Bertrand-Edgeworth oligopoly: Characterization of mixed strategy equilibria when some firms are large and the others are small

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Abstract

This paper studies Bertrand-Edgeworth competition among firms producing a homogeneous commodity under efficient rationing and constant (and identical across firms) marginal cost until full capacity utilization is reached. Our focus is on a subset of the no pure-strategy equilibrium region of the capacity space in which, in a well-defined sense, some firms are large and the others are small. We characterize equilibria for such subset. For each firm, the payoffs are the same at any equilibrium and, for each type of firm, they are proportional to capacity. While there is a single profile of equilibrium distributions for the large firms, there is a continuum of equilibrium distributions for the small firms: what is uniquely determined, for the latter, is the capacity-weighted sum of their equilibrium distributions and hence the union of the supports of their equilibrium strategies.

Keywords: Bertrand-Edgeworth oligopoly, mixed strategy equilibrium, large and small firms.

JEL Classification: C72, D43, L13

1 Introduction

Bertrand-Edgeworth competition among capacity-constrained sellers of a homogeneous product has been an active field of research since Levitan and Shubik's [13] reappraisal of such theoretical framework. Assume a given number of firms producing on demand a homogeneous good at constant and identical unit variable cost up to some fixed capacity. Further, assume that rationing takes place according to the surplus-maximizing rule and that demand is a continuous, non-increasing, and non-negative function defined on the set of non-negative prices and is positive, strictly decreasing, twice differentiable and such that the monopolist's profit function is strictly concave when positive. Then there are a few well-established facts about the equilibrium of this price game. First, at any pure strategy equilibrium the firms earn competitive profit. However, a pure strategy equilibrium need not exist. In this case existence of a mixed strategy equilibrium is guaranteed by the sufficient conditions of Theorem 5 of Dasgupta and Maskin [3]. Under similar assumptions on demand and cost, the set of mixed strategy equilibria was characterized by Kreps and Scheinkman [12] for the duopoly within a two-stage capacity and price game. This model was subsequently extended to allow significant convexities in the demand function (by Osborne and Pitchik, [14]) or differences in unit cost among the duopolists (by Deneckere and Kovenock, [10]). This led to the discovery of new phenomena, such as the possibility of the supports of the equilibrium strategies being disconnected and non-identical for the duopolists.

The characterization of equilibria of the price game among capacity-constrained sellers of a homogeneous product under general oligopoly is far from complete in the literature. An important result is that the equilibrium payoff of the largest firm (or any of the largest firms, if more than one firm has the largest size) is equal to the payoff of the Stackelberg follower when the rivals supply their entire capacity ([2] and [5]).[1] Based on this property, Ubeda [15] showed, among other things, that the maximum and minimum over all the supports of equilibrium strategies belong to the support of the equilibrium strategies of any firm with the largest capacity.[2] Other results were provided by De Francesco and Salvadori [6].

Progress on the characterization of equilibria of the price game under given capacities has been made along several directions. One direction was to restrict the number of competing firms. Hirata [11] and De Francesco and Salvadori [6], [8], and [9] have analyzed the triopoly price game with a decreasing and concave demand function, establishing independently a number of features of equilibria. In a recent study on price strategic interaction among capacity-unconstrained sellers facing "captive customers" and price-rigidity of market demand, Mark Armstrong and John Vickers [1] have also compared the resulting equilibria with equilibria in the more standard Bertrand-Edgeworth framework; such a task has been accomplished for the triopoly, providing a complete characterization of the equilibria arising in the Bertrand-Edgeworth price game with rigid demand.

A second direction of research focused on portions of the whole region of an oligopoly capacity space where no pure strategy equilibria exist (hereafter, the no-pure strategy equilibrium region, for brevity). Vives [16], amongst others, characterized the (symmetric) mixed strategy equilibrium of the price game for the subset in which all firms have the same capacity. De Francesco and Salvadori [7] generalized Vives' result: they established uniqueness of equilibrium in Vives' symmetric capacity case and, more generally, whenever the capacities of the largest and smallest firm are, in a precise sense, sufficiently close

[1] The proof in [2] is carried out along the lines in [12] for the analogous result under duopoly. After pointing out a mistake in the proof, [5] establishes the result correctly along the same lines.

[2] In a still unpublished paper Ubeda [15] compares discriminatory and uniform auctions among capacity-constrained producers and obtains a number of novel results on discriminatory auctions: a discriminatory auction could be designed in such a way as to be equivalent to Bertrand-Edgeworth competition under the efficient rationing rule.

to each other. Furthermore, they characterized the equilibrium in this "quasi-symmetric" oligopoly, showing that the supports of the equilibrium strategies of all firms are intervals, each with the same minimum price whereas the higher a firm's capacity, the higher the maximum price. Within an analysis concerning horizontal merging of firms, Davidson and Deneckere [4] characterized, for the case of linear demand, equilibria for the subset in which there are $z-1$ equally-sized small firms and one large firm with a capacity that is a multiple of the small firm's. Again, the attention was restricted to equilibria in which the strategies of equally-sized firms are symmetrical.

There is one result in Hirata [11][3] that extends straightforwardly to the oligopoly. Hirata [11] showed, for the triopoly but also for the oligopoly, that a continuum of equilibria exists in the subset of the no-pure strategy equilibrium region in which the largest firm can meet even the highest level of total demand possibly arising at an equilibrium. In fact, while there is one equilibrium strategy for the largest firm, there is a continuum of equilibrium strategies for smaller firms, in that there is a single equation determining the capacity-weighted sum of their cumulative distributions throughout the lowest price and the highest price. The present paper shows constructively that the subset of the no-pure strategy equilibrium region in which a continuum of equilibria exists is much wider.

We specifically analyze a subset of the no-pure strategy equilibrium region in which there are two groups of firms, "large" firms and "small" firms: the total capacity of the large firms can meet even the highest level of demand that can arise at an equilibrium of the price game whereas the total capacity of the small firms is so small that total industry capacity minus the capacity of any of the large firms does not exceed even the smallest level of total demand that can arise at an equilibrium. Incidentally, the combination of these two conditions means, amongst other things, that the capacities of the large firms are close enough to each other, in a similar fashion as in De Francesco and Salvadori [7].

Such a bipolarized industry structure has two interesting and intertwined implications. On the one hand, and similarly as in the mentioned case studied by Hirata [11], there is no "direct" strategic interaction among the small firms: more specifically, regardless of the prices being charged by the other small firms, each small firm either sells its entire capacity, if at least one of the large firms is more expensive, or sells nothing, if all the large firms are cheaper. On the other hand, each large firm sells its entire capacity if, and only if, at least one of the other large firms is more expensive. In the event of all the other large firms selling cheaper, the expected value of its residual demand falls short of total demand by an amount equal to the total capacity of the other large firms (as it would be in De Francesco and Salvadori [7]) plus the capacity-weighted sum of the probabilities of all the small firms charging a lower price. We will characterize the equilibria for such a bipolarized industry structure. It will be shown that the above implications are ultimately responsible for the existence of a continuum of equilibrium distributions for the small firms. What is uniquely

[3]The same result was independently reached by De Francesco and Salvadori [6].

determined, instead, are the equilibrium payoffs of all firms, the equilibrium distributions of the large firms and hence the supports of their equilibrium strategies, the union of the supports of the equilibrium strategies of the small firms, and the capacity-weighted sum of the equilibrium distributions of the small firms. Most importantly, characterizing the continuum of equilibria for any such bipolarized industry structure involves determining the lowest and highest price that small firms can ever charge in equilibrium: the former is generally higher than (in a limit case, equal to) the (uniform) minimum price each large firm will ever charge in equilibrium and the latter is always less than the maximum price any large firm will ever charge.

Although our interest is purely theoretical, as mentioned above, the present study is potentially relevant to a wide array of empiricists. First, the parameter region it covers appears fairly natural. Second, the unique results in terms of each firm's equilibrium payoff, the supports of the equilibrium strategies of the large firms and the minimum and maximum of the union of the supports of the small firms' equilibrium strategies provide a set of empirically testable predictions. Quite interestingly, carrying out such a test need not require detailed information on the individual capacities of each small firm, which might be more difficult to obtain than an approximate estimate of their total capacity, which is what actually matters for the equilibrium features.[4]

The remainder of the paper is organized as follows. Section 2 presents basic properties of the equilibrium of the price game in the no-pure strategy equilibrium region of the capacity space. Section 3 defines an industry containing "large" firms as well as "small" firms and then characterizes the continuum of equilibria arising under such circumstances; a numerical example is also provided at the end of the section to illustrate the theoretical findings and to clarify how the role of small firms may well be far from negligible. Section 4 briefly concludes.

2 Preliminaries

Denote by $\mathcal{Z} = \{1, ..., z\}$ the set of firms.[5] Each firm i produces to order a homogeneous commodity with the same constant marginal cost (with no loss of generality normalized to zero) up to its fixed capacity k_i. Denote by K total capacity and, with no loss of generality, let $k_1 \geqslant k_2 \geqslant ... \geqslant k_z$. A continuous demand function $D(p)$ which is strictly decreasing and such that $pD(p)$ is strictly concave over the price range in which $D(p) > 0$ is assumed to exist. Firm i's profit at strategy profile (p_i, p_{-i}) is $\Pi_i(p_i, p_{-i}) = p_i \min\{d_i(p_i, p_{-i}), k_i\}$, where $d_i(p_i, p_{-i})$ is the demand forthcoming to firm i at (p_i, p_{-i}), p_i is the price charged by firm i and p_{-i} is the vector of prices charged by all firms except firm i. Under efficient rationing and assuming that such demand is

[4]For the reader's information, a redistribution of total capacity among the small firms would not affect the total of their equilibrium payoffs.

[5]The assumptions and notation laid down in this section largely draw on De Francesco and Salvadori (2013).

proportional to capacity for equally priced firms, we have that $d_i(p_i, p_{-i}) = \max\{0, D(p_i) - \sum_{j:p_j<p_i} k_j\} \times \frac{k_i}{\sum_{r:p_r=p_i} k_r}$.

Denote by p^c the competitive price: $D(p^c) = K$ if $D(0) \geqslant K$ and $p^c = 0$ if $D(0) \leqslant K$. As is well known (see, e.g., De Francesco and Salvadori [6]), $(p_1, ..., p_z) = (p^c, ..., p^c)$ is an equilibrium of the price game if, and only if, either

$$K - k_1 \geqslant D(0) \text{ when } D(0) \leqslant K, \tag{1}$$

or

$$k_1 \leqslant -p^c \left[D'(p)\right]_{p=p^c} \text{ when } D(0) > K. \tag{2}$$

Holding (2), $(p^c, ..., p^c)$ is the unique equilibrium; holding (1), the competitive payoff is earned by each firm at any equilibrium. It is also known that there are no pure strategy equilibria if neither inequality (1) nor inequality (2) holds or, equivalently, if

$$\frac{k_1}{K} > \max\left\{1 - \frac{D(0)}{K}, \mid \varepsilon \mid_{p=p^c}\right\}. \tag{3}$$

where ε is the price elasticity of demand.

In the remainder, inequality (3) is assumed to hold. Denote by $\sigma_i : (0, \infty) \to [0, 1]$ a mixed strategy of firm i, where $\sigma_i(p) = \Pr_{\sigma_i}(p_i < p)$ is the probability that firm i charges a price lower than p under strategy σ_i. Note that $\sigma_i(p)$ is continuous except at any p° such that $\Pr_{\sigma_i}(p_i = p^\circ) > 0$. A mixed strategy equilibrium is denoted by $\phi = (\phi_1, ..., \phi_z) : (0, \infty)^z \to [0, 1]^z$, where $\phi_i(p) = \Pr_{\phi_i}(p_i < p)$. We denote by $\Pi_i(\sigma_i, \phi_{-i})$ firm i's expected profit when it follows strategy σ_i and the rivals are playing their equilibrium strategy profile ϕ_{-i}; in particular $\Pi_i(p, \phi_{-i})$ is firm i's expected profit when it charges p with certainty and the rivals are playing their equilibrium strategy profile ϕ_{-i}. We denote by Π_i^* firm i's expected profit at equilibrium ϕ, by S_i the support of ϕ_i, and by $p_M^{(i)}$ and $p_m^{(i)}$ the maximum and the minimum of S_i, respectively. Note that $p \in S_i$ when there is $\lambda > 0$ such that $\phi_i(p + h) > \phi_i(p - h)$ for each $h \in (0, \lambda)$. Clearly, $\Pi_i^* \geqslant \Pi_i(\sigma_i, \phi_{-i})$ (each i). For any $p \in S_i$, $\Pi_i^* = \Pi_i(p, \phi_{-i})$ almost everywhere, namely, whenever $\Pr_{\phi_j}(p_j = p) = 0$ (any $j \neq i$). In fact, $\Pi_i^* = \lim_{p_i \longrightarrow p-} \Pi_i(p_i, \phi_{-i})$ everywhere for $p \in S_i$ since, quite obviously, $\Pi_i^* \geq \lim_{p_i \longrightarrow p-} \Pi_i(p, \phi_{-i})$ (any p) and, furthermore, Π_i^* cannot be greater than $\lim_{p_i \longrightarrow p-} \Pi_i(p, \phi_{-i})$ for some $p \in S_i$: since $\lim_{p_i \longrightarrow p+} \Pi_i(p, \phi_{-i}) \leqslant \Pi_i(p, \phi_{-i}) \leqslant \lim_{p_i \longrightarrow p-} \Pi_i(p, \phi_{-i})$, that event would imply that $\Pi_i(p, \phi_{-i}) < \Pi_i^*$ on a neighbourhood of p, contrary to the fact that $p \in S_i$.

We now present a number of properties of mixed strategy equilibria.

Proposition 1 *Let inequality (3) hold. Then, in any equilibrium:*

1. $p_M^{(1)} = p_M > p^c$, *where*

$$p_M = \arg\max_p p(D(p) - \sum_{j \neq 1} k_j); \tag{4}$$

2. $\Pi_i^* = p_M(D(p_M) - \sum_{j\neq 1} k_j) > 0$ *for any* i *such that* $k_i = k_1$;

3. $p_m^{(1)} = p_m > p^c$, *where* $p_m = \max\{\widehat{p}, \widehat{\widehat{p}}\}$ *where* $\widehat{p} = \Pi_1^*/k_1$ *and* $\widehat{\widehat{p}}$ *is the lower solution of equation* $pD(p) = \Pi_1^*$;

4. $\Pi_2^* = p_m k_2$ *whether* $k_2 < k_1$ *or* $k_2 = k_1$; $\Pi_i^* = p_m k_i$, *each* $i \in \mathcal{L} - \{1\}$, *where* $\mathcal{L} := \left\{ i \in \mathcal{Z} \,\middle|\, p_m^{(i)} = p_m \right\}$;

5. $D(p_m) < \sum_{i\in\mathcal{L}} k_i$ *and* $\Pr_{\phi_i}(p_i = p_m) = 0$, *each* $i \in \mathcal{Z}$;

6. $\Pr_{\phi_1}(p_1 = p_M) > 0$ *if, and only if,* $k_1 > k_2$ *and* $\Pr_{\phi_i}(p_i = p_M) = 0$ *for any* $i \neq 1$.

For proof of these points, see Kreps and Scheinkman [12] for the duopoly and, e.g., De Francesco and Salvadori [6] for the oligopoly. Yet in the proof of the current Proposition 1(5) De Francesco and Salvadori [6] took for granted that $\Pr_{\phi_i}(p_i = p_m) = 0$ (each $i \in \mathcal{Z}$). For this reason we complete here the proof of this part.

Proof. (of Proposition 1(5))
By way of contradiction, let $D(p_m) > \sum_{i\in\mathcal{L}} k_i$. Then $\Pi_i(p, \phi_{-i}) = pk_i > p_m k_i = \Pi_i^*$ (each $i \in \mathcal{L}$) on a right neighbourhood of p_m. Next let $D(p_m) = \sum_{i\in\mathcal{L}} k_i$. Then $\frac{(p-p_m)k_1}{p\left[\sum_{j\in\mathcal{L}} k_j - D(p)\right]} > 1$ on a right neighbourhood of p_m since $\lim_{p\to p_m+} \frac{(p-p_m)k_1}{p\left[\sum_{j\in\mathcal{L}} k_j - D(p)\right]} = \frac{k_1}{-p_m D'(p_m)} > 1$: indeed, $p_m k_1 > -p^2 D'(p)$ over the range $[p_m, p_M)$, since $-p^2 D'(p)$ is strictly increasing over that range, by strict concavity of $pD(p)$, and $-p_M^2 D'(p) \mid_{p=p_M} = p_m k_1$.[6] Therefore, $\prod\limits_{j\in\mathcal{L}-\{1\}} \phi_j(p) \geq \frac{(p-p_m)k_1}{p\left[\sum_{j\in\mathcal{L}} k_j - D(p)\right]} > 1$ on a right neighbourhood of p_m - the weak inequality being certainly an equality if $p \in S_1$ on such a neighbourhood[7] - an obvious contradiction. Therefore, $D(p_m) < \sum_{j\in\mathcal{L}} k_j$. Should $\Pr_{\phi_i}(p_i = p_m)$ be greater than zero for some $i \in \mathcal{L}$, then $\Pi_j(p, \phi_{-j}) < \Pi_j(p_m, \phi_{-j})$ (any $j \in \mathcal{L} - \{i\}$) for p larger than and close enough to p_m, and hence $p \notin (\cup S_{j\in\mathcal{L}-\{i\}})$ for any such p. This means that $\Pr_{\phi_j}(p_j = p_m) > 0$ (each $j \in \mathcal{L} - \{i\}$), in its turn implying that $\Pi_j^* = \Pi_j(p_m, \phi_{-j}) < p_m k_j = \lim_{p\to p_m^-} \Pi_j(p, \phi_{-j}) = p_m k_j$. ■

3 Some firms are large and the others are small

We will focus on a specific subset of the region of no pure strategy equilibria, that in which

$$k_1 + ... + k_n \geqslant D(p_m) \tag{5}$$

[6] The last equality derives from equalities $D(p_M) - \sum\limits_{j\neq 1} k_j + p_M D'(p) \mid_{p=p_M} = 0$ and $p_m k_1 = p_M[D(p_M) - \sum\limits_{j\neq 1} k_j]$ (see Propositions 1(1)&(2)).

[7] Strict inequality might hold if, instead, $p \notin S_1$ for p higher than and close enough to p_m - which, by the way, implies that $\Pr_{\phi_1}(p_1 = p_m) > 0$.

$$D(p_M) \geqslant K - k_n. \tag{6}$$

The sets $\mathcal{N} = \{1, ..., n\}$ and $\mathcal{Z} - \mathcal{N}$ will be referred to as the set of "large" firms and the set of "small" firms, respectively. Let us look more deeply at these inequalities in order to grasp the rationale for this terminology. According to inequality (5), large firms as a whole can meet even the highest demand that can arise at an equilibrium of the price game, $D(p_m)$. If $n = 1$, inequality (5) coincides with the inequality that defines the subset of the no-pure strategy equilibrium region mentioned in the introduction as explored by Hirata [11] (and De Francesco and Salvadori [6]). According to inequality (6), total industry capacity minus the capacity of even the smallest of the large firms does not exceed the smallest level of demand possibly arising at an equilibrium of the price game, $D(p_M)$. If $n = 1$, inequality (6) coincides with inequality $D(p_M) \geqslant K - k_1$, which certainly holds as a strict inequality. Most importantly, since $K > D(p_m) > D(p_M)$, inequalities (5) and (6) imply that

$$k_n > k_{n+1} + ... + k_z, \tag{7}$$

$$k_1 - k_n \leqslant D(p_M) - \sum_{j \neq 1} k_j \tag{8}$$

consistent with the "small" labelling of firms from $n + 1$ to z and with the "large" labelling of firms from 1 to n.

Because of inequalities (5) and (6), almost everywhere in the range $[p_m, p_M]$ the payoff function of firm $i \in \mathcal{N}$ in the face of rivals' equilibrium strategies is equal to

$$\Pi_i(p, \phi_{-i}) = p \prod_{j \in \mathcal{N} - \{i\}} \phi_j(p) \left[D(p) - \sum_{j \in \mathcal{N} - \{i\}} k_j - \sum_{r \in \mathcal{Z} - \mathcal{N}} \phi_r(p) k_r \right] +$$
$$+ \left[1 - \prod_{j \in \mathcal{N} - \{i\}} \phi_j(p) \right] p k_i,$$

that is

$$\Pi_i(p, \phi_{-i}) = p k_i - p \prod_{j \in \mathcal{N} - \{i\}} \phi_j(p) \left[\sum_{j \in \mathcal{N}} k_j + \sum_{r \in \mathcal{Z} - \mathcal{N}} \phi_r(p) k_r - D(p) \right], \tag{9}$$

whereas almost everywhere in the same range the payoff function of firm $r \in \mathcal{Z} - \mathcal{N}$ in the face of rivals' equilibrium strategies is equal to[8]

$$\Pi_r(p, \phi_{-r}) = F(p) k_r \tag{10}$$

[8]There are two reasons for the "almost everywhere" qualification. First, thus far we have not ruled out the event that, for some $p^\circ \in (p_m, p_M)$, $\phi_j(p^{\circ +}) > \phi_j(p^\circ)$ (some $j \in \mathcal{N}$): under that event, for instance, $\Pi_i(p^\circ, \phi_{-i}) < \lim_{p \to p^\circ -} \Pi_i(p, \phi_{-i})$, the RHS being actually the RHS of (9). Second, because of Proposition 1(6), if $k_1 > k_2$ then $\lim_{p \to p_M -} \Pi_i(p, \phi_{-i}) > \Pi_i(p_M, \phi_{-i})$ (any $i \in \mathcal{Z} - \{1\}$).

where

$$F(p) = \left[1 - \prod_{j \in \mathcal{N}} \phi_j(p)\right] p. \tag{11}$$

As will become apparent below, the main features of equilibria derive from the properties of the payoff functions, $\Pi_i(p, \phi_{-i})$ (each $i \in \mathcal{N}$) and $\Pi_r(p, \phi_{-i})$ (each $r \in \mathcal{Z} - \mathcal{N}$). The payoff function of each small firm does not depend on the equilibrium distributions of the other small firms: each small firm has no residual demand when all the large firms are selling at a lower price, while selling its entire capacity when at least one of the large firms is more expensive. The payoff function of each large firm depends on the joint equilibrium cumulative distribution of the other large firms and the capacity-weighted sum of the equilibrium cumulative distributions of the small firms. In fact, each large firm sells its entire capacity when at least one of the other large firms is more expensive, whereas, under the complementary event in which all the other large firms are less expensive, its expected residual demand is equal to total demand minus the total capacity of all the other large firms minus a weighted sum of the small firms' capacities, with weights equal to the values of the respective cumulative distributions.

Before going any further, it is helpful to note that

$$\frac{\partial \Pi_1(p, \phi_{-1})}{\partial p} = \prod_{j \in \mathcal{N}-\{1\}} \phi_j(p) \left[D(p) + pD'(p) - \sum_{j \in \mathcal{N}-\{1\}} k_j - \sum_{r \in \mathcal{Z}-\mathcal{N}} \phi_r(p) k_r\right] + \left[1 - \prod_{j \in \mathcal{N}-\{1\}} \phi_j(p)\right] k_1 > 0 \tag{12}$$

since $D(p) + pD'(p) - \sum_{j \neq 1} k_j > 0$ because of Proposition 1(2) and the strict concavity of $pD(p)$. We can now determine the equilibrium payoff of each large firm (and each small firm in a special case) and prove properties concerning the supports of the strategies, the payoffs and the equilibrium distributions of the large firms.

Proposition 2 *In any equilibrium*

(i) $\mathcal{L} \supseteq \mathcal{N}$, $\Pi_i^* = p_m k_i$ *(each* $i \in \mathcal{N}$*) and* $\phi_i(p) k_i = \phi_j(p) k_j$ *everywhere for* $p \in S_i \cap S_j$ *(any* $i, j \in \mathcal{N}$*); moreover,* $k_j \Pi_i(p, \phi_{\_i}) = k_i \Pi_j(p, \phi_{\_j})$ *almost everywhere for* $p \in S_i \cap S_j$ *(any* $i, j \in \mathcal{N}$*);*[9]

(ii) $\Pi_r^*/k_r = \Pi_s^*/k_s$ *(each* $r, s \in \mathcal{Z} - \mathcal{N}$*);*

(iii) if $k_1 + ... + k_n > D(p_m)$*, then* $\mathcal{L} = \mathcal{N}$ *and* $\Pi_r^* > p_m k_r$ *(each* $r \in \mathcal{Z} - \mathcal{N}$*);*

(iv) if $k_1 + ... + k_n = D(p_m)$*, then* $\mathcal{L} \supset \mathcal{N}$ *and* $\Pi_i^* = p_m k_i$ *(each* $i \in \mathcal{Z}$*);*

(v) $S_i = [p_m, p_M^{(i)}]$ *(each* $i \in \mathcal{N}$*);* $S_1 = S_2 \supseteq S_3 \supseteq ... \supseteq S_n$*; moreover,* $S_i \supset S_{i+1}$ *(each* $i \in \mathcal{N} - \{1, n\}$*) if and only if* $k_i > k_{i+1}$*;*

[9] Because of part (vi), not yet proved, $k_j \Pi_i(p, \phi_{\_i}) = k_i \Pi_j(p, \phi_{\_j})$ *everywhere* for $p \in S_i \cap S_j - \{p_M\}$ (any $i, j \in \mathcal{N}$).

(vi) $\Pr_{\phi_i}(p_i = p) = 0$ *(any* $p \in [p_m, p_M)$ *and any* $i \in \mathcal{Z}$*);*

(vii) $\overline{\overline{p}} := \max \bigcup_{r \in \mathcal{Z}-\mathcal{N}} S_r \leqslant p_M^{(n)}$*;*

(viii) if either any of inequalities (5) and (6) is satisfied as a strict inequality or $k_2 > k_n$*, then* $\overline{\overline{p}} < p_M^{(n)}$ *and* $\Pi_r(p_M^{(n)}, \phi_{-r}) < \Pi_r^*$*.*

Proof. (i) Since $D(p_m) > D(p_M) \geqslant K - k_n \geqslant \sum_{i \in \mathcal{L}} k_i$, if $\mathcal{L} \not\supseteq \mathcal{N}$, then Proposition 1(5) is contradicted. $\mathcal{L} \supseteq \mathcal{N}$ implies $\Pi_i^* = p_m k_i$ (each $i \in \mathcal{N}$), because of Proposition 1(4), and hence

$$\prod_{j \in \mathcal{N}-\{i\}} \phi_j(p) = \frac{(p - p_m)k_i}{p\left[\sum\limits_{j \in \mathcal{N}} k_j + \sum\limits_{r \in \mathcal{Z}-\mathcal{N}} \phi_r(p)k_r - D(p)\right]}, \qquad \forall p \in S_i \tag{13}$$

because of equation (9). As a consequence, for any $p \in S_i \cap S_j$ $(i, j \in \mathcal{N})$, $\phi_i(p)k_i = \phi_j(p)k_j$. Moreover, from equations (9) we obtain that, almost everywhere throughout $[p_m, p_M]$,

$$\frac{\Pi_j(p, \phi_{-j})}{k_j} = \frac{\Pi_i(p, \phi_{-i})}{k_i} + \frac{\phi_j(p)k_j - \phi_i(p)k_i}{\phi_j(p)\phi_i(p)k_j k_i} p \prod_{l \in \mathcal{N}} \phi_l(p) \left[\sum_{l \in \mathcal{N}} k_l + \sum_{r \in \mathcal{Z}-\mathcal{N}} \phi_r(p)k_r - D(p)\right].$$

(ii) Equations (10) imply that $\Pi_r(p, \phi_{-r})k_s = \Pi_s(p, \phi_{-s})k_r$ (any $r, s \in \mathcal{Z} - \mathcal{N}$), almost everywhere throughout $[p_m, p_M]$.[10] Then the claim follows straightforwardly. Indeed, if $\Pi_r^*/k_r < \Pi_s^*/k_s$, then $S_r \cap S_s = \emptyset$ since, at any $p \in S_r \cap S_s$, $\Pi_r(p, \phi_{-r}(p)) = \Pi_r^*$ and $\Pi_s(p, \phi_{-s}(p)) = \Pi_s^*$; but then firm r's strategy would not be a best response to ϕ_{-r}, since a payoff of $\Pi_r(p, \phi_{-r}) = (k_r/k_s)\Pi_s^* > \Pi_r^*$ is obtained by quoting any $p \in S_s$.

(iii) If $\mathcal{L} \supset \mathcal{N}$, then, by Proposition 1(4) and part (ii), $\Pi_r^* = p_m k_r$ (each $r \in \mathcal{Z} - \mathcal{N}$. Then, according to Proposition 1(5), $p \in S_r$ (some $r \in \mathcal{L} - \mathcal{N}$) for p larger than and close enough to p_m. Hence equation (10) implies $\prod\limits_{i \in \mathcal{N}} \phi_i(p) = \frac{p - p_m}{p}$. Thus, on a right neighbourhood of p_m, $\sum\limits_{j \in \mathcal{N}} k_j + \sum\limits_{r \in \mathcal{Z}-\mathcal{N}} \phi_r(p)k_r - D(p) = \phi_i(p)k_i$, because of equation (13) (each $i \in \mathcal{N}$); but then it follows from $\lim_{p \to p_m+} \phi_i(p) = 0$ that $\lim_{p \to p_m+} \sum\limits_{r \in \mathcal{Z}-\mathcal{N}} \phi_r(p)k_r < 0$. Thus $\mathcal{L} = \mathcal{N}$. Further, since $\lim_{p \to p_m+} F'(p) = 1$, $F(p)$ is increasing on a right neighbourhood of p_m. As a consequence, $\Pi_r^* > p_m k_r$ (any $r \in \mathcal{Z} - \mathcal{N}$). Otherwise firm r would have failed to make a best response given that $\Pi_r(p, \phi_{-r}) > p_m k_r$ for p close enough to p_m.

(iv) If $\mathcal{L} = \mathcal{N}$, then Proposition 1(5) is contradicted. Therefore, by Proposition 1(3) and part (ii), $\Pi_r^* = p_m k_r$ (each $r \in \mathcal{Z} - \mathcal{N}$).

(v) The claim is obviously equivalent to:

$$S_{n-u} = \left[p_m, p_M^{(n-u)}\right] = \cap_{h \in [1, n-u]} S_h \qquad u = 0, 1, ..., n-2 \tag{14}$$

[10] The argument in the text would work even if, contrary to part (vi), not yet proved, $\Pr_{\phi_i}(p_i = p^\circ) > 0$ (some $p^\circ \in S_i$ and some $i \in \mathcal{N}$), except for $\Pi_r(p^\circ, \phi_{-r})$ and $\Pi_s(p^\circ, \phi_{-s})$ being replaced by $\lim_{p \to p^\circ -} \Pi_r(p, \phi_{-r})$ and $\lim_{p \to p^\circ -} \Pi_s(p, \phi_{-r})$, respectively.

Property (14) will be proved by induction. Let us first prove that (14) holds for $u = 0$. Because of part (i) and Proposition 1(5), there is $\widetilde{p}$ such that $[p_m, \widetilde{p}] \subseteq \cap_{i \in \mathcal{N}} S_i$. If there is $\widetilde{\widetilde{p}} > \widetilde{p}$ such that $(\widetilde{p}, \widetilde{\widetilde{p}}) \cap (\cap_{i \in \mathcal{N}} S_i) = \emptyset$, then either (a) $\widetilde{p} = p_M^{(n)}$, or (b) $\Pr_{\phi_i}(p_i = \widetilde{p}) > 0$ for some $i \in \mathcal{Z}$, or (c) there is a gap $(\widetilde{p}, p^\circ)$ in S_j (some $j \in \mathcal{N}$ and some $p^\circ > \widetilde{p}$): namely, $\phi_j(p^\circ) = \phi_j(\widetilde{p})$, while $\phi_j(p)$ is increasing in both $\widetilde{p}$ and p°. Let us first exclude the event (b). By way of contradiction, let $\Pr_{\phi_r}(p_r = \widetilde{p}) > 0$ (some $r \in \mathcal{Z} - \mathcal{N}$). As a consequence, there is $p^\circ \in (\widetilde{p}, p_M)$ such that $(\widetilde{p}, p^\circ) \cap (\cup_{j \in \mathcal{N}} S_j) = \emptyset$, since $\lim_{p \to \widetilde{p}+} \Pi_j(p, \phi_{-j}) < \lim_{p \to \widetilde{p}-} \Pi_j(p, \phi_{-j}) = \Pi_j^*$, each $j \in \mathcal{N}$. But then it follows from equation (10) that $\Pi_r(p, \phi_{-r}) > \Pi_r(\widetilde{p}, \phi_{-r}) = \Pi_r^*$ over the range $(\widetilde{p}, p^\circ)$. Quite similarly, if $\Pr_{\phi_i}(p_i = \widetilde{p}) > 0$ (some $i \in \mathcal{N}$), then $(\widetilde{p}, p^\circ) \cap (\cup_{j \in \mathcal{Z} - \{i\}} S_j) = \emptyset$ for some $p^\circ \in (\widetilde{p}, p_M)$, but then $\Pi_i(p, \phi_{-i}) > \Pi_i(\widetilde{p}, \phi_{-i}) = \Pi_i^*$ over the range $(\widetilde{p}, p^\circ)$ because of part (i) and inequality (12). Let us now exclude the event (c). If there is $p^{\circ\circ} \in (\widetilde{p}, p^\circ]$ such that $(\widetilde{p}, p^{\circ\circ}) \cap (\cup_{i \in \mathcal{Z}} S_i) = \emptyset$, a contradiction is found by following an argument similar to that used to exclude the event (b). Then there is $h \neq j$ such that $(\widetilde{p}, p^\circ) \cap S_h \neq \emptyset$ and $\phi_j(p^\circ) k_j = \phi_j(\widetilde{p}) k_j = \phi_h(\widetilde{p}) k_h < \phi_h(p^\circ) k_h$, and therefore $\Pi_j(p^\circ, \phi_{-j}) < \frac{k_j}{k_h} \Pi_h(p^\circ, \phi_{-h}) \leqslant \frac{k_j}{k_h} \Pi_h^* = \Pi_j^*$, contrary to the fact that $p^\circ \in S_j$. Now assume that property (14) holds for $u = v < n-2$; then there is $\widetilde{p} \geqslant p_M^{(n-v)}$ such that $[p_m, \widetilde{p}] = \ cap_{h \in \mathcal{N}; h \leqslant n-v-1} S_h$. Hence the same argument used above proves that property (14) holds for $u = v + 1$. Note that $p_M^{(n-v)} = p_M^{(n-v-1)}$ if and only if $k_{n-v-1} = k_{n-v}$, because of part (i).

(vi) It is an obvious consequence of part (v).

(vii) By way of contradiction, let $\overline{\overline{p}} > p_M^{(n)}$. Then $\phi_1(\overline{\overline{p}}) > \phi_1(p_M^{(n)}) = \frac{k_n}{k_1}$, the equality being a consequence of part (i). Therefore,

$$\prod_{j \in \mathcal{N}} \phi_j(\overline{\overline{p}}) = \phi_1(\overline{\overline{p}}) \frac{\overline{\overline{p}} - p_m}{\overline{\overline{p}}} \frac{k_1}{K - D(\overline{\overline{p}})} > \frac{\overline{\overline{p}} - p_m}{\overline{\overline{p}}} \frac{k_n}{K - D(\overline{\overline{p}})} \geqslant \frac{\overline{\overline{p}} - p_m}{\overline{\overline{p}}} : \qquad (15)$$

the equality is a consequence of equation (9) and part (i) since $\sum_{r \in \mathcal{Z} - \mathcal{N}} \phi_r(\overline{\overline{p}}) k_r = \sum_{r \in \mathcal{Z} - \mathcal{N}} k_r$; the second inequality is a consequence of inequality (6). Thus $\Pi_r(\overline{\overline{p}}, \phi_{-r})) < p_m k_r$ because of equation (10) and the definition of $\overline{\overline{p}}$ is contradicted.

(viii) By way of contradiction, let $\overline{\overline{p}} = p_M^{(n)}$. Then instead of (15) we have

$$\prod_{j \in \mathcal{N}} \phi_j(\overline{\overline{p}}) = \phi_1(\overline{\overline{p}}) \frac{\overline{\overline{p}} - p_m}{\overline{\overline{p}}} \frac{k_1}{K - D(\overline{\overline{p}})} = \frac{\overline{\overline{p}} - p_m}{\overline{\overline{p}}} \frac{k_n}{K - D(\overline{\overline{p}})} \geqslant \frac{\overline{\overline{p}} - p_m}{\overline{\overline{p}}}. \qquad (16)$$

It follows that $\Pi_r(\overline{\overline{p}}, \phi_{-r})) \leqslant p_m k_r$. Hence, if inequality (5) holds as a strict inequality, part (iii) is contradicted; if either inequality (6) holds as a strict inequality or $k_2 > k_n$ (or both), then the weak inequality in (16) is satisfied as a strict inequality and hence $\Pi_r(\overline{\overline{p}}, \phi_{-r})) < p_m k_r$. Finally, it follows from $\overline{\overline{p}} < p_M^{(n)}$ that

$$\prod_{j \in \mathcal{N}} \phi_j(p_M^{(n)}) = \phi_1(p_M^{(n)}) \frac{(p_M^{(n)} - p_m) k_1}{p_M^{(n)} \left[K - D(p_M^{(n)})\right]} = \frac{(p_M^{(n)} - p_m) k_n}{p_M^{(n)} \left[K - D(p_M^{(n)})\right]} \geqslant \frac{p_M^{(n)} - p_m}{p_M^{(n)}},$$

implying that $\Pi_r(p_M^{(n)}, \phi_{-r})) \leqslant p_m k_r \leqslant \Pi_r^*$, with at least one strict inequality: the last inequality is strict if inequality (5) is strict and the first inequality is strict if either $k_2 > k_n$, or inequality (6) is strict (or both). ■

Proposition 2 allows segment $[p_m, p_M]$ to be partitioned into three parts: $[p_m, \overline{p})$, $[\overline{p}, \overline{\overline{p}}]$, $(\overline{\overline{p}}, p_M]$, where $\overline{p} = \min \bigcup\limits_{r\in\mathcal{Z}-\mathcal{N}} S_r$ and $\overline{\overline{p}} = \max \bigcup\limits_{r\in\mathcal{Z}-\mathcal{N}} S_r$. The first part is empty if and only if $k_1 + ... + k_n = D(p_m)$; the second part contains $\bigcup\limits_{r\in\mathcal{Z}-\mathcal{N}} S_r$. In the first and third parts the equilibrium distributions are easily determined.

3.1 The equilibrium distributions in $[p_m, \overline{p})$

In this and in the following subsection we assume that $k_1 + ... + k_n > D(p_m)$. In the range $[p_m, \overline{p})$ the equilibrium distributions are: $\phi_r(p) = 0$ for each $r \in \mathcal{Z} - \mathcal{N}$ and

$$\phi_l(p) = \frac{1}{k_l}\left(\frac{p - p_m}{p}\frac{\prod\limits_{j\in\mathcal{N}} k_j}{\sum\limits_{j\in\mathcal{N}} k_j - D(p)}\right)^{\frac{1}{n-1}} \tag{17}$$

for each $l \in \mathcal{N}$, because of equations (9) and Proposition 2. It is easily recognized that the RHS of equation (17) is quasi-concave throughout $[p_m, p_M]$.[11] Moreover, it is larger than 1 for $p = p_M$ and $l = n$ since $p_M\left[D(p_M) - \sum\limits_{j\in\mathcal{N}-\{1\}} k_j\right] > p_m k_1$. Hence there is $\widetilde{p}_M^{(n)} \in (p_m, p_M)$ such that in the range $[p_m, \widetilde{p}_M^{(n)}]$ the RHS of equation (17) is increasing and no larger than 1 for each $i \in \mathcal{N}$. Hence the functions $F(p)$ and $\Pi_r(p, \phi_{-r}) = F(p)k_r$ are well-defined in the range $[p_m, \overline{p}]$ if and only if $\overline{p} \leqslant \widetilde{p}_M^{(n)}$ and this inequality can easily be proved by following the same procedure used to prove Proposition 2(vii)-(viii).

3.2 Determining $\overline{p}$ and the equilibrium payoffs of small firms

In order to determine $\overline{p}$ and the equilibrium payoff of each small firm, the functions $\phi_l(p)$ (each $l \in \mathcal{N}$) and $F(p)$, as calculated in the range $[p_m, \overline{p}]$ - that is, *by keeping* $\phi_r(p) = 0$ (each $r \in \mathcal{Z} - \mathcal{N}$) - need to be extended somewhat beyond $\overline{p}$. Let us call these extended functions $\phi_l^g(p)$ and $G(p)$, respectively. In the range $[p_m, \widetilde{p}_M^{(n)}]$, $\phi_l^g(p)$ consists of the RHS of equation (17) and $G(p) = \left[1 - \prod\limits_{j\in\mathcal{N}} \phi_l^g(p)\right] p$. The functions $\phi_l^g(p)$ and $G(p)$ are well-defined in the

[11]The sign of its first derivative coincides with the sign of function $p_m\left[\sum\limits_{j\in\mathcal{N}} k_j - D(p)\right] + (p - p_m)\, pD'(p)$ which is decreasing in the mentioned range, is positive for $p = p_m$ and negative for $p = p_M$.

mentioned range. As we will see, $\overline{p}$ equals the argument of a maximum of $G(p)$ in the range $(p_m, \widetilde{p}_M^{(n)})$. We will show that such a maximum exists, but we were not able to prove that it is unique, even if all our simulations suggest that it is so. That said, we prove that $\overline{p}$ coincides with the largest argument in which such a maximum is obtained.

Proposition 3 *Let $k_1+...+k_n > D(p_m)$. Then $\overline{p} = \max \arg\max_{p \in (p_m, \widetilde{p}_M^{(n)})} G(p)$ and*

$$\Pi_r^* = \left[1 - \left(\frac{(\overline{p} - p_m)\left(\prod_{j \in \mathcal{N}} k_j\right)^{\frac{1}{n}}}{\overline{p}\left[\sum_{j \in \mathcal{N}} k_j - D(\overline{p})\right]}\right)^{\frac{n}{n-1}}\right] \overline{p} k_r. \tag{18}$$

Proof. Since $G(\widetilde{p}_M^{(n)}) = \left[1 - \prod_{j \in \mathcal{N}} \phi_j^g(\widetilde{p}_M^{(n)})\right] \widetilde{p}_M^{(n)} < p_m$, as can easily be checked, $G(p)$ has a maximum at some $p \in (p_m, \widetilde{p}_M^{(n)})$. By way of contradiction, let $G(p) \geqslant F(\overline{p})$ for some $p \in (\overline{p}, \widetilde{p}_M^{(n)})$. Then, $\left[1 - \prod_{j \in \mathcal{N}} \phi_j(p)\right] p \leqslant F(\overline{p}) = G(\overline{p}) \leqslant G(p) = \left[1 - \prod_{j \in \mathcal{N}} \phi_j^g(p)\right] p$, where the first weak inequality is certainly an equality for $p \in \cup S_r$. Therefore, $\prod_{j \in \mathcal{N}} \phi_j(p) \geqslant \prod_{j \in \mathcal{N}} \phi_j^g(p)$ and, as a consequence of equation (13) and the definition of functions $\phi_j^g(p)$'s, $\sum_{r \in \mathcal{Z} - \mathcal{N}} \phi_r(p) k_r \leqslant 0$, an obvious contradiction. Next, again by way of contradiction, let $G(p) > G(\overline{p})$ for some $p \in (p_m, \overline{p})$. Under such an event, firm r would get $G(p)k_r > \Pi_r^* = G(\overline{p})k_r$ by charging a price somewhat less than $\overline{p}$. Finally, equation (18) derives straightforwardly from $\Pi_r^* = F(\overline{p})k_r$ (each $r \in \mathcal{Z} - \mathcal{N}$) and equation (17). ■

A simple intuition can be gained if we spell out the procedure whereby, whenever $\overline{p} > p_m$, we have determined $\overline{p}$ and correspondingly the equilibrium payoff per unit of capacity $F(\overline{p})$ of each small firm: $\overline{p}$ is such as to maximize firm r's payoff function, when the strategy profile of the large firms is such as to yield them their equilibrium payoffs $p_m k_i$ in the event of small firms charging a higher price.

3.3 The equilibrium distributions in $(\overline{\overline{p}}, p_M]$

In the range $(\overline{\overline{p}}, p_M]$, $\phi_r(p) = 1$ for each $r \in \mathcal{Z} - \mathcal{N}$ and equations (9) can thus be written

$$\Pi_i(p, \phi_{-i}) = pk_i - \prod_{j \in \mathcal{N} - \{i\}} \phi_j(p) p \left[K - D(p)\right]. \tag{19}$$

Taking into account Proposition 2(i)&(iv), these equations are enough to determine all the ϕ_i's in the range $(\overline{\overline{p}}, p_M]$. This is done straightforwardly if $k_2 = k_n$.

In this case $[\overline{\overline{p}}, p_M] \subset (\cap_{j\in\mathcal{N}} S_j)$: then it follows from equations (19) that, for each $i \in \mathcal{N}$,

$$\phi_i(p) = \frac{1}{k_i}\left(\frac{p-p_m}{p}\frac{\prod\limits_{j\in\mathcal{N}} k_j}{K-D(p)}\right)^{\frac{1}{n-1}} \tag{20}$$

throughout $(\overline{\overline{p}}, p_M]$. If, instead, $k_2 > k_n$, then $(\overline{\overline{p}}, p_M]$ can be partitioned in a number of non-empty intervals $(p_M^{(i+1)}, p_M^{(i)}]$, where each $i < n$ is such that $k_i > k_{i+1}$ and, by definition, $p_M^{(n+1)} = \overline{\overline{p}}$. In each range $(p_M^{(i+1)}, p_M^{(i)}]$, $\phi_l(p) = 1$ for $l = i+1, ..., n$; then equations (19) lead to

$$\phi_l(p) = \frac{1}{k_l}\left(\frac{p-p_m}{p}\frac{\prod\limits_{j\leqslant i} k_j}{K-D(p)}\right)^{\frac{1}{i-1}} \tag{21}$$

for each $l = 1, ..., i$. The RHS of equation (21) (each $l = 1, ..., i$) is in fact strictly increasing over the range $(p_M^{(i+1)}, p_M]$, its derivative being strictly decreasing over that range and equal to zero at $p = p_M$: hence, $p_M^{(i)}$ is the unique solution of the equation $(p-p_m)\prod\limits_{j\leqslant i} k_j = p\,[K - D(p)]\, k_i^{i-1}$ over the range $(p_M^{(i+1)}, p_M]$. Thus $p_M^{(i)} = p_M$ if $k_i = k_2$, since $p_m k_1 = p_M\left[D(p_M) - \sum\limits_j k_{j\neq 1}\right]$; if $k_i < k_2$, then $p_M^{(i)} < p_M$ and $\phi_l(p_M^{(i)}) = \frac{k_i}{k_l} < 1$ for any $l < i$ such that $k_l > k_i$.

Next we prove that any large firm l with $k_l < k_2$ would earn strictly less than Π_l^* by charging any price higher than $p_M^{(l)}$. In the next subsection, we prove that any small firm r would earn strictly less than Π_r^* by charging more than $\overline{\overline{p}}$. This will complete the analysis of the range $(\overline{\overline{p}}, p_M]$.

Proposition 4 *For any $l \in \mathcal{N} - \{1, 2\}$ such that $k_l < k_2$, $\Pi_l(p, \phi_{-l}) < \Pi_l^*$ over the range $(p_M^{(l)}, p_M]$.*

Proof. It is enough to remark that over any non-empty range $(p_M^{(i+1)}, p_M^{(i)}]$, $\Pi_l(p, \phi_{-l})/k_l < \Pi_i(p, \phi_{-i})/k_i = p_m$ for any $l \geqslant i+1$, since $\phi_i(p) > \frac{k_l}{k_i}$. ■

3.4 The equilibrium distributions in $[\overline{p}, \overline{\overline{p}}]$

Let $p \in \bigcup\limits_{r\in\mathcal{Z}-\mathcal{N}} S_r$. Then we obtain from equations (10) and Proposition 3 that

$$\prod_{j\in\mathcal{N}} \phi_j(p) = \frac{p - F(\overline{p})}{p} \tag{22}$$

and, afterwards, from equations (9) and Proposition 2 that

$$\phi_l(p) = \frac{1}{k_l}\frac{p - F(\bar{p})}{p - p_m}\left[\sum_{j\in\mathcal{N}} k_j + \sum_{r\in\mathcal{Z}-\mathcal{N}} \phi_r(p)k_r - D(p)\right] \quad l \in \mathcal{N}. \tag{23}$$

As a consequence, also by using equation (22) again,

$$\sum_{r\in\mathcal{Z}-\mathcal{N}} \phi_r(p)k_r = \left[\frac{p}{p - F(\bar{p})}\right]^{\frac{n-1}{n}} \frac{p - p_m}{p}\left(\prod_{j\in\mathcal{N}} k_j\right)^{\frac{1}{n}} - \left[\sum_{j\in\mathcal{N}} k_j - D(p)\right] \tag{24}$$

Finally, from equations (23) and (24) we obtain

$$\phi_l(p) = \frac{1}{k_l}\left[\frac{p - F(\bar{p})}{p}\right]^{\frac{1}{n}}\left(\prod_{j\in\mathcal{N}} k_j\right)^{\frac{1}{n}} \quad l \in \mathcal{N}. \tag{25}$$

Remark 1. *By construction the RHS of equation (17) equals the RHS of equation (25) for $p = \overline{p}$, whereas it is larger than the latter for $p > \overline{p}$. As a consequence, the RHS of equation (24) equals zero for $p = \overline{p}$ and is positive for $p > \overline{p}$.*

Another remark concerns a constant finding of our simulations, according to which the RHS of equation (24) is strictly increasing over the relevant subset. Whenever this is the case, equations (24) and (25) hold throughout $[\overline{p}, \overline{\overline{p}}]$ and $[\overline{p}, \overline{\overline{p}}] = \bigcup_{r\in\mathcal{Z}-\mathcal{N}} S_r$. On the other hand, we have not been able to establish theoretically the generality of the above finding, except for the special case in which $k_1 + ... + k_n = D(p_m)$ (see Proposition 5(v) below). Nevertheless, a general characterization of equilibria is possible. This is what is done in the following proposition.

Proposition 5 *If $k_1 + ... + k_m > D(p_m)$, in any equilibrium*

(i) $\overline{\overline{p}}$ is the largest solution of the equation

$$\left[\frac{p - F(\bar{p})}{p}\right]^{\frac{n-1}{n}} = \frac{p - p_m}{p\,[K - D(p)]}\left(\prod_{j\in\mathcal{N}} k_j\right)^{\frac{1}{n}} \tag{26}$$

over the range $(\overline{p}, p_M^{(n)})$;

(ii) the set of equilibrium distributions of the small firms is any set of non-negative, continuous and non-decreasing functions no larger than 1 *such that*

$$\sum_{r\in\mathcal{Z}-\mathcal{N}} \phi_r(p)k_r = \min_{y\in[p,\overline{\overline{p}}]} \left\{ \left[\frac{y}{y-F(\overline{p})}\right]^{\frac{n-1}{n}} \frac{y-p_m}{y} \left(\prod_{j\in\mathcal{N}} k_j\right)^{\frac{1}{n}} - \left[\sum_{j\in\mathcal{N}} k_j - D(y)\right]\right\} \quad (27)$$

over the range $[\overline{p}, \overline{\overline{p}}]$*;*

(iii) the equilibrium distributions of the large firms are uniquely determined by the equations

$$\phi_i(p) = \frac{1}{k_i}\left(\frac{p-p_m}{p}\frac{\prod_{j\in\mathcal{N}} k_j}{\sum_{j\in\mathcal{N}} k_j + \sum_{r\in\mathcal{Z}-\mathcal{N}} \phi_r(p)k_r - D(p)}\right)^{\frac{1}{n-1}} \quad (28)$$

over the range $[\overline{p}, \overline{\overline{p}}]$*;*

(iv) $\Pi_r(p, \phi_{-r}) < \Pi_r^*$ *over the range* $(\overline{\overline{p}}, p_M]$*, each* $r \in \mathcal{Z} - \mathcal{N}$*.*

(v) If $k_1 + ... + k_n = D(p_m)$*, then the RHS of equation (24) is increasing in the whole range* $[\overline{p}, \overline{\overline{p}}]$*, so that* $[\overline{p}, \overline{\overline{p}}] = \bigcup_{r\in\mathcal{Z}-\mathcal{N}} S_r$*, and* $\overline{\overline{p}} < p_M^{(n)} \leqslant p_M$*;* $\overline{\overline{p}}$ *is the single solution of the equation*

$$\left(\frac{p-p_m}{p}\prod_{j\in\mathcal{N}} k_j\right)^{\frac{1}{n}} - [K - D(p)] = 0 \quad (29)$$

over the range $(\overline{p}, p_M^{(n)})$*; the set of equilibrium distributions of the small firms is, over the range* $[\overline{p}, \overline{\overline{p}}]$*, any set of non-negative, continuous and non-decreasing functions no larger than* 1 *such that*

$$\sum_{r\in\mathcal{Z}-\mathcal{N}} \phi_r(p)k_r = \left(\frac{p-p_m}{p}\prod_{j\in\mathcal{N}} k_j\right)^{\frac{1}{n}} - \left[\sum_{j\in\mathcal{N}} k_j - D(p)\right]; \quad (30)$$

the equilibrium distributions of the large firms are uniquely determined by the equations (28) over the range $[\overline{p}, \overline{\overline{p}}]$*;* $\Pi_r(p, \phi_{-r}) < \Pi_r^*$ *over the range* $(\overline{\overline{p}}, p_M]$*, each* $r \in \mathcal{Z} - \mathcal{N}$*.*

Proof. (i) By definition $\overline{\overline{p}}$ is a solution to equation (26) and $\overline{p} < \overline{\overline{p}} < p_M^{(n)}$ because of Proposition 2(v)&(viii). Note, furthermore, that the RHS of (26) is lower (higher) than the LHS at any p where the RHS of (24) is lower (higher) than $\sum_{j\in Z-\mathcal{N}} k_r$. Over $(\overline{p}, p_M^{(n)})$ equation (26) has an odd number of solutions. Indeed, since the RHS of equation (24) equals zero at $\overline{p}$ (Remark 1), the RHS of equation (26) is smaller than the LHS at $\overline{p}$ too. On the other side, the RHS of equation (26) is larger than the LHS at $p_M^{(n)}$. In order to recognize this fact, we obtain from equations (11) and (21) for $i = n$, that

$$F(p_M^{(n)}) = \left[1 - \frac{1}{\prod\limits_{j\in\mathcal{N}} k_j}\left(\frac{p_M^{(n)} - p_m}{p_M^{(n)}}\frac{\prod\limits_{j\in\mathcal{N}} k_j}{K - D(p_M^{(n)})}\right)^{\frac{n}{n-1}}\right] p_M^{(n)}$$

and since $F(p_M^{(n)}) < F(\bar{p})$ because of Proposition 2(viii), we obtain that the RHS of equation (26) is larger than the LHS at $p_M^{(n)}$.

Let us say that a solution is odd if there is a left neighbourhood in which the RHS of equation (26) is smaller than the LHS, whereas a solution is even if there is a left neighbourhood in which the RHS of equation (26) exceeds the LHS. Let p' be an odd solution differing from the largest one and p'' be the lowest even solution larger than p'. Clearly, $\overline{\overline{p}} \neq p''$ since the RHS of equation (24) is decreasing for p less than and close enough to p'' whereas, because of Proposition 2(vi), $p \in \bigcup\limits_{r\in\mathcal{Z}-\mathcal{N}} S_r$ on some left neighbourhood of $\overline{\overline{p}}$. Nor can it be that $\overline{p} = p'$. Under such an event, $\sum\limits_{r\in\mathcal{Z}-\mathcal{N}} \phi_r(p)k_r = \sum\limits_{r\in\mathcal{Z}-\mathcal{N}} k_r$ is larger than the RHS of equation (24) in a right neighbourhood of p'' that is part of $\bigcap\limits_{i\in\mathcal{N}} S_i$ (see Proposition 2(v)-(vi)) and therefore $\phi_j(p)$ is lower than the RHS of equation (25) (each $j \in \mathcal{N}$), but then $F(p) = F(\overline{p})$, an obvious contradiction.

(ii)-(iii) The RHS of equation (27) is a non-decreasing function that equals 0 at $\overline{p}$, also because of Remark 1, and equals $\sum\limits_{r\in\mathcal{Z}-\mathcal{N}} k_r$ at $\overline{\overline{p}}$. Whenever the RHS of equation (27) is increasing, it equals the RHS of equation (24) and the RHS of equation (28) equals the RHS of equation (25). Therefore $F(p) = F(\overline{p})$ whenever the RHS of equation (27) is increasing. Over any range $(p', p'') \subset [\overline{p}, \overline{\overline{p}}]$ in which the RHS of equation (27) is constant, it is lower than the RHS of equation (24) and the RHS of equation (28) is higher than the RHS of equation (25). Therefore $F(p) < F(\overline{p})$, consistent with the fact that $(p', p'') \cap (\cup_{r\in\mathcal{Z}-\mathcal{N}} S_r) = \emptyset$.

(iv) Since $\sum\limits_{r\in\mathcal{Z}-\mathcal{N}} \phi_r(p)k_r = \sum\limits_{r\in\mathcal{Z}-\mathcal{N}} k_r$ is lower than the RHS of equation (24) over the range $(\overline{\overline{p}}, p_M^{(n)}]$ (see the proof of part (i)), over the same range $\phi_l(p)$ is larger than the RHS of equation (25), each $l \in \mathcal{N}$, and, as a consequence, $F(p) < F(\overline{p})$. If $k_n < k_2$, so that $p_M^{(n)} < p_M$, then $F(p) = p\left[1 - \phi_1(p)\frac{p-p_m}{p}\frac{k_1}{K-D(p)}\right] < p_m < F(\overline{p})$ over the range $(p_M^{(n)}, p_M)$. The first inequality derives from $\phi_1(p) > \phi_1(p_M^{(n)}) = \frac{k_n}{k_1} > \frac{K-D(p)}{k_1}$, whereas the last inequality holds since inequality (5) is strict and Proposition 2(iii) holds.

(v) Because of Proposition 2(iii), $\overline{p} = F(\overline{p}) = p_m$. As a consequence, equation (24) can be written as equation (30) and equation (26) can be written as equation (29). The derivative of the RHS of equation (30) is positive if, and only if,

$$p_m \left(\prod_{j \in \mathcal{N}} k_j \right)^{\frac{1}{n}} + m \left(\frac{p - p_m}{p} \right)^{\frac{n-1}{n}} p^2 D'(p) > 0. \tag{31}$$

The LHS of inequality (31) is a strictly decreasing function in the range $[p_m, p_M]$ since the second addend is strictly decreasing due to the strict concavity of $pD(p)$. This is enough since the LHS of inequality (31) is by definition non-negative for $p = \overline{\overline{p}}$. Now we will prove that $\overline{\overline{p}} < p_M^{(n)} = p_M$. Because of Proposition 2(viii) we can concentrate on the case in which $k_2 = k_n$ and $K - k_n = D(p_M) > K - k_1$ (the inequality is a consequence of Proposition 1(2)). In this case the RHS of equation (30) equals $\sum_{r \in \mathcal{Z} - \mathcal{N}} k_r$ also at $p_M = p_M^{(n)}$. Nevertheless, at $p = p_M$, the LHS of inequality (31) is negative since, because of the fact that $p_M^2 D'(p_M) = -p_m k_1$, it equals $p_m k_1^{\frac{1}{n}} k_2^{\frac{n-1}{n}} - n \left(\frac{p_M - p_m}{p_M} \right)^{\frac{n-1}{n}} p_m k_1 = p_m k_1^{\frac{1}{m}} \left[k_2^{\frac{n-1}{n}} - n \left(\frac{p_M - p_m}{p_M} k_1 \right)^{\frac{n-1}{n}} \right] = p_m k_1^{\frac{1}{n}} (1 - m) \left[K - D(p_M) \right]^{\frac{n-1}{n}} < 0$, the last equality deriving since $\frac{p_M - p_m}{p_M} = \phi_1(p_M) = \frac{k_2}{k_1}$. Hence, because of quasi-concavity of the RHS, equation (29) has two solutions in the range $[\overline{p}, p_M]$: the former is $\overline{\overline{p}}$, the latter is p_M. Clearly, over the range $(\overline{\overline{p}}, p_M)$, the RHS of equation (30) is higher than $\sum_{r \in \mathcal{Z} - \mathcal{N}} k_r$. Therefore, $\Pi_r(p, \phi_{-r}) < \Pi_r^*$ over that range and $\Pi_r(p_M, \phi_{-r}) < \Pi_r^*$. ■

Remark 2. *There is a continuum of profiles of equilibrium distributions for the small firms, and this is so whether or not the RHS of equation (24) is strictly increasing over the relevant subset. The continuum of equilibria includes one in which the equilibrium distributions are the same for each small firm: at the "symmetric" equilibrium, the equilibrium distribution is*

$$\phi_r(p) = \frac{\left[\frac{p}{p - F(\overline{p})} \right]^{\frac{n-1}{n}} \frac{p - p_m}{p} \left(\prod_{j \in \mathcal{N}} k_j \right)^{\frac{1}{n}} - \left[\sum_{j \in \mathcal{N}} k_j - D(p) \right]}{\sum_{r \in \mathcal{Z} - \mathcal{N}} k_r}$$

for any $p \in \bigcup_{r \in \mathcal{Z} - \mathcal{N}} S_r$ *(each* $r \in \mathcal{Z} - \mathcal{N}$*).*

Some considerations are in order about the role played by firms $r \in \mathcal{Z} - \mathcal{N}$. Although the total capacity of these firms is fairly small, their impact on the equilibrium may well be sizeable. Simple comparative statics will help to see this point. Take the number and capacities of the small firms as an independent variable while keeping fixed the number and capacities of the large firms. Of course, mere reshuffling of capacities among the small firms would not affect the continuum of equilibria: so long as $\sum_{r \in \mathcal{Z} - \mathcal{N}} k_r$ does not change, S_i and Π_i^* (each $i \in \mathcal{N}$), $\sum_{r \in \mathcal{Z} - \mathcal{N}} \Pi_r^*$ and $\cup S_{r \in \mathcal{Z} - \mathcal{N}}$ remain unchanged. On the other hand, there is room for a significant (upward or downward) change in $\xi := \sum_{r\in\mathcal{Z}-\mathcal{N}} k_r$ that does not violate inequalities (5) and (6): any such change would have a considerable impact on the equilibria. The resulting change of the equilibrium payoff is $\Delta\Pi_1^* \approx -p_M\Delta\xi$ for the largest firm and $\Delta\Pi_i^* \approx -\frac{k_i}{k_1}p_M\Delta\xi$ for any large firm; thus, for each large firm, the proportional change in the equilibrium payoff is $\frac{\Delta\Pi_i^*}{\Pi_i^*} \approx -\frac{\Delta\xi}{D(p_M)-\sum_{j\neq 1} k_j}$, which may be far from negligible, as the example in the following subsection illustrates.

3.5 Numerical example

A numerical example will be helpful in several respects, first of all to illustrate our theoretical findings. Let $n = 5$, $D(p) = 22 - p$, $k_1 = 9.2$, $k_2 = 8.5$, $k_3 = 6$, $k_4 = 0.4, k_5 = 0.2$. Then $p_M = 3.45$, $p_m = \widehat{p} = 1.29375$, $\Pi_1^* = p_m k_1 = 11.9025$, and $\Pi_2^* = p_m k_2 = 10.996875$. Since $k_1+k_2+k_3 = 23.7 > D(p_m) = 20.70625$ and $D(p_M) = 18.55 > K - k_3 = 18.3$, then firms 1, 2, and 3 are "large" firms, consequently $\Pi_3^* = p_m k_3 = 7.7625$, and firms 4 and 5 are "small" firms. Inequality (5) is strict and hence $L = \{1, 2, 3\}$. According to equations (17), over the range $[p_m, \overline{p}]$, $\phi_1(p) = \frac{1}{9.2}\sqrt{469.2\frac{p-1.29375}{p(1.7+p)}}$, $\phi_2(p) = \frac{1}{8.5}\sqrt{469.2\frac{p-1.29375}{p(1.7+p)}}$ and $\phi_3(p) = \frac{1}{6}\sqrt{469.2\frac{p-1.29375}{p(1.7+p)}}$; hence, $G(p)k_r = p\left[1 - 21.66102489\left(\frac{p-1.29375}{p(1.7+p}\right)^{\frac{3}{2}}\right]k_r$ ($r = 3, 4$) over the range $[p_m, \widetilde{p}_M^{(3)}] = [1.29375; 1.761639635]$. Then it is found that $\arg\max_{p\in(p_m,\widetilde{p}_M^{(3)})} G(p) = 1.330357324$, implying that $F(\overline{p}) = 1.305422514$ and hence $\Pi_4^* = F(\overline{p})k_4 = 0.5221690056$ and $\Pi_5^* = F(\overline{p})k_5 = 0.2610845028$. Over the set $\cup_{r\in\{4,5\}}S_r$, $(\phi_4(p), \phi_5(p))$ is any pair of continuous and non-decreasing functions such that equation (24) holds, namely:

$$0.4\phi_4(p)+0.2\phi_5(p) = \frac{p-1.29375}{p}\left(\frac{p}{p-1.305422514}\right)^{\frac{2}{3}} 469.20^{\frac{1}{3}} - 1.7 - p \quad (32)$$

The RHS of equation (32) is strictly increasing throughout $[\overline{p}, \widetilde{p}_M^{(3)}]$, implying that $\cup_{r\in\{4,5\}}S_r = [\overline{p}, \overline{\overline{p}}]$, where $\overline{\overline{p}} = 1.423433842$, the single value of $p \in [\overline{p}, \widetilde{p}_M^{(3)}]$ such that the RHS of (32) is equal to 0.6. According to equations (25), over the range $[\overline{p}, \overline{\overline{p}}] = [1.330357324; 1.423433842]$, $\phi_1(p) = \frac{1}{9.2}\left(469.2\frac{p-1.305422514}{p}\right)^{\frac{1}{3}}$, $\phi_2(p) = \frac{1}{8.5}\left(469.2\frac{p-1.305422514}{p}\right)^{\frac{1}{3}}$, and $\phi_3(p) = \frac{1}{6}\left(469.2\frac{p-1.305422514}{p}\right)^{\frac{1}{3}}$.

Over the range $(\overline{\overline{p}}, p_M^{(3)}] = (1.423433842; 1.911346695]$, $\phi_1(p) = \frac{1}{9.2}\sqrt{\frac{p-1.29375}{p(2.3+p)}469.2}$, $\phi_2(p) = \frac{1}{8.5}\sqrt{\frac{p-1.29375}{p(2.3+p)}469.2}$ and $\phi_3(p) = \frac{1}{6}\sqrt{\frac{p-1.29375}{p(2.3+p)}469.2}$; $p_M^{(3)} = 1.911346695$. Over the remaining range $(p_M^{(3)}, p_M] = (1.911346695; 3.45]$, $\phi_1(p) = 8.5\frac{p-1.29375}{p(2.3+p)}$ and $\phi_2(p) = 9.2\frac{p-1.29375}{p(2.3+p)}$: of course, $\phi_2(p_M) = 1$ while $\phi_1(p_M) = \frac{k_2}{k_1} = \frac{8.5}{9.2} = 0.9239130437$.

A few variants of this numerical example also allow us to assess the role played by the small firms. Suppose that, other things being equal, the total capacity of the small firms decreases from 0.6 to zero. This would result in a sizeable increase in p_M, p_m, and Π_i^* (each $i \in \mathcal{N}$): Π_1^* would rise to 14.0625, meaning that Π_i^* (each $i \in \mathcal{N}$) would increase approximately by 18.15 percent. Or let total capacity of the small firms increase from 0.6 to 1.1, no matter whether their number change. Note that firms 1, 2, and 3 are still "large" firms while the remaining firms are still "small" firms in that inequalities (5) and (6) still hold. By straightforward computation it is found that the equilibrium payoff of firm 1 would now fall to 10.24, meaning a fall by approximately 13.97 percent for each large firm, compared to the initial industry structure.

4 Concluding remarks

This paper is a further contribution to the analysis of equilibria of the price game in a setting of given capacities. We in fact characterized the equilibria in a specific subset of the no-pure strategy equilibrium region of the capacity space, the subset where, according to a well-defined distinction, there are "large" firms along with "small" firms. It was found that, with an industry structure like this, the interval between the minimum price p_m and maximum price p_M being quoted in equilibria can be partitioned into three intervals, $[p_m, \overline{p})$, $[\overline{p}, \overline{\overline{p}}]$, and $(\overline{\overline{p}}, p_M]$, where $\overline{p}$ and $\overline{\overline{p}}$ are, respectively, the minimum and the maximum of the union of the supports of the small firms. The first part is empty in a limit case, whereas the other two are never so. As is the case for all firms in the "quasi-symmetric" oligopoly (De Francesco and Salvadori [7]), it was shown, first of all, that the minimum price p_m is the minimum of the support of the equilibrium strategy for any large firm and, second, that for any large firm the support is an interval which, for any large firm smaller than firm 2, is smaller than for any larger firm. We determined the equilibrium payoffs for all firms and we saw that, for firms of the same type, the equilibrium payoffs are proportional to capacities. Except in the limit case in which $\overline{p} = p_m$, the equilibrium payoff per unit of capacity is larger for the small firms and we saw that $\overline{p}$, and correspondingly the equilibrium payoff of each small firm, is the solution of a virtual maximization problem facing any small firm. Finally, although a continuum of equilibrium distributions exist for the small firms, the capacity-weighted sum of these distributions is the same at each equilibrium and hence the union of the supports of their equilibrium strategies is also the same.

To conclude, there is undoubtedly still a long way to go before the equilibria of the price game among capacity-constrained sellers across the whole region of no-pure strategy equilibria are characterized. Yet it is encouraging that such a task could be performed for the bipolarized distribution of total capacity assumed in the present paper. It seems reasonable to expect that the findings obtained - most notably, the procedure to determine the equilibrium payoff and the minimum price for the relatively small firms - may also be helpful to characterize equilibria in parts of that region that lie somewhere in between the symmetric and "quasi-symmetric" case (De Francesco and Salvadori [7]) and the bipolarized industry structure of this paper.

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