# Checking the Evidence for Declining Discount Rates

Szekeres, Szabolcs

IID Gazdasági Tanácsadó Kft

4 August 2020

**Checking the Evidence for Declining Discount Rates**

by Szabolcs Szekeres*

**Abstract:** A numerical model is used to experimentally compute certainty equivalent discount rates (CERs) of risk neutral and risk-averse decision makers. Investors are characterized by utility functions of the constant-intertemporal-elasticity-of-substitution (CIES) type. Stochastic interest rates are generated using a Cox, Ingersoll & Ross (CIR) type model, calibrated to 1992-2017 US three-month Treasury Bill rates. The paper replicates empirical studies providing evidence for declining discount rates (DDRs) and tests claims regarding risk averse CERs in a descriptive discounting context. It is shown that DDRs as proposed by Weitzman are based on a fallacy. The reviewed papers seeking empirical evidence of DDRs repeat the mistake. Risk averse CERs can be decline with time because of portfolio effects. If these are low, risk averse CERs are slightly lower than risk neutral ones but not secularly declining.

**Keywords:** Weitzman-Gollier puzzle; declining discount rates; discounting

**JEL classification:** D61; H43

The literature on the Weitzman-Gollier puzzle, based on Weitzman (1998 and 2001) and Gollier (2004), centered on the question on whether certainty equivalent discount rates should be growing or declining functions of time in capital markets with autocorrelated interest rates. It "did not converge to a consensus" according to Gollier (2016), despite the claimed resolution of the puzzle by Gollier and Weitzman (2010).

On the one hand, a number of papers attempted to lend support to Weitzman's prescription of declining discount rates (DDR) through empirical investigations that mainly focused on the observable autocorrelation of market interest rates, given that such autocorrelation is a sine-qua-non assumption of Weitzman's DDR prescription.

On the other hand, many papers sought a solution of the Weitzman-Gollier puzzle outside its original risk-neutral context and shifted the focus of discussion

* Independent researcher
https://orcid.org/0000-0003-3903-5377

to risk-averse investors. Gollier and Weitzman (2010) analyze the utility preserving certainty equivalent discount rate for a decision maker with a constant-intertemporal-elasticity-of-substitution (CIES) type utility function. Going further, Gollier (2016) claimed that "the risk-neutrality assumption underlying the two discounting rules is technically incompatible with an uncertain interest (or discount) rate […] Thus, in order to reconcile the basic ingredient of the gamma discounting approach (*i.e.*, uncertain interest rates with economic theory), a model with a risk-averse representative agent must be considered."

These strands of research were instrumental in the adoption of the rule of declining discount rates by the UK, France, and Norway. The OECD (2018) Cost-Benefit Analysis manual cites Weitzman's work and states "it can be shown that the certainty-equivalent decreases with time."

Referring to the familiar descriptive and prescriptive classification of approaches to discounting, William Nordhaus (2019) observed that the debate about discounting is "just as unsettled as it was when first raised three decades ago." It is useful to keep the distinction of the two approaches in mind, because often pronouncements are made in the Weitzman-Gollier puzzle literature in the context of one of the approaches which may not necessarily be valid when transposed to the other.

Weitzman's model belongs to the descriptive discounting approach because the proposed DDRs are meant to measure the opportunity cost of investment projects by reference to yields in capital markets. Some of the ensuing papers, however, belong to the prescriptive discounting approach. A clear example of this is Gollier (2016), which states "In this paper, we refer to the discount rate as the rate at which future changes in consumption are discounted" and "If the rate of pure preference for the present is not zero, all interest rates discussed in this paper should simply be reinterpreted as net of this rate." Therefore, the interest rates referred to are not market interest rates, but rather "efficient discount rates" which in that paper's terminology are the utility preserving certainty equivalent rates for the representative agent described in the paper.

The purpose of defining a certainty equivalent hurdle rate (the minimum return a project should yield for it to be worth undertaking) in Gollier (2016) is to determine if the representative agent should fund the project at the expense of present consumption, for the sake of the investment induced increase in future consumption. This is essentially a savings decision. Should the agent move away from his initial consumption path? This is why Gollier (2016) noted that interest rates are to be taken to be net of the rate of pure time preference, as the hurdle rate should compensate for that as well.

In the descriptive discounting approach, however, the pure rate of time preference plays no role because the aim of discounting under this approach is to determine the opportunity cost of investing in a given project. In this decision setting the investor has already decided to forgo some current consumption to make an investment, so he is not evaluating a savings decision. What he wants to know is whether the project in question will be a better investment than an alternative, assumed in this paper, as in all of Weitzman's, to be the capital market. As both the project under review and the equivalent investment in the capital market will result in consumption increases in the same future period, the pure rate of time preference, whatever its value may be, makes no difference in the comparison, so it can be ignored.

The purpose of this paper is to examine the behavior of the certainty equivalents of uncertain interest rates of capital markets, following the descriptive approach, as assumed in Weitzman (1998 and 2001), from the point of view of both risk-neutral and risk-averse investors.

The empirical studies seeking to corroborate Weitzman's DDR conclusions concentrated on measuring the degree of interest rate autocorrelation that is a key assumption of Weitzman's DDR prescription. We will replicate their efforts in this paper. The replicated papers all assumed risk-neutral investors, so we will check if risk neutral CERs are declining as claimed in those papers. But subsequently we will also examine the behavior of risk-averse CERs and try to replicate some of the findings of the literature that is based on that assumption, thereby testing the validity of its claims in the context of the descriptive discounting approach.

We do this experimentally with the help of a numerical model specifically built for this purpose. Stochastic market interest rates are generated using Monte Carlo simulation of a model of the Cox, Ingersoll & Ross (CIR) type. The model is used to compute certainty equivalent discount rates of risk neutral and risk-averse decision makers under a variety of alternative assumptions.

We consider that the simulated interest rates of the CIR interest rate model are used by the modeled investors as a subjective forecast of future interest rates, used by them to identify actions that maximize their expected utility. Because all present values and the corresponding CERs are computed by numerical methods, that is, searching for the present value that will compound to the safe or expected certainty equivalent future value specified, our analysis will show conclusively what the correct risk neutral present value calculation method is. For risk averse CERs the model permits checking the validity, in a more realistic context, of claims made by papers assuming very special circumstances, such as decision makers living in the Lucas tree economy or having special clairvoyance.

This paper proceeds as follows. Section 1 describes the simulation model that is used as a tool to compute CERs under various circumstances. Section 2 addresses the question of whether risk neutral CERs are declining and reviews some of the empirical studies that sought to estimate the degree of decline that is supported by evidence. Section 3 computes optimal CERs for risk averse investors of different degrees of risk aversion, distinguishing two cases: one, in which there is an initial stock of wealth to be allocated between the present and the future, and another in which the future period also has a source of wealth, and therefore borrowing against it is possible. Section 4 contains conclusions.

Appendix A describes the simulation model and its operation in greater detail. The model, created in Excel, can be downloaded[1] and the reader can replicate the results of this paper or analyze alternative cases.

# 1. The simulation model

## 1.1 Interest rates and compound factors

The simulation model uses Monte Carlo simulation to generate up to 10,000 interest rate scenarios from a model of the Cox, Ingersoll & Ross (CIR) type, with parameters that were calibrated by Yajie Zhao and Boru Wang (2017) with reference to a monthly data series of US three-month Treasury Bill rates spanning the period 1992 to 2017. Interest rates are generated monthly for 133 months, but only the month 13 interest rate distribution is used. This is done to obtain a distribution that is sufficiently distant from the model's initial conditions, but that is unaffected by the mean and variance drift inherent in the recursive model. The entire series is used, however, to measure the degree of correlation over alternative compounding periods that can be specified.

Once a compounding frequency is selected (from 1 to 120 months), a new interest rate distribution is generated for each compounding period. These distributions have the cumulative distribution of the standard month-13 distribution and are correlated to the distribution of the previous compounding period by the correlation coefficient that is observed for the time distance between them within the 133 monthly distributions simulated.

Based on this set of distributions, compound factors are calculated for each compounding period and scenario, information that is used for the optimization decisions of the investors modeled. The model calculates results for 30 different

[1] https://doi.org/10.3886/E120537V1

years. If the compounding period is 12 months or less, results will be shown for years 1-30. If the compounding period is 120 months, results for years 10-300 will be shown, 10 years apart. Figure 1 describes the distributions of CERs (certainty equivalent rates) of the compound factors simulated in the latter case.

Figure 1
**Frequency distributions of compound factor CERs**
(compounding frequency 120 months)

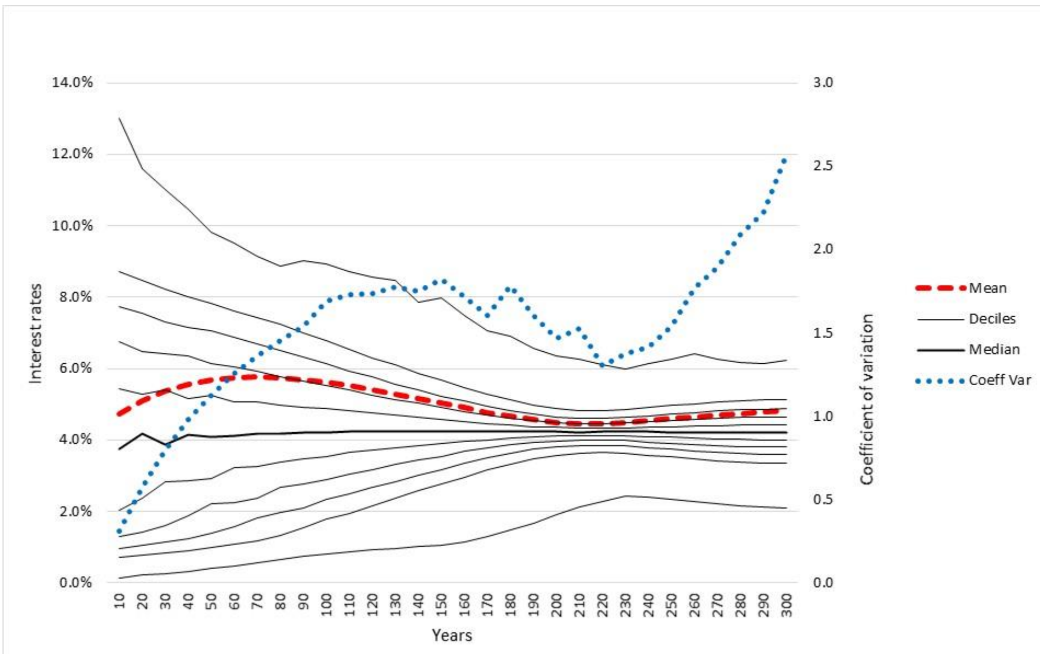



Figure 1 shows the limits of the deciles of the distributions and their expected values. In this case the correlation coefficient between interest rate distributions used in consecutive compounding periods is 0.77. It is this fact that explains the initial rise in the expected values of CERs. The coefficients of variation shown are those of the compound factors, not of the CERs derived for them. It should be borne in mind that the compound factors grow exponentially.

## 1.2 Investors modeled

The investment decision optimization framework adopted in this paper is like the one proposed in Gollier and Weitzman (2010), as described in Szekeres (2017). We assume that investors aim to maximize a welfare function of the following form:

$$V(C) = \sum_0^t \frac{U(C_t)}{(1+\rho)^t} \tag{1}$$

where $\rho > 0$ is the pure rate of time preference and $U(C_t)$ is a utility function of the constant-intertemporal-elasticity-of-substitution (CIES) type:

$$U(C) = \frac{C^{1-\sigma} - 1}{1-\sigma} \tag{2}$$

where consumption $C > 0$, and the elasticity of marginal utility with respect to consumption $\sigma > 0$ but not equal to 1. This is also the measure of the decision maker's constant proportional risk-aversion. When $\sigma = 1$, the utility function takes the form $U(C) = ln(C)$. Notice that when $\sigma = 0$, which defines risk-neutrality, expression (2) effectively becomes $U(C) = C$.

The optimization will only involve two time periods, the present and time $t$, which will be calculated alternatively for the 30 periods of the model.

The above utility function will be maximized in two alternative forms. In one we follow Gollier and Weitzman (2010) and assume that inherited capital $K_0$ is the investors' only source of income. Then $C_t$, consumption in time $t$, will be a function of $C_0$, the amount consumed at $t = 0$, and the uncertain yield of investing $K_0 - C_0$ at the stochastic interest rate $r_i$ of the capital market, where the subscript $i$ refers to the up to 10,000 interest rate scenarios simulated. We can therefore only specify $C_t$ as an expectation:

$$E[C_t] = E[(1 + r_i)](K_0 - C_0) \tag{3}$$

The optimization problem to solve is to maximize (1) taking budget constraint (3) into account. This problem has a separate analytical solution, shown in Szekeres (2017), for each possible interest rate $r_i$, and to make that possible Gollier and Weitzman (2010) make the implausible assumption that the uncertainty about $r$ is resolved an instant after the investment decision is made, so that the savings optimization can take place without uncertainty. We do not make that assumption in this paper, however, but rather find the savings amount that results in the highest expected welfare through numerical methods. Our model finds, through an iterative process, the value of $C_0$ that maximizes (1) and shows the resulting optimal savings amount.

In a second calculation form we assume that instead of investors only having an initial endowment $K_0$, they have an initial wealth level $W_0$ which grows exogenously to $W_t = W_0 (1+g)^{t,}$ where $g$ is the annual growth rate of wealth, which may be uncertain. In that case the budget constraint becomes:

$$E[C_t] = E[(1 + r_i)](W_0 - C_0) + E[W_0(1 + g)^t] \tag{4}$$

The level of savings at $t = 0$ is solved by numerical methods in this case as well.

It is important to realize that in Gollier and Weitzman (2010) investors are always on the utility maximizing consumption path thanks to clairvoyance being bestowed upon them just after the decision whether to invest in a very small project has been made. In other words, the interest rate uncertainty only applies to the investment decision and not to the optimal savings decision. In our case, in contrast, investors will not be on the optimal consumption path they would have chosen had they been clairvoyant. Rather, they will be as they would be in reality: living in the state of the world that actually occurred, having taken the best possible decision with the information they had when they had to decide.

The model simulates four decision makers characterized by $\sigma = 0$, $\sigma = 0.5$, $\sigma = 1$, and $\sigma = 1.5$. The latter three values can be modified in the accompanying Excel workbook.

## 2. Risk-neutral investors

Gollier *et al* (2008) "Declining Discount Rates: Economic Justifications and Implications for Long-Run Policy" is an excellent starting point for checking the evidence for DDRs under the risk neutrality assumption, both because it treats separately the prescriptive and descriptive discounting methods and because it gives numerical examples of the latter, which can be readily verified.

Gollier *et al* (2008) imply that the certainty equivalent discount factor (the present value of €1) at time *t* is the following when short term interest rates *r* are perfectly autocorrelated in time:

$$D(t) = E[e^{-rt}] \tag{4}$$

A numerical example is provided: "the rate could be either 3 percent or 5 percent with equal probability. Note that the average expected rate is 4 percent (=0.5*0.03+0.5*0.05). In this case, the expected PV of €1,000 received after *t* years is $0.5*1000*e^{-0.03t}+0.5*1000*e^{-0.05t}$." In their Table B1 the result of this calculation for *t* =100 is stated to be €28.2625.

If €28.2625 is the present value of €1,000 received after 100 years, then, according to the definition of present value, €28.2625 should compound back to

€1,000 using the same interest rate probabilities. Verifying, we get $0.5*28.2625*e^{0.03t} + 0.5*28.2625*e^{0.05t} = 2{,}381.0972$ when $t = 100$.

Therefore €28.2625 is *not* the present value of €1,000 under the conditions stated. A shown in Szekeres (2013) the correct present value can be readily derived from the expected future value, using the definition of present value:

$$PV\ (0.5e^{0.03\ 100} + 0.5e^{0.05\ 100}) \equiv 1{,}000 \tag{5}$$

$$PV = \frac{1000}{0.5e^{0.03\ 100}+0.5e^{0.05\ 100}} = 11.8695 \tag{6}$$

Thus the correct PV is not the one calculated by Gollier *et al* (2008). The basic problem lies with definition (4), which is a fallacy, as it does not comply with the definition of present value[2]. The correct present value of any future value *FV(t)* is:

$$PV(t) = \frac{FV(t)}{E[e^{rt}]} \tag{7}$$

Expression (4) overstates the correct present values (7). It is generally true that

$$\sum p_i e^{-rt} > \frac{1}{\sum p_i e^{rt}} \tag{8}$$

Because the fallacy in Weitzman's proposition lies in assuming that the expectation of the inverses is the same as the inverse of the expectation, we can use a well-known statistical relationship to measure the difference between the correct and incorrect ways of computing PVs. Let random variable $X$ be $e^{rt}$ and random variable $Y$ be $1/e^{rt}$. The expected values of X and Y relate as follows:

$$E[XY] = \ E[X]E[Y] + cov(X,Y) \tag{9}$$

As $E[XY] = 1$ because $Y$ is the reciprocal of $X$, we can rewrite (9) as follows:

$$E[Y] = \ \frac{1 - cov(X,Y)}{E[X]} \tag{10}$$

Which becomes the following if we replace X and Y by what they stand for:

$$\sum p_i e^{-rt} = \ \frac{1 - cov(\,e^{rt},\ e^{-rt})}{\sum p_i e^{rt}} \tag{11}$$

[2] For an explanation of what Weitzman discounting really computes see Szekeres (2019)

This is illustrated in the following table, taking the Scenario B data from Gollier *et al* (2008), from which the above numerical example was also taken:

Table 1
**Relationship between the correct and incorrect present values of €1**

| $t$ | E[exp(-rt]] | E[exp($rt$]] | Correct PV | cov($e^{rt}$, $e^{-rt}$) | Incorrect result |
|---|---|---|---|---|---|
| (1) | (2) | (3) | (4) | (5) | (6) |
| 1 | 0.960837 | 1.040862815 | 0.960741 | -0.0001 | 0.960837 |
| 10 | 0.673674 | 1.499290039 | 0.666982 | -0.01003 | 0.673674 |
| 50 | 0.152608 | 8.332091516 | 0.120018 | -0.27154 | 0.152608 |
| 100 | 0.028263 | 84.24934801 | 0.01187 | -1.3811 | 0.028263 |
| 150 | 0.005831 | 949.0297729 | 0.001054 | -4.53383 | 0.005831 |
| 200 | 0.001262 | 11214.94729 | 8.92E-05 | -13.1541 | 0.001262 |
| 300 | 6.19E-05 | 1638560.228 | 6.1E-07 | -100.358 | 6.19E-05 |
| 400 | 3.07E-06 | 242663975.1 | 4.12E-09 | -744.74 | 3.07E-06 |

Column (1) shows the years displayed in Table 2B of Gollier *et al* (2008), column (2) contains the latter's corresponding Scenario B PVs divided by 1,000, to make the future value equal to €1; column (3) contains the corresponding compound factors; column (4) contains the reciprocals of the values in column (3), which are therefore the correct PVs. Column (5) contains the covariances as defined in the text above, while column (6) contains the calculated incorrect results, as in expression (11) above in the text. Notice that column (6) values equal column (2) values.

For either the correctly or incorrectly calculated present values, certainty equivalent discount rates (CER) can be computed by the following expression:

$$CER(t) = -\frac{ln(PV(t))}{t} \tag{11}$$

The CERs corresponding to the correct and incorrect PVs are plotted in the following Figure 2. These correspond to the values in columns (4) and (6) of Table 1, respectively.

Figure 2
**Correct and Incorrect Certainty Equivalent Rates**

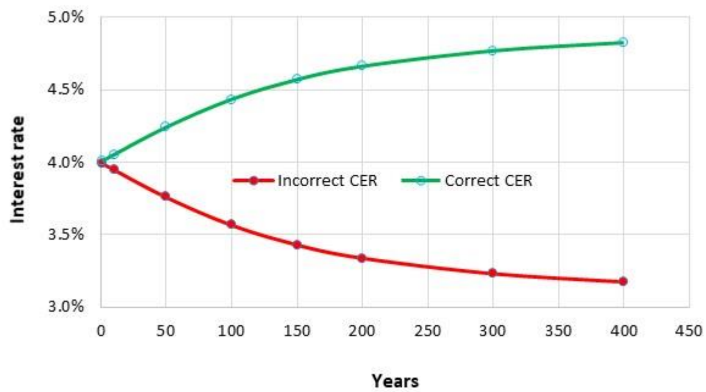
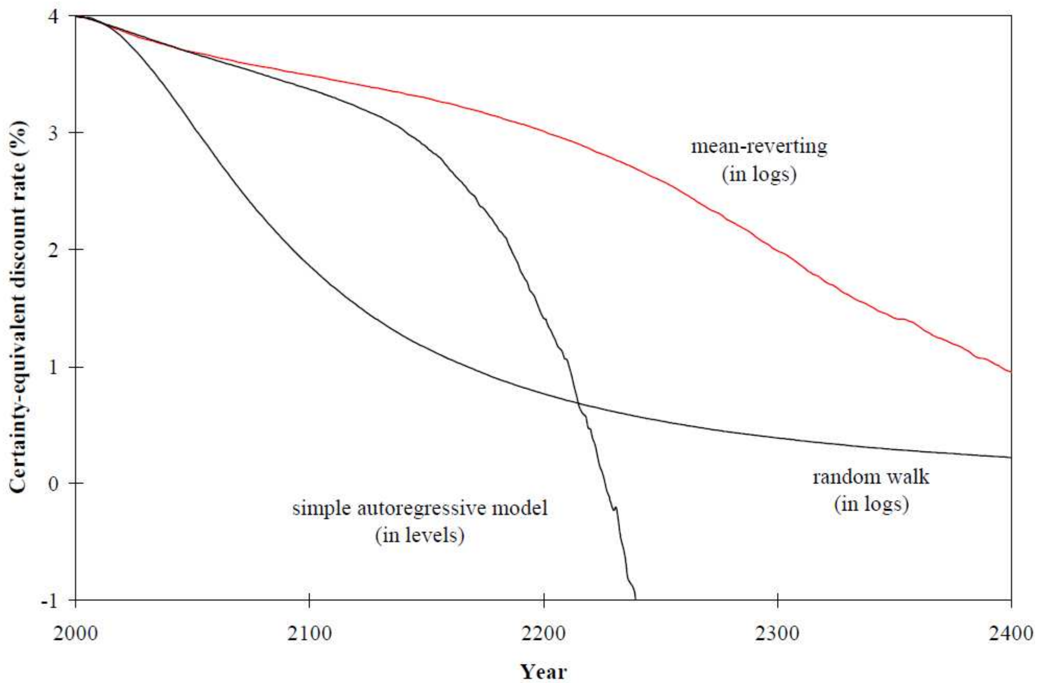



Figure B2 in Gollier *et al* (2008) shows a CER corresponding to its Scenario B that is like the incorrect CER plotted above. The discrepancy between the correct and incorrect CERs is due to the to the incorrect definition of present value on which the latter is based. If interest rates are perfectly autocorrelated, CERs are a growing, not a declining function of time.

Those who embraced the DDR recommendation realized that the perfect autocorrelation assumption is not realistic, so several investigations were launched into the degree of autocorrelation that could be supported empirically. Possibly the earliest to be undertaken was Newel and Pizer (2000). It estimates three models of interest rate behavior: random walk, mean reverting and simple autoregressive, and based on the results obtained, computes the certainty equivalent discount rates shown in Figure 3.

Figure 3
**Newel and Pizer (2000) forecast alternative CERs**



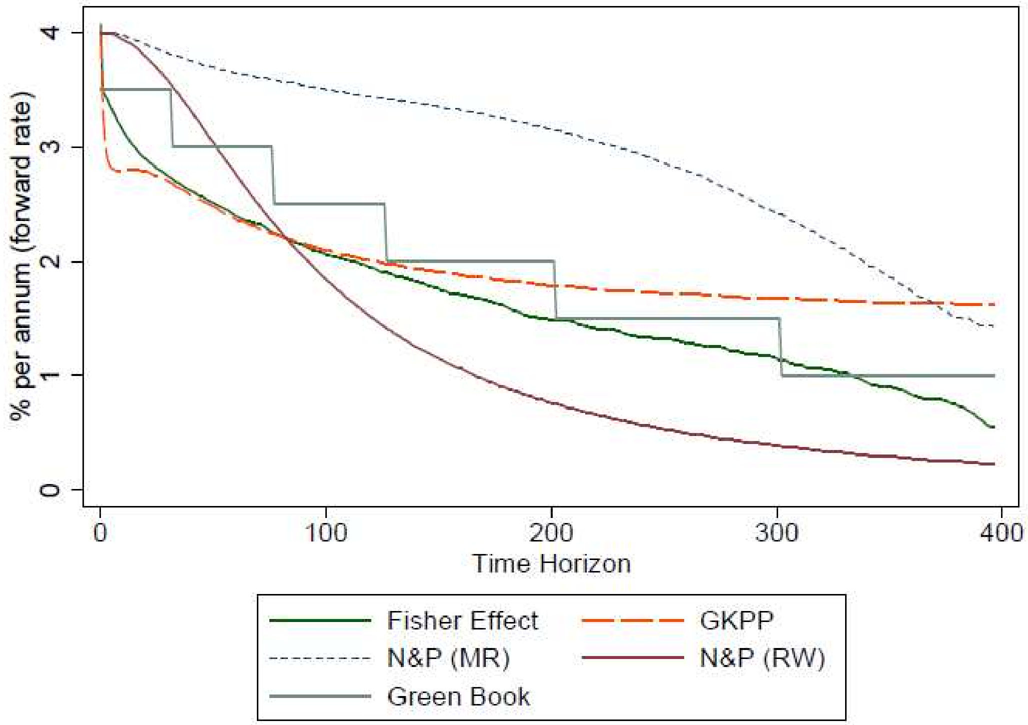

We do not go into the details of the models employed, because it is made clear in Newel and Pizer (2000) that the computation of CERs was done using the fallacious formula proposed by Weitzman (1998). Therefore, the derived results cannot be right. A numerical example in the introductory text, also used in Gollier *et al* (2008), makes this unambiguously clear. Had the correct PV formula been used instead, the CERs in Figure 3 would have been increasing with time, not declining.

Groom *et al* (2007), focusing on finding a sufficiently reliable DDR estimate that could inform public policy decisions, tests five alternative models to forecast interest rates: mean reverting, random walk, AR IGARCH, regime switching and state space. In calculating CERs Groom *et al* (2007) first derives discount factors following the fallacious Weitzman (1998) formula, but then commits another discounting error by defining CERs as the marginal changes in discount factors: "Following Weitzman (1998) we define (1) as the certainty equivalent discount factor, and the corresponding certainty-equivalent forward rate for discounting between adjacent periods at time $t$ as equal to the rate of change of the expected discount factor." Such certainty equivalent discount factors would constitute a violation of the definition of present value even if computed from correct discount factors because the PVs computed thereby would not compound back to the originally discounted future amount. Their numerical results, presented in tabular form, are therefore incorrect and not quoted here.

Freeman *et al* (2015) "rather than following previous work which used a single series of real U.S. Treasury bond returns, [treats] nominal interest rates and inflation as cointegrated series and [estimates] the empirical term structure of discount rates via the `Fisher Effect'. This nests previous empirical models and is more flexible. It also addresses an irregularity in previous work which used data on nominal interest rates until 1950, and real interest rates thereafter. As [it shows], the real and nominal data have very different time series properties. This paper therefore provides a robustness check on previous discounting advice and updated methodological guidance at a time when governments around the world are reviewing their guidelines on social discounting."

Figure 4 presents its results and compares them to that of others, including those of Newel and Pizer (2000) and Groom *et al* (2007), in addition to showing the guidance adopted by HM Treasury Green Book (2003).

Figure 4
**Freeman *et al* (2015) forecast alternative CERs**




The problem with these conclusions is the same as before. The model results are fed into the discounting formula of Weitzman (1998). The use of the Weitzman CER formula naturally results in declining discount rates, instead of the increasing ones that would have been obtained had the calculation method dictated by the definition of present value been used.

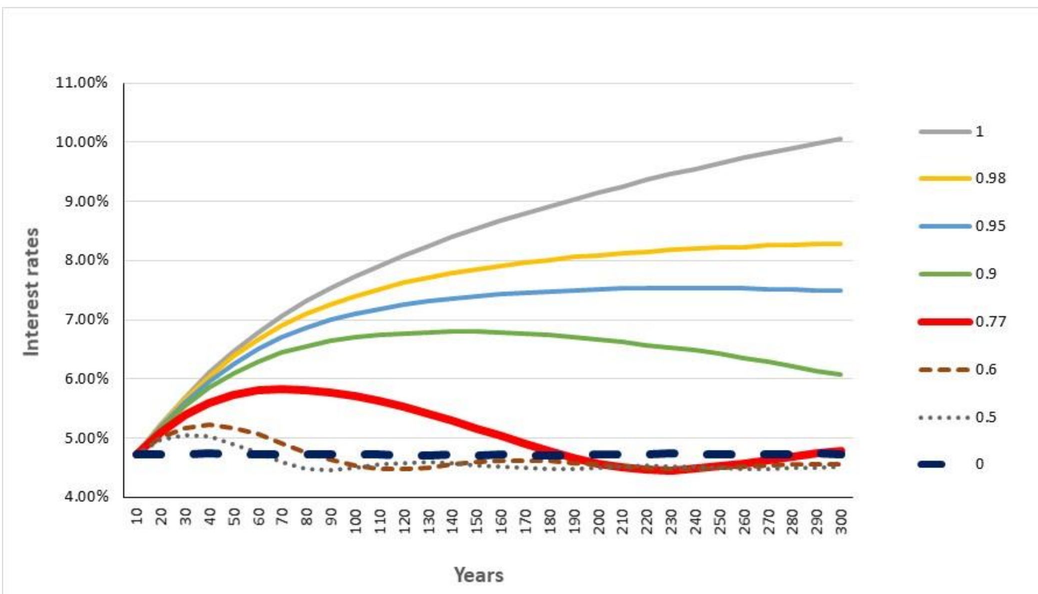
As we have seen in the foregoing, all the cited papers, seeking empirical evidence of DDRs, used a calculation method that guarantees it. The only question of substance on which they differ is the choice of data and modeling techniques used to measure interest rate autocorrelation.

There is neither a point nor an easy way to replicate the cited studies while correcting only their calculation method, but this paper will fulfill the intention of their authors by showing how correctly calculated CERs behave as a function of alternative degrees of autocorrelation.

Our results, incidentally, corroborate that the correct calculation method indeed computes the right PV, because our model does not calculate PVs analytically, but rather finds them through numerical methods. The fact that the results so found exactly correspond to those derived from expected compound factors, however, corroborate the conclusions of the discussion at the beginning of this section.

The following Figure 5 shows simulated risk neutral CERs as a function of the compounding factor autocorrelation coefficients indicated in the figure. In the case of perfect correlation, as assumed by Weitzman, CERs are growing functions of time and tend to the highest possible interest rate because of the yield boosting effects of this level of autocorrelation. When the autocorrelation coefficient is zero, the term structure of CERs is flat. This is explained by the fact that high rates can be followed by low rates, which counters the yield boosting effect.

Figure 5
**Risk-neutral CERs as a function of compounding period correlation**




When the compounding period is set to 3 months, our CIR model yields an autocorrelation coefficient of 0.99; for 12 months it is 0.97 and for 120 months it is 0.77. As the correlation coefficient declines, the yield boosting effect fades and CERs begin to gravitate towards the expected value of interest rates.

We can conclude from the preceding that there is no evidence supporting the conclusion that the phenomenon of autocorrelated discount rates, sometimes referred to as permanent shocks to interest rates, cause risk neutral CERs to decline.

Weitzman's results are due to the use of an incorrect expected value calculation method. The cited studies, which sought to empirically corroborate DDRs, repeat the mistake.

## 3. Risk-averse investors

When calculating a CER, it is necessary to specify the consumption path of investors so that their utility functions are properly parametrized. Given interest rate uncertainty, the full consumption path cannot be established with certainty, however, given that consumption at time $t$ depends wholly or in part on the yield of investing the savings set aside at time zero. The consumption path will therefore have a known $C_o$ and an expectation at time $t$, $E[C_t]$.

Our model establishes such consumption paths by maximizing the expected value of total welfare as defined by (1) above, subject to the constraint of investors' present and/or future wealth. As already noted, we examine separately two cases: (1) in which there is only an initial wealth endowment, as in Gollier and Weitzman (2010) and $C_t$ is purely the return on the savings made at time zero, and (2) the case in which there is an additional source of wealth at time $t$.

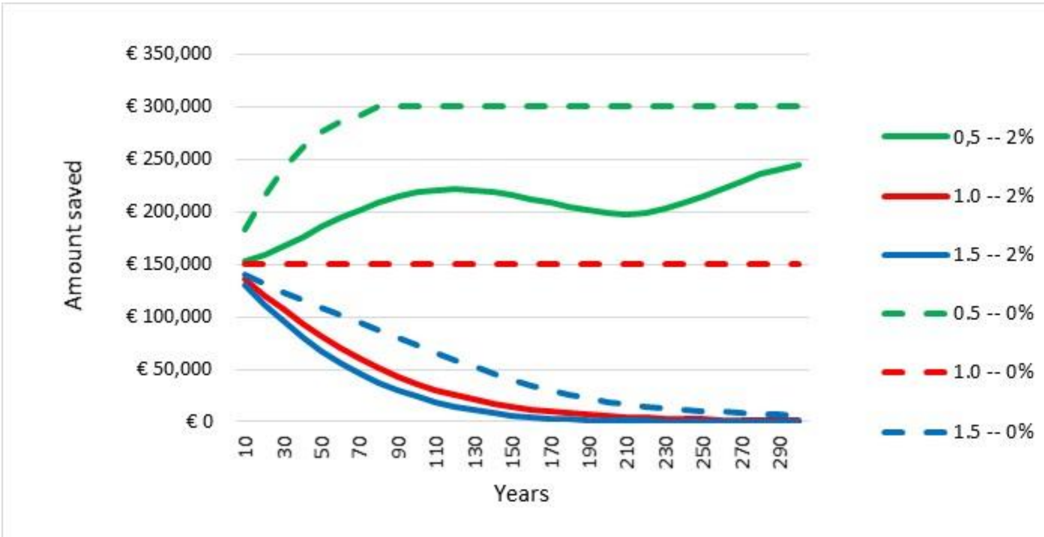
In all the simulations performed in this section we assume that compounding takes place at 120-month intervals and that the autocorrelation coefficient of interest rates over the compounding periods is 0.77.

### 3.1 Case 1, CERs of a safe asset with initial wealth endowment only

For the numerical example presented below we have assumed a wealth endowment of €300,000 and a pure rate of time preference of 2%. These numbers are arbitrary, of course. It is interesting to note, however, that the choice of endowment amount has no impact on the resulting CERs. Neither does, as expected, the pure rate of time preference. However, the rate of time preference has an important effect on the level of savings.

The following Figure 6 shows the optimal savings levels for the three degrees of risk-aversion modelled. Solid lines correspond to time preference of 2%, dashed lines to no time preference.

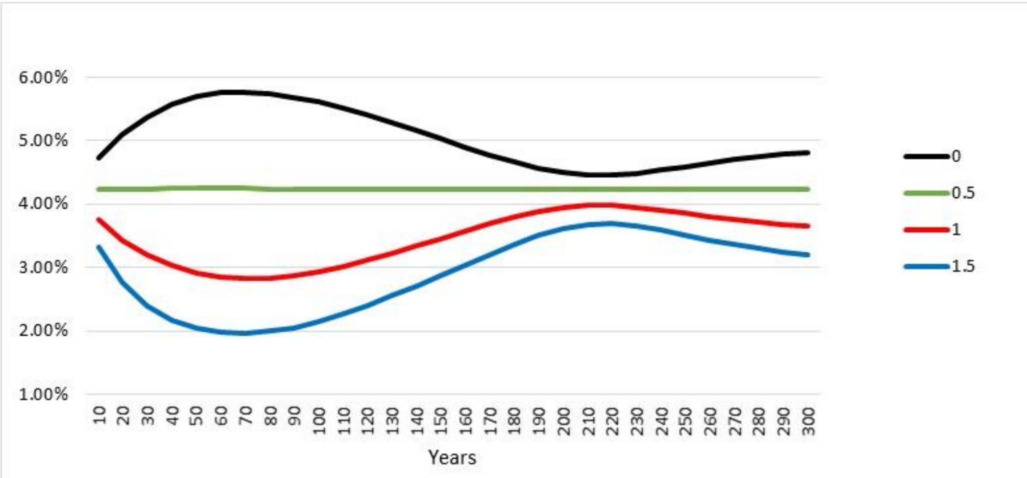
Figure 6
**Optimal savings amount for 2% and 0% time preference**




We can see that the lower the degree of risk aversion, the higher the level of savings. When the pure rate of time preference is zero, future utilities are not discounted, therefore the optimal level of savings is higher than when present consumption is preferred.

Having defined the consumption paths through this optimization process, we can proceed to find the PVs of €1 that is due in the future periods modelled and compute the corresponding CERs. Figure 7 shows the results. The CERs of risk-neutral investors are shown for reference.

Figure 7
**CERs as a function of time and degree of risk aversion**




We can see that for degrees of risk aversion higher that 0.5 CERs decline initially and then rebound, but risk averse CERs are always lower than risk neutral CERs. The reason for this is that *ceteris paribus* risk averse investors always prefer safe assets, so for a risky asset to be equally desirable it must yield more. This effect increases with the degree of risk aversion. We can analyze this with the help of the following table.

Table 2
**Analysis of CER calculations for a safe €1 due in year 100**

| | **Coefficients of risk aversion** | | | |
|---|---|---|---|---|
| | **0** | **0.5** | **1** | **1.5** |
| Present value of asset (PV) | 0.004 | 0.016 | 0.056 | 0.120 |
| Expected future value of PV | 1.000 | 3.703 | 13.106 | 28.139 |
| Future CE of investing PV | 1.000 | 1.000 | 1.000 | 1.000 |
| Future CE of asset | 1.000 | 1.000 | 1.000 | 1.000 |
| CE Discount rate | 5.61% | 4.24% | 2.93% | 2.15% |

Given that the €1 face value asset is safe, its certainty equivalent in the future period is €1for all investors. In computing its present value, we seek the amount invested at time zero that will compound to a future value such that its certainty equivalent be €1 for all investors. This depends on the degree of risk aversion. A risk neutral investor only needs an expected payoff of €1 to be indifferent between it and the safe asset, which requires an investment of €0.004 at time zero. This is therefore the PV of the safe asset. It is obtained by dividing the required future payoff by the expected compound factor for year 100.

With a risk aversion coefficient of 1.5, however, it takes an expected payoff of €28.139 to make its certainty equivalent equal to €1. The PV in this case, also obtained by dividing the required payoff by the expected discount factor (not shown in the table, but equal to 235.29), is €0.12, triple that of the risk-neutral investor. CERs computed from the higher PVs to the same future asset value of €1 are lower than that those computed from lower PVs.

Notice that the definition of present value is always respected, as the PV amounts always compound to future values that investors consider to be of equal value as the safe asset.

It is interesting to point out that generally discounting is taken to be a yardstick by which the value of an asset is measured, but in this case the safe asset is the standard of value and the CER measures the size of the investment that yields a certainty equivalent value. So, it is the yardstick that is being measured.

A very strong portfolio effect makes itself felt in the above results, because in this first case the investors' only wealth at time $t$ comes from the yield of the amounts saved at time zero, which is perfectly correlated with the yield of the PV of the asset computed above. This is illustrated by the following table, built on the basis of the cumulative distribution of future values of possible present values at the start of the numerical search.

Table 3

**Selected values of the cumulative distribution of the future values of the PV of the asset and the corresponding disposable incomes Year 100, σ = 1.5**

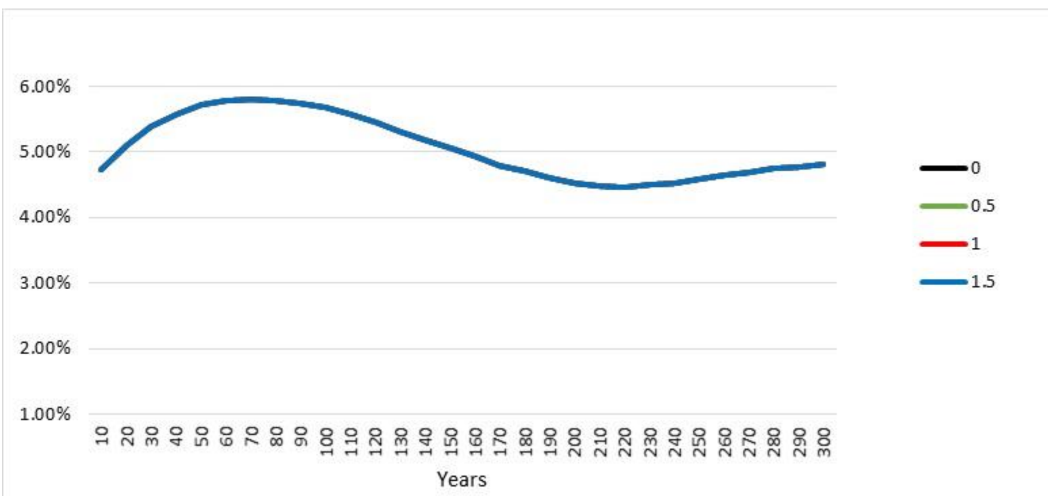
| Decile | FV of PV | Disposable Income | Relative marginal utility |
|---|---|---|---|
| 1 | 122.72 | 24,351,224 | 1% |
| 2 | 61.03 | 12,109,598 | 4% |
| 3 | 34.88 | 6,920,691 | 9% |
| 4 | 19.37 | 3,843,526 | 21% |
| 5 | 10.12 | 2,007,107 | 57% |
| 6 | 5.45 | 1,081,971 | 143% |

| | | | |
|---|---|---|---|
| 7 | 2.88 | 570,812 | 374% |
| 8 | 1.57 | 311,554 | 927% |
| 9 | 0.90 | 177,888 | 2,149% |
| 10 | 0.49 | 96,964 | 5,341% |

We can see in Table 3 that the fluctuation in the future value of present value of the asset is accompanied by a very large, positively correlated fluctuation in the investors' disposable income, which affects the obtained result because low FVs are evaluated at very different segments of the utility function from where high FV values are. The last column shows the relative marginal utility that applies to each decile (100% corresponds approximately to the median value of FVs). The lowest values are given over 5,000 times the weight of the highest.

We can eliminate the strong portfolio effect present by replacing the fluctuating time *t* wealth distribution shown in Table 3 by its certainty equivalent. In that case we obtain the following CER results.

Figure 8
**CERs as a function of time and degree of risk aversion without portfolio effect**




In this case all CERs are the same, regardless of the degree of risk aversion. None is a secularly declining function of time. The equivalent of Table 3 becomes the following:

Table 4

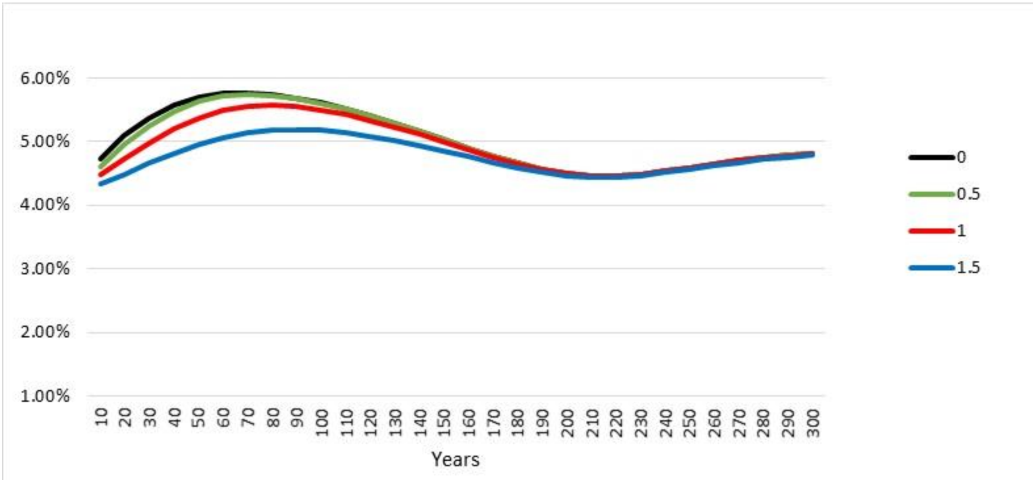
**Selected values of the cumulative distribution of the future values of the PV using certainty equivalent disposable income Year 100, σ = 1.5**

| Decile | FV of PV | Disposable Income | Relative marginal utility |
|---|---|---|---|
| 1 | 4.36 | 715,284 | 100% |
| 2 | 2.17 | 715,284 | 100% |
| 3 | 1.24 | 715,284 | 100% |
| 4 | 0.69 | 715,284 | 100% |
| 5 | 0.36 | 715,284 | 100% |
| 6 | 0.19 | 715,284 | 100% |
| 7 | 0.10 | 715,284 | 100% |
| 8 | 0.06 | 715,284 | 100% |
| 9 | 0.03 | 715,284 | 100% |
| 10 | 0.02 | 715,284 | 100% |

The result shown in Figure 8 might appear startling at first, for it seems that risk-averse investors display no risk aversion, but the explanation lies in that there is really no perceptible risk in deciding upon a €1 asset when investors' certainty equivalent wealth is €715,284.

If we take the safe asset to be worth €200,000, however, a sizable proportion of the certainty equivalent of future wealth, then we obtain the following result:

Figure 9
**CERs as a function of time and degree of risk aversion without portfolio effect (safe asset value of €200,000)**

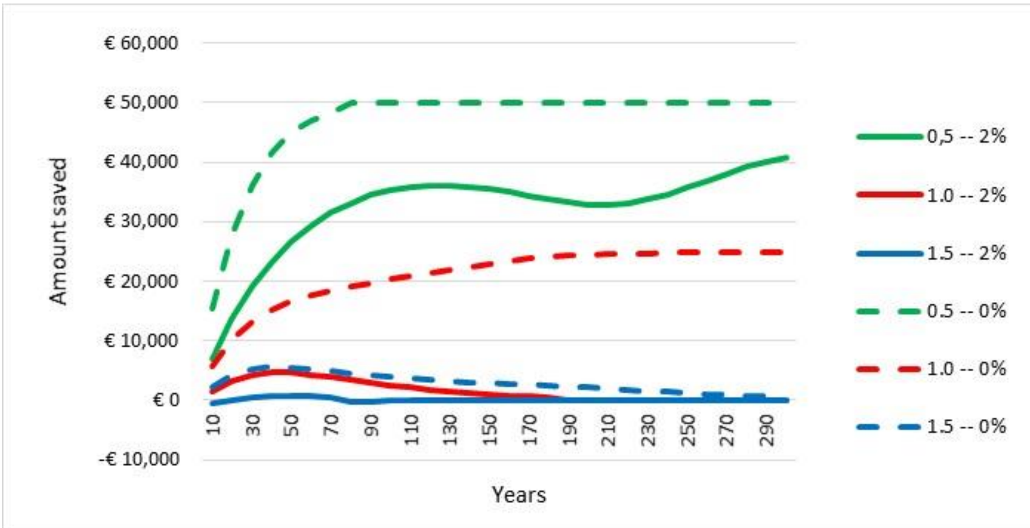



In this case the effects of risk aversion become visible, but even so no CERs is a secularly declining function of time.

### 3.1 Case 2, CERs of a safe asset with growing wealth

For the purposes of these calculations we assume an initial wealth of €50,000, growing at an annual rate of 2% and retain the 2% pure rate of time preference assumption. Again, the first task is to optimize the level of consumption at time zero and thus establish the amount saved. The results obtained are the following:

Figure 10
**Optimal savings amount for 2% and 0% time preference**

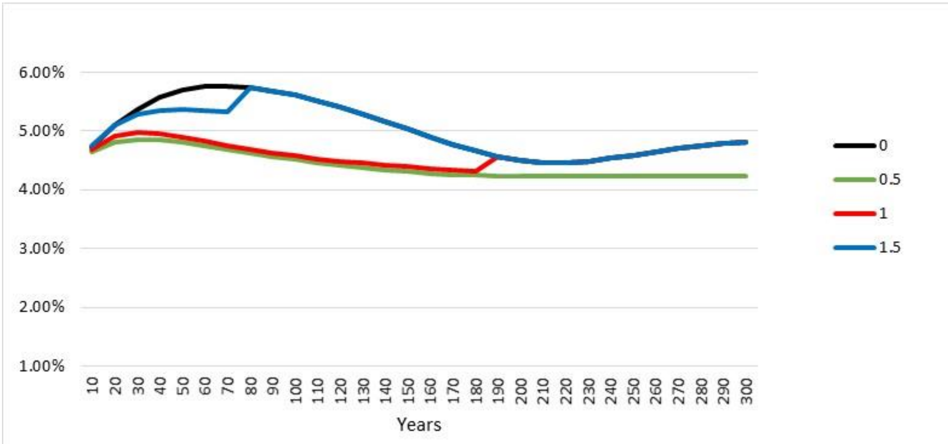



Just as in Case 1, savings are higher when the pure rate of time preference is zero. In this case, given that there is also an independent future source of wealth, borrowing is possible (negative savings), which occurs for degrees of risk aversion of 1 and 1.5 in the 2% time preference case. The amounts borrowed are constrained, however, to exclude the possibility of bankruptcy of investors. The model caps the amount that can be borrowed to prevent events of default even in the most adverse circumstances.

When the pure rate of time preference is 2%, the following CER values are obtained for a safe future €1:

Figure 11

**CERs as a function of time and degree of risk aversion**




All CERs initially increase with time due to the yield boosting effect of autocorrelation, with risk-averse CERs falling below the risk neutral ones because of the portfolio effect, which is initially of a lesser magnitude given that in this case investment income is not early on the main source of future wealth. CERs do not fall appreciably below the expected value of interest rates, which is 4.3%. For the more risk averse investors this is caused in good part by the fact their borrowing constraints keep their portfolio effects low.

Table 5

**Analysis of CER calculations for year 20**

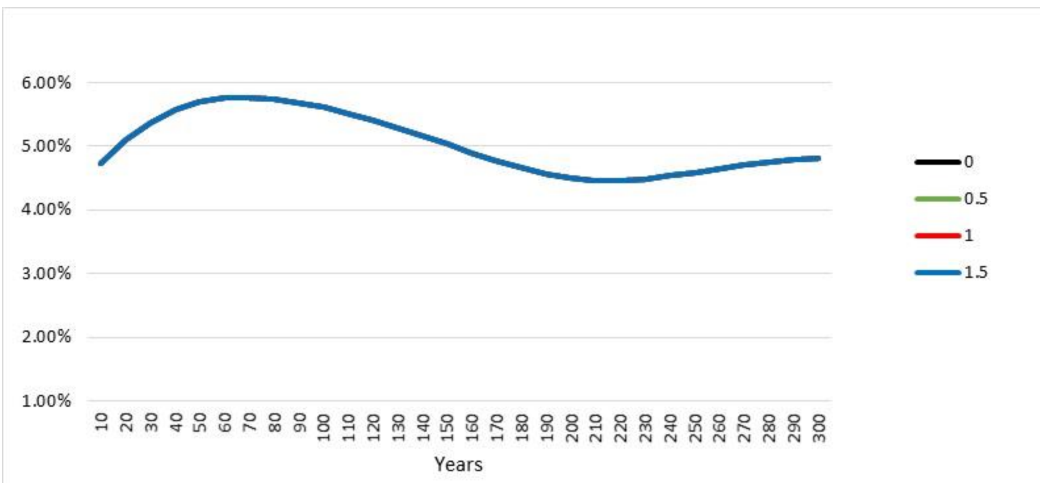
| | **Coefficients of risk aversion** | | | |
|---|---|---|---|---|
| | **0** | **0.5** | **1** | **1.5** |
| **Present value of asset** | 0.3698 | 0.3901 | 0.3826 | 0.3701 |
| **Expected future value of PV** | 1.0000 | 1.0549 | 1.0345 | 1.0009 |
| **Future CE of investing PV** | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| **Future CE of asset** | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| **CE Discount rate** | 5.10% | 4.82% | 4.92% | 5.09% |

The discontinuities seen in Figure 11 identify points at which risk averse CERs move to the level of risk neutral CERs. This occurs when borrowing constraints reduce borrowing capacity (for detailed values see the Plots worksheet of the accompanying Excel workbook). When that happens, the portfolio effect is removed.

The portfolio effect is what explains the fact that risk averse CERs in this case are a positive function of risk aversion, unlike in Case 1 (Compare Table 5 to Table 2) and contrary to what one would at first expect. But in Case2, we can see that savings levels are a negative function of risk aversion. Indeed, investors with risk aversion coefficients equal to or higher than one eventually borrow rather than save. Consequently, the portfolio effect is lower in their case.

We can remove the portfolio effect altogether by replacing total future wealth by its certainty equivalent, in which case we obtain the following:

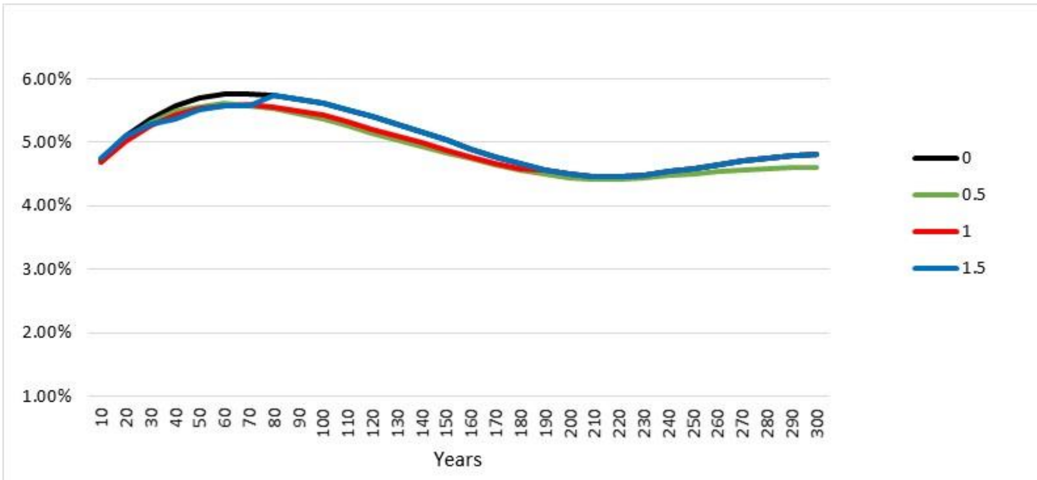
Figure 12
**CERs as a function of time and degree of risk aversion,**
**(using CE future wealth)**




Without portfolio effects, as in Case 1, risk averse investors behave risk neutrally for low value assets, which means that risk averse CERs are not secularly declining functions of time.

We can also check what happens if we restrict, rather than eliminate the portfolio effect. For example, if income from market investments is limited to 5% of total future disposable income. (This is achieved by apportioning part of savings to investments in assets that yield returns that are equal to those of the market but uncorrelated to them). In that case the computed CERs are the following:

Figure 13
**CERs as a function of time and degree of risk aversion,**
**(market correlated income limited**
**to 5% of total disposable income)**

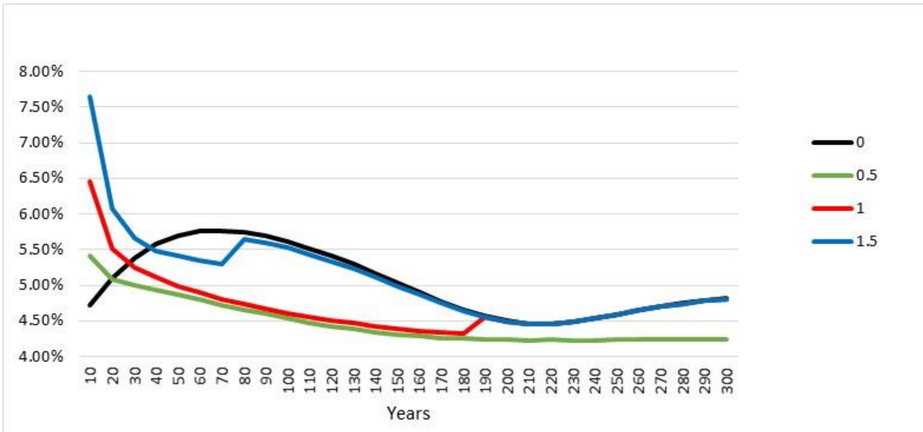



## 3.3 Risky assets

In all the preceding cases, the asset the PV of which was used to compute CERs was a safe €1 due in the future period. Replacing that by a lottery with a 50% chance of getting €2 and a 50% chance of getting nothing results in no changes in CERs in either the with or the without portfolio effects cases, compared to the corresponding safe asset CERs, because the amounts involved are too small to create perceptible risk.

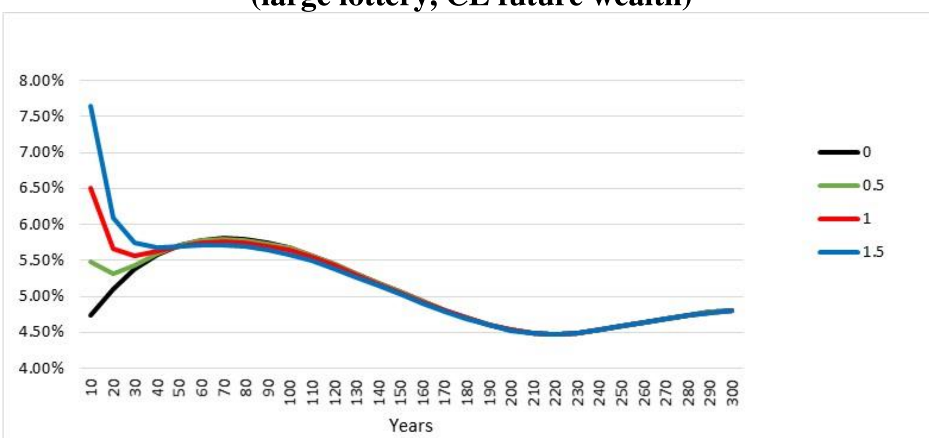
We can force the display of risk-averse behavior again by greatly increasing the risk. If the asset is a lottery with an even chance of getting €60,000 or nothing, we obtain the following CERs for the base case, with portfolio effect:

Figure 14
**CERs as a function of time and degree of risk aversion,**
**(large lottery, with portfolio effect)**




And we find the following with no portfolio effect:

Figure 15
**CERs as a function of time and degree of risk aversion,**
**(large lottery, CE future wealth)**




These are interesting results. Risk-averse CERs decline with time initially, but from values that are higher than risk-neutral CERs, and then eventually virtually coincide with the latter when no portfolio effect is present or remain below that if it is. We can find an explanation for this analyzing the values obtained for year 20.

Table 6
**Analysis of CER calculations for a large lottery due in year 20**

| | Coefficients of risk aversion | | | |
|---|---|---|---|---|
| | **0** | **0.5** | **1** | **1.5** |
| **Present value of asset** | 11,094 | 10,676 | 9,984 | 9,218 |
| **Expected future value of PV** | 30,000 | 28,868 | 26,999 | 24,926 |
| **Future CE of investing PV** | 30,000 | 28,376 | 25,930 | 23,445 |
| **Future CE of asset** | 30,000 | 28,376 | 25,930 | 23,445 |
| **CE Discount rate** | 5.10% | 5.30% | 5.65% | 6.08% |

The certainty equivalent of the lottery declines as a function of the coefficient of risk aversion. Consequently, the amount to be invested in the present to obtain those CE values also decline. Given that CERs are computed by reference to the expected monetary value of the asset (€30,000 in this case), CERs are a positive function of the degree of risk aversion.

Observe that risk averse investors apply two adjustments in valuing the risky asset. Let's take $\sigma = 1.5$ as an example. The future CE of the risky asset is €23,445, down from the expected value of €30,000. But there is a second source of risk: the uncertainty of the interest rates used to calculate present values. This investor requires an expected market yield for the present value of the asset of €24,926. This is the amount that has a CE of €23,445. These are the reasons why the PV of the asset is €9,218. The result complies with the definition of PV. Investing €9,218 has an expected FV of €24,926, which has a CE of €23,445, which is the CE of the asset for this investor.

It is often said that risky assets should be discounted at a risk adjusted rates. Continuing with the same example, discounting €30,000 by 6.08% yields the correct PV of €9,218. But it should be clear that this "discount" is a combination of a reduction of value of 21.85% (1-23,445/30,000) attributable to the intrinsic riskiness of the asset, which is unrelated to time value discounting, and a further reduction that is, the reduction from €23,445 to €9,218. The CER of a safe asset with a face value of €23,445 is $4.77% when $\sigma = 1.5$.

The discount rates shown in Table 6 therefore reflect both the riskiness of the asset itself and that of the capital market that is used to compute the opportunity cost of investing in the asset. However, even though risk averse behavior is displayed in the early years, this fades with the passage of time, as Figure 15 shows. This is because the value of the lottery is constant, so its size relative to growing wealth diminishes, reducing the significance of the risk involved.

We also examined the effect of assuming perfect correlation between the return of the small stochastic asset and market interest rates. The impact of this in the case with portfolio effect is to partially counteract it, bringing risk neutral CERs closer to risk neutral ones. Negative perfect correlation slightly enhances the portfolio effect. In the case with no portfolio effect neither type of correlation has any impact.

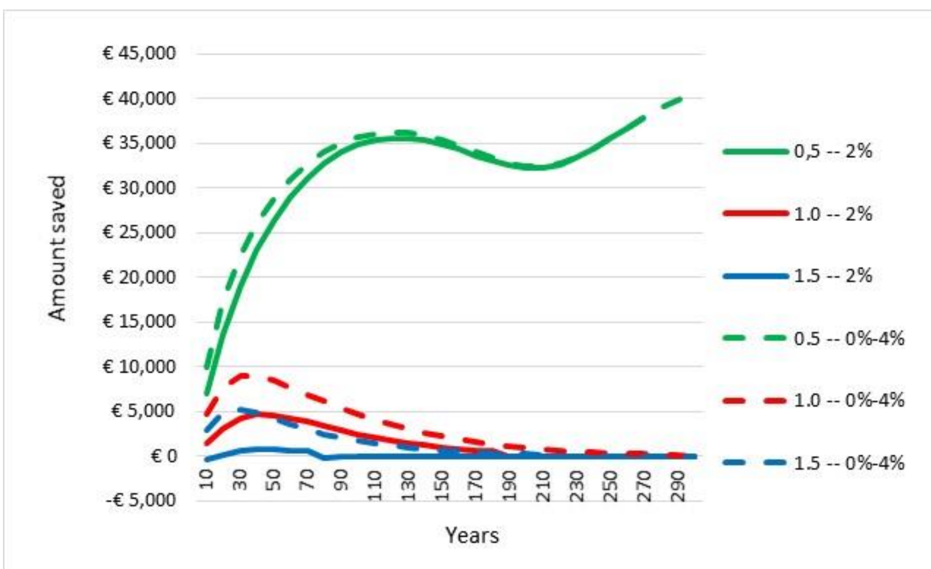
### 3.3 The effect of uncertainty of future wealth

States Gollier and Hammitt (2014) state as follows. "It is intuitive that uncertainty surrounding the future should induce society to take more care of it, i.e., to reduce the discount rate. At the micro level, this intuition is founded on the concepts of precautionary saving and prudence."

We can test this suggestion in the context of descriptive discounting. Using the assumptions of our Case 2 we first see how optimal savings levels change when uncertainty about future wealth growth is introduced. In the preceding section we assumed that wealth would grow at 2% p. a. Here we will assume that there is a 50% chance of that doubling to 4% and a 50% chance of wealth not growing at all. Time preference is left at 2% in both cases.

Figure 16

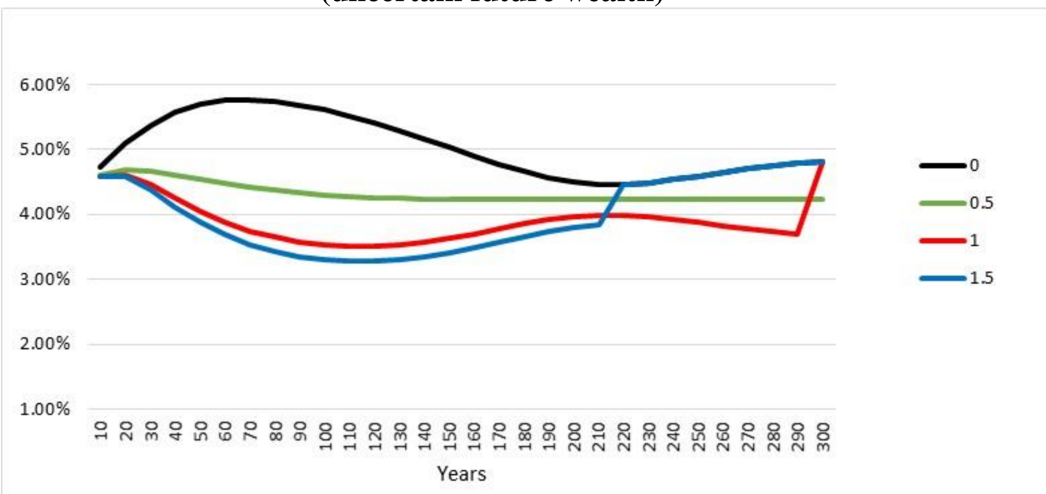
**Optimal savings amount for certain an uncertain future wealth**




Optimal savings increase indeed, with the effect declining over time. The comparison is marred, however, by the fact that borrowing constraints have a greater effect in the no uncertainty case.

Regarding the CERs of a safe €1, the future wealth uncertainty has the effect of increasing the portfolio effect that we have already seen, primarily because of the increased levels of savings brought about by the wealth uncertainty shown in Figure 16. Compounding makes investment income far outweigh exogenous wealth increases early in the timeframe analyzed. The increased portfolio effect lowers the CERs of the investors with higher degrees of risk aversion.

Figure 17
**CERs as a function of time and degree of risk aversion, (uncertain future wealth)**

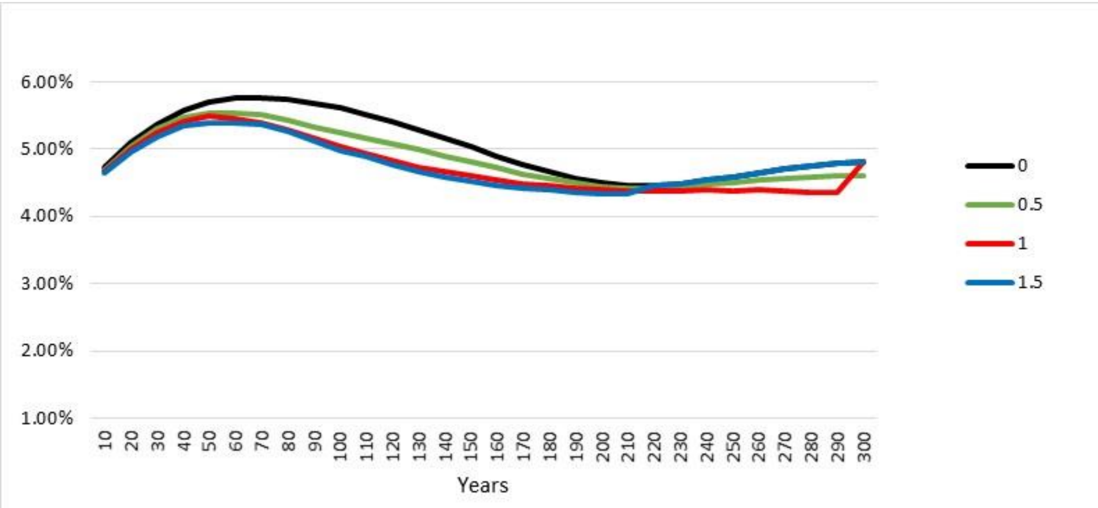



Removing the portfolio effect, however, all risk averse investors behave as if they were risk neutral, because the risk of the market return that is equivalent to the safe asset is imperceptibly small: we obtain the CERs of Figure 12. Thus, other than through its portfolio effects, the increase in savings have no effect on CERs.

Limiting the portfolio effect by reducing market linked income to 5% of the total we obtain the following CERs.

Figure 18

**CERs as a function of time and degree of risk aversion, (market correlated income limited to 5% of total disposable income)**




We also tested what happens if we force savings levels to be suboptimal. Plus/minus 25% deviations have only minimal impact on CERs if portfolio effects are considered and none if they are not.

Exactly the same happens when we make the small asset (uncorrelated) stochastic. Correlation of the small asset's yield with market interest rates has minimal effect when portfolio effects are present, while correlation with wealth uncertainty has an even lesser effect.

Therefore, considering the correlation between asset yields and either market interest of future wealth does not appear to be warranted for relatively small assets.

### 3.4 Conclusions concerning risk averse investors

For high enough degrees of risk aversion risk averse CERs can be declining for a number of years due to the portfolio effect. In the absence of portfolio effects, however, risk averse investors will behave like risk neutral ones for relatively small investments. If portfolio effects are reduced to a low level, risk neutral CERs are slightly lower than risk neutral ones but are not secularly declining.

## 4. Conclusions

This paper has analyzed the behavior of both risk neutral and risk averse CERs in the framework of descriptive discounting, that is, assuming that the opportunity cost of investing in public sector projects is investing in a capital market in which interest rates are stochastic. We represented this market with a model of interest rates of the Cox, Ingersoll & Ross (CIR) type calibrated to historically observed data.

Despite the experimental approach taken in this paper, its most important finding is not empirical but analytical (although empirically corroborated):

- Weitzman's proposition that CERs should be a declining function of time when interest rates are perfectly autocorrelated is wrong. In fact, the opposite is true: CERs should be a growing function of time.
- As the fallacy in Weitzman's proposition lies in assuming that the expectation of the inverses equals the inverse of the expectation. The discrepancy between declining and growing CERs is not a paradox (even though it gave rise to the large Weitzman-Gollier paradox literature) but the measurable consequence of an incorrect calculation.

All the cited papers presenting empirical evidence for DDRs have chosen some way of modeling the uncertainty in interest rates and proceeded to derive incorrect CERs by repeating the mistake present in Weitzman formulation. In attempting to verify their conclusions, we have not replicated their diverse methods of simulating market interest rates. Rather, we have used our own interest rate simulation model and, following their intentions, simulated the resulting CERs. No CERs where found to be declining.

- When the coefficient of autocorrelation of simulated interest rates is equal to one, CERs increase with time and tend to the highest possible interest rate.
- When the coefficient of autocorrelation of simulated interest rates is equal to zero, the term structure of CERs is flat.
- For moderate autocorrelation values CERs increase with time initially but eventually tend towards the expected value of interest rates.

We also examined the behavior of CERs derived from risk averse utility functions. We found that:

- Risk averse CERs can be declining functions of time, but only in the presence of the portfolio effect, that is, if investors' future disposable income depends significantly on the yield from savings invested at the same uncertain interest rate that is used to calculate the opportunity cost of investing in the asset for which the CER is being computed.

- Absent the portfolio effect, risk averse CERs are the same as risk neutral CERs for assets that are small relative to investors' wealth.
- If the portfolio effect is present but is low
    - Risk averse CERs are slightly lower than risk neutral ones but move in parallel with them.
    - Changing the level of savings has no effect on CERs
    - The correlation between asset yield and capital market yield has no perceptible impact on CERs
    - The correlation between asset yield and uncertain future wealth has no perceptible impact on CERs

Following the Arrow-Lindt (1970) theorem, it is generally accepted that the public sector should behave in a risk neutral manner. Should anyone none the less wish to attribute some degree of risk aversion to public sector decision makers, this paper suggests that because capital market interest income is not a typical source of government revenues, the CERs observed in practice would be close to risk neutral CERs.

The experimental results reported on in this paper derive from arbitrarily chosen numerical examples applied to theoretical utility functions. The results are robust to changes in the assumed values, however, as the reader can see for himself using the associated Excel worksheet. Considering that Weitzman's DDR proposal, based on an equally abstract and simplified model, was adopted as official policy by three countries and endorsed by the OECD CBA manual, pointing out the error in Weitzman's DDR recommendation, at the same level of abstraction, is warranted.

Should we recommend that growing CERs replace DDRs based on our findings? Gollier and Hammitt (2014) states "because of the absence of realism in Weitzman's assumption about the permanency of shocks on interest rates, it is not appropriate to use his famous rule [DDRs] to recommend a specific term structure, as did the UK government, for example." We could go further and point out that it is unrealistic to assume that the opportunity cost of long-term public-sector investment projects is the exploitation of autocorrelation induced yield boosting effects in the financial instruments market. Consequently, we find no reason to jump from a plausible experimental result to a policy recommendation.

What ought to be clear, however, is that there is no valid evidence justifying the use of DDRs for descriptive discounting in cost-benefit analyses of public sector projects.

## REFERENCES

Arrow, Kenneth J. and Robert C. Lind (1970), "Uncertainty and the Evaluation of Public Investment Decisions" The American Economic Review, Vol. 60, No. 3 (Jun., 1970), pp. 364-378, https://www.jstor.org/stable/1817987.

Freeman, M.C. *et al*. (2015), "Declining Discount Rates and the Fisher Effect: Inflated Past, Discounted Future," Journal of Environmental Economics and Management, Vol. 73, pp. 32-49, https://dx.doi.org/10.1016/j.jeem.2015.06.003.

Gollier, Christian and James K. Hammitt (2014) "The Long-Run Discount Rate Controversy," *The Annual Review of Resource Economics*, 2014. 6:273–95, https://doi.org/10.1146/annurev-resource-100913-012516

Gollier, Christian, Phoebe Koundouri, Theologos Pantelidis (2008) "Declining Discount Rates: Economic Justifications and Implications for Long-Run Policy" *Economic Policy*, Volume 23, Issue 56, 1 October 2008, Pages 758–795, https://doi.org/10.1111/j.1468-0327.2008.00211.

Groom, B., P. Koundouri, E. Panopoulou, & T. Pantelidis (2007) "Discounting the Distant Future: How much does model selection affect the certainty equivalent rate? *Journal of Applied Econometrics*, 22, 641-656.

HMT (2003). Guidelines on Cost Benefit Analysis. HM Treasury, UK.

Newell, Richard and William Pizer (2000) "Discounting the Distant Future: How Much Do Uncertain Rates Increase Valuations?" Discussion Paper 00-45, Resources for the Future, Washington, DC.

Nordhaus, William (2019). "Climate Change: The Ultimate Challenge for Economics." *American Economic Review* 2019, 109(6): 1991–2014, https://doi.org/10.1257/aer.109.6.1991

Szekeres, Szabolcs (2013). The "Weitzman-Gollier puzzle" is not a paradox but a mistake, and it is most likely moot. *Open Science Repository Economics*, Online (open-access), e23050448.

http://www.open-science-repository.com/economics-23050448.html

Szekeres, Szabolcs (2017) Checking Gollier and Weitzman's solution of the "Weitzman–Gollier puzzle". Economics Discussion Papers, No 2017-11, *Kiel Institute for the World Economy.* http://www.economics-ejournal.org/economics/discussionpapers/2017-11

Szekeres, Sz. (2019) Why expected discount factors yield incorrect expected present values. *Journal of Applied Economic Sciences*, Volume XIV, Spring, 1(63): 282-292. http://cesmaa.org/Extras/JAESArchive

Weitzman, M.L. (1998), "Why the far distant future should be discounted at its lowest possible rate." *Journal of Environmental Economics and Management*, 36, 201-208.

Weitzman, M. L. (2001) Gamma discounting. *American Economic Review* 91:260–271

# APPENDIX A

## 1. Description of the model

The simulation model that generated the results reported on in this paper is implemented in the accompanying Excel workbook. It contains three primary worksheets, described in the following sections, on which the sequence of necessary calculations is performed. There are additional auxiliary worksheets as well.

### 1.1 Interest rates and compound factors

This worksheet simulates stochastic monthly the interest rates using a model of the Cox, Ingersoll & Ross (CIR) type, with parameters that were calibrated by Yajie Zhao and Boru Wang (2017) with reference to a monthly data series of US three-month Treasury Bill rates spanning the period 1992 to 2017. Zhao and Wang state that for this period their Chi-Square test shows that “we cannot reject the CIR model” and that the model “only fails when the volatility of the real data is very large.” This fact does not really affect the conclusions of this paper, as those do not depend on any such accuracy, given that in this model the probability distribution of interest rates is taken to be the expectation of the investors whose behaviors are being explored.

Columns A-D contain the assumptions of the Zhao-Wang model. Rows 1-19 contain the frequency distribution of interest rates derived from the Zhao-Wang dataset, which is used as a time-0 distribution. Because the model is recursive, it requires a *t-1* period distribution for each time period simulated. This is distribution used for the simulation of the first month. Subsequent monthly simulations use the preceding month’s results.

Below line 20 we have the parameters of the fitted model and its simulation for 124 months, and descriptive statistics of the results. As these calculations are performed in the excel worksheet, the calculation formulas can be inspected. Random normal deviates are provided by Excel, so the COX interest rates are recalculated every time Excel updates the worksheet. These values are for reference and inspection only; they are not used in the simulations of the model.

The model includes a Monte Carlo simulation algorithm written as an Excel macro. It generates 133 months of monthly interest rates, running the user specified number of simulations (500 to 10,000). The average and extreme values obtained are shown in columns G-I below line 12. The average and the range over the entire period are reported on in line 7.

The generated interest rates are not used directly in the behavioral simulation part of the model, because as an inspection of the parameters of the simulated monthly interest rates will reveal, the Zhao-Wang model has a very small drift in its mean, and an asymmetrically slightly expanding range. As this paper aims to analyze the impact of the passage of time on CERs, it is best to avoid secular changes in the distribution of expected interest rates. Also, this paper will examine the behavior of long term CERs, for up to 300 years, and it would be unwise to push the Zhao-Wang model so far away from the period for which it has been calibrated. Therefore, the Zhao-Wang model is primarily used to derive a realistic monthly interest rate probability distribution. This is taken to be that of month 13. This choice is somewhat arbitrary but was chosen on the ground that by the $13^{th}$ month the effects of the time zero initial distribution would have been lost, and the simulation had not yet gone too far from its calibration data.

There is another important datum obtained from the Zhao-Wang model, however: the degree of autocorrelation that can be measured over the length of the compounding period chosen by the user. That observed correlation coefficient is reported on in cell S8.

The simulated interest rates for the years being analyzed are shown in columns P-S below line 12. Data for 30 different years are shown, Column P identifies the year, column Q contains the average for that year and columns R-S show the range. These values are always the same, as they replicate the distribution of the $13^{th}$ month of the Zhao-Wang model. In columns T-U we see the correlation with the distributions of the preceding displayed year and the correlation with the initial period. The latter value always decays quickly.

To explore the behavior of decision makers a forecast of compound factors is required. It is by reference to the probability distributions of these that the decision makers analyzed will optimize their savings and consumption decisions and compute certainty equivalent yields of investments in the market. Compounding takes place with a frequency established in months entered by the user into cell O9. Only multiples and submultiples of 12 no greater than 120 can be specified. The choice of this value has two effects:

1. It determines the time horizon of the forecasting period. As the number of displayed years is 30, setting the compounding period to 120 months, that is 10 years, the time horizon will be 300 years. Increasing the frequency of compounding reduces the time horizon to a minimum of 30 years.

2. It affects the degree of correlation between adjacent compounding factors. The value is taken from the observed correlation between interest rates simulated the specified number of months apart in the Zhao-Wang model starting with month 13. Longer compounding periods yield less correlated interest rates and therefore reduce the yield-boosting effect of their autocorrelation.

The frequency of compounding does not affect the effective interest rate because the compounding formulas keep the effective interest rate independent of the frequency of compounding.

The calculated expected compound factors are shown in columns J-O below line12. We see average values, extreme values, the coefficient of variation for each year, and the correlation to the factor of the preceding displayed year. It should be noted that it is normal for the compound factors to be highly correlated even if the distributions of interest rates are not. This phenomenon can be empirically tested with a simplified example found in the CF Correlation test worksheet.

Compound factors must be generated before any further analyses can be performed. The macro is launched by clicking on the "Simulate" button. There are five running options:

0 normal run – as described above

1 perfect correlation – subsequent compound factors are perfectly correlated

2 no correlation – subsequent compound factors are uncorrelated

3 specify correlation – subsequent compound factors are correlated to a user specified extent

4 test deterministic interest rate – no simulation, a user specified interest rate is held constant.

### 1.2 Optimal savings

On this worksheet the amounts saved by the four decision makers are optimized. The four decision makers are defined by their degree of constant proportional risk aversion. Coefficient Index 0 is reserved for the risk neutral decisionmaker whose coefficient is equal to zero. The others are set to 0.5, 1, and 1.5. The latter three values can be changed.

The optimization problem to be solved is to maximize expected utility given exogenous initial and future wealth amounts, and the savings/borrowing opportunities in the capital market, where the compound factors simulated previously determine the yield of savings or the cost of borrowing. The optimization is always between the present and one of 30 alternative future periods. These pairs are to be viewed as alternative problems that are unrelated to one another.

There are two ways to specify wealth levels. In the endowment only option there is no exogenous income in the future period, so the decisionmaker will only be able to consume in the future if he saves. Under this option it is not possible to borrow.

If the endowment option only is not specified, there will be exogenous wealth in the future period as well. Its level is defined by an annual growth factor with respect to initial wealth. Consequently, its absolute amount will grow for later periods. It is also possible to make future wealth uncertain by specifying an equiprobable deviation to the growth rate. E. g., a 20% deviation means that in half of the simulations the wealth growth rate will be multiplied by 0.8, while in the other half the factor will be 1.2.

A pure rate of time preference, understood to be annual, can be specified.

When borrowing is possible, it may be constrained to prevent the possibility bankruptcy even in the worst-case scenario. Cases of constrained borrowing are highlighted in yellow on the worksheet.

The optimal saving/borrowing amount is numerically found using a hill climbing algorithm that detects having passed the optimal point and changes direction taking smaller steps. This is repeated until a solution of acceptable precision is found. Corner solutions may develop, as it invariably does for risk neutral decision makers. The process starts by clicking on the "Optimize savings" button. The results, for the years under analysis, are found in columns D-G.

A way to manually verify the optimal amount for any case is provided. The user can specify ranges of savings to examine and will see the behavior of expected utility within that range. By changing the size of the range, the optimal point can be found manually. For each attempt click on the "Verify savings" button.

Once the solutions have been found, the levels of expected welfare are displayed for each case (columns I-L).

The final block of results displayed (columns N-Q) shows the expected value of the interest rate implicit in the budget line (compound factor) for each period, and the average marginal rate of substitution (MRS) between present and future consumption at the optimal consumption point. The optimal consumption point is optimal on an expected value basis but will not necessarily be optimal for any of the scenarios simulated. The MRS will probably never equal the slope of the budget line in any scenario, but the expected value of the MRSs is nonetheless close to the slope of the expected compound factor.

## 1.3 Present values

On this worksheet the present values and corresponding CERs are calculated. These calculations pertain to the asset defined by the option parameter in cell N1, under the assumptions further specified in the block that starts there. Three asset type options exist:

| | |
|---|---|
| fixed | Fixed face value |
| growing | Face value multiplied by the expected compound factor |

The face value of the asset is specified in cell N2. The asset value can be made stochastic trough multiplication by an equiprobable deviation factor, cell N3. E.g., if this factor is 25%, in half of the simulations the face value of the asset will be multiplied by 1.25, and in the other half by 0.75. The resulting coefficient of variation in the asset's value is computed.

If the asset value is stochastic, it is possible to specify a correlation coefficient. The choice, made in cell N4, is the following:

| | |
|---|---|
| none | No correlation |
| interest rate | Correlation with the compound factors distribution |
| future wealth | Correlation with exogenous future wealth |

The desired coefficient of correlation can be specified in cell O4.

The options to treat exogenous future income are the following:

| | |
|---|---|
| Stochastic | Unchanged from the specification in the optimal savings worksheet, so it can be stochastic. |
| Cert equiv | This option replaces stochastic total disposable incomes by their certainty equivalents. |

The portion of investment income in the future period that results from savings invested in the market can be limited. The limitation is specified as a percent in cell N6. Specifying 100% means that there is no constraint. If a lower percent is chosen, investments on the market will be limited to comply with the constraint. If the constraint is binding, the savings amount that cannot be invested in the market will be invested in an asset that has the exact same probability of yield, but with a distribution that is uncorrelated with that of the market. This is a way of limiting the portfolio effect affecting CER calculations.

Savings suboptimality can be specified. This will reduce or increase savings from their optimal levels by a percent given in cell N7.

Based on the options chosen, a present value is calculated for the asset for each future year considered and for each decision maker. An iterative procedure seeks to find a present value amount such that the expected utility of receiving the proceeds of compounding it to the future equals the expected utility of holding the asset. This is in accordance with the definition of present value. The decision maker is indifferent between two identical expected utility values: that of holding the asset and that of receiving the proceeds of investing the present value in the market.

The calculations are launched by clicking on the "Compute present values" button. The results are shown in columns B-E. Column F shows the value obtained through the theoretically correct calculation for risk neutral investors. It is the asset face value (which is always the monetary expected value of the asset because any uncertainty about it is symmetrical) divided by the expected compound factor for the corresponding year. The values in column B and F always coincide, empirically proving the correctness of the correct present value calculation formula. The CERs corresponding to the present values are shown in columns G-J. A plot of these CERs is produced.

The calculation of any present value can be manually verified using the table at the left below row 34. Clicking on the "Verify PV" button compares the expected welfare of holding the asset and of investing in the market amounts between the minimum and maximum values specified. It is possible to approximate the solution to any desired degree of precision by successively narrowing the limits and recalculating.

It is possible to analyze the results of all decision makers for any given year, specified in cell M29, by clicking on the Analysis button. See the table starting in cell L28. The resulting table is interesting. It provides data for each of the decision makers in the following variables:

*Present value of asset:* This is the computed present value

*Expected future value of PV:* This is the expected monetary value of investing the present value in the market. Notice that this is higher for higher degrees of risk aversion, because the higher the risk aversion, the more the decision maker must be compensated for the risk.

*Future CE of investing PV:* This is the certainty equivalent of investing the present value in the market. This value will be the same as the previous one for risk neutral investors, but lower for risk averse investors, who discount the value because of the uncertainty inherent in investing at stochastic yields in proportion to their coefficient of risk aversion.

*Future CE of asset:* This is the certainty equivalent of holding the asset. It should be equal to the preceding, as this is the equality that the present value calculation algorithm is attempting to achieve.

*CE Discount rate:* Recalls the calculated CER for reference.

There is a final table available on this worksheet that starts with cell O38. The purpose of this table is to illustrate the search for present values. It works in conjunction with the previously described analysis table and will always be run for the year specified there. The only selection the use can make is the choice of coefficient index, identifying the degree of risk aversion to analyze. The present value of the asset and its future certainty equivalent will be copied from the analysis table. To fill the table the user should click on the button labeled "Table."

The column labeled FV of PV contains selected values of the cumulative distribution of the product of possible present values of the asset multiplied by simulated compound factors. The values shown are obtained by sorting all simulated compound factors and choosing the right one for each decile in descending order.

The column labeled Disp. Income is the future period disposable income of the decision maker for the same simulation that defined the corresponding FV of PV decile.

Column U(FV) shows the utility of the future value of the computed present value, column U(CE) shows the utility of the certainty equivalent of the asset, and column ΔU() shows the difference between the latter two. Column Relative Marg Util shows the relative value of the marginal utility for each displayed case. The base of the Relative Marg Util values calculated is the median observation.

The point of the illustration is to show how widely disposable income swings in tandem with small changes in incremental future value. In computing utility changes, very distant parts of the utility function come into play. The relative marginal utilities serve to emphasize this point.

### 1.4 Auxiliary worksheets

*Plots:* Used to generate some of the plots displayed in the paper. New plots can be generated by clicking on the generate button. Ensure that the right options have been specified first.

*CF correlation test.* Serves to illustrate that perfectly uncorrelated interest rates will result in correlated compound factors.

*Gollier et al example.* Contains the calculations and plot for Section 2 of the paper.

*Debug.* Stores search results when debugging output is generated by a macro. This can be turned on in the code. Normally empty, for use by developers only.

## 2. Macros

The macros for the calculations of the three primary worksheets are stored in modules 1-3 respectively. The code is available for inspection and is documented via comments.

Macros used in auxiliary worksheets are stored in the code areas of the respective worksheets.