in the main body of the paper. Specifically, they argue that for this approximation to be reasonable, there need be large n ("most accurate for n large") for the equilibrium. The issue is that any exogenous variable that increases n tends to decrease ϕ in the Grossman and Shapiro (1984) model, potentially further restricting the applicability of the model. That is, exogenous changes to the model to make the equilibrium n large, tend to make the equilibrium ϕ small.

Finally, recall that to explicitly solve their model, Grossman and Shapiro (1984) assume that the value for n is large enough such that $(1-\phi)^{n-1}$ is effectively zero so that the demand (Eq. 2)

$$x(p,\phi) = \frac{\delta\phi(\overline{p}-p)}{t}[1-(1-\overline{\phi})^{n-1}] + \frac{\delta\phi}{n\overline{\phi}}[1-(1-\overline{\phi})^{n}]$$

is approximated by an expression that inflates the quantity sold by a firm:

$$x(p,\phi) = \frac{\delta\phi(\overline{p}-p)}{t} + \frac{\delta\phi}{n\overline{\phi}}.$$ [10]

As shown in the above proposition, in their more general model, that is, without using the approximation, as n goes to infinity, a necessary condition for a firm not to deviate to the supercompetitive price, that is, for an equilibrium, is that $\phi > 1/2$. However, as their main results use this approximation instead of the actual demand, in Appendix C, a necessary condition for existence – when firms would deviate to the supercompetitive price – is established using a similar but slightly stronger version of their approximation, finding again that a necessary condition is that $\phi > 1/2$.

4 Further Restrictions

Grossman and Shapiro (1984, Fn.9) assume that at the equilibrium price $p(\phi, n)$ the market

[10](Grossman and Shapiro, 1984, Fn. 10) "The approximation is most accurate for n large [...and large t or low marginal cost of advertising]."

is "covered," that is, that all consumers who receive an ad, buy from some firm: $v - p(\phi, n) - t/2 > 0$. The assumption is needed for the existence of the equilibrium, else a firm would deviate to a lower price. To see why intuitively, note that the equilibrium $p(\phi, n)$ is a best response to all the other firms setting $p(\phi, n)$, specifically the derivative of profits with respect to price is zero. However, this profit expression assumes that at this price all consumers who receive an ad at that price are willing to purchase (the market is "covered"). If the covered market assumption does not hold at that price, then the furthest consumers do not purchase as their surplus is negative ($v - p(\phi, n) - t/2 < 0$). Thus, quantity demand is less than if the market is covered. In addition, an increase in price results in the marginal (furthest) consumers no longer purchasing from the firm even if they receive no other ads. Thus, the derivative must be negative at $p(\phi, n)$ if the covered market condition does not hold. While the general point is well-known, for completeness, in Appendix B can be found the proof to the following lemma for Grossman and Shapiro's 1984 specific model.

Lemma 3 *The competitive equilibrium of Grossman and Shapiro (1984) does not exist if the covered market assumption ($p(\phi, n) < v - t/2$) does not hold.*

Their assumption is weaker than the standard assumption that at the monopoly price the market is covered, that is, that $v - t - c > 0$. Indeed, in their simulation results, that standard assumption is violated (that is, $v - t - c < 0$).[11] However, if $v - t - c < 0$, then it is possible that the competitive equilibrium price violates *their* covered market assumption ($v - p(\phi, n) - t/2 > 0$) and as a result is greater than the monopoly price.

Lemma 4 *When $v - t - c < 0$, if*

$$0 > 1 - \frac{2 - 2(1-\phi)^n}{\phi n[1 - (1-\phi)^{n-1}]},$$

then the competitive equilibrium of Grossman and Shapiro (1984) does not exist (it is greater *than the monopoly price).*

[11] In such a case, the monopoly price is not $v - t/2$, but rather $(v + c)/2 > v - t/2$.

Proof. If

$$0 > 1 - \frac{2 - 2(1-\phi)^n}{\phi n[1-(1-\phi)^{n-1}]},$$

then,

$$0 > \frac{t}{2}\left[1 - \frac{2 - 2(1-\phi)^n}{\phi n[1-(1-\phi)^{n-1}]}\right] = t/2 - \frac{t}{\phi n}\frac{1-(1-\phi)^n}{1-(1-\phi)^{n-1}}.$$

Add c to both sides and rearranging yields

$$c + \frac{t}{\phi n}\frac{1-(1-\phi)^n}{1-(1-\phi)^{n-1}} > \frac{t}{2} + c,$$

where the LHS is the competitive price of Grossman and Shapiro (1984). By assumption $t + c > v$, or $t/2 > (v-c)/2$. Thus,

$$c + \frac{t}{\phi n}\frac{1-(1-\phi)^n}{1-(1-\phi)^{n-1}} > \frac{v-c}{2} + c = \frac{v+c}{2},$$

which is the monopoly price when $t + c > v$ as then $(v+c)/2 > v - t/2$, which implies that the covered market assumption does not hold with the competitive price. ■

Simple calculation obtains these cases:

- If $n = 2$, then their equilibrium does not exist (the competitive price is greater than the monopoly price).
- If $n = 3$, then their equilibrium does not exist if $\phi < \sqrt{6}(\sqrt{6}-1)/5 \approx 0.71$ (at which 98% of the consumers receive at least one ad).
- if $n = 4$, then their equilibrium does not exist if

$$\phi < \frac{10}{9} + \frac{8 - (134 + 18\sqrt{57})^{\frac{2}{3}}}{9(134 + 18\sqrt{57})^{\frac{2}{3}}} \approx 0.531.$$

Thus, with Proposition 1 and Lemma 4 and the added implicit assumption in Grossman and Shapiro (1984, Appendix), their equilibrium price violates their covered market assumption when there are two or three firms. That is, with the restriction used, there are no equilibria when ϕ is approximately less than one-half. In addition, there may be no equilibrium when ϕ is greater than one-half and there are four or fewer firms in the market.[12] The intuition is straightforward: in their simulations, consumers' value (v) is small relative to production and transportation costs. However, for the covered market condition to exist requires a sufficiently large v relative to the competitive price. When there are few firms in the market and little advertising, the competitive price is high and hence, the covered market condition does not hold. Finally, note this is a sufficient condition: for any $v < c + t$. Thus, since the covered market condition requires $p(\phi, n) < v - t/2$, the condition here holds when $v = c + t$; for any small v the competitive price is well above $v - t/2$. A different way to interpret the results of this section is that the more common, stronger assumption that $v > c + t$ is more appropriate.

If instead it is assumed that $v > c + t$ contrary to the simulations, then it can be shown there is little restriction on the mode. Specifically, for when $v - t/2 > p(\phi, n)$ it can be shown that a firm never deviates to the monopoly price (while with $v < c + t$ this may not be true). Similarly, when considering the other extreme – at the monopoly price the firm only sells to consumers who receive no other ads – it can be shown that for any ϕ and n, there always exists a v such that a firm would not deviate.

5 Conclusion

It has been shown that though Grossman and Shapiro (1984) assume that deviating to the supercompetitive price is not profitable, for ϕ roughly less than one-half, it actually is a profitable deviation unless it is a duopoly. That is, equilibria in which there are a large number of firms ($n > 3$) do not exist if they require small advertising reach (small ϕ). There has been other work that examines the question of existence in Grossman and Shapiro (1984) and "large" deviations. Christou and Vettas (2008) in analyzing advertising

[12] For $n = 5$, the assumption rules out equilibria when $\phi < 0.421$, so it does not rule out further equilibria.

with a random utility model (Perloff and Salop (1985)) have already derived examples in which the competitive equilibrium of Grossman and Shapiro (1984) does not exist because a firm would deviate to the monopoly price. While they note the possibility that "lower prices may be profitable," no examples are provided. In contrast, here, by extending the analysis in Grossman and Shapiro (1984) to allow for supercompetitive pricing, explicit conditions are given for the existence of Grossman and Shapiro's 1984 competitive equilibrium based on deviating down to the supercompetitive price.

Appendix A Proof of Lemma 1

Lemma 1 *If it is (not) profitable to deviate in price alone, then it is (not) profitable to deviate to in price (p) and corresponding optimal advertising reach (ϕ) given the deviation price.*

Proof. Consider a deviation price p'. First, if it is profitable to deviate to p' without adjusting advertising intensity, then it is profitable to deviate to p' with ϕ chosen to maximize profits given the deviation to the supercompetitive price (denoted ϕ'). Consider next an unprofitable deviation to p'. To analyze this, it is convenient to use demand condition on consumers receiving an ad from the firm (and so not dependent on the firm's ϕ), defined as

$$\tilde{x}(p, \overline{\phi}) = \left\{ \frac{\delta(\overline{p} - p)}{t}[1 - (1 - \overline{\phi})^{n-1}] + \frac{\delta}{n\overline{\phi}}[1 - (1 - \overline{\phi})^n] \right\},$$

that is, $\phi\tilde{x}(p, \overline{\phi}) \equiv x(p, \phi)$. An unprofitable deviation in price without adjusting advertising, that is, when $\pi(p', \phi) \leq \pi(p^e, \phi)$, then can be expressed as

$$(p' - c)\delta\phi^e\tilde{x}(p', \phi^e) - F - \delta A(\phi^e) \leq (p^e - c)\delta\phi^e\tilde{x}(p^e, \phi^e) - F - \delta A(\phi^e),$$

(Recall that the second argument of $\tilde{x}$ is the advertising intensity of the other firms.) This is true if and only if $(p' - c)\delta\tilde{x}(p', \phi^e) \leq (p^e - c)\delta\tilde{x}(p^e, \phi^e)$. Then it follows that

$$(p' - c)\delta\phi'\tilde{x}(p', \phi^e) - F - \delta A(\phi') \leq (p^e - c)\delta\phi'\tilde{x}(p^e, \phi^e) - F - \delta A(\phi').$$

As ϕ^e maximizes profit $(p^e - c)\delta\phi\tilde{x}(p^e, \phi^e) - F - \delta A(\phi)$, then

$$(p^e - c)\delta\phi'\tilde{x}(p^e, \phi^e) - F - \delta A(\phi') \leq (p^e - c)\delta\phi^e\tilde{x}(p^e, \phi^e) - F - \delta A(\phi^e).$$

Thus,

$$(p' - c)\delta\phi'\tilde{x}(p^e, \phi^e) - F - \delta A(\phi') \leq (p^e - c)\delta\phi^e\tilde{x}(p^e, \phi^e) - F - \delta A(\phi^e).$$

■

Appendix B Proof of Lemma 3

Lemma 3 *The competitive equilibrium of Grossman and Shapiro (1984) does not exist if the covered market assumption ($p(\phi, n) < v - t/2$) does not hold.*

Proof. In the construction of demand in section 3.1, the last group of remaining consumers was given as

$$N_n = \frac{\delta}{n} - \frac{\delta(\overline{p} - p)}{t}. \tag{10}$$

This is derived by subtracting all of the previous groups from δ (the population size). However, if the covered market condition does not hold then this last group is smaller (if zero, then the argument recurses to the penultimate group, etc.; for ease, only the case of the last group being positive is presented). Specifically, the furthest consumer with positive surplus from buying from the firm is defined by $v - tx - p$, that is $x = (v - p)t$. As this occurs in both directions from the firm, the population size is now $\delta[1 - 2(v - p)/t]$. Subtracting the consumers in the other groups yields the final group of consumers having size

$$\delta[1 - \frac{1 - 2(v-p)}{t}] - \delta\left[\frac{\overline{p} - p}{t} + \frac{1}{n}\right] + \delta \sum_{k=2}^{n-1} \frac{1}{n} = \delta \frac{2nv - t(n-1) - n(p + \overline{p})}{tn}.$$

Thus, the correct demand in this case is

$$\delta\left[\frac{\overline{p} - p}{t} + \frac{1}{n}\right]\phi + \delta \sum_{k=2}^{n-1} \phi(1-\phi)^{k-1}\frac{1}{n} + \delta\phi(1-\phi)^{n-1}\frac{2nv - t(n-1) - n(p + \overline{p})}{tn}.$$

The profit expression, then, is

$$(p - c)\left\{\delta\left[\frac{\overline{p} - p}{t} + \frac{1}{n}\right]\phi + \delta \sum_{k=2}^{n-1} \phi(1-\phi)^{k-1}\frac{1}{n} + \delta\phi(1-\phi)^{n-1}\frac{2nv - t(n-1) - n(p + \overline{p})}{tn}\right\}.$$

Differentiating the profit expression with respect to p and evaluating at $p = \overline{p}$ obtains

$$\delta\left[\frac{\phi}{n} - \frac{(1-\phi)^n}{(1-\phi)n} + \frac{(1-\phi)^2}{(1-\phi)n} + \frac{\phi(1-\phi)^{n-1}}{n}\frac{2n(v-\overline{p})-t(n-1)}{t} - (\overline{p}-c)\phi\frac{1+(1-\phi)^{n-1}}{t}\right]. \tag{B.1}$$

In contrast, if the covered market condition held, differentiating the profit expression in that case and evaluating $p = \overline{p}$ obtains

$$\delta\left[\frac{\phi}{n} - \frac{(1-\phi)^n}{(1-\phi)n} + \frac{(1-\phi)^2}{(1-\phi)n} + \frac{\phi(1-\phi)^{n-1}}{n} - (\overline{p}-c)\phi\frac{1+(1-\phi)^{n-1}}{t}\right]. \tag{B.2}$$

The difference between the two expressions is in (B.1) term $[2n(v-\overline{p})-t(n-1)]/t$ post-multiplying $(\phi(1-\phi)^{n-1})/n$ while in (B.2) it is post-multiplying 1. Note that if $\overline{p} = v - t/2$ so that the market was covered, the expressions are identical. However, if the market is not covered ($\overline{p} > v - t/2$), then $[2n(v-\overline{p})-t(n-1)]/t$ is clearly less than 1. The equilibrium in Grossman and Shapiro (1984) requires that (B.2) is zero, however, this implies that (B.1) is negative if the market is not covered: at the proposed equilibrium in Grossman and Shapiro (1984) if the market is not covered, a firm would set a lower price in response to $p(\phi, n)$ from the other firms. It is not an equilibrium. ∎

Appendix C Existence under approximation

To explicitly solve their model, Grossman and Shapiro (1984) assume that the value for n is large enough such that $(1-\phi)^{n-1}$ is effectively zero. As a result, the equilibrium price in Grossman and Shapiro (1984) is $c + t/(\phi n)$, and equilibrium profits gross of advertising and entry costs are

$$\frac{\delta t}{\phi n^2}. \tag{C.3}$$

Given this equilibrium price, the supercompetitive deviation price is $c + t/(\phi n) - t/n = c + (1-\phi)t/(\phi n)$. To make the analysis more tractable, if the approximation (the assumption of large n) is that $(1-\phi)^{n-2}$ is approximately zero, rather than $(1-\phi)^{n-1}$ in Grossman and

Shapiro (1984), the deviation demand (15) can be approximated as

$$x(p - t/n, \phi) = \frac{\delta}{n}\left[1 + 2\phi\right].$$

The expression is intuitive: the firm has its market share (δ/n) plus captures all of the consumers it reaches in its two closest rivals' markets $\delta\phi/n$. As a result, deviation profits are

$$\frac{(1-\phi)t}{\phi n}\frac{\delta}{n}\left[1 + 2\phi\right]. \tag{C.4}$$

Deviation profits (C.4) are greater than equilibrium profits (C.3) whenever $\phi < 1/2$.

Lemma 5 *Under "large n" assumption/approximation, a necessary condition for an equilibrium is that* $\phi > 1/2$.

The corollary is consistent with the proposition which states that if n is greater than 4 and ϕ is less than one-half, then the equilibrium does not exist. This is logical as Grossman and Shapiro (1984) interpret the approximation as assuming "large n" in their model.

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