Discussing copulas with Sergey Aivazian: a memoir

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Abstract

Sergey Aivazian was the head of my department at the Moscow School of Economics, but he was much more than that. He played an important role in my life, and he contributed to my studies devoted to copula modelling. This small memoir reports how this amazingly polite and smart scientist helped me to develop my academic skills and to further stimulate my interest in multivariate modelling and risk management. Some open questions related to multivariate discrete models that were among the last topics I discussed with Sergey are reported, hoping they can be of interest to young researchers for further studies.

Keywords: Copula, multivariate models, market risk, operational risk, discrete distribution, risk management.

JEL classification: C32, C51, C53, C58, G17, G32, G33.

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1 Introduction

Sergey Artemievich Aivazian (June 24, 1934 - March 12, 2019) was a Soviet and Russian economist. He was the recipient of several awards and honors including (among the many), the prize and medal of the French National Congress of Statisticians (1986), the prize of the Council of Ministers of the USSR (1986), the medal of the European Econometric Society (1988), the title of Honored Scientist of the Russian Federation (2002), and the L.V. Kantorovich Prize (2017) for the monograph *"Quality of Life and Living Standards Analysis: An Econometric Approach"*. The last prize is awarded by the Economics Department of the Russian Academy of Sciences for outstanding work in economics and mathematical models and methods.

This small memoir wants to present the personality and work of Sergey Aivazian through the eyes of an Italian researcher like me, who settled in Moscow at the beginning of the 21st century. This memoir is organized as follows: Section 2 reports some interesting moments that I spent with Sergey, while Section 3 briefly introduces the topic of copulas, which was the main topic I discussed with him during the time we worked together. Section 4 describes in more detail the multivariate modelling of operational risks with copulas, which was one of the last topics I discussed with Sergey and which still has some open questions that can be of interest to young researchers. Section 5 briefly concludes.

2 Sergey and me

One of the most vivid memories that I have of Sergey Aivazian was our first meeting at the Central Economic Mathematical Institute (CEMI) in Moscow in August 2007 (the CEMI is an economic research institute of the Russian Academy of Sciences which focuses on econometrics, economic theory and mathematical economics). His politeness and gentle methods immediately impressed me: he knew that I arrived in Moscow for the first time in my life and I needed some time to settle in. He offered me immediately a pretty large assortment of biscuits and sweets together with black tea: this tradition would have characterized our meetings for years to come.

Sergey Aivazian was like a grandfather-like figure for me and he accompanied me in several steps of my professional life: from writing several articles about copulas, to our joint textbook mainly dedicated to econometric methods for finance (Aivazian, S. and Fantazzini, D. (2014), *Methods of Econometrics, Vol 2: Advanced Advanced course with applications in Finance*, Master, Infra-M, [in Russian]), to my Candidate of Science in Economics, till the preparation of my defense as Doctor of Science in Economics: the Candidate of Science is the first of two doctoral-level scientific degrees in Russia, while the second and highest doctoral degree is the title of Doctor of Science.

I always admired the way Sergey communicated with younger colleagues. He was a living legend in econometrics; however, he was always open to new ideas and new people. He treated me with respect and attention. The difference in age and the difference in titles between us seemed to disappear the moment we took our tea.

Probably, the best episode that characterized him as a man for me was the speech he gave at my wedding party on 17/05/2008. At the time of the toasts for the newlyweds, the party organizer gave him the microphone and presented him as the head of my department and as my "boss". He then replied: "*there are no bosses here, we are both men of science*". We were colleagues in the best meaning of this word: two men interested in developing the same topic for mutual benefit and -hopefully- to the benefit of econometrics and our students.

3 Copulas (or Copulae?)

The topic of multivariate modelling with copulas was by far the main topic I discussed with Sergey during the time we worked together. I still remember, as if it happened yesterday, the first time when he asked me which plural form we should use with the term copula: "copulas" or "copulae"? The first one is the regular English plural, while the latter is the irregular Latin plural for the nominative case. Both of them are correct in the English language, but "copulae" is usually considered the most professional-looking, and a sign of respect for the Latin language. So, how come that copulae became such an intensive field of collaboration with Sergey Aivazian, which culminated with the publication of our joint textbook in 2014?

Multivariate statistical analysis was one of Sergey's main research interests and he taught and mentored several generations of specialists in multivariate statistical analysis and econometrics, see for example https://www.hse.ru/en/org/persons/314460253/. Moreover, he organized and supervised the famous weekly seminar in "*Multivariate Statistical Analysis and Probabilistic Modeling of Real Processes*", which began its work at the CEMI in March 1969 and functioned continuously every Wednesday during the winter and spring semesters (http://www.cemi.rssi.ru/activity/seminars/index.php#2). Therefore, it should not come as a surprise that he got very interested when I started discussing with him my research work with copulas. In this regard, it is important to remark that the evidence of lack of multivariate normality for the joint distribution of many economic and financial variables has been one of the main drivers behind the development of copula theory. For example, evidence that economic variables are non-normal has been widely reported and discussed as far back as Mills (1927), while the most reported deviations are excess kurtosis and skewness in univariate distributions, as well as asymmetric dependence, see e.g. Patton (2006), Fantazzini (2008), Fantazzini (2010) and references therein.

The theory of copulas dates back to Hoeffding (1940) and Sklar (1959), but its application in statistical modelling is far more recent: see Joe (1997) and Nelsen (1999) for an introduction to copula theory, while Cherubini et al. (2004), Aivazian and Fantazzini (2014), and Fantazzini (2019) provide a discussion of copula techniques for financial applications.

What is a copula? An n-dimensional copula is a multivariate cumulative distribution function with uniform distributed margins in [0,1]. Particularly important is the ***Sklar's theorem*** (1959):

Let H denote a n-dimensional distribution function with margins $F_1, \ldots F_n$. Then, there exists a n-copula C such that for all real $(x_1, \ldots, x_n)$, we have that $H(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n))$. If all the margins are continuous, then the copula is unique; otherwise C is uniquely determined on $RanF_1 \times RanF_2 \ldots \times RanF_n$, where Ran is the range of the marginals. Conversely, if C is a copula and $F_1, \ldots F_n$ are distribution functions, then the function H defined above is a joint distribution function with margins $F_1, \ldots F_n$.

Proof: The proof of the Sklar's theorem was not given in Sklar (1959), but a sketch of it was provided in Sklar (1973), and finally showed in details by Schweizer and Sklar (1974). See also Joe (1997), Nelsen (1999), and Durante and Sempi (2015).

The Sklar's theorem implies that we can join together any $n \geq 2$ univariate distributions, of any type (not necessarily from the same family), with any copula to get a valid bivariate or multivariate distribution. Copulas allow to break the distribution of a (continuous) random vector into individual components (the marginals) with a dependence structure among them modelled by a copula, without losing any information. Needless to say, this decomposition considerably simplifies the estimation of a multivariate model. Moreover, we can use this theorem to extract copulae from well known multivariate distributions: for example, the Normal copula from the multivariate Normal distribution, the Student's t copula from the multivariate Student's t and so on, see Aivaizian and Fantazzini (2014) for a detailed discussion.

4 Copulae and Operational risks

The modelling of operational risks with copulas was one of the last topics that I discussed with Sergey and it was examined in the penultimate section (7.3) of our joint textbook published in 2014. This theme has become important following the development of the Basel II and then the Basel III accords, which are recommendations on bank capital adequacy issued by the Basel Committee on Banking Supervision. The latest reforms introduced with Basel III want to simplify the Basel II framework by proposing a single "Standardised Measurement Approach" (SMA) to assess operational risk. Such an approach combines a refined measure of gross income with the bank's own internal loss history over the last 10 years. Moreover, it allows the bank to consider net losses after recoveries and insurance, see e.g. Chernobai et al. (2007), Ramirez (2017), Basel Committee (2017) and Akkizidis and Kalyvas (2018) for more details. I provide below a brief introduction to the multivariate modelling of operational risks and then present some open questions that I was discussing with Sergey before his illness, unfortunately, took over.

4.1 A brief review of the theory

The term "operational risks" is used to define all financial risks that are not classified as market or credit risks. They may include all losses due to human errors, technical or procedural problems, etc. One of the most common classes of models for operational risks is the Loss Distribution Approach (LDA), see Chapelle (2019) and Naim and Condamin (2019) for a discussion at the textbook level. This approach employs a distribution to describe the frequency of the risky events, and another disctribution to describe the severity of the losses. Formally, for each type of risk $i = 1, ..., R$ and for a given time period, operational losses can be defined as a sum (S_i) of the random number (n_i) of the losses (X_{ij}):

$$S_i = X_{i1} + X_{i2} + \ldots + X_{in_i}$$

A widespread statistical model within the LDA class of models is the *actuarial model*, where the probability distribution of S_i is defined as $F_i(S_i) = F_i(n_i) \cdot F_i(X_{ij})$, where $F_i(S_i)$ is the probability distribution of the expected loss for risk i, $F_i(n_i)$ is the probability of the event (frequency) for risk i, while $F_i(X_{ij})$ is the loss given the event (severity) for risk i, where $j = 1, \ldots, n_i$. The actuarial model assumes that the losses are random variables, independent and identically distributed (i.i.d.), and the distribution of n_i (frequency) is independent of the distribution of X_{ij}. I want to remark that in the most general case, the Basel II accord divides banks' activities into a matrix of eight business lines (BLs) and seven event types (ETs), for a total of $R = 56$ BLs/ETs risk combinations, see Basel Committee (2002), Chernobai et al. (2007) and Karam and Planchet (2012) for more details.

In general, the frequency is modelled with a Poisson or a Negative Binomial distribution, while the severity is modelled with an Exponential or a Pareto or a Gamma distribution, or using the lognormal for the body of the distribution and the Extreme Value Theory (EVT) approach for the tail, see Kudrov

(2008) and Fantazzini and Kudrov (2010). The distribution F_i of the losses S_i for each intersection i among business lines and event types is then obtained by the convolution of the frequency and severity distributions: given that the analytic representation of this distribution is computationally difficult or impossible, Monte Carlo methods are usually employed.

Once the risk measures for each loss S_i are estimated -either the Value at Risk (VaR) or the Expected Shortfall (ES)-, the global risk measure is then usually computed as the simple sum of these individual measures, thus assuming a *perfect dependence among the different losses* S_i. If we use the Sklar's theorem (1959) and the Frechet-Hoeffding bounds, the multivariate distribution among the R losses at time t would be given by

$$H(S_{1t}, \ldots, S_{R,t}) = \min\left(F_1(S_{1,t}), \ldots, F_R(S_{R,t})\right)$$

where H is the joint distribution of the vector of losses S_{it}, $i = 1, \ldots, R$, and $F_i(\cdot)$ are the cumulative distribution functions of the losses' marginals. Needless to say, such an assumption in quite unrealistic. Fantazzini et al. (2008) proposed to use *copulas to model the dependence among operational risk losses*: by using again the Sklar's Theorem, the joint distribution H of a vector of losses S_{it}, $i = 1, \ldots, R$ can be expressed simply as the copula of the cumulative distribution functions of the losses' marginals:

$$H(S_{1t}, \ldots, S_{R,t}) = C(F_1(S_{1,t}), \ldots, F_R(S_{R,t}))$$

Monte Carlo methods are again used to compute the required total capital for operational risk. This approach is also known as the *canonical aggregation model via copulas*.

Lindskog and McNeil (2003), Embrechts and Puccetti (2008) and Rachedi and Fantazzini (2009) proposed a different aggregation model (known as the Poisson shock model) where the dependence is modelled among severities and among frequencies using Poisson processes. Suppose there are m different types of shock or event and, for $e = 1, \ldots, m$, let n_t^e be a Poisson process with intensity λ^e recording the number of events of type e occurring in $(0, t]$. Assume further that these shock counting processes are independent. Consider losses of R different types and, for $i = 1, \ldots, R$, let n_{it} be a counting process that records the frequency of losses of the i^{th} type occurring in $(0, t]$. At the r^{th} occurrence of an event of type e the Bernoulli variable $I_{i,r}^e$ indicates whether a loss of type i occurs. The vectors

$$\mathbf{I}_r^e = (I_{1,r}^e, \ldots, I_{R,r}^e)' \quad \text{for} \quad r = 1, \ldots, n_t^e$$

are considered to be independent and identically distributed with a multivariate Bernoulli distribution. According to the Poisson shock model, the loss processes $n_{it}, i = 1, \ldots, R$, are clearly Poisson themselves, since they are obtained by superpositioning m independent Poisson processes generated by the m underlying event processes. Therefore, $(n_{1t}, \ldots, n_{Rt})$ can be thought of as having a multivariate Poisson distribution. However, it follows that the total number of losses is not itself a Poisson process, but rather a compound Poisson process:

$$n_t = \sum_{e=1}^{m} \sum_{r=1}^{n_t^e} \sum_{i=1}^{R} I_{i,r}^e$$

These shocks cause a certain number of losses in the i-th BL/ET, whose severity is (X_{ir}^e), $r = 1, \ldots, n_t^e$, where (X_{ir}^e) are i.i.d. with distribution function F_{it} and independent with respect to n_t^e. As it may appear immediately from the previous discussion, the key point of this approach is to identify the underlying m Poisson processes: unfortunately, this field of studies is quite recent and more research has to be made with this regard. Moreover, the paucity of data limits any precise identification. A simple approach is to identify the m processes with the R risky intersections (Business Lines or Event Types or both), so that we are back to the standard framework of the LDA approach. This is the "*soft-model*" proposed in Embrechts and Puccetti (2008) and later applied to a real dataset by Rachedi and Fantazzini (2009). Embrechts and Puccetti (2008) and Rachedi and Fantazzini (2009) allow for positive/negative dependence among the shocks (n_{it}) and also among the loss severities (X_{it}) using copulas, but the number of shocks and loss severities are independent to each other:

$$\begin{aligned} H^{frequency}(n_{1t}, \ldots, n_{Rt}) &= C^{frequency}(F_1(n_{1t}), \ldots, F_R(n_{Rt})) \\ H^{severity}(X_{1j}, \ldots, X_{Rj}) &= C^{severity}(G_1(X_{1j}), \ldots, G_R(X_{Rj})) \\ H^{frequency} &\perp H^{severity} \end{aligned}$$

where $H^{frequency}(\cdot)$ is the joint distribution function of the random vector $(n_{1t}, \ldots, n_{Rt})$, $F_i(\cdot)$ is the cumulative distribution function of the random variable n_{it}, while $C^{frequency}(\cdot)$ is the copula of the multivariate distribution function $H^{frequency}(\cdot)$; $H^{severity}(\cdot)$ is the joint distribution function of the random vector $(X_{1,j}, \ldots, X_{R,j})$, $G_i(\cdot)$ is the cumulative distribution function of the random variable X_{ij}, and $C^{severity}(\cdot)$ is the copula of the multivariate distribution function $H^{severity}(\cdot)$. Usual risk measures such as the VaR and ES can then be computed using simulation methods, see Rachedi and Fantazzini (2009), Aivazian and Fantazzini (2014) for more details.

4.2 Open questions

The development of multivariate models with discrete marginals poses serious problems and it was a topic that I discussed with Sergey in some occasions, particularly when preparing the section of our textbook dedicated to the Poisson shock model. For example, during one of our meetings in Armenia for the traditional summer conference in "Multivariate statistical analysis and econometrics", Sergey told me that the knowledge that a multivariate discrete distribution does not possess a unique copula representation was also known in the Russian statistical literature before Marshall (1996), which is the first publication in the English literature discussing this topic: see e.g. Blagoveschensky (2012 p. 114-115) who mentions the article titled "Classification and visualization algorithms based on quantile analysis", published in 1989 in the journal *Computer software, BIM-M Application Program Library*, Issue 20, p. 60-76, Minsk (the author want to thank the Guest Editor for pointing him this reference). I report below some open questions related to multivariate discrete models in general, and Poisson shock models in particular, that can be of interest to young researchers for further studies.

Issue 1: Estimating copulas with discrete marginals by Maximum Likelihood (ML).
It is well known that if the marginal distribution functions are all continuous then the copula C is unique, while this is not true when the marginal distributions are discrete: in this case, the copula is only uniquely identified on $\bigotimes_{i=1}^{K} \text{Range}(F_i)$, a K-dimensional set, which is the Cartesian product of the range of all marginals. As Genest and Neslehova (2007) said, "*despite the unidentifiability issue, copula models for discrete distributions are valid constructions. They are helpful, e.g., in the context of simulation and robustness studies*". However, when we work with discrete distributions, the Probability Integral Transformation Theorem (PITT) of Fisher (1932) does not apply, and the uniformity assumption does not hold, regardless of the quality of the specification of the marginal model. As a consequence, the model estimates obtained with maximum likelihood are no more consistent, see Genest and Neslehova (2007), Heinen and Rengifo (2007), and Trivedi and Zimmer (2017) for a detailed discussion. Let's see an example:

$$Example: \begin{cases} \text{Copula:} & \text{Bivariate T-copula with} \quad \rho = -0.5, \ \nu = 3 \\ \text{Marginals:} & Poisson_1(\lambda_1 = 1.5) \\ & Poisson_2(\lambda_2 = 3) \\ T: & 100000 \end{cases}$$

$\Rightarrow \hat{\rho} = \mathbf{-0.45}, \ \hat{\nu} = \mathbf{17}$

$\Rightarrow \hat{\lambda}_1 = 1.50, \ \hat{\lambda}_2 = 3.00$

The 2-step ML estimation provides consistent estimates for the marginals, but the dependence parameters are misspecified: the estimated t-copula is quite close to a normal copula.

Issue 2: Does the "continuous extension" of discrete marginals with uniform marginals help the ML estimation?

Stevens (1950), Denuit and Lambert (2005), Heinen and Rengifo (2007), and Trivedi and Zimmer (2007) proposed the continuous extension of discrete random variables to overcome the previous misspecification problem. More specifically, they proposed to generate artificially continued variables $X^* = X + (U - 1)$, by adding $Uniform[0,1]$ random variables to the discrete variables X with domain $\mathbf{X}$. Note that this method does not change the concordance measure between the variables. Moreover, they state a discrete analog of the PITT, by showing that

$$F^*(s) = Pr(X^* \leq s) = \sum_{x \in \mathbb{X}: x \leq [s]} f_x + (s - [s]) f_{[s+1]} = F([s]) + (s - [s]) f_{[s+1]}$$

is uniformly distributed on $[0, 1]$, where $[s]$ is the integer part of $s \in \mathbb{R}$, and $f_x = Pr(X = x)$, $x \in \mathbf{X}$. Does it work? Let's continue to use the previous numerical example:

$$\begin{array}{ll} & \left\{ \begin{array}{lll} \text{Copula:} & \text{Bivariate T-copula with} & \rho = -0.5, \ \nu = 3 \\ \text{Marginals :} & Poisson_1(\lambda_1 = 1.5) & \\ & Poisson_2(\lambda_2 = 3) & \\ T: & 100000 & \end{array} \right. \end{array}$$

Example (continued) :

$\Rightarrow \hat{\rho} = \mathbf{-0.41}$, $\hat{\nu} = \mathbf{6}$

$\Rightarrow \hat{\lambda}_1 = 1.50$, $\hat{\lambda}_2 = 3.00$

There is indeed an improvement, particularly for the degrees of freedom coefficient (6 is closer to 3 than 17). However, the dependence structure is still misspecified.

Issue 3: Dealing with zero losses when computing the dependence structure of severities in the Poisson shock model.

Suppose to observe the following severities in four Business Lines:

1st year	BL_1	BL_2	BL_3	BL_4
January	20	100	13	77
February	10	134	18	56
March	0	98	0	87
April	0	0	0	101
		...		
2nd year	BL_1	BL_2	BL_3	BL_4
January	15	92	13	57
February	27	122	18	66
March	4	0	23	99
April	0	0	0	71
		...		
nth year	BL_1	BL_2	BL_3	BL_4
		...		

If we want to measure the dependence among the four severities, that is $C^{Severity}\left(F(X_{1,t}), \ldots, F(X_{4,t})\right)$, we have to consider only the rows where all $X_{i,t}$ are different from zero,

1st year	BL_1	BL_2	BL_3	BL_4
January	**20**	**100**	**13**	**77**
February	**10**	**134**	**18**	**56**
March	0	98	0	87
April	0	0	0	101
		...		
2nd year	BL_1	BL_2	BL_3	BL_4
January	**15**	**92**	**13**	**57**
February	**27**	**122**	**18**	**66**
March	4	0	23	99
April	0	0	0	71
		...		
nth year	BL_1	BL_2	BL_3	BL_4
		...		

while we have to remove the remaining ones where zeros are present. If we do not exclude the zeros, the dependence is underestimated (and the marginals are misspecified as well). Let' s see a small bivariate example:

$$
Example: \begin{cases} \text{Frequency:} & n_{1,t} \sim Poisson_1(\lambda_1 = 1.5) \quad \perp \quad n_{2,t} \sim Poisson_2(\lambda_2 = 3) \\ \hline \text{Severity:} & \text{Bivariate T-copula with} \quad \rho = -0.5, \;\; \nu = 3 \\ & \text{Marginals} : \Gamma_1(0.2, 90000), \quad \Gamma_2(0.2, 90000) \\ \hline T: & 100000 \end{cases}
$$

$\Rrightarrow \hat{\rho} = \mathbf{-0.32}$, $\hat{\nu} = \mathbf{31}$

$\Rrightarrow \widehat{\boldsymbol{\Gamma}}_{\mathbf{1}}(\mathbf{0.08, 98823})$, $\widehat{\boldsymbol{\Gamma}}_{\mathbf{2}}(\mathbf{0.18, 92997})$

Instead, if we remove the zeros everything is fine:

$$
Example: \begin{cases} \text{Frequency:} & n_{1,t} \sim Poisson_1(\lambda_1 = 1.5) \quad \perp \quad n_{2,t} \sim Poisson_2(\lambda_2 = 3) \\ \hline \text{Severity:} & \text{Bivariate T-copula with} \quad \rho = -0.5, \;\; \nu = 3 \\ & \text{Marginals} : \Gamma_1(0.2, 90000), \quad \Gamma_2(0.2, 90000) \\ \hline T: & 100000 \end{cases}
$$

$\Rrightarrow \hat{\rho} = \mathbf{-0.50}$, $\hat{\nu} = \mathbf{3}$

$\Rrightarrow \widehat{\boldsymbol{\Gamma}}_{\mathbf{1}}(\mathbf{0.20, 90275})$, $\widehat{\boldsymbol{\Gamma}}_{\mathbf{2}}(\mathbf{0.20, 90804})$

Everything solved? Well, not really. Unfortunately, already with only 8 operational risk BLs, it is not very common to have 8 severities different from zero at the same time t, particularly for small-medium financial institutions. If we consider the whole 56 BLs/ETs used for operational risks, then this situation is close to impossible (for time frequencies higher or equal to monthly observations). A potential solution could be to compute the dependencies pairwise and then putting them together, see Fantazzini (2010) and references therein for more details. Such a procedure may not lead to a positive definite correlation matrix in elliptical copulas, and the eigenvalue method by Rousseeuw et al. (1993) would have to be used. Fantazzini (2010) found that the effects of such a method on the coverage rates and the parameters estimates in the case of a T-copula with one eigenvalue of the correlation matrix close to zero or negative are rather limited. However, the previous solution with very scarce data like in operational risk datasets may lead to several negative eigenvalues in the correlation matrix. More research work would definitely be needed in this regard.

The previous discussion clarifies why I mainly suggested using the loss distribution approach with comonotonic losses or (better) with the canonical aggregation model via copulas, whereas I suggested to handle the Poisson shock model with care: the latter model may deliver underestimated risk measures due to poor estimates of the distribution tails.

5 Conclusions

One year has passed from the death of Sergey Aivazian and his presence is sorely missed. This memoir wanted to recount his life from a different perspective, reporting some interesting moments that the author spent with Sergey. Moreover, the general topic of copulae and some open questions related to the multivariate modelling of operational risks were discussed, with the hope to inspire some young researchers to deal with the increasingly important and fascinating topic of multivariate statistical analysis and econometrics.

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