Identification of Time Preferences in Dynamic Discrete Choice Models: Exploiting Choice Restrictions

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Abstract

I study the identification of time preferences in dynamic discrete choice models. Time preferences play a crucial role in these models, as they affect inference and counterfactual analysis. Previous literature has shown that observed choice probabilities do not identify the exponential discount factor in general. Recent identification results rely on specific forms of exogenous variation that impact transition probabilities but not instantaneous utilities. Although such variation allows for set identification of the respective parameter, point identification is only achieved in limited cases. To circumvent this shortcoming, I focus on models in which economic decision-makers might be restricted in their choice sets. I show that time preferences can be identified provided that there is variation in the probability of being restricted that does not affect utilities or transition probabilities. The derived exclusion restrictions are easy to interpret and potentially fulfilled in many empirical applications.

JEL: C14, C23, C61

Keywords: discount factor; identification; dynamic discrete choice

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1 Introduction

Dynamic discrete choice models are used to estimate the behaviour of economic agents and analyse counterfactual policies in numerous fields.[1] The discount factor is crucial in these models, as it determines the reactions to expected future events. In empirical applications, researchers are often forced to set the discount factor, as it cannot be identified from observed probabilities without further restrictions (see Magnac & Thesmar, 2002; Rust, 1994). Time preferences appear to be stable over time (Meier & Sprenger, 2015) but vary substantially across contexts and populations (Frederick, Loewenstein, & O'Donoghue, 2002). Thus, setting the discount factor to a predetermined value can lead to incorrect inference and misleading counterfactual analysis. Recent literature (Abbring & Daljord, 2020b; Fang & Wang, 2015) on the identification of time preferences has focused on variation in transition probabilities but is only able to point identify the discount factor in limited cases.[2]

This paper provides a new class of instruments – changes in probabilistic choice restrictions – to point identify the discount factor. A choice restriction occurs when economic agents are not able to choose from all potential alternatives. Examples of such restrictions are common. In labor economics, individuals can be restricted in their choices: When individuals are unemployed, they need to receive a job offer to be able to choose a positive number of working hours. In industrial organizations, choices might be restricted when mergers are subject to approval. In marketing, the availability of products might be restricted when products are no longer sold or out of stock. In environmental economics, the amount of emissions a firm can produce in a given period might be restricted with uncertainty stemming from changes in environmental regulations.

[1] Keane and Wolpin (2009) provide an overview of these fields. Recent examples include Blundell, Costas-Dias, Meghir, and Shaw (2016) in labor economics, Miravete and Palacios-Huerta (2014) in industrial organizations, De Groote and Verboven (2019) in environmental economics, and Blevins, Khwaja, and Yang (2018) in marketing.

[2] Although Fang and Wang (2015) claim to *generically* identify various parameters related to time preferences, Abbring and Daljord (2020a) show that their proposed exclusion restriction is not sufficient for identification.

I show that exogenous variation in the probability of being restricted in one's choice directly point identifies the discount factor of dynamic discrete choice models. Identification is achieved without requiring the normalization of one alternative's utility. Although this is a standard approach in empirical applications, it can lead to misleading counterfactual policy simulations in some cases (see, e.g., Kalouptsidi, Scott, & Souza-Rodrigues, 2019; Norets & Tang, 2014). Furthermore, the identification approach does not require that models are stationary. In contrast to Magnac and Thesmar (2002), my proposed exclusion restriction is formulated on instantaneous utilities and does not rely on future value functions. This simplifies the interpretation and makes it easier to find variables that satisfy the exclusion restriction in empirical applications.

Exploiting variation in choice restrictions also overcomes the issue of set identification of previous proposed exclusion restrictions (as in Abbring & Daljord, 2020b) by directly point identifying the discount factor. The derived formulas for the exponential discount factor are easy to interpret and align with economic intuition. For finite horizon models, time preferences can also be identified in short panels and for models in which the reachable part of the state space changes over time.

Dynamic discrete choice models that include probabilistic choice restrictions nest the standard dynamic discrete choice models as discussed by Magnac and Thesmar (2002) or Abbring and Daljord (2020b). Standard models limit the probabilities to be restricted to zero in all cases. The framework presented here also permits the probability of being restricted in the future to be either one or zero. Thus, settings where choice sets vary non-stochastically depending on previous choices are nested.

The vast majority of dynamic discrete choice models can be easily adjusted to include choice restrictions. One possibility of a minimal adjustment is the following: In principle, choices are never restricted, and economic agents can always choose from all alternatives. To identify the exponential discount factor in such settings, it is sufficient that a state exists that provides the same utility and the same transition probabilities as another state. This state must then lead to a choice set that is reduced by at least one alternative.

Consequently, it is easy to adjust the vast majority of dynamic discrete choice models in such a way that the exclusion restriction is fulfilled. For instance, in the classic bus-engine replacement example of Rust (1987), the following adjustments lead to the identification of the exponential discount factor. Assume that one bus engine is in stock. Once a bus engine is replaced, Harald Zurcher has to order a new engine. A restriction is present if there are random strikes or other reasons that a new bus engine cannot be delivered immediately. As a result, Harold Zurcher no longer has the option to replace a bus engine in two periods in a row. This variation would be sufficient to point identify the exponential discount factor in this model since it neither affects the transition probabilities of the buses' mileage nor Harold Zurcher's instantaneous utilities.

In most other contexts, it is also possible to find potential restrictions that permit point identification of time preferences. Models in labor economics often include choice restrictions in the form of job offers (see, for example, Adda, Dustmann, & Stevens, 2017). When explaining home buying choices for different neighbourhoods (see, for example Bayer, McMillan, Murphy, & Timmins, 2016), the availability of homes in some neighbourhoods might be restricted. Exogenous variation in this availability can be sufficient for the identification of households' time preferences. For De Groote and Verboven (2019), who study the adoption of solar photovoltaic systems, the availability of specific systems might vary due to strikes or supplier shortages. Such exogenous variation can then be exploited to identify discount functions.

The literature on the identification of dynamic discrete choice models (e.g., Abbring, 2010; Arcidiacono & Miller, 2020; Chen, 2017; Hu & Sasaki, 2018; Srisuma, 2015) has discussed various aspects of these models, with only a few papers providing conditions to identify time preferences. The identification of time preferences is pivotal for the identification of alternative-specific utilities. Magnac and Thesmar (2002) demonstrate that it is not possible to identify utilities if the discount factor is not known. Conversely, once the discount factor is identified and given the distribution of preference shocks and the normalization of one utility, all other utilities can be uniquely determined.[3]

Magnac and Thesmar (2002) are among the first to examine how the exponential discount factor can be recovered from observed choices. Their derived exclusion restriction requires two states that have different expected streams of utilities for some choices but equal expected streams of utilities for at least one choice. The states should also provide the same instantaneous utilities. The requirements on future expected utility streams increase the difficulty of finding such states in empirical applications.

Abbring and Daljord (2020b) extend the identification approach of Magnac and Thesmar (2002). They develop an exclusion restriction on instantaneous utilities, avoiding requirements on future utility streams. Their exclusion restriction requires two states that have equal instantaneous utilities but different transition probabilities. The identifying equation includes an infinite geometric sum in the exponential discount factor. As a result, time preferences are only set identified. Excluding discount factors that are close to one, the set consists of a finite number of discount factors. Abbring and Daljord (2020b) discuss several examples that allow for a reduction in the number of potential solutions, for example, by relying on the concept of finite dependence (see Arcidiacono & Miller, 2019).

Overall, variations in transition probabilities only lead to point identification if at least one additional requirement is met. Abbring and Daljord (2020b) also have to rely on the normalization of one alternative's utility. The exclusion restriction proposed in this paper overcomes these issues and directly point identifies the exponential discount factor.

[3]This identification result carries over to the setting with choice restrictions and is discussed in section 5.

The remainder of the paper is structured as follows: Section 2 introduces the general model. Sections 3 and 4 discuss identification in infinite and finite horizon models, respectively. Section 5 provides sufficient conditions to identify instantaneous utilities once time preferences are identified. Section 6 concludes the paper.

2 Model

In each period $t \in \{0, \ldots, T\}$, where T is either finite or infinite, each agent has to choose an alternative from a finite set of alternatives. The set of alternatives is denoted by $\mathcal{D}$ and contains K elements. With some probability, agents are forced to choose an alternative from a subset of $\mathcal{D}$, i.e., from a *restricted* choice set. The restricted choice set that agents might face in a given period depends on the alternative d chosen in the previous period. Restricted choice sets are denoted by $\widetilde{\mathcal{D}}(d)$. The probability that agents have to choose from $\widetilde{\mathcal{D}}(d)$ instead of $\mathcal{D}$ depends on d and the state $x \in \mathcal{X}$ in the previous period, where $\mathcal{X}$ contains a finite number of J elements. In particular, if the state is currently x and agents choose d, then in the next period, they have to choose from $\widetilde{\mathcal{D}}(d)$ with probability $\pi(d, x)$ and can choose from $\mathcal{D}$ with probability $1 - \pi(d, x)$. The probabilities $\pi(d, x)$ are called *restriction probabilities*.[4] I assume that $\pi(d, x) > 0$ for at least one pair $(d, x) \in \mathcal{D} \times \mathcal{X}$.

Choosing alternative d, given state x, provides agents with the instantaneous utility $u^\star(d, x, \eta_d)$, where η_d denotes an alternative-specific preference shock. Preference shocks are assumed to be mean-zero type-1 extreme value distributed and are assumed to be independent over all $d \in \mathcal{D}$ and all $t \in \{0, \ldots, T\}$.[5] It is assumed that instantaneous utility is additively separable in the preference shock,

[4]Note that these restrictions are not the same as those discussed in McFadden (1978). The probabilistic choice restrictions in McFadden (1978) are introduced to reduce the computational burden for the researcher in problems with a vast number of alternatives. Restricted choice sets should also not be interpreted as *consideration sets*, as for instance, in Shocker, Ben-Akiva, Boccara, and Nedungadi (1991) or Goeree (2008). Consideration sets exclude alternatives that agents are not aware of and thus do not consider. This paper assumes that agents are fully aware of potential restrictions and can base their decisions on future probabilities to be restricted.

[5]The results extend to any other continuous distribution.

i.e., $u^{\star}(d, x, \eta_d) = u(d, x) + \eta_d$. The vector $\boldsymbol{\eta} = \{\eta_1, \ldots, \eta_K\}$ contains the preference shocks related to all alternatives.

The probability of observing a certain state x_t in period t depends on both the choice d_{t-1} and the state x_{t-1} in period $t-1$. In particular, state x_t occurs with the transition probability $q(x_t | d_{t-1}, x_{t-1})$. It is assumed that transition probabilities (and restriction probabilities) do not depend on preference shocks $\boldsymbol{\eta}_t$. This assumption is one variant of the conditional independence assumption (see Rust, 1987).

The order of events in each period t is the following. First, the choice set is determined, and agents observe the state x_t and preference shocks $\boldsymbol{\eta}_t$. Second, agents choose one alternative d_t. Finally, agents collect instantaneous utility $u^{\star}(d_t, x_t, \eta_{d_t,t})$, and the period ends.

In period t, agents maximize their total expected discounted stream of instantaneous utilities $v^{\star}(d_t, x_t, \eta_{d_t,t})$. The additive separability of $u(d_t, x_t)$ and the conditional independence assumption imply $v^{\star}(d_t, x_t, \eta_{d_t,t}) = v(d_t, x_t) + \eta_{d_t,t}$, with

$$\begin{aligned} v(d_t, x_t) = u(d_t, x_t) & \\ + \beta \sum_{\substack{x_{t+1} \\ \in \mathcal{X}}} \Bigg[& (1 - \pi(d_t, x_t)) \mathbb{E}\left[\max_{j \in \mathcal{D}} \{ v(j, x_{t+1}) + \eta_{j,t+1} \} \right] \\ & + \pi(d_t, x_t) \mathbb{E}\left[\max_{j \in \widetilde{\mathcal{D}}(d_t)} \{ v(j, x_{t+1}) + \eta_{j,t+1} \} \right] \Bigg] q(x_{t+1} | d_t, x_t), \end{aligned} \tag{1}$$

where β denotes the discount factor. Under the assumption made about the distribution of the preference shocks, (1) can be expressed as

$$\begin{aligned} v(d_t, x_t) = u(d_t, x_t) & \\ + \beta \sum_{\substack{x_{t+1} \\ \in \mathcal{X}}} \Bigg[& (1 - \pi(d_t, x_t)) \ln\left(\sum_{j \in \mathcal{D}} \exp(v(j, x_{t+1})) \right) \\ & + \pi(d_t, x_t) \ln\left(\sum_{j \in \widetilde{\mathcal{D}}(d_t)} \exp(v(j, x_{t+1})) \right) \Bigg] q(x_{t+1} | d_t, x_t). \end{aligned} \tag{2}$$

Further discussion distinguishes between *observed* and *genuine* choice probabilities. The observed choice probabilities describe the probability that agents choose alternative d_t given the current period's state x_t and the choice d_{t-1} and state x_{t-1} in period $t-1$. They reflect a combination of preferences and possible choice restrictions. The observed choice probabilities are denoted by $\Pr\left(d_t \,|x_t, d_{t-1}, x_{t-1}\right)$. Genuine choice probabilities describe the probability that agents choose alternative d_t from a specific choice set $\hat{\mathcal{D}} \in \left\{\mathcal{D}, \widetilde{\mathcal{D}}\left(d_{t-1}\right); d_{t-1} \in \mathcal{D}\right\}$, given state x_t. They exclusively reflect preferences over the alternatives included in the respective choice set. Genuine choice probabilities are denoted by $\mathrm{GPr}\left(d_t \,\middle|\hat{\mathcal{D}}, x_t\right)$.

I concentrate on the identification of the time preference parameter β in (2) because it is essential to the identification of the model.[6] Suppose that data are available, such that all transition, restriction, and genuine choice probabilities can be derived. Then, the model is point identified if and only if utilities can be uniquely determined from these probabilities. As discussed in section 5, an adapted version of Proposition 2 of Magnac and Thesmar (2002) is fulfilled: Given the distribution of preference shocks and the normalization of one alternative's utility, all other utilities depend on the discount factor β. Thus, without knowing β, utilities cannot be uniquely determined, and the model is not identified.

The presented model is closely related to the dynamic discrete choice model discussed by Magnac and Thesmar (2002). The key difference between the two frameworks is that Magnac and Thesmar (2002) limits all restriction probabilities to zero. I highlight two potential paths to minimally adjust the standard model to fit the discussed framework. One potential adjustment introduces a non-zero restriction probability for exactly one choice-state combination. The respective choice set includes all but one alternative of $\mathcal{D}$. Another potential adjustment makes an additional alternative available after one choice-state combination. To do so, it is assumed that all restriction probabilities are equal to one except for one choice-state

[6] Note that I focus on infinite-horizon models and finite-horizon models, for which the last period cannot be used to identify instantaneous utilities.

combination. All restricted choice sets exclude the same single alternative. Note that in addition to these two paths, restriction probabilities can be limited to either equal one or zero. Thus, the presented framework also allows for settings with deterministic changes in the set of alternatives.

For ease of exposition, I drop the subscript t henceforth and denote variables for period $t+1$ with a prime. The following sections also assume that restriction and genuine choice probabilities are known. Appendix A discusses the identification of restriction probabilities when these cannot be recovered from other data sources. Appendix B discusses the identification of genuine choice probabilities.

3 Infinite horizon

I assume that the data $\{\mathrm{GPr}(d|\mathcal{D},x),\, q(\cdot|d,x),\, \pi(d,x);\, (d,x)\in\mathcal{D}\times\mathcal{X}\}$ are known for at least two consecutive periods. Note that the genuine choice probabilities conditioned on choice set $\mathcal{D}$ identify the genuine choice probabilities conditioned on any restricted choice set $\widetilde{\mathcal{D}}(\cdot)$.[7] I do not make any assumptions about stationarity.

Fix $n'\in\mathcal{D}$. Subtract the value function $v(n',x')$ from both terms within the square brackets of (2), and add it once to neutralize the subtraction. As a result, the following can be derived:

$$\begin{aligned} v(d,x) = u(d,x) + \beta \sum_{x'\in\mathcal{X}} \Big[&(1-\pi(d,x))\, m(\mathcal{D},n',x') \\ &+ \pi(d,x)\, m\left(\widetilde{\mathcal{D}}(d),n',x'\right) + v(n',x') \Big] q(x'|d,x), \end{aligned} \tag{3}$$

where $m\left(\hat{\mathcal{D}},n',x'\right) = \ln\left(\sum_{j\in\hat{\mathcal{D}}} \exp(v(j,x'))\,(\exp(v(n',x')))^{-1}\right)$ for a given choice set $\hat{\mathcal{D}} \in \left\{\mathcal{D}, \widetilde{\mathcal{D}}(d); d\in\mathcal{D}\right\}$. Each $m(\cdot,\cdot,\cdot)$ can be directly recovered from a transformation of the respective genuine choice probability

$$m\left(\hat{\mathcal{D}},n',x'\right) = -\ln\left(\mathrm{GPr}\left(n'\left|\hat{\mathcal{D}},x'\right.\right)\right),$$

[7]For details, see appendix B.

as long as $n' \in \hat{\mathcal{D}}$ (see Arcidiacono & Miller, 2011). If $n' \notin \hat{\mathcal{D}}$, the assumption of the independence of irrelevant alternatives can be exploited, and $m\left(\hat{\mathcal{D}}, n', x'\right)$ is identified by

$$m\left(\hat{\mathcal{D}}, n', x'\right) = \ln\left(\frac{\sum_{j\in\hat{\mathcal{D}}} \mathrm{GPr}\left(j\,|\mathcal{D}, x'\right)}{\mathrm{GPr}\left(n'\,|\mathcal{D}, x'\right)}\right).$$

Let $\boldsymbol{m}\left(\hat{\mathcal{D}}, n'\right)$, $\boldsymbol{q}\left(d, x\right)$ and $\boldsymbol{v}\left(n'\right)$ denote vectors of size $J \times 1$, for which the j-th element is $m\left(\hat{\mathcal{D}}, n', x_j\right)$, $q\left(x_j\,|d, x\right)$ and $v\left(n', x_j\right)$, respectively. Using this notation, (3) can be expressed as

$$\begin{aligned} v\left(d, x\right) = u\left(d, x\right) + \beta\Big[&\left(1 - \pi\left(d, x\right)\right)\boldsymbol{q}\left(d, x\right)^{\top}\boldsymbol{m}\left(\mathcal{D}, n'\right) \\ &+ \pi\left(d, x\right)\boldsymbol{q}\left(d, x\right)^{\top}\boldsymbol{m}\left(\widetilde{\mathcal{D}}\left(d\right), n'\right) + \boldsymbol{q}\left(d, x\right)^{\top}\boldsymbol{v}\left(n'\right)\Big], \end{aligned} \tag{4}$$

where the superscript $\top$ denotes the transpose.

Hotz and Miller (1993) show that for a given state x, the difference between the value functions of two alternatives can be identified using a function of their choice probabilities. For the presented model, the difference in the logarithms of the genuine choice probabilities of two alternatives $\ell, r \in \mathcal{D}$ determines the difference between their value functions:

$$\ln\left(\mathrm{GPr}\left(\ell\,|\mathcal{D}, x\right)\right) - \ln\left(\mathrm{GPr}\left(r\,|\mathcal{D}, x\right)\right) = v\left(\ell, x\right) - v\left(r, x\right). \tag{5}$$

By combining (4) and (5), the following can be derived:

$$\begin{aligned} \ln\left(\mathrm{GPr}\left(\ell\,|\mathcal{D}, x\right)\right) - \ln\left(\mathrm{GPr}\left(r\,|\mathcal{D}, x\right)\right) &= u\left(\ell, x\right) - u\left(r, x\right) \\ + \beta\Big[&\left(1 - \pi\left(\ell, x\right)\right)\boldsymbol{q}\left(\ell, x\right)^{\top}\boldsymbol{m}\left(\mathcal{D}, n'\right) + \pi\left(\ell, x\right)\boldsymbol{q}\left(\ell, x\right)^{\top}\boldsymbol{m}\left(\widetilde{\mathcal{D}}\left(\ell\right), n'\right) \\ &- \left(1 - \pi\left(r, x\right)\right)\boldsymbol{q}\left(r, x\right)^{\top}\boldsymbol{m}\left(\mathcal{D}, n'\right) - \pi\left(r, x\right)\boldsymbol{q}\left(r, x\right)^{\top}\boldsymbol{m}\left(\widetilde{\mathcal{D}}\left(r\right), n'\right) \\ &+ \left(\boldsymbol{q}\left(\ell, x\right) - \boldsymbol{q}\left(r, x\right)\right)^{\top}\boldsymbol{v}\left(n'\right)\Big]. \end{aligned} \tag{6}$$

Many elements in (6) are identified. The remaining unknown elements are β, the difference in instantaneous utilities between the two alternatives ℓ and r, and the value functions of alternative n'. Based on this observation, I formulate the following exclusion restriction:

Exclusion Restriction. *There exist two different states $x^A, x^B \in \mathcal{X}$ and two different alternatives $\ell, r \in \mathcal{D}$, such that*

(1) $u(\ell, x^A) = u(\ell, x^B)$ *and* $u(r, x^A) = u(r, x^B)$*;*

(2) $q(x|d, x^A) = q(x|d, x^B)$ *for* $d \in \{\ell, r\}$*, and* $x \in \mathcal{X}$*;*

(3) $\pi(\ell, x^A) < \pi(\ell, x^B)$*.*

The exclusion restriction is formulated for two states and two alternatives that must fulfil three conditions. First, each alternative must provide the same instantaneous utility for both states. Second, for each alternative, the transition probabilities must be equal for both states. Third, for at least one alternative, the restriction probabilities must differ between the two states.

Assume that the exclusion restriction is fulfilled for states $x^A, x^B \in \mathcal{X}$ and alternatives $\ell, r \in \mathcal{D}$. Subtracting (6) using $x = x^B$ from the same equation using $x = x^A$ results in

$$\begin{aligned}\ln\left(\frac{\mathrm{GPr}(\ell|\mathcal{D}, x^A)}{\mathrm{GPr}(\ell|\mathcal{D}, x^B)}\right) - \ln\left(\frac{\mathrm{GPr}(r|\mathcal{D}, x^A)}{\mathrm{GPr}(r|\mathcal{D}, x^B)}\right) = \\ \beta\Big[\left(\pi\left(\ell, x^B\right) - \pi\left(\ell, x^A\right)\right)\boldsymbol{q}\left(\ell, x^A\right)^{\top}\boldsymbol{\Theta}(\ell) \\ - \left(\pi\left(r, x^B\right) - \pi\left(r, x^A\right)\right)\boldsymbol{q}\left(r, x^A\right)^{\top}\boldsymbol{\Theta}(r)\Big],\end{aligned} \tag{7}$$

where $\boldsymbol{\Theta}(d) = \boldsymbol{m}(\mathcal{D}, n') - \boldsymbol{m}\left(\widetilde{\mathcal{D}}(d), n'\right)$. Note that the j-th element of $\boldsymbol{\Theta}(d)$ is given by

$$\ln\left(1 + \frac{\sum_{k\in\mathcal{D}\setminus\widetilde{\mathcal{D}}(d)}\exp(v(k, x_j))}{\sum_{k\in\widetilde{\mathcal{D}}(d)}\exp(v(k, x_j))}\right),$$

and is independent of alternative n'. For the identification of the time preference parameter β, the following rank condition is required.

Rank Condition.

$$\left(\pi\left(\ell,x^B\right)-\pi\left(\ell,x^A\right)\right)\boldsymbol{q}\left(\ell,x^A\right)^{\top}\boldsymbol{\Theta}\left(\ell\right)-\left(\pi\left(r,x^B\right)-\pi\left(r,x^A\right)\right)\boldsymbol{q}\left(r,x^A\right)^{\top}\boldsymbol{\Theta}\left(r\right)\neq 0. \quad (8)$$

Assuming that rank condition (8) is fulfilled, β is point identified by

$$\beta=\frac{\ln\left(\frac{\mathrm{GPr}\left(\ell|\mathcal{D},x^A\right)}{\mathrm{GPr}\left(\ell|\mathcal{D},x^B\right)}\right)-\ln\left(\frac{\mathrm{GPr}\left(r|\mathcal{D},x^A\right)}{\mathrm{GPr}\left(r|\mathcal{D},x^B\right)}\right)}{\left(\pi\left(\ell,x^B\right)-\pi\left(\ell,x^A\right)\right)\boldsymbol{q}\left(\ell,x^A\right)^{\top}\boldsymbol{\Theta}\left(\ell\right)-\left(\pi\left(r,x^B\right)-\pi\left(r,x^A\right)\right)\boldsymbol{q}\left(r,x^A\right)^{\top}\boldsymbol{\Theta}\left(r\right)}. \quad (9)$$

If the ratios of the genuine choice probabilities of ℓ and r are identical for both states, the numerator on the right-hand side of (9) equals zero. As a result, the discount factor β also equals zero. Intuitively, if agents' choices do not differ, although they lead to different expected futures, they do not place any value on future utilities when choosing an alternative. They are myopic. Note that the rank condition (8) guarantees that the expected futures for states x^A and x^B differ for choices ℓ and r.

For a more detailed discussion of (9), two exhaustive cases are distinguished.

Case 1. *The restriction probabilities after choice $\ell\in\mathcal{D}$ differ, such that $\pi\left(\ell,x^A\right)<\pi\left(\ell,x^B\right)$. Furthermore, there exists at least one other alternative $r\in\mathcal{D}$ that fulfils conditions (1) and (2) of the exclusion restriction, for which the restriction probabilities are equal for states x^A and x^B, i.e., $\pi\left(r,x^A\right)=\pi\left(r,x^B\right)$.*

For case 1, rank condition (8) simplifies to

$$\boldsymbol{q}\left(\ell,x^A\right)^{\top}\boldsymbol{\Theta}\left(\ell\right)\neq 0. \quad (10)$$

The sum in (10) consists exclusively of terms larger than or equal to zero. The j-th value in $\boldsymbol{\Theta}\left(\ell\right)$ only becomes zero if none of the alternatives excluded from $\widetilde{\mathcal{D}}\left(\ell\right)$ provides any value other than negative infinity for the respective state. Thus, the rank condition is violated if and only if there is no real restriction. As a result, rank condition (10) is always fulfilled for correctly specified models.

In this case, (9) simplifies to[8]

$$\beta = \frac{\ln\left(\mathrm{GPr}\left(\ell\,|\mathcal{D}, x^A\right)\right) - \ln\left(\mathrm{GPr}\left(\ell\,|\mathcal{D}, x^B\right)\right)}{\left(\pi\left(\ell, x^B\right) - \pi\left(\ell, x^A\right)\right) \boldsymbol{q}\left(\ell, x^A\right)^{\top} \boldsymbol{\Theta}\left(\ell\right)}, \tag{11}$$

and has a clear economic interpretation.

First, if the genuine choice probabilities of choosing $\ell \in \mathcal{D}$ are equal for both states, the right-hand side of (11) is zero. As a result, β is also zero, and agents are myopic. This result has an economic meaning, as the rank condition guarantees that there is a utility loss when restricted, while the exclusion restriction guarantees that the probability of being restricted differs between states x^A and x^B. Thus, if economic agents make the same decisions in both states, they ignore future consequences.

Second, because $\pi\left(\ell, x^B\right) > \pi\left(\ell, x^A\right)$ and because the left-hand side of (10) is positive, the denominator on the right-hand side of (11) must be positive. For a positive value of β, the numerator must also be positive. That is fulfilled if there is a higher chance of choosing ℓ for state x^A than state x^B. The economic interpretation is the following: Holding everything else equal, agents should prefer an option A to an option B if option A results in a lower chance of being restricted in the next period. In contrast, if agents prefer to be restricted in future periods, the discount factor is negative. Thus, a negative value of β is only possible if the economic model is violated.

Third, the size of β depends on the relation of three elements: the difference in genuine choice probabilities, the difference in the restriction probabilities, and the additional expected value of being able to choose freely. A greater difference in the genuine choice probabilities coincides with a greater value of β when holding the other two elements constant. Intuitively, the larger the reaction to a given difference in restriction probabilities and a given difference in the expected future values, the

[8]Note that because utilities and transition and restriction probabilities are the same for x^A and x^B after choosing r, the logarithm of the ratio of their genuine choice probabilities equals zero.

more weight agents place on their future. In contrast, the larger the difference in the restriction probabilities, ceteris paribus, the lower β is. Intuitively, the greater the probability difference of being restricted has to be to cause a fixed difference in behaviour, the lower the weight agents place on their expected futures. Similarly, the larger the expected surplus of not being restricted, holding everything else constant, the lower β is. Intuitively, the larger the surplus of being able to choose freely has to be to cause a fixed difference in behaviour, the weaker the influence of the expected future is on current decisions.

Finally, note that (11) does not restrict β to be larger than 0 or lower than 1. A negative β violates the economic meaning of the model and is only possible if agents prefer to be restricted. In contrast, because no assumption about stationarity is imposed, a β value above one does not necessarily violate the economic model.

Case 2. *The restriction probabilities after choice $\ell \in \mathcal{D}$ differ, such that $\pi(\ell, x^A) < \pi(\ell, x^B)$. Furthermore, there exists no other alternative $r \in \mathcal{D}$ that fulfils conditions (1) and (2) of the exclusion restriction, for which restriction probabilities are equal for states x^A and x^B. Thus, $\pi(r, x^A) \neq \pi(r, x^B)$.*

For the discussion of the second case, an exhaustive division into three further cases is helpful.

Case 2.a. $\pi(\ell, x^B) - \pi(\ell, x^A) = \pi(r, x^B) - \pi(r, x^A)$.

For case 2.a, rank condition (8) simplifies to

$$\left(\pi\left(\ell, x^B\right) - \pi\left(\ell, x^A\right)\right)\left(\boldsymbol{q}\left(\ell, x^A\right)^{\top}\boldsymbol{\Theta}(\ell) - \boldsymbol{q}\left(r, x^A\right)^{\top}\boldsymbol{\Theta}(r)\right) \neq 0. \qquad (12)$$

Rank condition (12) is fulfilled if the weighted surplus of not being restricted after choosing alternative ℓ differs from that after choosing alternative r. If the exclusion restriction is fulfilled for additional alternatives besides ℓ and r, it is only necessary that one pair of alternatives exists for which the expected surpluses of not being restricted differ.

For case 2.a, (12) represents the denominator on the right-hand side of (9). If the expected surplus of not being restricted after choosing ℓ is larger than that after choosing r, the denominator will be positive. Given that economic agents are less likely to be restricted in state x^A, both logarithms of (9) should be positive. This is fulfilled as long as individuals do not prefer to be restricted in their choice. Otherwise, it would be a violation of the economic model. Further, if economic agents gain more by not being restricted after choosing ℓ instead of r, the first logarithm of the numerator of (9) should be greater than the second logarithm. As a result, β has to be greater than zero.

In contrast, if the expected surplus of not being restricted after choosing ℓ is lower than that after choosing r, the denominator will be negative. Arguing along the same lines as before, the numerator of (9) should be negative as long as agents do not prefer to be restricted. As a result, β is positive.

The size of β depends on the interaction of three elements: the difference in the logarithms of the ratio of the genuine choice probabilities, the difference in the restriction probabilities, and the difference in the weighted expected surplus between alternatives ℓ and r when not restricted. Similar arguments as for case 1 can be made to discuss the size of β.

Case 2.b. $\pi\left(r, x^A\right) > \pi\left(r, x^B\right)$.

In this case, the denominator on the right-hand side of (9) is positive. Consequently, β is positive if and only if the numerator on the right-hand side of (9) is also positive. As long as individuals prefer not being restricted to being restricted, the first term in the numerator of (9) is positive and the second term is negative, guaranteeing a positive β.

The size of β is driven by multiple factors, which can be divided according to whether they depend either ℓ or r. For elements depending on ℓ, the discussion of case 1 applies when holding all elements depending on r constant. The elements depending on r enter with a negative sign in the numerator and with a positive sign in the denominator of (9). Because $\pi(r, x^A) > \pi(r, x^B)$, the genuine choice probability of choice-state combination (r, x^A) should be smaller than the genuine choice probability of state-choice combination (r, x^B). This is fulfilled as long as individuals do not prefer to be restricted. As a consequence, the logarithm of the ratio of the two genuine choice probabilities in (9) should be negative. This allows us to apply the same discussion as for case 1 for the elements depending on r when holding elements depending on ℓ constant.

Case 2.c. $\pi(r, x^B) > \pi(r, x^A)$ *and* $\pi(\ell, x^B) - \pi(\ell, x^A) \neq \pi(r, x^B) - \pi(r, x^A)$.

In this case, the sign of the denominator on the right-hand side of (9) is ambiguous. The sign of β depends on the different value functions for choices not included in the two restricted choice sets. It also depends on the transition probabilities and the relation of the restriction probabilities. A detailed discussion of all possible cases is not productive without knowing the signs of most of these elements. With more information on the different signs, similar arguments as before can be made.

3.1 Discussion

Restriction probabilities can be interpreted as external factors that only affect agents' choice sets. Thus, the requirements of the exclusion restriction are potentially fulfilled in many empirical contexts. For instance, in the context of labor supply, the job market might be hit by a negative demand shock, and the likelihood of receiving a job offer decreases. Although such a shock affects the restriction probabilities (i.e., the job offer probabilities), it does not affect the instantaneous utilities or the transition probabilities. The instantaneous utilities depend on agents' leisure and consumption trade-offs and should be unaffected by most labor demand shocks. Similarly, the probabilities of transitioning from state to state should not be affected, as for example, agents' human capital develops independently of labor demand shocks.[9]

In the context of firm entries into regulated markets, administrative variations leading to different probabilities of the approval of mergers and acquisitions can lead to the identification of time preferences. For identification, firms must time their mergers due to administrative variations. Further, these variations are not allowed to affect firms' instantaneous utilities (or payoffs) and transition probabilities.

When estimating dynamic models of product demand, random shocks to the supply of such products can identify the discount factor of consumers. For example, if an exogenous shock affects the supply of one specific car brand, the probability that certain car models are available changes. If consumers have the option to delay the purchase of a car by one period until their preferred model becomes available again, time preferences can be identified. Supply shocks should not affect consumers' utilities from cars or their transition probabilities.

The exclusion restriction leads to point identification of the time preference parameter β as long as the rank condition is fulfilled. The rank condition is only violated in rare cases: either alternatives that are excluded from restricted choice sets are of no value for agents or there exist no two alternatives with different expected surpluses when choosing freely.

All identifying equations have a clear economic meaning. Nevertheless, the exclusion restriction does not restrict β to be between zero and one. Negative values are only possible if at least one model assumption is violated. Values above one are possible without clear violations of the economic model, as no assumption about stationarity is made.

[9] Note that the transition probabilities also determine the process of potential factors that only affect restriction probabilities.

4 Finite horizon

In finite-horizon models, the identification of β can potentially be achieved under less restrictive assumptions, as the last period can be exploited for identification. If the reachable part of the state space does not change over time, the genuine choice probabilities in the last period can be used to identify all differences in instantaneous utilities. Once the instantaneous utility of an arbitrary alternative is normalized, all other instantaneous utilities are identified. Then, the discount factor can be point identified using the genuine choice probabilities from the second-to-last period.

Relying on the exclusion restriction of the previous section, β can also be identified for models for which the data do not include the last period such as in short panels. Furthermore, time preferences can be recovered for models in which the reachable part of the state space changes over time. The discussion of the required rank condition and the identifying equation carries over from the previous section, as long as the current period t is part of the state space. This inclusion is necessary, as all value functions are period specific and period t determines their distance to the last period.

For the finite horizon model, the assumption of identical transition probabilities, as stated in the exclusion restriction, can be relaxed. The non-stationary nature allows for subsequent periods of different restriction probabilities as long as these do not lead to different reachable state spaces. This allows for settings where different groups of economic agents are differently restricted in their choices for a limited number of periods.

In the labor context, this is the case when a randomly selected group of unemployed individuals takes part in a program that supports job searchers in finding jobs while another random group does not receive such support. The former group should have greater job offer probabilities (i.e., lower restriction probabilities) than the latter group.

To identify β in such settings, differences between the groups must be limited in time. Furthermore, the last period for which differences exist and the period after that have to be observed. Time preferences can then be identified from these two periods in the same fashion as in section 3.

In contrast, time preferences are not necessarily point identified if researchers observe the data for two groups of economic agents that exclusively differ in their restriction probabilities. To show this, let $\delta \in \{A, B\}$ denote a group indicator. Restriction probabilities depend in this case not only on state-choice combinations but also on the group indicator. I denote these by $\pi(d, x, \delta)$. Because of the different restriction probabilities, value functions differ between the two groups. I denote the vectors $\boldsymbol{m}\left(\hat{\mathcal{D}}, d\right)$ and $\boldsymbol{v}(d)$ by $\boldsymbol{m}\left(\hat{\mathcal{D}}, d, \delta\right)$ and $\boldsymbol{v}(d, \delta)$ for this demonstration, where $\hat{\mathcal{D}} \in \left\{\mathcal{D}, \widetilde{\mathcal{D}}(d); d \in \mathcal{D}\right\}$. Similarly, I denote the genuine choice probabilities by $\mathrm{GPr}\left(d \middle| \hat{\mathcal{D}}, x, \delta\right)$. In this case, (6) can be written as

$$\begin{aligned}
\ln\left(\frac{\mathrm{GPr}(\ell \,|\mathcal{D}, x, \delta)}{\mathrm{GPr}(r \,|\mathcal{D}, x, \delta)}\right) = {} & u(\ell, x) - u(r, x) \\
& + \beta \boldsymbol{q}(\ell, x)^{\top}\Big[(1 - \pi(\ell, x, \delta))\, \boldsymbol{m}(\mathcal{D}, n', \delta) + \pi(\ell, x, \delta)\, \boldsymbol{m}\left(\widetilde{\mathcal{D}}(\ell), n', \delta\right) \\
& \qquad + \boldsymbol{v}(n', \delta)\Big] \\
& - \beta \boldsymbol{q}(r, x)^{\top}\Big[(1 - \pi(r, x, \delta))\, \boldsymbol{m}(\mathcal{D}, n', \delta) + \pi(r, x, \delta)\, \boldsymbol{m}\left(\widetilde{\mathcal{D}}(r), n', \delta\right) \\
& \qquad + \boldsymbol{v}(n', \delta)\Big].
\end{aligned} \tag{13}$$

Taking the difference of (13) between the two groups A and B results in

$$\begin{aligned}
&\ln\left(\frac{\mathrm{GPr}\left(\ell\,|\mathcal{D},x,A\right)}{\mathrm{GPr}\left(r\,|\mathcal{D},x,A\right)}\right)-\ln\left(\frac{\mathrm{GPr}\left(\ell\,|\mathcal{D},x,B\right)}{\mathrm{GPr}\left(r\,|\mathcal{D},x,B\right)}\right)=\\
&\beta\boldsymbol{q}\left(\ell,x\right)^{\top}\Big[\left(1-\pi\left(\ell,x,A\right)\right)\boldsymbol{m}\left(\mathcal{D},n',A\right)\\
&\qquad+\pi\left(\ell,x,A\right)\boldsymbol{m}\left(\widetilde{\mathcal{D}}\left(\ell\right),n',A\right)-\left(1-\pi\left(\ell,x,B\right)\right)\boldsymbol{m}\left(\mathcal{D},n',B\right)\\
&\qquad-\pi\left(\ell,x,B\right)\boldsymbol{m}\left(\widetilde{\mathcal{D}}\left(\ell\right),n',B\right)+\boldsymbol{v}\left(n',A\right)-\boldsymbol{v}\left(n',B\right)\Big]\\
&-\beta\boldsymbol{q}\left(r,x\right)^{\top}\Big[\left(1-\pi\left(r,x,A\right)\right)\boldsymbol{m}\left(\mathcal{D},n',A\right)\\
&\qquad+\pi\left(r,x,A\right)\boldsymbol{m}\left(\widetilde{\mathcal{D}}\left(r\right),n',A\right)-\left(1-\pi\left(r,x,B\right)\right)\boldsymbol{m}\left(\mathcal{D},n',B\right)\\
&\qquad-\pi\left(r,x,B\right)\boldsymbol{m}\left(\widetilde{\mathcal{D}}\left(r\right),n',B\right)+\boldsymbol{v}\left(n',A\right)-\boldsymbol{v}\left(n',B\right)\Big].
\end{aligned}\tag{14}$$

Similar to the infinite horizon framework, all $\boldsymbol{m}\left(\hat{\mathcal{D}},n',\delta\right)$ can be identified from the data. However, as future restrictions might differ, future value functions also differ. Hence, the value functions for choice n' do not cancel out between the two groups. Consequently, changes in the genuine choice probabilities cannot recover β because they might reflect differences in future values. Only when the differences in value functions between the two groups are known can β be uniquely determined in cases in which individuals only differ in their restriction probabilities.

5 Identification of instantaneous utilities

Magnac and Thesmar (2002) show that in a stationary dynamic discrete choice model with an infinite horizon and without choice restrictions, alternative specific utilities are identified under the following conditions:

(1) The distribution of preference shocks η is known.

(2) The instantaneous utility of one alternative is normalized.

(3) The discount factor β is known.

This result does not necessarily carry over when restriction probabilities are non-zero. Theorem 1 states a sufficient condition for the identification of instantaneous utilities for a stationary dynamic discrete choice model with an infinite horizon and at least one strictly positive restriction probability.

Theorem 1. *Given the data* $\{\mathrm{GPr}(d|\mathcal{D},x),\ q(\cdot|d,x),\ \pi(d,x);\ (d,x)\in\mathcal{D}\times\mathcal{X}\}$*, all instantaneous utilities can be recovered under the following conditions:*

(1) The distribution of preference shocks η *is known.*

(2) The instantaneous utility of one alternative is normalized.

(3) The discount factor β *is known.*

(4) There exists an alternative $d\in\mathcal{D}$*, such that* $d\in\widetilde{\mathcal{D}}(d)$ *or/and the assumption of independence of irrelevant alternatives is fulfilled.*

Proof. Fix an arbitrary $\bar{d}\in\mathcal{D}$. Let $\mathbf{\Pi}(d)$ be the diagonal matrix with i-th diagonal entry $\pi(d,x_i)$. Let $\boldsymbol{Q}(d)$ denote a matrix with the $\{i,j\}$-th element being $q(x_j|d,x_i)$. Finally, let $\boldsymbol{I}_J$ denote an identity matrix of size J. With this notation, (3) can be rewritten for the full state space as

$$\begin{aligned}\boldsymbol{v}(\bar{d}) = \boldsymbol{u}(\bar{d}) + \beta\big[\,&(\boldsymbol{I}_J-\mathbf{\Pi}(\bar{d}))\,\boldsymbol{Q}(\bar{d})\,\boldsymbol{m}(\mathcal{D},\bar{d})\\ &+\mathbf{\Pi}(\bar{d})\,\boldsymbol{Q}(\bar{d})\,\boldsymbol{m}(\widetilde{\mathcal{D}}(\bar{d}),\bar{d})+\boldsymbol{Q}(\bar{d})\,\boldsymbol{v}(\bar{d})\,\big].\end{aligned}$$

Minor manipulation results in

$$\begin{aligned}\boldsymbol{v}(\bar{d}) &= \left[\boldsymbol{I}_J-\beta\boldsymbol{Q}(\bar{d})\right]^{-1}\boldsymbol{u}(\bar{d})\\ &\quad+\left[\boldsymbol{I}_J-\beta\boldsymbol{Q}(\bar{d})\right]^{-1}\beta\big[\,(\boldsymbol{I}_J-\mathbf{\Pi}(\bar{d}))\,\boldsymbol{Q}(\bar{d})\,\boldsymbol{m}(\mathcal{D},\bar{d})\\ &\qquad+\mathbf{\Pi}(\bar{d})\,\boldsymbol{Q}(\bar{d})\,\boldsymbol{m}(\widetilde{\mathcal{D}}(\bar{d}),\bar{d})\,\big].\end{aligned}$$

Note that the j-th element of $\boldsymbol{m}\left(\hat{\mathcal{D}}, d\right)$ with $\hat{\mathcal{D}} \in \left\{\mathcal{D}, \widetilde{\mathcal{D}}(d); d \in \mathcal{D}\right\}$ can be identified even if $d \notin \widetilde{\mathcal{D}}(d)$ due to the assumption of irrelevant alternatives. With the assumption that preference shocks are independently and identically distributed following a mean-zero type-I extreme value distribution, the j-th element of $\boldsymbol{m}\left(\hat{\mathcal{D}}, d\right)$ is identified by[10]

$$m\left(\hat{\mathcal{D}}, d, x_j\right) = \ln\left(\frac{\sum_{i \in \hat{\mathcal{D}}} \mathrm{GPr}\left(i \,|\mathcal{D}, x_j\right)}{\mathrm{GPr}\left(d \,|\mathcal{D}, x_j\right)}\right).$$

Given β and normalizing $\boldsymbol{u}\left(\bar{d}\right) = \mathbf{0}$ uniquely determines $\boldsymbol{v}\left(\bar{d}\right)$. With $\boldsymbol{v}\left(\bar{d}\right)$, all other value functions are determined by a combination of the genuine choice probabilities and $\boldsymbol{v}\left(\bar{d}\right)$. For the assumption that η is the mean-zero type-I extreme value distributed

$$v(d, x) = \ln\left(\frac{\mathrm{GPr}\left(d \,|\mathcal{D}, x\right)}{\mathrm{GPr}\left(\bar{d} \,|\mathcal{D}, x\right)}\right) - v\left(\bar{d}, x\right)$$

can be derived. Finally, utilities are uniquely determined by

$$\begin{aligned}\boldsymbol{u}(d) = \boldsymbol{v}(d) - \beta\Big[&\left(\boldsymbol{I}_J - \boldsymbol{\Pi}(d)\right)\boldsymbol{Q}(d)\,\boldsymbol{m}\left(\mathcal{D}, \bar{d}\right) \\ &+ \boldsymbol{\Pi}(d)\,\boldsymbol{Q}(d)\,\boldsymbol{m}\left(\widetilde{\mathcal{D}}(d), \bar{d}\right) + \boldsymbol{Q}(d)\,\boldsymbol{v}\left(\bar{d}\right)\Big], \ \forall d \in \mathcal{D}\backslash\bar{d}.\end{aligned}$$

■

6 Conclusion

This paper presents a new exclusion restriction to identify the exponential discount factor in dynamic discrete choice models. It relies on differences in restriction probabilities. Restriction probabilities describe the probability that agents are restricted in their choice and cannot choose from all alternatives. The new exclusion restriction requires two states that exclusively cause different restriction probabilities. These states are not allowed to cause differences in instantaneous utilities or transition

[10] Note that if the assumption of the independence of irrelevant alternatives is not fulfilled, it is sufficient that there exists an alternative d such that $d \in \widetilde{\mathcal{D}}(d)$.

probabilities. With these conditions, the exponential discount factor is point identified.

Relative to that in Magnac and Thesmar (2002), this paper presents an exclusion restriction that is easy to interpret. The exclusion restriction depends exclusively on instantaneous utilities, transition rates, and restriction probabilities. In contrast to Abbring and Daljord (2020b), the presented exclusion restriction leads directly to point identification. As it is only necessary that agents are potentially restricted after one state-choice combination, most models can be easily adapted to the presented framework. I discuss two potential avenues for adaptations to models as discussed in Magnac and Thesmar (2002) or Abbring and Daljord (2020b).

Restriction probabilities can be interpreted to be caused by external factors. As such external factors might exclusively impact agents' possibilities to choose from all alternatives, the presented exclusion restriction might be fulfilled in many applications. As a result, it might be easier to find variables satisfying the presented exclusion restriction than those presented in Magnac and Thesmar (2002) or Abbring and Daljord (2020b).

To point identify β, neither stationarity nor the normalization of the utility of one alternative is necessary. In the simplest case, economic agents are only potentially restricted in their choice set after a single state-choice combination. If the exclusion restriction is fulfilled, a combination of observed choice probabilities leads to point identification of the discount factor. Identification for other cases is achieved by the use of genuine choice probabilities, which can be derived from the observed choice probabilities in almost all possible cases. All derived identification equations are economically intuitive.

Identification within an infinite- and a finite-horizon model is discussed. It is shown that for finite-horizon models, time preferences can also be identified for short panels. Due to the non-stationary nature of the finite-horizon model, point identification is also possible if there exist two groups with different restriction probabilities over multiple periods. To identify β in such cases, it is sufficient that this difference is limited in time.

Many empirical examples are provided for which the presented exclusion restriction is fulfilled. In the labor market, an active labor market policy that temporarily supports unemployed individuals in finding employment can be exploited for identification. Such a policy decreases the restriction probability to remain unemployed by increasing job offer rates. If a comparable group of unemployed individuals is found that does not receive such support, time preferences can be identified from differences in the observed choice probabilities of the two groups.

Potential extensions include more than one restricted choice set per alternative. Identification seems possible in these cases but complicates the recovery of restriction and genuine choice probabilities. Future research might also derive conditions to identify parameters of hyperbolic discounting.

Appendix A: Restriction probabilities

In some cases, restriction probabilities are unknown. For instance, in the context of labor supply, job offers are rarely observed. In such cases, the researcher has to disentangle observed choices made due to restrictions and due to preferences. Under certain circumstances, it is possible to recover the probability of being restricted. The discussion is reduced to the following setting:

Restriction Probability Assumption 1. *All restricted choice sets $\widetilde{\mathcal{D}}(d)$ with $d \in \mathcal{D}$ are known. In particular, depending on the previous period's choice d and state x, the researcher knows the available alternatives of each restricted choice set $\widetilde{\mathcal{D}}(d)$. Only the probability that forces agents to choose from $\widetilde{\mathcal{D}}(d)$ is unknown.*

The following presents three different sets of additional assumptions; each alone is sufficient for the identification of all restriction probabilities.

Restriction Probability Assumption Set 1. *There exists an alternative $d \in \mathcal{D}$ after which agents are not restricted in their choice set, such that $\pi(d, x) = 0$, $\forall x \in \mathcal{X}$.*

Let $d \in \mathcal{D}$ denote an alternative after which choices are not restricted in the subsequent period. Let $r \in \mathcal{D}$ denote an alternative after which choices are potentially restricted to the set $\widetilde{\mathcal{D}}(r) \subset \mathcal{D}$. Consider the following observed choice probabilities:

$$\Pr\left(i' \notin \widetilde{\mathcal{D}}(r) \,|d, x', x\right) = \mathrm{GPr}\left(i' \,|\mathcal{D}, x'\right),$$
$$\Pr\left(i' \notin \widetilde{\mathcal{D}}(r) \,|r, x', x\right) = \left(1 - \pi(r, x)\right) \mathrm{GPr}\left(i' \,|\mathcal{D}, x'\right).$$

Dividing the two equations and rearranging leads to

$$\pi(r, x) = 1 - \frac{\Pr\left(i' \notin \widetilde{\mathcal{D}}(r) \,|r, x', x\right)}{\Pr\left(i' \notin \widetilde{\mathcal{D}}(r) \,|d, x', x\right)},$$

identifying the restriction probability $\pi(r, x)$. In this manner, and with the help of the unrestricted choice set after choosing d, it is possible to recover all restriction probabilities from the data.

Restriction Probability Assumption Set 2. *There exists an alternative $d \in \mathcal{D}$, the subsequent restricted choice set $\widetilde{\mathcal{D}}(d)$ of which includes only one choice $s \in \mathcal{D}$. The respective restriction probability $\pi(d,x)$ is known for all states $x \in \mathcal{X}$.*

Let $d \in \mathcal{D}$ denote the choice related to a restricted choice set that includes only one alternative $s' \in \widetilde{\mathcal{D}}(d)$. The observed choice probability of a choice $i' \neq s$ is

$$\Pr\left(i' \notin \widetilde{\mathcal{D}}(d) \middle| d, x', x\right) = (1 - \pi(d,x)) \operatorname{GPr}(i' | \mathcal{D}, x').$$

Knowing $\pi(d,x)$ makes it possible to recover $\operatorname{GPr}(j' | \mathcal{D}, x')\ \forall j' \notin \widetilde{\mathcal{D}}(d)$. Because $\widetilde{\mathcal{D}}(d)$ is a singleton, it is also possible to recover $\operatorname{GPr}(s' | \mathcal{D}, x')$ for choice $s' \in \widetilde{\mathcal{D}}(d)$:

$$\operatorname{GPr}(s' | \mathcal{D}, x') = 1 - \sum_{j' \notin \widetilde{\mathcal{D}}(d)} \operatorname{GPr}(j' | \mathcal{D}, x') \text{ with } s' \in \widetilde{\mathcal{D}}(d).$$

With the help of the genuine choice probabilities, all other restriction probabilities can be identified using the observed choice probabilities of an alternative not included in the restricted choice set:

$$\pi(k,x) = 1 - \frac{\Pr\left(i' \notin \widetilde{\mathcal{D}}(k) \middle| k, x', x\right)}{\operatorname{GPr}\left(i' \in \mathcal{D} \middle| \mathcal{D}, x'\right)}, \quad \forall k \notin \widetilde{\mathcal{D}}(d).$$

Restriction Probability Assumption Set 3. *Each restricted choice set $\widetilde{\mathcal{D}}(d)$, $d \in \mathcal{D}$, excludes at least one choice $j' \in \mathcal{D}$ that is also excluded in another restricted choice set. Additionally, at least one restriction probability is known.*

The observed choice probability of alternative $k' \notin \widetilde{\mathcal{D}}(d)$ after having selected d in the previous period is

$$\Pr(k' | d, x', x) = (1 - \pi(d,x)) \operatorname{GPr}(k' | \mathcal{D}, x').$$

As long as k' is also excluded in another set $\widetilde{\mathcal{D}}(j)$ with $j \neq d$, it is possible to divide the observed choice probabilities for the two choices j and d in the previous period:

$$\frac{\Pr\left(k' \,|d, x', x\right)}{\Pr\left(k' \,|j, x', x\right)} = \frac{\left(1 - \pi\left(d, x\right)\right)}{\left(1 - \pi\left(j, x\right)\right)}.$$

Since it is assumed that one restriction probability is known, it is possible to derive the restriction probabilities for all choices $d \in \mathcal{D}$.

Appendix B: Genuine Choice Probabilities

Lemma 1 states conditions sufficient to identify all genuine choice probabilities from the observed choice and restriction probabilities.

Lemma 1. *Given the data* $\{\Pr\left(d' \,|d, x', x\right), \pi\left(d, x\right); d', d \in \mathcal{D}, x', x \in \mathcal{X}\}$*, one of the following conditions is sufficient to uniquely determine the genuine choice probabilities* $\mathrm{GPr}\left(d' \,\middle|\widetilde{\mathcal{D}}\left(d\right), x'\right)$*, for all* $(d', x') \in \widetilde{\mathcal{D}}\left(d\right) \times \mathcal{X}$*:*

1. *At least one restriction probability is zero for all states* $x \in \mathcal{X}$*.*
2. *All alternatives, or all but one, are excluded at least once from one of the restricted choice sets* $\widetilde{\mathcal{D}}\left(d\right)$ *with* $d \in \mathcal{D}$*.*
3. *There exist two identical restricted choice sets* $(\widetilde{\mathcal{D}}\left(l\right) = \widetilde{\mathcal{D}}\left(r\right))$ *with different restriction probabilities* $(\pi\left(l, x\right) \neq \pi\left(r, x\right)$ *for all* $x \in \mathcal{X})$*.*
4. *The restriction probabilities for alternatives, which are common among all restricted choice sets, differ across these alternatives.*

Proof. The observed choice probabilities have one of these two forms:

$$\Pr\left(d' \,|d\right) = \left(1 - \pi\left(d\right)\right) \mathrm{GPr}\left(d' \,|\mathcal{D}\right) \qquad \text{if } d' \notin \widetilde{\mathcal{D}}\left(d\right) \tag{A.15}$$

or

$$\Pr\left(d' \,|d\right) = \left(1 - \pi\left(d\right)\right) \mathrm{GPr}\left(d' \,|\mathcal{D}\right) + \pi\left(d\right) \mathrm{GPr}\left(d' \,\middle|\widetilde{\mathcal{D}}\left(i\right)\right) \quad \text{if } d' \in \widetilde{\mathcal{D}}\left(d\right), \tag{A.16}$$

where x' and x are dropped to ease the exposition.

If a genuine choice probability of choosing d' from the general set is identified, the respective genuine choice probabilities of choosing d' from any of the restricted choice sets are uniquely determined by (A.16). Thus, to prove lemma 1, it is sufficient to demonstrate the identification of the genuine choice probability when choosing from the unrestricted choice set $\mathcal{D}$.

Proof of condition 1: The observed choice probabilities after choosing an alternative without a subsequent restriction are equal to their respective genuine choice probability of choosing from $\mathcal{D}$.

Proof of condition 2: For an alternative that is at least once excluded from a restricted choice set, there exists an observed choice probability that is takes the form of (A.15). This directly identifies its genuine choice probability. If one alternative (d') is included in all restricted choice sets, its genuine choice probability can be identified from the genuine choice probabilities of all the other alternatives

$$\mathrm{GPr}\left(d' \,|\mathcal{D}\right) = 1 - \sum_{l' \in \mathcal{D}} \mathrm{GPr}\left(l' \,|\mathcal{D}\right).$$

Thus, for models in which all alternatives but one are at least once excluded from a restricted choice set, all genuine choice probabilities are identified. Note that this includes cases in which one restricted choice set is a singleton.

Proof of condition 3: Without the loss of generality, assume that $\widetilde{\mathcal{D}}(j) = \widetilde{\mathcal{D}}(i) = \widetilde{\mathcal{D}}$. For alternatives included in $\widetilde{\mathcal{D}}$, the observed choice probabilities are

$$\begin{aligned}
\Pr\left(d' \,|j\right) &= \left(1 - \pi\left(j\right)\right) \mathrm{GPr}\left(d' \,|\mathcal{D}\right) + \pi\left(j\right) \mathrm{GPr}\left(d' \,\middle|\widetilde{\mathcal{D}}\right) \\
\Pr\left(d' \,|i\right) &= \left(1 - \pi\left(i\right)\right) \mathrm{GPr}\left(d' \,|\mathcal{D}\right) + \pi\left(i\right) \mathrm{GPr}\left(d' \,\middle|\widetilde{\mathcal{D}}\right).
\end{aligned}$$

The two linear equations are independent as long as $\pi(j) \neq \pi(i)$. The genuine choice probability of each alternative included in $\widetilde{\mathcal{D}}$ is identified by

$$\mathrm{GPr}(d'|\mathcal{D}) = \frac{\pi(j)\Pr(d'|i) - \pi(i)\Pr(d'|j)}{\pi(j) - \pi(i)}$$

For each alternative not included in $\widetilde{\mathcal{D}}$, an equation in the form of (A.15) exists that identifies the respective genuine choice probability.

Proof of condition 4: Denote the set of alternatives that are common among all restricted choice sets by $\hat{\mathcal{D}} = \{\hat{d}_1, \ldots, \hat{d}_{\hat{N}}\}$. Denote the set of alternatives that are part of $\widetilde{\mathcal{D}}(d)$ but not common among all sets by $\check{\mathcal{D}}(d) = \{\check{d}_1^d, \ldots, \check{d}_{\check{N}^d}^d\}$. The following system of equations can be derived for each $d \in \mathcal{D}$:

$$\Pr\left(\hat{d}_1 \in \hat{\mathcal{D}}|d\right) = (1 - \pi(d))\,\mathrm{GPr}\left(\hat{d}_1|\mathcal{D}\right) + \pi(d)\,\mathrm{GPr}\left(\hat{d}_1\middle|\widetilde{\mathcal{D}}(d)\right) \tag{A.17.1}$$

$$\vdots$$

$$\Pr\left(\hat{d}_{\hat{N}-1} \in \hat{\mathcal{D}}|d\right) = (1 - \pi(d))\,\mathrm{GPr}\left(\hat{d}_{\hat{N}-1}|\mathcal{D}\right) + \pi(d)\,\mathrm{GPr}\left(\hat{d}_{\hat{N}-1}\middle|\widetilde{\mathcal{D}}(d)\right) \tag{A.17.N-1}$$

$$\begin{aligned}\Pr\left(\hat{d}_{\hat{N}} \in \hat{\mathcal{D}}|d\right) = {} & (1 - \pi(d))\left[1 - \sum_{l=1}^{\hat{N}-1}\mathrm{GPr}\left(\hat{d}_l|\mathcal{D}\right) - \sum_{j\notin\hat{\mathcal{D}}}\mathrm{GPr}(j|\mathcal{D})\right] \\ & + \pi(d)\left[1 - \sum_{l=1}^{\hat{N}-1}\mathrm{GPr}\left(\hat{d}_l\middle|\widetilde{\mathcal{D}}(d)\right) - \sum_{l=1}^{\check{N}^i}\mathrm{GPr}\left(\check{d}_l^d\middle|\widetilde{\mathcal{D}}(d)\right)\right].\end{aligned} \tag{A.18}$$

The system consists of $\hat{N}$ independent equations. (A.17.1) – (A.17.N-1) feature two unknowns each: $\mathrm{GPr}\left(\hat{d}|\mathcal{D}\right)$ and $\mathrm{GPr}\left(\hat{d}\middle|\widetilde{\mathcal{D}}(d)\right)$. (A.18) does not include any additional unknowns, as choices not included in $\widetilde{\mathcal{D}}(d)$ can be directly identified with an equation similar to (A.15). In total, the system includes $2\left(\hat{N} - 1\right)$ unknowns. An additional system for a choice $k \neq d$ only adds $\hat{N} - 1$ unknowns because the genuine choice probabilities choosing from $\mathcal{D}$ are already included in the system of choice d. Furthermore, each additional system adds $\hat{N}$ independent equations as long as $\pi(k) \neq \pi(d)$. In total, there are $(1 + J)\left(\hat{N} - 1\right)$ unknowns and $J\hat{N}$ independent equations, leading to $(J - N + 1)$ more equations than unknowns. Thus, as long as the restriction probabilities differ for alternatives that are common among all restricted choice sets, all genuine choice probabilities can be recovered.

■

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